PROPs and Symbolic dynamics

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Plan

Conjugacy

Relation on integral nonnegative matrices:

Definition

M is *Strong Shift Equivalent* to N, if $M \sim N$ where $\sim$ is the smallest equivalence relation s.t. $RS \sim SR$ for all nonsquare integral nonnegative matrices R, S

Main open problem of symbolic dynamics: If SSE decidable ?

Invariants

To partially solve this problem, one uses invariants:

Definition

An invariant is a quantity $\phi(M)$, easy to compute s.t.
If $M \sim N$ then $\phi(M) = \phi(N)$.

This talk: how to get invariants for free.

Warning

- I will cheat and obtain invariants for a related notion, called flow equivalence, that I won't define.
- SSE just needs ten more minutes.

Plan

PROPs

Definition

PROPs are symmetric monoidal categories generated by a single object

Wait, what ?

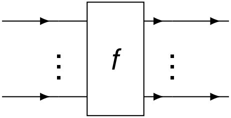
PROPs

PROPs come from category theory, and correspond to structures with two composition rules: a sequential and a parallel composition rule.

No need to know category theory, but universal algebra might help.

PROPs

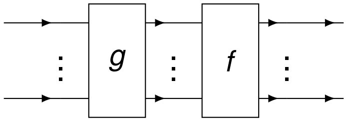
In a prop, we have functions that have m inputs and n outputs.
We write $f : m \to n$.



PROPs

We can compose sequentially $g : a \to b$ and $f : b \to c$ if $\#outputs_g = \#inputs_f$.

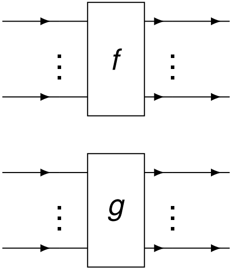
$f \circ g$



PROPs

We can compose "in parallel" $g : a \to b$ and $f : c \to d$.

$f \otimes g$:



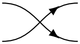
PROPs

We have access to the identity function : $id : 1 \to 1$:

id :

And to a swap function: $\sigma : 2 \to 2$.

σ



Example

If we have access to $f : 2 \to 1$ and $g : 2 \to 2$, we can e.g. write:

$$(g \otimes id) \circ (f \otimes \sigma) \circ (id \otimes g \otimes id)$$

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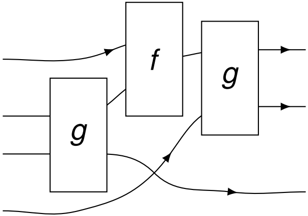

PROPs

Everything satisfy the natural equations they should satisfy:

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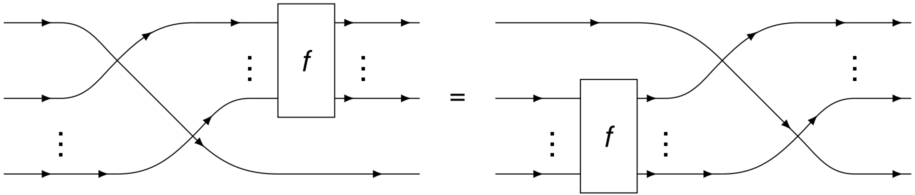
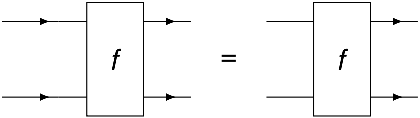


PROPs

Everything satisfy the natural equations they should satisfy:

$$f \circ (\mathrm{id} \otimes \mathrm{id}) = f$$



PROPs

Everything satisfy the natural equations they should satisfy:

$$(f_1 \circ g_1) \otimes (f_2 \circ g_2) = (f_1 \otimes f_2) \circ (g_1 \otimes g_2)$$

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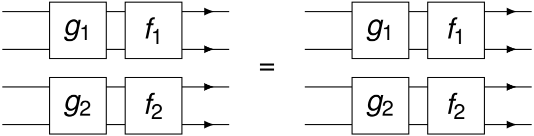


PROPs

(Coherence theorem)

The axioms imply that reasoning with pictures is OK!

