Tilings robust to errors

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Abstract We study the error robustness of tilings of the plane. The fundamental question is the following: given a tileset, what happens if we allow a small probability of errors? Are the objects we obtain close to an error-free tiling of the plane?
We prove that tilesets that produce only periodic tilings are robust to errors. For this proof, we use a hierarchical construction of islands of errors (see [6,7]). We also show that another class of tilesets, those that admit countably many tilings is not robust and that there is no computable way to distinguish between these two classes.

Introduction

Tilings are a basic and intuitive way to express geometrical constraints. They have been used as static geometrical models of computation since Berger proved the undecidability of the so-called domino problem [2] by capturing geometric aspects of computation [19,2,11,14,5,10,3]. The model assumes the reliability of local color-constraints of tilings, hence several research tracks were aimed at constructing tilesets that are reliable to errors, be it as a computing model [7,8] or as a model for DNA self-assembly [22,18].

In this paper we are interested in error robustness of tilesets. We can see an erroneous tiling as an usual tiling by Wang tiles [21] where we allow a small proportion of colors on adjacent edges of tiles not to match. We give a more formal definition of this notion in Section 1. Some constructions using fixed point methods construct tilesets that are robust to a small proportion of errors [7,8]. Our goal here is to focus on general properties that imply error-robustness or non-error-robustness of some classes of tilesets.

In Section 2 we give a construction of "error-cleaning functions" for tilesets that allow only periodic tilings. We prove that for this class of tilesets it is possible to apply well known hierarchical constructions [9,6] so that we can repair every erroneous tiling with probability one granted the probability of errors is sufficiently small.

On the other hand, we prove that the family of tilesets that produce only a countable number of tilings are not locally robust: the correction of a finite number of errors in those tilings may always require a modification of an infinite number of tiles. This is incompatible with local error correction and thus robustness. Locally robust tilesets can be expressed as properties of their ground state configurations as used to model crystals [16,17].

Finally, by using classical constructions [3,19,15] for encoding Turing machines into tilesets, we prove in Section 4 that these two classes of tilesets are recursively inseparable, which shows that it is not possible to obtain a simple (recursive) characterization of tilesets robust to errors.

1 Definitions

We present notations and definitions for tilings; we focus our study on tilings of the plane but most of the results naturally extend to higher dimensions. We define tilings by *local constraints* as they give the most straightforward definitions for tiling errors. Different models are used in literature, such as Wang tiles [20] or subshifts of finite type [12], but one can easily transform one formalism into another (see [4] for more details and proofs).

In our definition of tilings, we first associate a color to each cell of the plane. Then we impose a local constraint on them, that is we describe which colorings are allowed and which are not. More formally, Q is a finite set, called the *set of colors*. A *configuration* c consists of cells of the plane with colors, thus c is an element of $Q^{\mathbb{Z}^2}$. We denote by $c(i)$ the color of c at the cell $i \in \mathbb{Z}^2$. For an element x of $\mathbb{Z}^2$ we denote by $c + x$ the configuration $i \to c(i + x)$.

Definition 1 (Patterns). *A* pattern *P is a finite restriction of a configuration i.e., an element of Q^V for some finite domain V of $\mathbb{Z}^2$. A pattern* appears *in a configuration c if it can be found somewhere in c; i.e., if there exists a vector $t \in \mathbb{Z}^2$ such that $c(x + t) = P(x)$ on the domain of P.*

Definition 2 (Local constraints). *A local constraint is a pair $\tau = (V, \delta)$. V is a finite domain of $\mathbb{Z}^2$ and is called the neighborhood. δ is the constraint function from Q^V to $\{0, 1\}$.*

The idea behind this formalism of local constraints is to define which patterns are allowed and which are not. A pattern is allowed if and only if it maps to 0 by the constraint function. The local constraint $\tau = (V, \delta)$ naturally extends to a global constraint function $\Delta_\tau : Q^{\mathbb{Z}^2} \to \{0, 1\}^{\mathbb{Z}^2}$ by applying it uniformly on every cell of the plane: $\Delta_\tau(c)(x) = \delta((c + x)_{|V})$. This corresponds to a sliding block code [13, Chapter 1, § 5].

