Symbolic dynamics as a categorical notion

E. Jeandel

Université de Lorraine, France
Plan

1. Introduction
2. The simplification
3. Categories
4. Examples
5. Conclusion
Main open problem of symbolic dynamics:

Decide if two subshifts of finite type are conjugate.

Subshifts of finite type (SFT) can be defined in various ways. Here we focus on the graph approach.
Given a finite graph $G$, the subshift of finite type $X_G$ associated to $G$ is the set of all biinfinite paths on $G$.

We may think either of $G$ as a graph, or equivalently as a matrix with nonnegative coefficients.
Conjugacy

- We say that two SFTs are conjugate if the dynamical systems they represent are conjugate.
- If we write the biinfinite paths as words over some infinite alphabet, then the conjugacy is a cellular automaton.

Main problem of symbolic dynamics: decide conjugacy.
Conjugacy

In terms of matrices:

\( M \) is *Strong Shift Equivalent* to \( N \), if \( M \sim N \) where \( \sim \) is the smallest equivalence relation s.t. \( RS \sim SR \) for all nonsquare integral nonnegative matrices \( R, S \).

In terms of graph:

\( G \) is conjugate to \( G' \) if \( G \) can be obtained from \( G' \) by a series of incoming/outgoing splits and amalgamations.

Incoming split: transform one vertex \( u \) into two vertices \( u_1, u_2 \), split the inputs and share the outputs.
Examples

All pictures from Kitchen’s book (Symbolic Dynamics):
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Figure 2.1.6
I will use “Strong Shift equivalence” (SSE) instead of conjugacy

- Williams 1973: SSE is introduced
- Williams 1973: SSE is decidable for one-sided SFTs (only incoming splits/amalgamations)
- Franks 1984: Flow equivalence (a variant of SSE) is decidable
- Kim-Roush 1988: Shift equivalence (a variant of SSE) is decidable
- Kim-Roush 1992: Shift equivalence is not the same as SSE
- Folklore: SSE is decidable for matrices in $\mathbb{Z}$ rather than in $\mathbb{Z}_+$ (graphs with negative edges)

Conclusion: while SSE is not known to be decidable, there are a lot of variants that are.
This talk

- SSE is complicated because the split/amalgamation stuff is complicated
- We will introduce a simplified version of the split/amalgamation
- The equations we obtain will remind us of category theory, and we will use category theory to obtain some results
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We will first focus on *flow equivalence*, a variant of SSE.

Flow equivalence is just SSE with a looser notion of time.

i.e. we can now stretch a vertex:
We will reformulate flow equivalence with simpler equations
Then we will go back to the original problem
Goal: get rid of the split/amalgamations equations.
Represent the graph in a new formalism with two kinds of vertices:

- Vertices that collect incoming edges

- Vertices that distribute outgoing edges:
Symbolic dynamics as a categorical notion
How does flow equivalence translate into rules for red-blue graphs?
We want to think of the blue vertex as gathering incoming edges:

- Gathering one incoming edge is the same as doing nothing
- Gathering three incoming edges is the same as gathering the first two, then gathering the result with the third

We only need blue vertices of incoming degree 2

(technically we also need vertices of incoming degree 0)

The same is true for red vertices
Two rules
Two rules

\[
\begin{align*}
\text{Diagram 1} & \quad = \quad \text{Diagram 2}
\end{align*}
\]
What axioms do we need to take into account amalgamations/split?
We only need ONE additional axiom:
Theorem 1

Flow equivalence, when expressed on bicolored graphs is entirely given by the following equations;

(plus other axioms for degenerate graphs, i.e. graphs with sources and sinks)
As an example, how to do the following split?

\[ \begin{array}{c}
\text{Left side} \\
\text{Right side}
\end{array} \]
Proof
Proof
Proof
Flow equivalence, when expressed on bicolored graphs is entirely given by the following equations:

(plus other axioms for degenerate graphs, i.e. graphs with sources and sinks)
Strong Shift Equivalence

How to go back to strong shift equivalence (conjugacy)?

Flow equivalence is just SSE with a looser notion of time.

SSE is just flow equivalence with a stronger notion of time.

(formal statement uses results from Boyle and Wagoner)
We will add a new vertex that represents one unit of time.
Theorem 2

**Theorem**

*SSE, when expressed on bicolored graphs is entirely given by the following equations;*

(plus other axioms for degenerate graphs)
Idea: Do not see these boxes as nodes in a graph, but as operators:

Typically, the blue node takes two inputs, and converts them to one output, similarly for the others.
What do we need to represent graphs?

- A way to compose these operators sequentially
- A way to compose these operators in parallel

What we need is a symmetric monoidal category.
A *prop* is the data, for each pair \((n, m)\) of a set \(P[m, n]\). Think of elements of \(P[m, n]\) as boxes with \(m\) inputs and \(n\) outputs. We write \(f : m \rightarrow n\).

