The Domino Problem: Raiders of the lost proof

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What

- A proof of the undecidability of the domino problem
- To prove the undecidability of a very specific fragment of $\forall\exists\forall$
- Where ?
  Harry Lewis, *Unsolvables Classes of Quantificational Formulas*.
  Stal O. Aanderaa and Harry R. Lewis. *Linear Sampling and the $\forall\exists\forall$ Case of the Decision Problem*
  (Wild Guess) in Aanderaa's PhD Thesis.

Warning

- A sketch of the proof
- From a superficial reading of the book
- No technical details
- I cannot answer any technical questions.

(Informal) Abstract Tiling Problem

- A concept of configuration
- A concept of constraints
- A tiling is a configuration which respect the constraints

Problem

Can we decide, given the constraints, if a tiling exists ?

Origin-constrained domino problem

- A configuration is a coloring of the quarter plane $\mathbb{N}^2$ with colors in Q

Constraints

- Two vertically-aligned consecutive cells must be in $V \subset Q^2$
- Two horizontally-aligned consecutive cells must be in $H \subset Q^2$
- The cell at position $(0, 0)$ is $q_0 \in Q$

Theorem

The origin-constrained domino problem is undecidable

Proof: (easy) reduction from the Halting problem

Domino problem

- A configuration is a coloring of the plane $\mathbb{Z}^2$ with colors in Q

Constraints

- Two vertically-aligned consecutive cells must be in $V \subset Q^2$
- Two horizontally-aligned consecutive cells must be in $H \subset Q^2$

Plan

1-System

b	a	a	a	b	c	c	a	b	b	a
a	b	a	c	b	a	b	c	a	a	c

Definition

A configuration is a pair (x, x) of colorings of $\mathbb{Z}$ with colors in Q.

1-System

b	a	a	a	b	c	c	a	b	b	a
a	b	a	c	b	a	b	c	a	a	c

Definition

A *local constraint* is a subset L of Q^k for some k.
The configuration (x, x) satisfy L if

$$\forall i\ (x_i, x_{i+1}, \dots x_{i+k}) \in L$$

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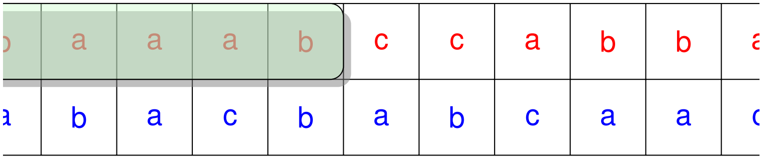
1-System



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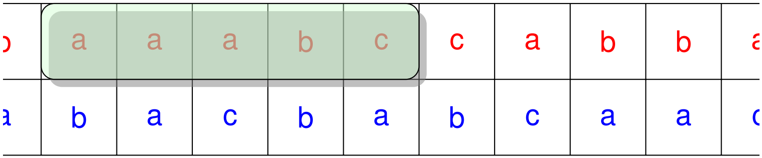

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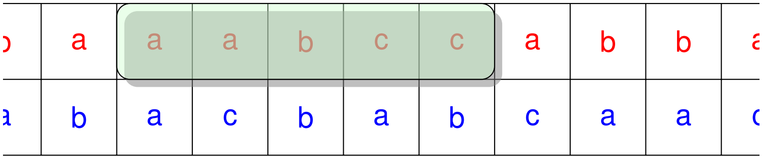

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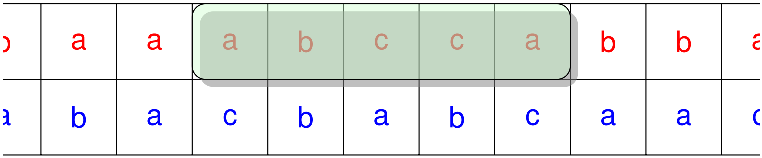

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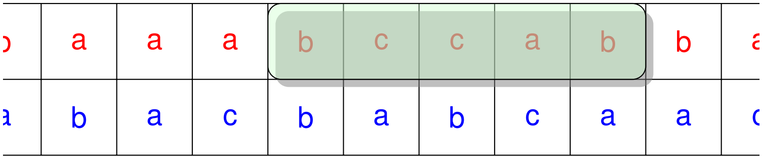