Definition 3 (Tilings). *A configuration $c \in Q^{\mathbb{Z}^2}$ is said to be valid for $\tau = (V, \delta)$ (or a tiling by τ, or allowed by τ) when it satisfies the local constraint everywhere, that is for every cell x, $\Delta_\tau(c)(x) = 0$.*

The set of tilings by τ is denoted by $\mathcal{T}_\tau$. In this paper we only consider tilesets that can tile the plane, thus $\mathcal{T}_\tau \neq \emptyset$ is an implicit condition of all the results.

As we want to study tilings with some errors, this definition of classical tilings naturally extends to tilings with errors by considering an error repartition where some cells are not correctly tiled:

Definition 4 (Tiling with errors). *Let e be an element of $\{0,1\}^{\mathbb{Z}^2}$ and c a configuration, we say that c is a tiling by τ with error repartition e if $\Delta_\tau(c) = e$.*

One may remark that with this definition, repairing an error is different from replacing the erroneous tile with a correct one as errors may have consequences that require replacing other tiles, even if those tiles are locally correct.

In this paper we first prove that the consequences of such a correction are not problematic in the case of tilesets that allow only periodic tilings (Section 2) but may have consequences on infinitely many cells in the case of tilesets that allow countably many tilings (Section 3).

Before entering the core of the problem of tilings with errors we need to recall a couple of structural results on regular tilings of the plane. We embed Q with the discrete topology, $Q^{\mathbb{Z}^2}$ with the induced product topology. A classical result on the set of configurations is its compactness, as a direct application of Tychonoff's theorem:

Proposition 1. *$Q^{\mathbb{Z}^2}$ is a compact perfect metric space (a Cantor space).*

The metric we consider, as induced by the product topology, is defined by $d(x,y) = 2^{-|i|}$ where $|(a,b)|$ denotes $|a| + |b|$ and i is the point closest to $(0,0)$ (i.e., with minimal norm) such that $x(i) \neq y(i)$. The ball of center x and radius k is the set of points y such that $|x - y| \leq k$. In this paper we reformulate the compactness of sets of tilings in a way that suits our needs:

Lemma 1. *For each finite subset D of $\mathbb{Z}^2$, there exists C such that if a pattern defined on D can be extended to a pattern on C while respecting the local constraints then $P_{|D}$ appears in a tiling of $\mathbb{Z}^2$.*

This means that if we can tile sufficiently large but finite patterns we are sure that a small part of it will appear in a valid tiling of the whole plane. The function that is given a tileset τ and outputs the set C, even with a fixed $D = \{(0,0)\}$, is uncomputable since this would allow us to decide the domino problem which is a well known non-recursive problem [2,15,19].

Proof. We prove that for every pattern P there exists a finite domain C_P of $\mathbb{Z}^2$ such that there exists a valid pattern defined on C_P that contains P if and only if P appears in a tiling of the plane. This proves the lemma by taking C to be the finite union of all C_P where P is a pattern of domain D.

If P appears in a tiling of the plane take C_P to be the domain of P. Suppose that there is a pattern P that does not appear in a tiling of the plane but such that there exists arbitrary large valid extensions of it. By compactness, this sequence of patterns can be extracted to converge towards a configuration. This configuration contains P and is a valid tiling of the plane. Therefore, if P does not appear in a tiling of the plane, there exists a finite domain such that any correct tiling of this domain does not contain P; we take C_P to be this domain. □

2 Robustness

In this section we show how, in the case of tilesets that allow only periodic tilings, it is possible to reconstruct a valid tiling from one with a small proportion of errors by a local modification. The method is the same that was used to prove the error robustness of strongly aperiodic tilesets in [7,8]; since sets of periodic tilings have a very simple structure we are able to easily apply the same methods. We consider error distributions e_ε such that each cell of $\mathbb{Z}^2$ has probability ε to be equal to 1 independently of other points, i.e., a Bernoulli distribution.