Examples

- $f : m \to n$ are functions from A^m to A^n
- $\circ$ is the composition of functions
- SWAP: $(x, y) \to (y, x)$
- $f \otimes g$ is the cartesian product

Examples

- $f : m \to n$ are functions from $[1, m]$ to $[1, n]$.
- $\circ$ is the composition of functions
- SWAP: $1 \leftrightarrow 2$
- If $f_1 : [1, m_1] \to [1, n_1]$ and $f_2 : [1, m_2] \to [1, n_2]$, then

$$f_1 \otimes f_2 : \quad \begin{array}{rcl} [1, m_1 + m_2] & \to & [1, n_1 + n_2] \\ x \leq m_1 & \mapsto & f_1(x) \\ x > m_1 & \mapsto & f_2(x - n_1) + n_2 \end{array}$$

Examples

- $f : m \to n$ are matrices from $M_{n,m}(\mathbb{R})$
- $\circ$ is the matrix product
- swap: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- $M \otimes N$ is the Kronecker sum: $\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}$

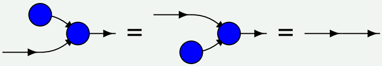
Plan

f.p. PROPs

Like with groups and other structures, we can look at props given by generators and relations

Generators:

Relations:



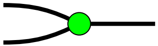
Plan

The Boolean Prop

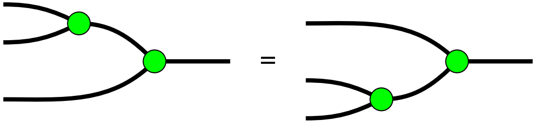
What are the generators and relations of the props corresponding to functions $\{0, 1\}^m \to \{0, 1\}^n$?

The Boolean prop

We have a generator: $AND : 2 \to 1$

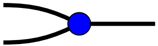


and some equations like



The Boolean prop

We have a generator: $OR : 2 \to 1$



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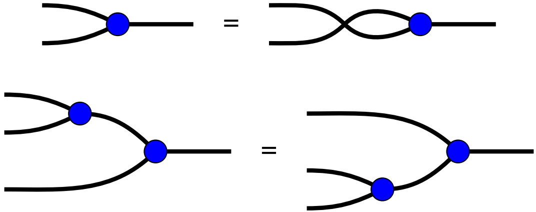
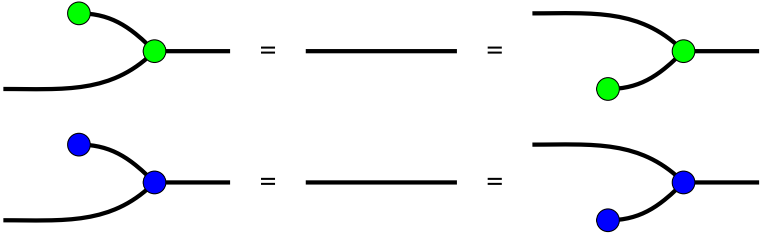


The Boolean prop

We have two generator: $T : 0 \to 1$ and $F : 0 \to 1$

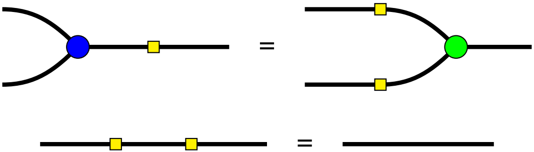
and some equations like



The Boolean prop

We have one generator: $NOT : 1 \to 1$

and some equations like



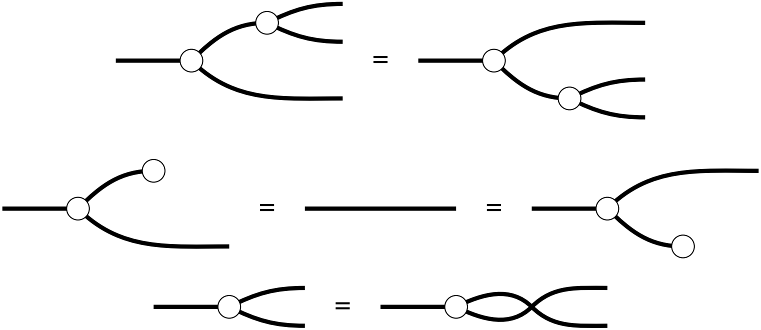
The Boolean Prop

We’re missing some equations but do we have all generators ?

