Research Article

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On the reconstruction of obstacles and of rigid bodies immersed in a viscous incompressible fluid

Abstract: We consider the geometrical inverse problem consisting in recovering an unknown obstacle in a viscous incompressible fluid by measurements of the Cauchy force on the exterior boundary. We deal with the case where the fluid equations are the nonstationary Stokes system and using the enclosure method, we can recover the convex hull of the obstacle and the distance from a point to the obstacle. With the same method, we can obtain the same result in the case of a linear fluid-structure system composed by a rigid body and a viscous incompressible fluid. We also tackle the corresponding nonlinear systems: the Navier–Stokes system and a fluid-structure system with free boundary. Using complex spherical waves, we obtain some partial information on the distance from a point to the obstacle.

Keywords: Geometrical inverse problems, fluid-structure interaction, Navier–Stokes system, enclosure method, complex geometrical solutions

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1 Introduction

This paper is devoted to reconstructing an unknown structure $\mathcal{S}$ included in a bounded cavity $\Omega \subset \mathbb{R}^N$ ($N = 2, 3$) filled by a viscous incompressible fluid. More precisely, we aim to obtain some geometrical information on $\mathcal{S}$ by measurement on the boundary $\partial\Omega$ of Ω. Such a geometrical inverse problem is important in several applied areas such as medicine (foreign bodies in the bloodstream), biology (fishes), naval engineering (submarines), etc.

We assume in what follows that $\mathcal{S}$ is a compact connected subset of Ω with nonempty interior and that $\mathcal{F} = \Omega \setminus \mathcal{S}$ is connected.

In the first part of the article, the fluid equations that we consider are the nonstationary Stokes system

$$\frac{\partial \boldsymbol{u}}{\partial t} - \mathbf{div}\,\boldsymbol{\sigma}(\boldsymbol{u}, p) = \mathbf{0} \quad \text{in } (0, T) \times \mathcal{F}, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } (0, T) \times \mathcal{F}, \tag{1.2}$$

$$\boldsymbol{u} = \mathbf{0} \quad \text{on } (0, T) \times \partial\mathcal{S}, \tag{1.3}$$

$$\boldsymbol{u} = \boldsymbol{f} \quad \text{on } (0, T) \times \partial\Omega, \tag{1.4}$$

$$\boldsymbol{u}(0, \cdot) = \mathbf{0} \quad \text{in } \mathcal{F}. \tag{1.5}$$

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In the above system, $(\boldsymbol{u}, p)$ are the velocity and the pressure of the fluid. Moreover, we have denoted by $\boldsymbol{\sigma}(\boldsymbol{u}, p)$ the Cauchy stress tensor, which is defined by the Stokes law as

$$\boldsymbol{\sigma}(\boldsymbol{u}, p) = -p\boldsymbol{I}_N + 2\boldsymbol{D}(\boldsymbol{u}),$$

where $\boldsymbol{I}_N$ is the identity matrix of $\mathcal{M}_N(\mathbb{R})$, with $\mathcal{M}_N(\mathbb{R})$ denoting the space of real square matrices of order N, and where $\boldsymbol{D}(\boldsymbol{u})$ is the strain tensor defined by

$$[\boldsymbol{D}(\boldsymbol{u})]_{kl} = \frac{1}{2}\Big(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\Big). \tag{1.6}$$

To simplify the writing, we take in this paper the kinematic viscosity of the fluid equal to 1.

The idea is to impose a condition $\boldsymbol{f}$ in (1.4) and to measure the corresponding Cauchy force

$$\boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n}_{|(0,T)\times\partial\Omega} \tag{1.7}$$

in order to deduce information on the obstacle $\mathcal{S}$. Here and in all what follows, $\boldsymbol{n}$ denotes the unit outer normal to the fluid domain.