We first describe a generic process [6,7] of sorting errors into "islands of errors" that given a repartition taken with a Bernoulli distribution transforms it into an error free repartition with probability one (Section 2.1). Then we present some structural results on tilesets that will allow us to apply this generic process to these tilesets (Section 2.2).

2.1 Iterative cleaning process

When ε is small the intuition is that cells with value 1 will be sparse; we will not have big clusters of 1's; however big clusters have a non-null probability to appear thus they almost surely appear on the infinite plane $\mathbb{Z}^2$. Even if there is such an unavoidable problem, we will see that we can decompose error repartitions into different layers then repair each layer incrementally so that eventually everything gets repaired. First of all, let us define what a layer is:

Definition 5. *For an error repartition $e \in \{0,1\}^{\mathbb{Z}^2}$ and a point $x \in \mathbb{Z}^2$ such that $e(x) = 1$, x is said to be in an $(i,j)-$*island *of e if there exists no point at distance between i and j in e that has value 1, that is:*

$$\forall y \in \mathbb{Z}^2, e(y) = 1 \Rightarrow |x - y| \notin [i;j]$$

We denote by $\mathcal{I}_{i,j}(e)$ the set of points of $\mathbb{Z}^2$ that are in an $(i,j)-$island of e. Figure 1 depicts some islands. These islands can be seen as isolated clusters of errors. The idea now is to remove the islands, hence our definition of an erasing function:

Definition 6 (Erasing function). *The function $\mathcal{E}_{i,j}$ from $\{0,1\}^{\mathbb{Z}^2}$ into itself erases the $(i,j)-$islands of an error repartition:*

$$\begin{array}{rl} \{0,1\}^{\mathbb{Z}^2} \rightarrow & \{0,1\}^{\mathbb{Z}^2} \\ e \quad \rightarrow c: c(x) = & \begin{cases} 0 & \text{if } x \in \mathcal{I}_{i,j}(e) \\ e(x) & \text{otherwise} \end{cases} \end{array}$$

Now if we consider integer sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$, we can think about applying successively the functions $\mathcal{E}_{\alpha_n,\beta_n}$ to an error repartition so that we obtain an iterative cleaning process [6,7] that erases small islands of errors, then bigger ones, then even bigger ones, etc.

Definition 7 (Iterative cleaning process). *Let e be an error repartition and consider two sequences α_n and β_n. We define the* iterative cleaning *of e by (α, β) denoted $(e_n^{\alpha,\beta})_{n\in\mathbb{N}}$ by:*

- $e_0^{\alpha,\beta} = e$
- $e_{n+1}^{\alpha,\beta} = \mathcal{E}_{\alpha_n,\beta_n}(e_n^{\alpha,\beta})$.

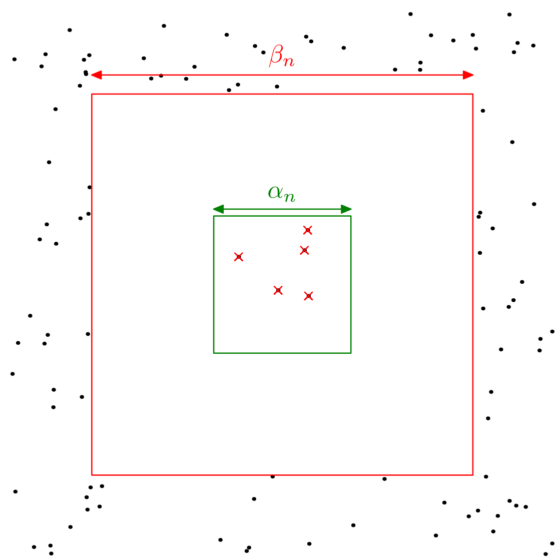


Figure 1. Islands of rank n

We call "*islands of rank n*" the points that we corrected at the n^{th} iteration. This operation is pictured on Figure 1 where the black points are the points with value 1, those that will be "cleaned" at this step are marked by a cross, and we can see that we have a kind of security belt between α_n and β_n. The important part of this process is that cleaning the islands of rank n at the n^{th} iteration creates more islands that we will catch at the $n+1^{th}$ step. The following theorem catches this phenomenon:

Theorem 1 ([6,7]). *If the sequences α_n and β_n match the conditions:*

- $\forall i, 8(\beta_1 + ... + \beta_i) < \alpha_{i+1} \leq \beta_{i+1}$
- $\sum_i \frac{\log(\beta_i)}{2^i} < \infty$

Then there exists $\varepsilon > 0$ (sufficiently small) such that, almost surely, for any element of $\{0,1\}^{\mathbb{Z}^2}$ taken with the Bernoulli distribution of probability ε, this iterative cleaning process removes all the ones.

We do not give a proof of this result and refer the reader to the original [6,7]. It is easy to check that the following sequences match the conditions of Theorem 1 for n sufficiently large:

$$\alpha_n = 2^{n^2}$$

$$\beta_n = n\alpha_n$$

Moreover, what interests us is the fact that $\lim_{n\to\infty} \beta_n - \alpha_n = \infty$, which means that we can have arbitrary large security belts.

2.2 Error-robustness of periodic tilesets

In this section we consider tilesets such that their only valid tilings are periodic. We describe how to use Theorem 1 for tilings.

With the process described in Section 2.1, when n is sufficiently large, $\alpha_n - \beta_n$ is also large. In our application to tilings with errors, this means that the islands of rank n are surrounded by a belt of width $\alpha_n - \beta_n$ that is correctly tiled. However, this zone is only locally tiled correctly. Lemma 1 ensures that when we have a sufficiently large correctly tiled pattern then a smaller part of it appears in a tiling of the whole plane. Hence, if n is sufficiently large, the islands of rank n are surrounded by a belt of large width, say k, where each pattern of size $k \times k$ appears in a tiling of the whole plane. One may remark that k may be much smaller than $\alpha_n - \beta_n$ but k can still be as big as we would like it to be if we take n sufficiently big. This is depicted on Figure 2.

Now we want to replace the erroneous zone by other tiles so that the whole zone is tiled correctly but we are facing a problem: even if we can ensure that the islands of rank n are surrounded by a belt of patterns that appear in a valid tiling of the whole plane, like on Figure 2, how can we be certain that the zone surrounded by the belt can be properly filled by tiles so that no error remains?

A pattern P defined on D is said to be $k-$extensible if there exists a decomposition of D in $(\mathcal{D}_i)_{1\leq i\leq n}$ such that:

- $\cup_{1\leq i\leq n}\mathcal{D}_i = D$
- For every i, $P_{|\mathcal{D}_i}$ appears in a valid tiling of the plane
- For every $i < n$, $\mathcal{D}_i \cap \mathcal{D}_{i+1}$ contains a ball of radius k.

With the help of the previous remarks, it is clear that for any k there exists a sufficiently large n such that the islands of rank n are surrounded by a belt that is $k-$extensible.

It is already known that a tileset allows only a finite number of tiling if and only if it allows only periodic tilings [1, Theorem 3.8], hence we can focus on this simpler case:

Lemma 2. *If a tileset τ admits only a finite number of tilings then there exists k such that any $k-$extensible pattern P can be found in a tiling of the whole plane.*

Proof. There exists $\varepsilon > 0$ such that the distance between two different tilings is greater than ε. Let k be an integer such that any two configurations that are equal at their center on the pattern defined on $[-k;k] \times [-k;k]$ are at distance strictly smaller than ε. We now prove that any $k-$extensible pattern can be found in a tiling of the whole plane.

Let n and $(\mathcal{D}_i)_{1 \leq i \leq n}$ be such that P is $k-$extensible over $(\mathcal{D}_i)_{1 \leq i \leq n}$. We may assume that $n = 2$, the result for any n being obtained by an easy induction of this simpler case.