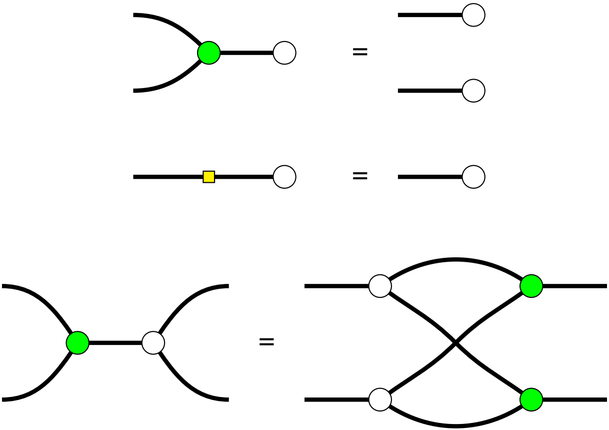
The Boolean prop

We have two generators: $CPY : 1 \to 2$ and $TRASH : 1 \to 0$.

and some equations like



Some of the remaining equations



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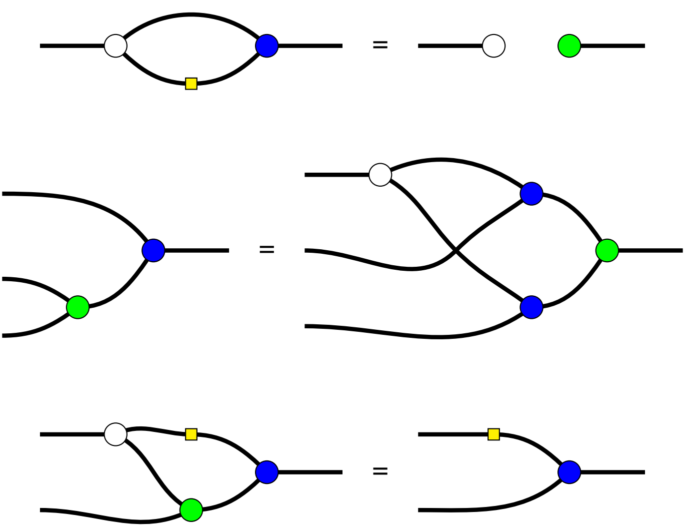


Theorem

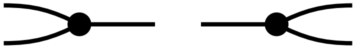
Theorem (Folklore)

The corresponding generators and equations correspond to the PROP of functions from $\{0,1\}^m \to \{0,1\}^n$.

Plan

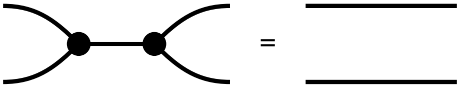
Example

What is this prop ?
Generators:



Equations:

and



(No commutativity or associativity)

Example

What is this prop ?

Theorem

It is the prop of bijective dyadic affine functions from $[1, m]$ *to* $[1, n]$.

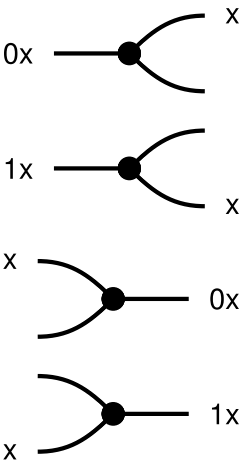
Example

Theorem

It is the prop of bijective "local" functions from $\{1,\ldots,m\} \times \{0,1\}^\omega$ *to* $\{1,\ldots,n\} \times \{0,1\}^\omega$

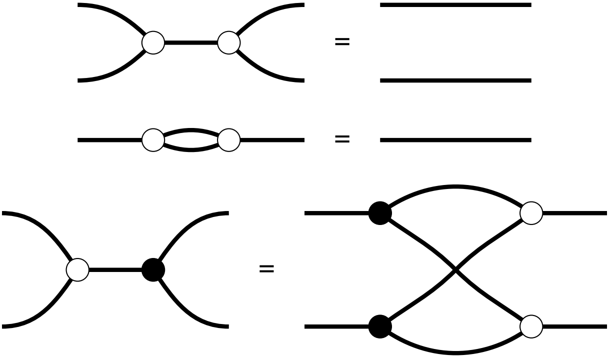
These are called "generalized shifts" in the vocabulary of Moore [1991].
Input (n, x) is to be interpreted as "I have the infinite word x in the n-th wire".