We also consider in this paper the following linear fluid–rigid body system:

$$\frac{\partial \boldsymbol{u}}{\partial t} - \operatorname{\mathbf{div}} \boldsymbol{\sigma}(\boldsymbol{u}, p) = \boldsymbol{0} \quad \text{in } (0, T) \times \mathcal{F}, \tag{1.8}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } (0, T) \times \mathcal{F}, \tag{1.9}$$

$$\boldsymbol{u} = \boldsymbol{f} \quad \text{on } (0, T) \times \partial\Omega, \tag{1.10}$$

$$\boldsymbol{u} = \boldsymbol{\ell} + \boldsymbol{\omega} \times \boldsymbol{y} \quad \text{on } (0, T) \times \partial\mathcal{S}, \tag{1.11}$$

$$m\boldsymbol{\ell}' + \int_{\partial\mathcal{S}} \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n}\, d\gamma = \boldsymbol{0} \quad \text{in } (0, T), \tag{1.12}$$

$$\boldsymbol{I}_0\boldsymbol{\omega}' + \int_{\partial\mathcal{S}} \boldsymbol{y} \times \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n}\, d\gamma = \boldsymbol{0} \quad \text{in } (0, T), \tag{1.13}$$

$$\boldsymbol{u}(0, \cdot) = \boldsymbol{0} \quad \text{in } \mathcal{F}, \tag{1.14}$$

$$\boldsymbol{\ell}(0) = \boldsymbol{0}, \quad \boldsymbol{\omega}(0) = \boldsymbol{0}. \tag{1.15}$$

Here $\boldsymbol{\ell}$ and $\boldsymbol{\omega}$ represents respectively the linear and angular velocity of the rigid body. Let us note that in this simplified fluid-rigid body system, the structure domain $\mathcal{S}$ is fixed. We assume that the density $\rho^{\mathcal{S}}$ of the rigid body is a positive constant. In particular, the mass m and the inertia tensor I_0 are defined as follows:

$$m = \rho^{\mathcal{S}}\mu_3(\mathcal{S}), \quad \boldsymbol{I}_0 = \rho^{\mathcal{S}} \int_{\mathcal{S}} |\boldsymbol{x}|^2\boldsymbol{I}_3 - (\boldsymbol{x} \otimes \boldsymbol{x})\, d\boldsymbol{x},$$

where μ_3 denotes the Lebesgue measure in $\mathbb{R}^3$ and where $\boldsymbol{I}_3$ is the 3×3 identity matrix.

In dimension $N = 2$, the above system is slightly modified: $\omega \in \mathbb{R}$, $I_0 \in \mathbb{R}$, equation (1.11) and equation (1.13) write respectively

$$\boldsymbol{u} = \boldsymbol{\ell} + \omega\boldsymbol{y}^{\perp}, \quad I_0\omega' + \int_{\partial\mathcal{S}} \boldsymbol{y}^{\perp} \cdot \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n}\, d\gamma = 0,$$

where

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^{\perp} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}.$$

Finally, I_0 is defined by

$$I_0 = \rho^{\mathcal{S}} \int_{\mathcal{S}} |\boldsymbol{x}|^2\, d\boldsymbol{x}. \tag{1.16}$$

System (1.1)–(1.5) is a linear simplification of the classical Navier–Stokes system

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \operatorname{\mathbf{div}} \boldsymbol{\sigma}(\boldsymbol{u}, p) = \mathbf{0} \quad \text{in } (0, T) \times \mathcal{F}, \tag{1.17}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } (0, T) \times \mathcal{F}, \tag{1.18}$$

$$\boldsymbol{u} = \boldsymbol{f} \quad \text{on } (0, T) \times \partial\Omega, \tag{1.19}$$

$$\boldsymbol{u} = \mathbf{0} \quad \text{on } (0, T) \times \partial\mathcal{S}, \tag{1.20}$$

$$\boldsymbol{u}(0, \cdot) = \mathbf{0} \quad \text{in } \mathcal{F}, \tag{1.21}$$

and system (1.8)–(1.15) is a linear simplification of the "full" fluid-rigid body system that can be written in dimension $N = 2$ as