![Diagram of a prop]

We also need:

- A composition \(P[n, p] \times P[m, n] \rightarrow P[m, p]\) satisfying the obvious axioms.
- An identity element:
- A tensor product : \(P[m_1, n_1] \times P[m_2, n_2] \rightarrow P[m_1 + m_2, n_1 + n_2]\) satisfying the obvious axioms
- A swap element:
A prop is the data, for each pair \((n, m)\) of a set \(P[m, n]\). Think of elements of \(P[m, n]\) as boxes with \(m\) inputs and \(n\) outputs. We write \(f : m \to n\).

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\[
\begin{array}{ccc}
& f & \\
\vdash & & \\
& & f
\end{array}
\]

- An identity element:

\[
\begin{array}{ccc}
& g & \\
\vdash & & \\
& & f
\end{array}
\]

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\[
\begin{array}{c}
\vdash f \vdash \\
\end{array}
\]

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\[
\begin{array}{c}
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\end{array}
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\[
\begin{array}{c}
\begin{array}{c}
\text{f}
\end{array}
\end{array}
\]

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\[
\sigma:
\]

\[
\begin{array}{c}
\begin{array}{c}
\rightarrow
\end{array}
\end{array}
\]
A *traced prop* is a prop that contains an operator: 
\[ [n + 1, m + 1] \rightarrow [n, m], \] 
called the trace satisfying obvious axioms.
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A *traced prop* is a prop that contains an operator: $[n + 1, m + 1] \to [n, m]$, called the trace satisfying obvious axioms.
Main idea

- Find a traced prop which contains a *bigebra*, that is
  - An element \(2 \to 1\) to represent the blue node
  - An element \(1 \to 2\) to represent the red node
  - An arrow \(1 \to 1\) to represent the square

- Suppose these three things satisfy the axioms we gave previously

- Then one can “interpret” graphs/matrice/SFTs in this category in such a way that SFTs that are conjugate corresponds to the same element of the prop.

- This gives a way to obtain *invariants*
Main idea

Start from a graph/matrix

\[ M = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \]
Main idea

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- Convert it into a red/blue graph:
Main idea

- Convert it into a red/blue graph:

- Interpret the nodes as operators in some category:

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The equations

These equations are incredibly common, and appear in many parts of math:
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The equations

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     - Bialgebras and Hopf Algebras
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Monoids

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\[ x + y = E \]
Let $M$ be a commutative monoid. Inputs and outputs are elements of $M$:
Monoids

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Monoids

Let $\mathcal{M}$ be a commutative monoid. Inputs and outputs are elements of $\mathcal{M}$:

$\mathcal{M} = (\mathbb{R}, +)$:

$\begin{align*}
\begin{array}{c}
\quad x + y \\
\quad x
\end{array}
\end{align*}$

$x$ $\quad x$
$y$ $\quad x$

$\begin{align*}
\begin{array}{c}
\quad 2 \\
\quad 5
\end{array}
\end{align*}$

$\begin{align*}
\begin{array}{c}
\quad ?
\end{array}
\end{align*}$
Monoids

Solution: monoids with multiplicities:

- Input of size $n$: an element of $\mathcal{M}^n \rightarrow \mathbb{N}_{\infty}$.
- The trace counts for how many elements of $\mathcal{M}$ the diagram makes sense.
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Solution: monoids with multiplicities:

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![Diagram showing a monoid with multiplicities]
Monoids

Solution: monoids with multiplicities:

- Input of size $n$: an element of $M^n \rightarrow \mathbb{N}_\infty$.
- The trace counts for how many elements of $M$ the diagram makes sense.

$M = (\mathbb{R}, +)$:

all elements of $\mathbb{R}$

$2$

$0$
Monoids

What about the square?

It’s just a morphism for the monoid (which will automatically work with the copy)
Let $R$ be a matrix and $M$ be a monoid, and $h$ an homomorphism. When interpreting the diagram in the previous category, $R$ represents the number of solutions of the equation $x = h(Rx)$ in the monoid $M$.

For all commutative monoids $M$ and all homomorphisms $h$ of $M$, the number of solutions of the equation $x = h(Rx)$ in $M$ is an invariant of conjugacy.
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Bialgebras and Hopf Algebras are well studied in representation theory and combinatorics.