Let a be a tiling containing $P_{|\mathcal{D}_1}$ and b one containing $P_{|\mathcal{D}_2}$. There exists a' and b' that are shifted forms of, respectively, a and b such that a' and b' are equal on $P_{|\mathcal{D}_1} \cap P_{|\mathcal{D}_2}$ at their center. a' and b' are both tilings by τ. Since $P_{|\mathcal{D}_1} \cap P_{|\mathcal{D}_2}$ contains a ball of radius k, by our choice of k we obtain that $a' = b'$ because the distance between them is strictly lower than ε. Therefore P appears in a' which is a tiling of the whole plane. □

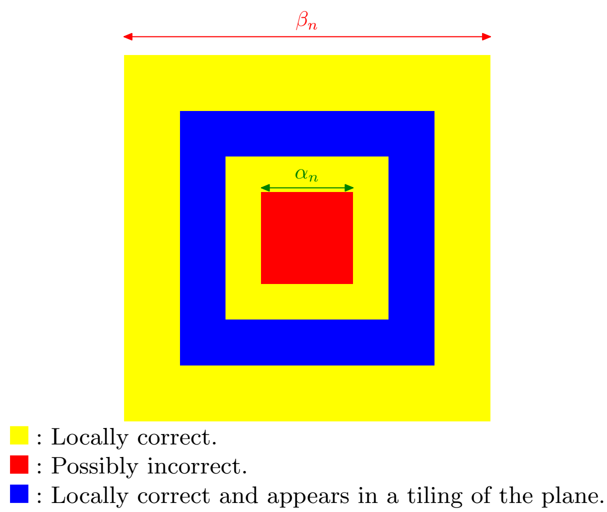


Figure 2. Example of what happens with a sufficiently large security belt.

Now, with n sufficiently large, Lemma 2 tells us a bit more on the belt surrounding islands of rank n: there exists such a surrounding belt that appears in a valid tiling of the plane. On Figure 2 this means that the outer (blue) belt is in fact part of a tiling of the plane. This allows us to state our main theorem of robustness:

Theorem 2. *If a tileset allows only periodic tilings then it is robust to a small probability of errors.*

Proof. By Theorem 1 from [1], tilesets that allow only periodic tilings are exactly the tilesets that allow only a finite number of tilings. Hence Lemma 2 applies to these tilesets. By Lemmas 1 and 2, there exists N such that any belt of width N contains a smaller belt that appears in a tiling of the plane.

If we take $\alpha_n = 2^{(n+N)^2}$ and $\beta_n = (n+N)\alpha_n$, for every n, the belt has width at least N, hence the (finite) set of points surrounded by this belt can be filled by tiles from the tiling where the belt appears. Moreover these sequences match the conditions of Theorem 1 thus we can repair every tiling taken with a small probability of errors. □

We remark that this method of obtaining a valid tiling by surrounding the errors with a security belt also works obviously for a tileset that allows everything ($\mathcal{T}_\tau = Q^{\mathbb{Z}^2}$). We can complicate it a little bit by considering $Q = \{0, 1, 2\}$ and the tilings may contain only 1's and 2's or only 0's. Moreover it is possible to use this method to obtain aperiodic tilings robust to errors [7,8].

3 Non robustness

In the previous section we proved that there exist ways of correcting errors for some tilesets. It would not make much sense to define exactly what we call a "tileset robust to errors" since there may exist other methods for correcting errors but it seems natural that every way of correcting errors has to be local, hence our definition of local robustness:

Definition 8 (Locally robust). *We say that a tileset is locally robust if for any finite repartition of errors (that contains only finitely many* 1*'s), any tiling with this error repartition can be transformed in a tiling without error by modifying only finitely many cells.*

This definition can be related to the ground state configurations used by C. Radin to model crystalline order [16,17]. Recall that a configuration is said to be ground state if whenever we modify a finite part of it we do not decrease the number of tiling errors in it. In that sense, locally robust tilesets are the tilesets for which ground state configurations are either tilings without any error or have an infinite number of tiling errors.