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \operatorname{\mathbf{div}} \boldsymbol{\sigma}(\boldsymbol{u}, p) = \mathbf{0}, \quad t \in (0, T),\ \boldsymbol{x} \in \mathcal{F}(t), \tag{1.22}$$

$$\operatorname{div} \boldsymbol{u} = 0, \quad t \in (0, T),\ \boldsymbol{x} \in \mathcal{F}(t), \tag{1.23}$$

$$\boldsymbol{u} = \boldsymbol{f}, \quad t \in (0, T),\ \boldsymbol{x} \in \partial\Omega, \tag{1.24}$$

$$\boldsymbol{u} = \boldsymbol{\ell} + \omega(\boldsymbol{x} - \boldsymbol{h})^{\perp}, \quad t \in (0, T),\ \boldsymbol{x} \in \partial\mathcal{S}(t), \tag{1.25}$$

$$m\boldsymbol{\ell}' + \int_{\partial\mathcal{S}(t)} \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n}\, d\boldsymbol{\gamma} = \mathbf{0}, \quad t \in (0, T), \tag{1.26}$$

$$I_0\omega' + \int_{\partial\mathcal{S}(t)} (\boldsymbol{x} - \boldsymbol{h})^{\perp} \cdot \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n}\, d\boldsymbol{\gamma} = 0, \quad t \in (0, T), \tag{1.27}$$

$$\boldsymbol{h}' = \boldsymbol{\ell}, \quad t \in (0, T), \tag{1.28}$$

$$\theta' = \omega, \quad t \in (0, T), \tag{1.29}$$

$$\boldsymbol{u}(0, \cdot) = \mathbf{0}, \quad \boldsymbol{x} \in \mathcal{F}(0), \tag{1.30}$$

$$\boldsymbol{\ell}(0) = \mathbf{0}, \quad \omega(0) = 0, \tag{1.31}$$

$$\boldsymbol{h}(0) = \boldsymbol{h}^0, \quad \theta(0) = \theta^0, \tag{1.32}$$

$$\mathcal{S}(t) = \boldsymbol{h}(t) + \boldsymbol{R}_{\theta(t)}\mathcal{S}_0. \tag{1.33}$$

Here $\boldsymbol{h}(t) \in \mathbb{R}^2$ and $\theta(t) \in \mathbb{R}$ are respectively the center of mass and the orientation of $\mathcal{S}(t)$. In particular, the center of mass of $\mathcal{S}_0$ is located at $\mathbf{0}$. We have denoted by $\boldsymbol{R}_\theta$ the matrix of rotation of angle θ. Contrary to system (1.8)–(1.15), here the solid is moving (equation (1.33)). Let us emphasize that system (1.8)–(1.15) is important to study system (1.22)–(1.33): for instance, this linear system is used in [36, 37] to prove the existence of strong solutions for system (1.22)–(1.33) with the aid of a fixed point argument. Let us also note that equations (1.26), (1.27) for the rigid body are the Newton laws.

As for the previous systems, the idea is to take some particular choice of $\boldsymbol{f}$ and to measure the corresponding Cauchy force given by (1.7) in order to obtain geometrical information on $\mathcal{S}(t)$. However, here there is an important difference: applying $\boldsymbol{f}$ at the boundary of $\partial\Omega$ makes the rigid body moves through the (unknown) trajectory $(\boldsymbol{h}(t), \theta(t))$. Moreover, with such a boundary condition, it could possible that the rigid body touches $\partial\Omega$ and it is not clear what happens after this contact (see [34]).