- Input of size $n$: an element of $V^\otimes n$ where $V$ is a vector space over some field $\mathbb{K}$
- If $V$ is a vector space with basis $e_i$, $V \otimes V$ is a vector space with basis $e_i \otimes e_j$
- Boxes are linear maps
Monoid ring: \( \mathbb{K}[\mathcal{M}] \), vector space with basis \( e_x, x \in \mathcal{M} \)

- Multiplication: \( e_x \otimes e_y \rightarrow e_{x+y} \)
- By the multiplication:
  \[ 3(e_2 \otimes e_3) - 2(e_1 \otimes e_4) + 3(e_1 \otimes e_5) \rightarrow e_5 + 3e_6 \]
- Comultiplication \( e_x \rightarrow e_x \otimes e_x \)
- By the comultiplication:
  \[ e_5 + 3e_6 \rightarrow e_5 \otimes e_5 + 3e_6 \otimes e_6 \]

Exactly the same example as before, presented differently.
The binomial bialgebra: $V = \mathbb{K}[X]$, basis $(X^n)_{n \geq 0}$

- **Multiplication:** $X^n \otimes X^m \rightarrow X^{n+m}$

- By the multiplication:
  $3(X^2 \otimes X^3) - 2(X^1 \otimes X^4) + 3(X^1 \otimes X^5) \rightarrow X^5 + 3X^6$

- **Comultiplication** $X^n \rightarrow \sum_k \binom{n}{k} X^k \otimes X^{n-k}$

- By the comultiplication: $X^2 \rightarrow 1 \otimes X^2 + 2X \otimes X + X^2 \otimes 1$

- **Homomorphism:** $X^n \rightarrow (\lambda X)^n$ for some $\lambda \in \mathbb{K}$
The canonical example $V = \mathbb{K}[X]$ does not have a trace, we need to tweak it:

- Coefficients in the complete semiring $\mathbb{R}_\infty$ rather than in $\mathbb{R}$
- We allow infinite sums: $V = \mathbb{R}_\infty[[X]]$

Trace: sum over all $n$ of the coefficient of $X^n$ of the output if the input is $X^n$
Example

The golden shift:

\[ M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \]
Example

The golden shift:

\[ M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \]

We look without the traces.
If we start from \( X^n \otimes X^m \), the output is

\[ \lambda^{n+m} \sum_k \binom{n}{k} X^{m+k} \otimes X^{n-k} \]
Example

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Example

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\[ M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \]

The coefficient of \( X^n \otimes X^m \) in this sum is \( \binom{n}{n-m} \lambda^{n+m} \)

The value of the graph is therefore

\[
\sum_{n,m} \binom{n}{n-m} \lambda^{n+m} = \frac{1}{1 - \lambda^2 - \lambda}
\]
Theorem

Let $M$ be a nonnegative matrix. The result of the computation is $\zeta_M(\lambda)$, with $\zeta_M(t) = \frac{1}{\det(I-tM)}$. Therefore $\zeta_M$ is an invariant of conjugacy.

Consequence of McMahon master’s theorem.
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Cospans

New, weird category:

- A box with \( n \) inputs and \( m \) outputs is a \textit{commutative group}, with at least \( n + m \) generators, and a finite presentation.

- Inputs and outputs are to be understood as generators that can still be plugged in into other generators.

- Composition is the new group obtained by identifying input and output generators that are plugged together (pushout)

\[
\begin{align*}
\langle r, s, t, u \mid u &= 2t + r \\
& a, b, c, d \mid a - b &= c + d \\
& c - 3d &= a \\
& t = b \rangle
\end{align*}
\]
Tensor product is the new group obtained by putting the two groups side by side (sum of the group)

\[
\langle r, s, t, u \mid u = 2t + r \rangle \quad \langle a, b, c, d \mid a - b = c + d, c - 3d = a \rangle \quad \langle a, b, c, d, r, s, t, u \mid a - b = c + d, u = 2t + r, c - 3d = a \rangle
\]
Groups

- Trace consists in equating input and output
- We have to look at groups up to isomorphism of the internal generators.

What is the red and blue node?
- Red node: group $\mathbb{Z} = \langle x, y, z | x = y = z \rangle$: all generators are equal
- Blue node: group $\mathbb{Z}^2 = \langle x, y, z | x + y = z \rangle$: output generator is equal to the sum of the input generators.

Note: the square is the trivial homomorphism
Starting from a matrix $M$ (or a graph $G$), this construction associates to $M$ the abelian group

$$G = \langle x \mid x = Mx \rangle$$

This is the Bowen-Franks group
We can do the same with things other than groups: if we look at \( \mathbb{Z}[t] \)-modules instead of groups, we can have a nontrivial interpretation of the square, and obtain:

**Theorem**

Starting from a matrix \( M \) (or a graph \( G \)), this construction associates to \( M \) the \( \mathbb{Z}[t] \) module:

\[
G = \langle x \mid x = tMx \rangle
\]

This is the dimension group (Krieger).
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Conclusion

A systematic way to obtain invariants for symbolic dynamics by looking at algebraic structures in some categories. We recover the classical invariants, which proves the method works:

- The Zeta function
- The Bowen-Franks group
- The Dimension group

Now: test other categories, to obtain new invariants!