The structural results from Section 2 ensure that tilesets which allow only periodic tilings are locally robust: find n such that all the errors are in the same island of rank n and replace the finite area surrounded by the belt by a valid one.

Theorem 3. *Tilesets that allow countably many tilings are not locally robust.*

First recall a structural result about such tilesets:

Theorem 4 ([1]). *If a tileset allows a countable number of tilings then it allows one with exactly one direction of periodicity.*

Corollary 1. *If a tileset* τ *allows a countable number of tilings then it allows one with exactly one direction of periodicity that can be seen as a bi-infinite word* ${}^{\omega}xyz^{\omega}$ *with* $|x| = |y| = |z|$ *and* $y \neq z$*.*

Proof. Take a configuration c with exactly one direction of periodicity v from Theorem 4. We can represent c as a bi-infinite word w on a finite alphabet Σ. By representation we mean that we can decode letters of Σ in blocks of tiles such that when the blocks from w are repeated along v we obtain c. It is easy to prove that we can obtain from τ a tileset τ' over $\mathbb{Z}$ such that any tiling by τ' can be decoded in such a way in a tiling by τ. Without loss of generality we may also assume that the neighborhood of τ' is $\{-1, 0, 1\}$: it suffices to code the tileset τ' into Wang tiles [4]. Even after all these codings, Σ is still finite.

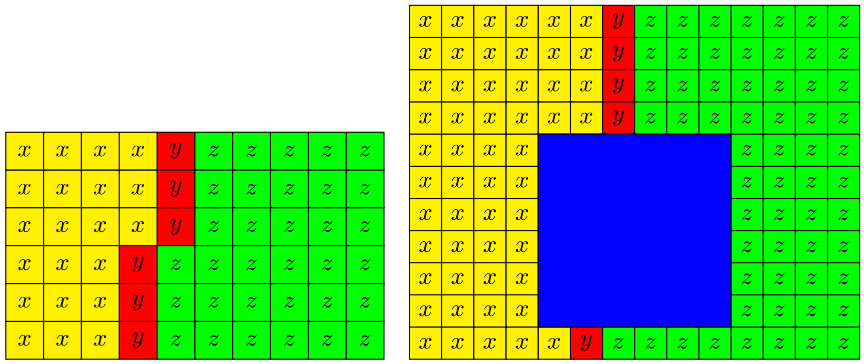
Since Σ is finite, there exists $j > i > 0$ such that $w(i) = w(j)$. Therefore, the word w is of the form $xayaz$ where a is a letter of Σ and x and z are infinite words.

If x is equal to ${}^{\omega}(ay)$, w is equal to ${}^{\omega}(ay)z'$. If x is not equal to ${}^{\omega}(ay)$, $x(ay)^{\omega}$ is also allowed by τ. In both cases, we obtain a bi-infinite word $w = x'(z')^{\omega}$ that is not periodic and x' is an infinite word (the case where x is equal to ${}^{\omega}(ay)$ being symmetric). Since w is not periodic it can be written as $w = xy(z)^{\omega}$ where $|y| = |z|$ and $y \neq z$. We repeat the same argument on the infinite x part of w to write it in the desired form. □

This corollary gives us a configuration that we prove to be incompatible with local robustness. Such a configuration is depicted on Figure 3. Imagine now that we shift half a plane horizontally by one cell. We obtain a configuration like on Figure 4(a). This configuration has only a finite number of tiling errors: around the cell where we broke the vertical line of y's. Now, what can we do in order to correct it? The only solution seems to shift back half a plane in order to restore the vertical line of y's. We now prove it is indeed the only solution by obtaining a contradiction with our hypothesis that the set of all possible tilings is countable.

Figure 3. Example of a tiling from Corollary 1.