We also consider a simplification of system (1.22)–(1.33) obtained by assuming that the Reynolds number is very small. In that case, neglecting the inertia forces, the 3D version of system (1.22)–(1.33) can be approximated by

$$-\operatorname{\mathbf{div}} \boldsymbol{\sigma}(\boldsymbol{u}, p) = \mathbf{0}, \quad t \in (0, T),\ \boldsymbol{x} \in \mathcal{F}(t), \tag{1.34}$$

$$\operatorname{div} \boldsymbol{u} = 0, \quad t \in (0, T),\ \boldsymbol{x} \in \mathcal{F}(t), \tag{1.35}$$

$$\boldsymbol{u} = \boldsymbol{f}, \quad t \in (0, T),\ \boldsymbol{x} \in \partial\Omega, \tag{1.36}$$

$$\boldsymbol{u} = \boldsymbol{\ell} + \boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{h}), \quad t \in (0, T),\ \boldsymbol{x} \in \partial\mathcal{S}(t), \tag{1.37}$$

$$\int_{\partial\mathcal{S}(t)} \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n}\, d\boldsymbol{\gamma} = \mathbf{0} \quad t \in (0, T), \tag{1.38}$$

$$\int_{\partial \mathcal{S}(t)} (\boldsymbol{x} - \boldsymbol{h}) \times \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n}\, d\gamma = \boldsymbol{0} \qquad t \in (0, T), \tag{1.39}$$

$$\boldsymbol{h}' = \boldsymbol{\ell}, \qquad t \in (0, T), \tag{1.40}$$

$$\boldsymbol{Q}' = \mathbb{A}(\boldsymbol{\omega})\boldsymbol{Q}, \qquad t \in (0, T), \tag{1.41}$$

$$\mathcal{S}(t) = \boldsymbol{h}(t) + \boldsymbol{Q}(t)\mathcal{S}_0, \qquad t \in (0, T). \tag{1.42}$$

The map $\mathbb{A}$ is defined as follows:

$$\mathbb{A}(\boldsymbol{r}) = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix}, \quad \boldsymbol{r} \in \mathbb{R}^3.$$

This map is related to the vector product by the formula

$$\mathbb{A}(\boldsymbol{r})\boldsymbol{x} = \boldsymbol{r} \times \boldsymbol{x}.$$

Let us remark that the above system is not linear since $\mathcal{S}(t)$ is not given. This system is studied in [7] where the identifiability of the rigid body is obtained through the measurement of the Cauchy forces on the boundary. Like system (1.22)–(1.33), the solid moves through the action of $\boldsymbol{f}$ on this system.

These geometrical inverse problems for fluid systems were already considered in [1] where the authors tackle the problem of recovering the shape and location of a fixed obstacle in a viscous incompressible fluid modeled by the Navier–Stokes system. They show the identifiability of the fixed obstacle: if $\boldsymbol{f}$ not identically equal to $\boldsymbol{0}$, then the mapping that associates to $\mathcal{S}$ the measurement given by (1.7) is one-to-one. They also prove a stability result: if two measurements are close, it implies that the two corresponding obstacles are close. Extensions of this result in the case of a fixed obstacle are obtained in [9] and in [10]. In [2], the authors consider a similar problem in the 2D case and for a fluid modeled by the Stokes system. They develop an integral method in order to recover the structure. The identifiability result of [1] is extended in [7] to the case of a moving rigid body, but only in the case of the stationary Stokes system. In the case of a potential fluid (thus inviscid), one can use, in 2D, complex analysis ([5, 6]) to detect a moving rigid body of particular shape (ball, ellipse) if the fluid fills the exterior of the structure domain.

Numerical aspects are considered in [3]: the authors use shape optimization techniques to detect a fixed obstacle in a viscous incompressible fluid. They prove in particular that the shape Hessian is compact and thus that the problem is ill-posed.