Example



Example

Harder example:



Example

Theorem

It is the prop of bijective "local" functions from $\{1,\dots,m\} \times \{0,1\}^\omega \times \{0,1\}^\omega$ *to* $\{1,\dots,n\} \times \{0,1\}^\omega \times \{0,1\}^\omega$

Theorem

It is the prop of reversible Turing machines.

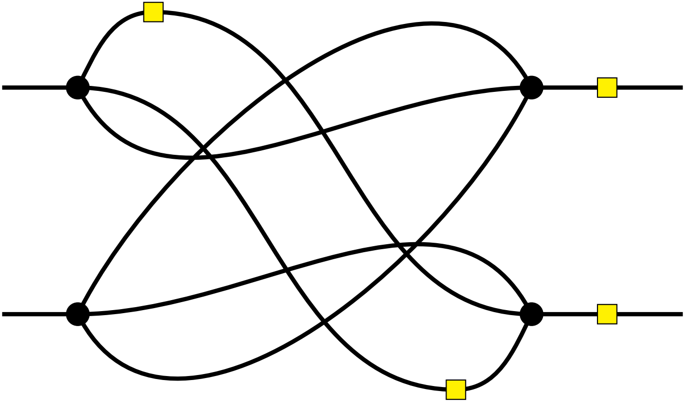
More precisely, maps $m \to m$ correspond to reversible Turing machines with m states.

Example

The following map:

is just the shift!

Example



Plan

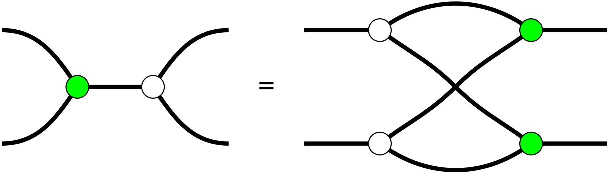
Matrices

- $f : m \to n$ are matrices from $M_{m,n}(\mathbb{N})$
- $\circ$ is the matrix product
- swap: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- $M \otimes N$ is the Kronecker sum: $\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}$

What are the generators and relations ?

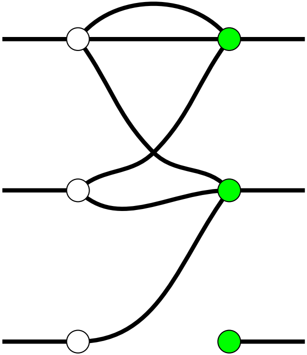
Matrices

Axioms: (co)associativity, (co)commutativity, (co)unit and:

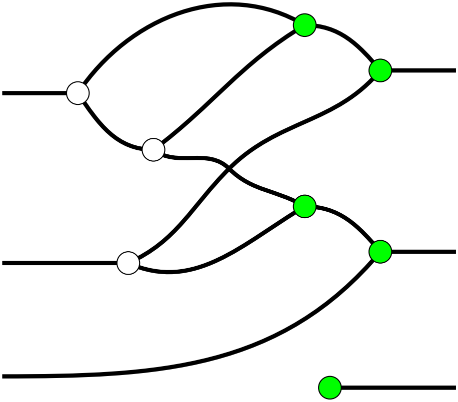


Matrices

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$



Matrices



$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

AND and matrices

All the previous properties were satisfied by AND and CPY. What does it mean ?