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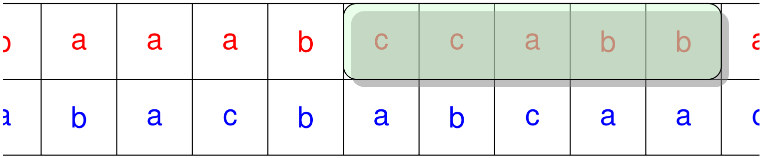

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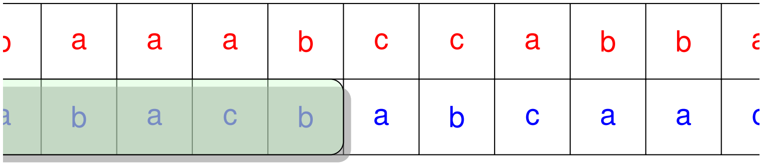

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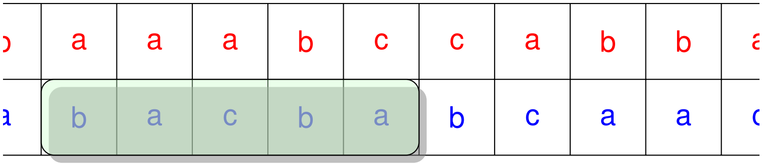

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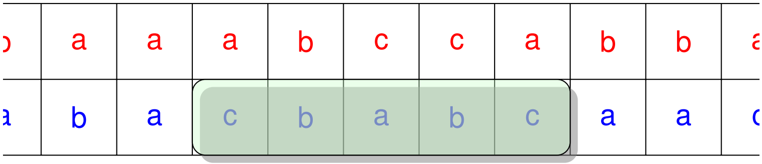

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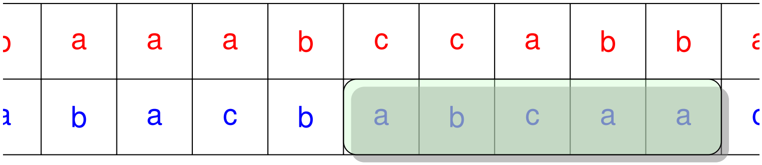

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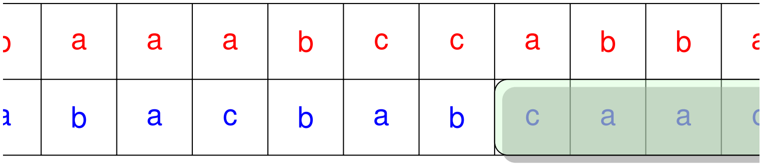


Definition

A *local constraint* is a subset L of Q^k for some k.
The configuration $({\color{red}x}, {\color{blue}x})$ satisfy L if

$$\forall i\ ({\color{red}x_i, x_{i+1}, \dots x_{i+k}}) \in L$$

$$\forall i\ ({\color{blue}x_i, x_{i+1}, \dots x_{i+k}}) \in L$$

1-System

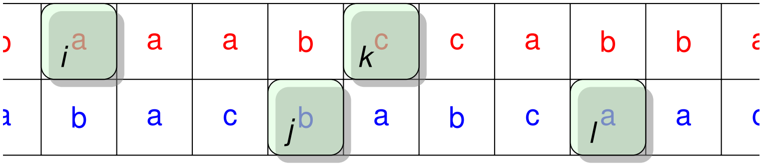
a	a	a	b	c	c	a	b	b	
b	a	c	b	a	b	c	a	a	

Definition

A *global constraint* is a subset G of Q^4.
The configuration (x, x) satisfy L if

$$\forall i, j, k, l,\ (j - i = k - l) \implies (x_i, x_j, x_k, x_l) \in G$$

1-System



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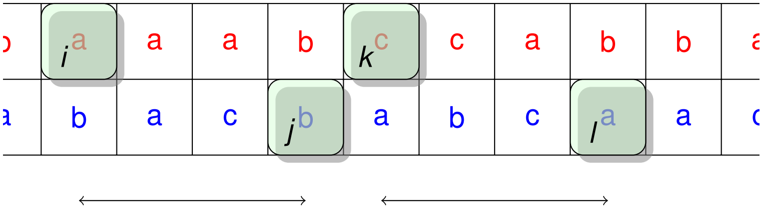