Here we are interested in obtaining geometrical information on the obstacle such as the distance from a fixed point to the obstacle or its convex hull. The problem of finding the distance from a fixed point was considered in [16], in the case of a fixed obstacle in a stationary Stokes fluid. In that study, they use a method based on complex geometrical solutions that was introduced in [35] and that has been applied in several inverse problems ([8, 12, 15, 29–31], etc.). In order to recover the convex hull of the obstacle, Ikehata introduced the enclosure method and used it in [21–23], etc. The above references were devoted to works on stationary problems. The case of the heat equation was considered in [13] with the use of complex geometrical solutions and [24, 27, 28] for the enclosure method.

In this work, we consider both methods to deal with nonstationary fluid or fluid-structure systems. More precisely, we use the approach in [27] in order to deal with the nonstationary Stokes system. A first step consists in considering the Laplace transform of the system in order to transform it into a stationary Stokes-type system. Then we show that if $(\hat{\boldsymbol{v}}_\alpha, \hat{q}_\alpha)$ is a family of solutions the same (stationary) system but on the whole domain Ω (see (2.1)–(2.2)), then a quantity (see (2.7)) based on the measurement given by (1.7) behaves in similar way as the H^1 norm of $\hat{\boldsymbol{v}}_\alpha$ on $\mathcal{S}$ as α goes to ∞ (Theorem 2.1). The idea is then to construct solutions $\hat{\boldsymbol{v}}_\alpha$ so that the H^1 norm on $\mathcal{S}$ gives geometrical information on the domain. One of the difficulties in this construction comes from the fact that here the test functions are divergence free. In particular, in the case of the distance of $\mathcal{S}$ to a point $\boldsymbol{x}_0$, we need to impose $\boldsymbol{x}_0 \notin \mathrm{ch}(\Omega)$ and $N = 3$. These hypothesis are not considered in the case of the heat equation (see [27]).

The above method can not be adapted to the case of nonlinear systems such as (1.17)–(1.21) and (1.22)–(1.33). As a consequence, for these nonlinear systems we use complex geometrical solutions constructed in [16]. This allows us to recover only some partial information, and more precisely, at the contrary to the linear case, we lose one of the inequalities. Nevertheless, these two approaches give some first results in the case of nonstationary fluid systems.

The plan of the paper is the following: in Section 2, we state our main results, for the linear systems and for the nonlinear systems. We recall some preliminaries in Section 3, that allow us to prove our first main result in Section 4: the relation between the measurement and the H^1 norm of $\hat{\boldsymbol{v}}_\alpha$ on $\mathcal{S}$, as explained above. Then in Section 5, we construct $\hat{\boldsymbol{v}}_\alpha$ in order to recover the convex hull of $\mathcal{S}$ and in Section 6, we construct $\hat{\boldsymbol{v}}_\alpha$ in order to recover the distance from a fixed point to $\mathcal{S}$. Section 7 is devoted to inverse problems for the nonlinear systems: we use there complex geometrical solutions.

2 Main results

Let us first describe the method used to recover geometric information on the obstacle $\mathcal{S}$ in the case of the linear systems (1.1)–(1.5) and (1.8)–(1.15).

First we consider a family $(\hat{\boldsymbol{v}}_\alpha, \hat{q}_\alpha) \in C^2(\overline{\Omega}) \times C^1(\overline{\Omega})$ of solutions of a Stokes system

$$\alpha\hat{\boldsymbol{v}}_\alpha - \mathbf{div}\,\boldsymbol{\sigma}(\hat{\boldsymbol{v}}_\alpha, \hat{q}_\alpha) = \mathbf{0} \quad \text{in } \widetilde{\Omega}, \tag{2.1}$$

$$\operatorname{div} \hat{\boldsymbol{v}}_\alpha = 0 \quad \text{in } \widetilde{\Omega}, \tag{2.2}$$

for some domain $\widetilde{\Omega} \supseteq \Omega$ and for $\alpha > 0$.