Proof (of Theorem 3). Let c be a tiling as in Corollary 1: c is of the form ${}^{\omega}xyz^{\omega}$ with $|x| = |y| = |z| = p$ and $y \neq z$. We modify c by shifting half a plane by p to obtain a configuration like depicted on Figure 4(a). Since this transformation breaks only finitely many tiling rules, suppose that we can correct this by modifying only a finite number of cells. We obtain a tiling like on Figure 4(b) where we have a semi-infinite line of y's in one direction and a shifted semi-infinite line of y's in the other direction, separated by a pattern that repaired the error.



(a) Small modification of c

(b) Corrected version of a small modification of c

Figure 4. Transformations of c

We obtained a transformation of the tiling and can repeat it on every sufficiently long vertical line of y's. This transformation gives us a different tiling each time we apply it since $y \neq z$. Such vertical lines of y's appear infinitely many times, therefore we obtain $2^{\aleph_0}$ valid tilings for τ, a contradiction. □

4 Recursive inseparability of robust and non-locally robust tilesets

Tilesets that allow countably many tilings are not locally robust (Theorem 3) while tilesets that allow only periodic tilings are robust to errors (Theorem 2). In this section we prove that those classes of tilesets are not recursively separable, hence neither are robust and non locally robust tilesets.

Theorem 5. *Robust tilesets and non-locally robust tilesets are recursively inseparable.*

Proof. We will assume the reader familiar with the encoding of Turing machines into tilesets *à la* Berger [2], if not please refer to the detailed constructions in [3,15]. Every tiling by such a tileset contains arbitrary large squares on which we can force the bottom left corner. This corner is where we put the start of a Turing machine computation. We can see the rows of these squares as the Turing machine's tape, time is going bottom-up. Then we do the following:

– If the machine halts and outputs 1 then we force a periodic tiling: when this halting state reaches the border of the square we force a new border such

that the only way to tile the plane is to repeat periodically this square. This is exactly what is done in [3, Appendix A].
- If the machine does not halt, the tileset tiles aperiodically with an infinite computation of the Turing machine inside.
- If the machine halts and outputs 2 then force the periodicity vertically but allow only a new color, blue, that forces a monochromatic half plane at its left and another color at its right, green, that also forces a monochromatic half plane.

This new tileset always tiles the plane with the uniform blue and green tilings. If the machine halts and outputs 1 then the tileset allows only periodic tilings. If it halts and outputs 2 then it allows countably many tilings because the vertical computation line can appear at a countable number of positions. If it does not halt then it allows an uncountable number of tilings.

The class $\mathcal{M}_1$ of Turing machines that halt and output 1 is recursively inseparable of the class $\mathcal{M}_2$ of TM that halt and output 2: consider the sets C_i ($i \in \{1,2\}$) of Turing machines that halt on i with their code as input, then a Turing machine M that outputs 2 if its input is not in C_2 and 1 if it is not in C_1; there exists such a machine that always halts if we suppose C_1 and C_2 to be recursively separable and $M(\langle M \rangle)$ gives the contradiction.

With the previous tileset construction, the class $\mathcal{M}_1$ is recursively encoded into tilesets that are robust to errors and the class $\mathcal{M}_2$ into tilesets that are not locally robust. □

5 Conclusions and open problems

In this paper we have shown how it is possible to repair tiling errors (as defined in Section 2) on very simple tilesets (the periodic ones). This correction process relies on the fact that we can surround errors by correct zones which we are sure appear in a valid tiling of the plane; while this is true for periodic tilesets we remark that there exists some other tilesets for which it is also true, it would be interesting to obtain a characterization of such tilesets.

We also proved that tilesets that allow countably many tilings are not locally robust; being locally robust is a necessary condition for being able to apply our iterative correction process. While this iterative correction process allows to repair periodic tilesets and even some aperiodic ones [7,8] we believe there exists some tilesets that would be locally robust but on which this iterative correction process will not work.

The recursive inseparability between tilesets that we are able to repair and the ones that are not locally robust shows that there is no simple characterization of error-robustness for tilesets.

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