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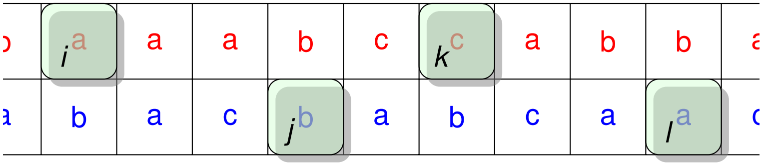

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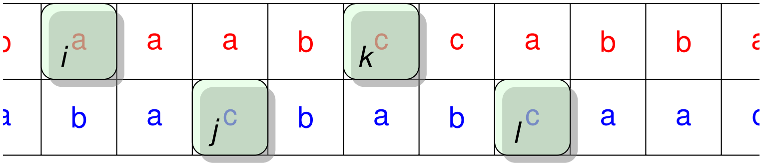

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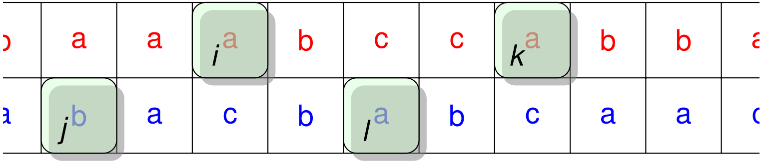

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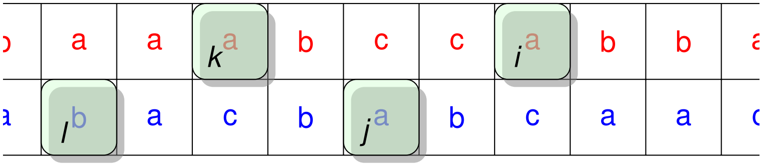

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1-System

a	a	a	b	c	c	a	b	b	a
b	a	c	b	a	b	c	a	a	c

Definition

A 1-System is given by a local constraint $L \subset Q^k$ and a global constraint $G \subset Q^4$.
A *tiling* is a configuration that satisfy both the local and global constraint.

Properties

- (x, x) is a tiling iff $(\sigma(x), x)$ is a tiling.
- The limit of a sequence of tilings is again a tiling.
- For each $d \in \mathbb{Z}$, let

$$P_d = \left\{(x_i, x_j) \middle| i - j = d\right\}$$

Then (x, x) is a tiling if and only if it verifies the local condition and

$$\forall (a, b), (c, d) \in P_d, (a, b, c, d) \in G$$

1-System tiling problem

Problem

Given a 1-System (L, G), is there a tiling ?

Note: if tilability implies that there exists always a periodic tiling, the problem is decidable.

Challenge

Find a 1-System with only aperiodic tilings.

(We will construct one in the next slides, but the proof is awful)

1-System tiling problem

Theorem

The 1-System tiling problem is undecidable

Challenge

Find an easier proof.

(We will give one in the next slides, but the proof is awful)

The Coding (Part 1)

- Let p be sufficiently large ($p > 10$).
- Associate to each i, the two first non-nul bits when we write i in base p. Call this $c(i)$. $c(0)$ is undefined.

Here $p = 3$.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$c(i)$	01	02	01	11	12	02	21	22	01	01	02	11	11	12

The Coding (Part 1)

Fix a number p.

Theorem

There exists a 1-System (G, L) in which every tiling (x, x) is roughly up to translation of the form

$$x_i = c(i)$$

$$x_i = c(i)$$

The alphabet is therefore $\{1 \leq x < p^2 | x \mod p \neq 0\}$.
Proof:

- Define G and L as what appears for c.
- Define “roughly”
- Prove.

Proof of the undecidability

Start from the Origin-Constrained Domino-Problem. We encode a tiling of the quarter-plane as follows

- Divide each tape (x, x) into three parts (a, b, d, a, b, d). Each tape has the same behaviour.
- a contains c_3
- b contains c_4
- d contains dominoes: Let $n \in \mathbb{Z}$. Let i (resp. j) be the largest power of 3 (resp. 4) divides n. Then d_n contains the tile at position (i, j).