We then consider $\boldsymbol{f}_\alpha$ defined by

$$\boldsymbol{f}_\alpha(t, \boldsymbol{x}) := \chi_\alpha(t)\hat{\boldsymbol{v}}_\alpha(\boldsymbol{x}), \tag{2.3}$$

with $\chi_\alpha \in C^\infty([0, T])$ such that $\chi_\alpha(0) = 0$ and $\chi_\alpha(t) > 0$ in $(0, T]$ and such that

$$\int_0^T e^{-\alpha t}\chi_\alpha(t)\,dt = 1.$$

For instance, in what follows, we take

$$\chi_\alpha(t) = \frac{\alpha^2 t}{1 - (1 + \alpha T)e^{-\alpha T}}, \quad t \in [0, T]. \tag{2.4}$$

In particular,

$$\hat{\boldsymbol{f}}_\alpha(\boldsymbol{x}) := \int_0^T e^{-\alpha t}\boldsymbol{f}_\alpha(t, \boldsymbol{x})\,dt = \hat{\boldsymbol{v}}_\alpha(\boldsymbol{x}), \quad \boldsymbol{x} \in \partial\Omega. \tag{2.5}$$

We can remark that since $\boldsymbol{f}_\alpha$ is given by (2.3), then it satisfies the condition

$$\int_{\partial\Omega} \boldsymbol{f}_\alpha \cdot \boldsymbol{n}\,d\gamma = 0 \quad \text{on } (0, T).$$

The above equation allows us to consider the solution $(\boldsymbol{u}_\alpha, p_\alpha)$ of the Stokes system (1.1)–(1.5), with the boundary condition

$$\boldsymbol{u}_\alpha = \boldsymbol{f}_\alpha \quad \text{on } (0, T) \times \partial\Omega. \tag{2.6}$$

Let us set

$$E_\alpha := \int_{\partial\Omega}\int_0^T e^{-\alpha t}\big(\hat{\boldsymbol{v}}_\alpha \cdot \boldsymbol{\sigma}(\boldsymbol{u}_\alpha, p_\alpha)\boldsymbol{n} - \boldsymbol{u}_\alpha \cdot \boldsymbol{\sigma}(\hat{\boldsymbol{v}}_\alpha, \hat{q}_\alpha)\boldsymbol{n}\big)\,dt\,d\gamma. \tag{2.7}$$

We are now in a position to state our first main result.

Theorem 2.1. *Assume* $(\hat{\boldsymbol{v}}_\alpha, \hat{q}_\alpha)$ *satisfies* (2.1)–(2.2) *and* $(\boldsymbol{u}_\alpha, p_\alpha)$ *is the solution of system* (1.1)–(1.5) *with* (2.3) *and* (2.6). *Then* E_α *defined by* (2.7) *satisfies*

$$\bigg(\int_{\mathcal{S}} \alpha|\hat{\boldsymbol{v}}_\alpha|^2 + 2|\boldsymbol{D}(\hat{\boldsymbol{v}}_\alpha)|^2\,d\boldsymbol{x}\bigg) - C\alpha^2 e^{-\alpha T}\|\hat{\boldsymbol{v}}_\alpha\|^2_{H^1(\Omega)} \leqslant E_\alpha$$
$$\leqslant C(\alpha+1)\bigg(\int_{\mathcal{S}} |\hat{\boldsymbol{v}}_\alpha|^2 + 2|\boldsymbol{D}(\hat{\boldsymbol{v}}_\alpha)|^2\,d\boldsymbol{x}\bigg) + C\alpha^2 e^{-\alpha T}\|\hat{\boldsymbol{v}}_\alpha\|^2_{H^1(\Omega)}. \quad (2.8)$$

The above result and the two corollaries below correspond the closure method associated with the evolutionary Stokes system. A general framework for this method in the case of heat type equations is developed in [25]. The first extension of this method to a system of partial differential equation was developed in [26].