AND and matrices

Proposition (Universal Property)

There exists a morphism (technically a functor) from matrices into boolean functions

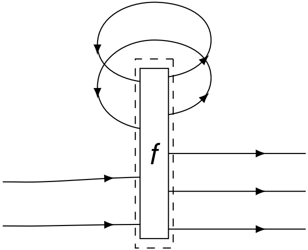
$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix} \quad (x, y, z) \mapsto (x \textit{ AND } z, x \textit{ AND } y \textit{ AND } y)$$

Plan

We are computer scientists, let’s add feedback loops!

Traces

A *traced prop* is a prop that contains an operator, called the trace, satisfying obvious axioms



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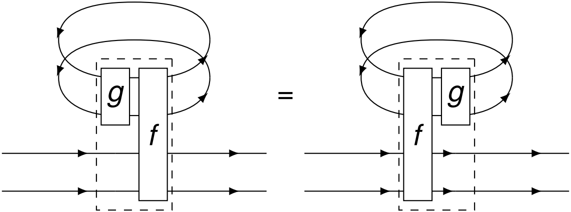

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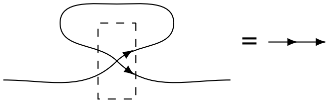


What becomes the prop of matrices when we add a loop ?

Theorem

Let M and N two matrices, represented as diagrams.
If we completely trace both matrices (we link all outputs to all inputs), then
$trM = trN$ iff M and N are flow equivalent.

Flow equivalence is an equivalence notion on matrices (technically on SFTs) coming from symbolic dynamics.

We have reformulated a notion from symbolic dynamics into a category notion. This is great!

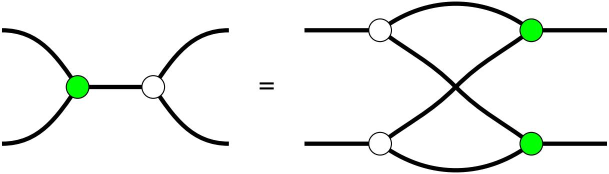
Why is this great ?

- Suppose we know of mathematical objects in some structure $\mathcal{C}$ that satisfy all properties I gave
- By the universal property, there is a morphism from (traced) matrices to these objects
- If we start from two matrices that are flow equivalent, then they should correspond to the same thing in $\mathcal{C}$

We have produced an invariant of flow equivalence!

Axioms again

Axioms: (co)associativity, (co)commutativity, (co)unit and:

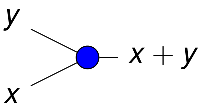


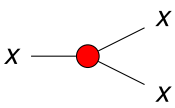
And there is a trace.

These equations are incredibly common, and appear in many parts of math.

Monoids

Let $\mathcal{M}$ be a commutative monoid. Inputs and outputs are elements of $\mathcal{M}$:





For the trace to make sense, we have to go with monoids with multiplicities rather than monoids.

Theorem

The number of solutions in $\mathcal{M}$ of the equation $Mx = x$ is an invariant of flow equivalence.
If M and N are flow equivalent, the equations $Mx = x$ and $Nx = x$ have the same number of solutions.

The binomial bialgebra

Inputs/Outputs are polynomials in n/m variables

- Green node: identify two variables $X_1^k X_2^m \to X^{k+m}$
- White node: divide two variables $X^n \to \sum_k \binom{n}{k} X_1^k X_2^{n-k}$
- Trace: sum over all n of the coefficient of X^n of the output if the input is X^n

Theorem

$det(I - M)$ is an invariant of flow equivalence.

Cospans

- A box with m inputs and n outputs is a *commutative group*, with at least $n + m$ generators
- Composition = identifying generators
- Green node: Group with three generators x, y, z s.t. the output z is the sum of the two inputs.
- White node: Group with three generators x, y, z s.t. the outputs y and z are equal to x

Theorem

The group

$$G = \langle x | x = Mx \rangle$$

is an invariant of flow equivalence

(This is the Bowen-Franks group)

Partial Conclusion

- Many invariants of symbolic dynamics can be recovered by trying to find structures (traced bialgebras) in some unrelated objects.
- This can also be done for conjugacy (strong shift equivalence)
  - We can recover the Zeta function
  - We can recover the dimension group

Now: test other categories, to obtain new invariants!