Proof of the undecidability

$t_{(i,j)}$ represent the tile at coordinate (i, j) of the quarterplane.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a	01	02	01	11	12	02	21	22	01	01	02	11	11	12	12	21
b	01	02	03	01	11	12	13	02	21	22	23	03	31	32	33	01
d	$t_{(0,0)}$	$t_{(0,0)}$	$t_{(1,0)}$	$t_{(0,1)}$	$t_{(0,0)}$	$t_{(1,0)}$	$t_{(0,0)}$	$t_{(0,1)}$	$t_{(2,0)}$	$t_{(0,0)}$	$t_{(0,0)}$	$t_{(1,1)}$	$t_{(0,0)}$	$t_{(0,0)}$	$t_{(1,0)}$	$t_{(0,2}$

Goal: Supposing a, b, a, b are correct (using some constraints), how can we enforce d (using more constraints) ?

Proof of the undecidability

To finish the encoding, we need to be able to encode the constraints of the tilings.

- If n represents the cell $(0,0)$, then $d_n = q_0$
- If n and n' represents the same cell of the quarterplane, then they must have the same tile : $d_n = d'_n$
- If n represents the cell at the north of n' then $(d_n, d_{n'}) \in V$
- If n represents the cell at the east of n' then $(d_n, d_{n'}) \in V$

Proof of the undecidability: The origin

x Decide if a given cell n represent the tile at the origin, that is decide if neither 3 nor 4 divides n.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$c(i)$	01	02	01	11	12	02	21	22	01	01	02	11	11	12

Proof of the undecidability (cont'd)

If n and n' represents the same tile, then they indeed contain the same tile. This is ruled with the following global rule:

$$\text{If } \begin{cases} a_j - a_i = a_l - a_k \mod 3 \\ a_j \neq a_i \mod 3 \\ a_j \neq a_l \mod 3 \\ a_i \neq a_l \mod 3 \\ a_j \neq a_k \mod 3 \\ b_j - b_i = b_l - b_k \mod 4 \\ b_j \neq b_i \mod 4 \\ b_j \neq b_l \mod 4 \\ b_i \neq b_l \mod 4 \\ b_j \neq b_k \mod 4 \end{cases} \text{ then } d_i = d_k$$

Proof of the undecidability (cont'd)

- The horizontal and vertical constraints
- Not given
- Abracadabra, the proof is finished.

Plan

2-System

b	a	a	a	b	c	c	a	b	b	a
a	b	a	c	b	a	b	c	a	a	c
b	b	c	c	a	c	b	a	b	c	a

Definition

A configuration is a tuple (x, x, x) of colorings of $\mathbb{Z}$ with colors in Q.

2-System

a	a	a	b	c	c	a	b	b	
b	a	c	b	a	b	c	a	a	
b	c	c	a	c	b	a	b	c	

Definition

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2-System

b	a	a	a	b	c	c	a	b	b	a
a	b	a	c	b	a	b	c	a	a	c
b	b	c	c	a	c	b	a	b	c	a

Definition

A *global constraint* is a subset G of Q^3.
The configuration (x, x, x) satisfy L if

$$\forall i, j, k,\ (j - i = k) \implies (x_i, x_j, x_k) \in G$$

This definition is origin-dependent !

2-System

a	a	a	b	c	c	a	b	b	
b	a	c	b	a	b	c	a	a	
b	c	c	a	c	b	a	b	c	

Definition

A 2-System is given by a local constraint $L \subset Q^k$ and a global constraint $G \subset Q^3$.
A *tiling* is a configuration that satisfy both the local and global constraint.

Properties

- (x, x, x) is a tiling iff $(\sigma(x), \sigma(x), x)$ is a tiling.
- (x, x, x) is a tiling iff $(x, \sigma(x), \sigma(x))$ is a tiling.
- The limit of a sequence of tilings is again a tiling.

2-System tiling problem

Theorem

The 2-System tiling problem is undecidable

Reduction from the 1-System tiling problem.
For a 1-system:
For each $d \in \mathbb{Z}$, let

$$P_d = \left\{ (x_i, x_j) \middle| i - j = d \right\}$$

Then (x, x) is a tiling if and only if it verifies the local condition and

$$\forall (a, b), (c, d) \in P_d, (a, b, c, d) \in G$$

Idea: The third tape will contain P_d.