The first corollary of Theorem 2.1 corresponds to the reconstruction of the support function $h_{\mathcal{S}}$ of $\mathcal{S}$. Let us recall that for any subset G of $\mathbb{R}^3$, the support function h_G of G is defined by

$$h_G(\boldsymbol{\kappa}) = \sup_{\boldsymbol{x}\in G} \boldsymbol{\kappa}\cdot\boldsymbol{x}, \quad \boldsymbol{\kappa} \in \mathbb{S}^2, \tag{2.9}$$

where $\mathbb{S}^2$ is the unit sphere of $\mathbb{R}^3$. This function is classically used in the theory of convex sets (see, for instance, [4, p. 26]). In particular, if G is convex,

$$G = \{\boldsymbol{x} \in \mathbb{R}^3 : \boldsymbol{x}\cdot\boldsymbol{\kappa} \leqslant h_G(\boldsymbol{\kappa}) \text{ for all } \boldsymbol{\kappa} \in \mathbb{S}^2\}.$$

Corollary 2.2 (Recovering the support function). *Assume* $\partial\mathcal{S}$ *is of class* C^2. *There exists a family of solutions* $(\hat{\boldsymbol{v}}_\alpha, \hat{q}_\alpha)$ *of* (2.1)–(2.2) *such that the solution* $(\boldsymbol{u}_\alpha, p_\alpha)$ *of* (1.1)–(1.5) *with* (2.6) *and* (2.3) *verifies*

$$\lim_{\alpha\to+\infty} \frac{1}{2\sqrt{\alpha}} \log(E_\alpha) = h_{\mathcal{S}}(\boldsymbol{\kappa}).$$

The second corollary of Theorem 2.1 allows us to obtain the distance $d(\boldsymbol{x}_0, \mathcal{S})$ of $\mathcal{S}$ to a point $\boldsymbol{x}_0 \notin \mathrm{ch}(\Omega)$ (the convex hull of Ω).

Corollary 2.3 (Recovering the distance to a point). *Assume* $N = 3$, $\partial\mathcal{S}$ *is of class* C^2 *and* $\boldsymbol{x}_0 \notin \mathrm{ch}(\Omega)$. *There exists a family of solutions* $(\hat{\boldsymbol{v}}_\alpha, \hat{q}_\alpha)$ *of* (2.1)–(2.2) *such that the solution* $(\boldsymbol{u}_\alpha, p_\alpha)$ *of* (1.1)–(1.5) *with* (2.6) *and* (2.3) *verifies*

$$\lim_{\alpha\to+\infty} \frac{1}{2\sqrt{\alpha}} \log(E_\alpha) = -d(\boldsymbol{x}_0, \mathcal{S}).$$

Remark 2.4. In contrast to [27, 28], in the above result, we have to assume that $\boldsymbol{x}_0 \notin \mathrm{ch}(\Omega)$. This restriction comes from the fact that we need in our construction that the family $(\hat{\boldsymbol{v}}_\alpha, \hat{q}_\alpha)$ satisfies the condition $\operatorname{div}\hat{\boldsymbol{v}}_\alpha = 0$. In [27, 28], the authors also manage to reconstruct the smallest sphere centered at a point and enclosing the obstacle. Here, we cannot extend their construction since we need the free divergence condition for $\hat{\boldsymbol{v}}_\alpha$.

We have similar results for the linear system (1.8)–(1.15):

Theorem 2.5. *Assume* $\partial\mathcal{S}$ *is of class* C^2.

(1) *There exists a family of solutions* $(\hat{\boldsymbol{v}}_\alpha, \hat{q}_\alpha)$ *satisfying* (2.1)–(2.2) *such that the solution* $(\boldsymbol{u}_\alpha, p_\alpha, \boldsymbol{\ell}_\alpha, \boldsymbol{\omega}_\alpha)$ *of* (1.8)–(1.15) *with* (2.3), (2.6) *verifies*

$$\lim_{\alpha\to+\infty} \frac{1}{2\sqrt{\alpha}} \log(E_\alpha) = h_{\mathcal{S}}(\boldsymbol{\kappa}).$$