Starting from a 1-System (L, G) on a alphabet Q,consider the 2-System (L', G') where:

$$Q' = Q \cup \mathcal{P}(Q \times Q)$$

Local condition

$(x_1 \dots x_k) \in L'$ if

- either $x_i \in Q$ for all i and $(x_1 \dots x_k) \in L$ (two first tapes)
- or $x_i \in \mathcal{P}(Q \times Q)$ for all i. (the last tape)

Global condition

$(x_1, x_2, x_3) \in G'$ if

- $x_1, x_2 \in Q$
- $x_3 \in \mathcal{P}(Q \times Q)$
- $(x_1, x_2) \in x_3$
- $\forall (a, b), (c, d) \in x_3, (a, b, c, d) \in G$

2-System tiling problem

Theorem

The 1-System (L, G) has a solution if and only if the 2-System (L', G') has a solution.

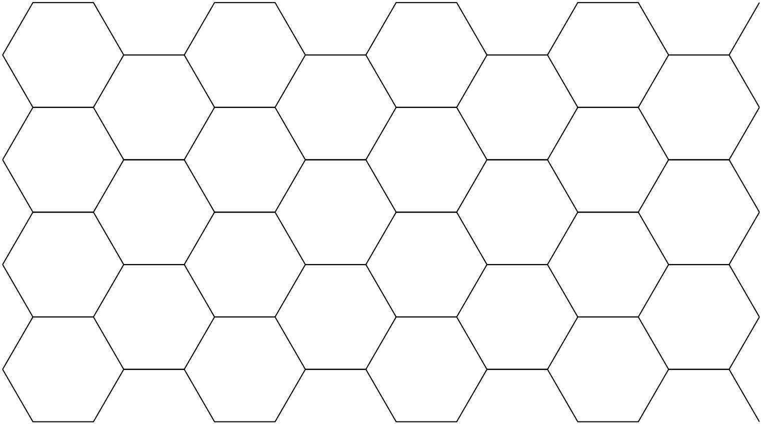
Easy.

Plan

Hex grid tiling problem

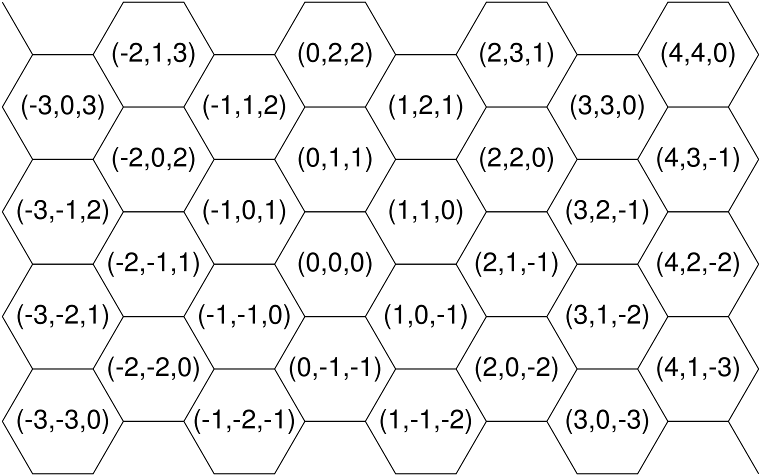
Problem

Given some tiles, and local constraints on a hex grid, is there a tiling of the whole infinite grid ?



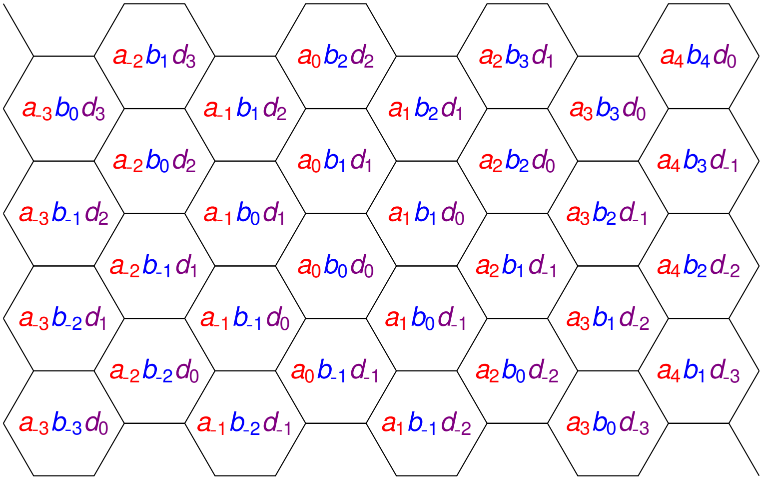
Hex grid tiling problem

We can give to each cell a coordinate (i, j, k) with $j - i = k$



Hex grid tiling problem

In the cell (i, j, k) encode $a_i b_j d_k$



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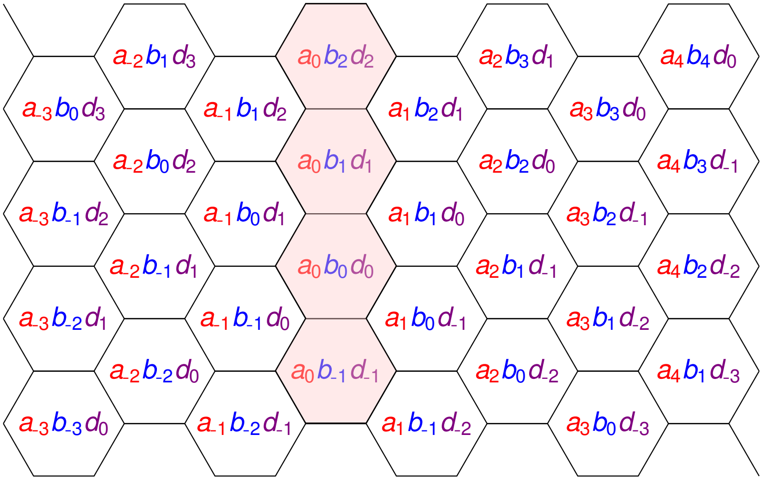
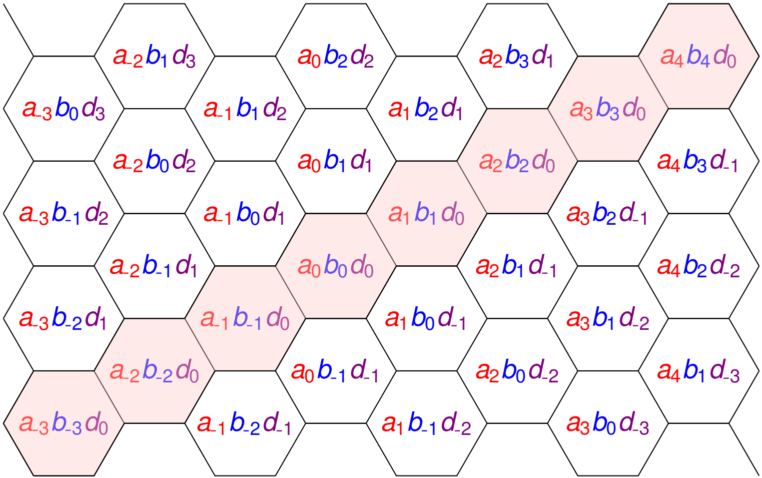


Hex grid tiling problem

In the cell (i, j, k) encode $a_i b_j d_k$



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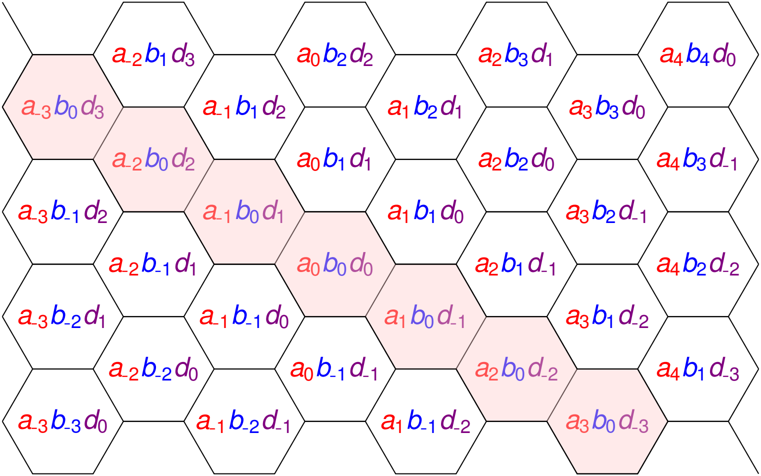

The encoding

The encoding of a configuration (a, b, d) of a 2-System into a tiling of the hex grid is sound. Using the local constraints, we may ensure that

- Two different cells (i, j, k) and (i, j', k') agree on the value of a_i
- The local constraints L of the 1-System is verified
- The global constraint G of the 1-System is verified

Corollary

The Tiling Problem for the hex Grid is undecidable.

The Domino Problem

Corollary

The Domino Problem is undecidable.