(2) *Assume* $\boldsymbol{x}_0 \notin \mathrm{ch}(\Omega)$ *and* $N = 3$. *There exists a family of solutions* $(\hat{\boldsymbol{v}}_\alpha, \hat{q}_\alpha)$ *satisfying* (2.1)–(2.2) *such that the solution* $(\boldsymbol{u}_\alpha, p_\alpha, \boldsymbol{\ell}_\alpha, \boldsymbol{\omega}_\alpha)$ *of* (1.8)–(1.15) *with* (2.3), (2.6) *verifies*

$$\lim_{\alpha\to+\infty} \frac{1}{2\sqrt{\alpha}} \log(E_\alpha) = -d(\boldsymbol{x}_0, \mathcal{S}).$$

The proof of the previous theorem is completely similar to the proof of Theorem 2.1, with the same families constructed in Corollary 2.2 and Corollary 2.3. Therefore, we omit its proof.

In the case of the nonlinear system (1.34)–(1.42), we use a family of solutions $(\boldsymbol{v}_\alpha, q_\alpha) \in C^2(\overline{\Omega}) \times C^1(\overline{\Omega})$ of

$$-\operatorname{\mathbf{div}} \boldsymbol{\sigma}(\boldsymbol{v}_\alpha, q_\alpha) = \mathbf{0} \quad \text{in } \widetilde{\Omega}, \tag{2.10}$$

$$\operatorname{div} \boldsymbol{v}_\alpha = 0 \quad \text{in } \widetilde{\Omega}, \tag{2.11}$$

for some domain $\widetilde{\Omega} \supseteq \Omega$. Here $\alpha > 0$ is a parameter in the construction of these solutions that eventually goes to ∞. We then consider $\boldsymbol{f}_\alpha$ defined by

$$\boldsymbol{f}_\alpha(\boldsymbol{x}) := \boldsymbol{v}_\alpha(\boldsymbol{x}). \tag{2.12}$$

As in the linear case, we then consider the solution $(\boldsymbol{u}_\alpha, p_\alpha)$ of systems (1.34)–(1.42) (respectively (1.22)–(1.33), and (1.17)–(1.21)), with the boundary condition

$$\boldsymbol{u}_\alpha = \boldsymbol{f}_\alpha \quad \text{on } (0, T) \times \partial\Omega. \tag{2.13}$$

We set

$$F_\alpha := \int_{\partial\Omega} (\boldsymbol{v}_\alpha \cdot \boldsymbol{\sigma}(\boldsymbol{u}_\alpha, p_\alpha)\boldsymbol{n} - \boldsymbol{u}_\alpha \cdot \boldsymbol{\sigma}(\boldsymbol{v}_\alpha, q_\alpha)\boldsymbol{n})\, d\gamma. \tag{2.14}$$

As explained in the previous section, one difficulty for stating result for this system is that the rigid body can touch $\partial\Omega$. We thus assume that for all regular $\boldsymbol{f}$,

$$d(\mathcal{S}(t), \partial\Omega) > 0 \quad \text{for all } t \in [0, T]. \tag{2.15}$$

Such an hypothesis is satisfied for instance in the case where $\mathcal{S}_0$ and Ω are balls (see [17–19]).

We fix $\boldsymbol{x}_0 \notin \operatorname{ch}(\Omega)$ (the convex hull of Ω) and $d > 0$. Then, we have the following results.

Theorem 2.6. *Assume $\partial\mathcal{S}$ is of class C^2, $d > 0$ and $\boldsymbol{x}_0 \notin \operatorname{ch}(\Omega)$. Assume also that (2.15) holds. Then, there exists a family of solutions $(\boldsymbol{v}_\alpha, q_\alpha)$ satisfying (2.10)–(2.11) such that the solution $(\boldsymbol{u}_\alpha, p_\alpha, \boldsymbol{\ell}_\alpha, \boldsymbol{\omega}_\alpha)$ of (1.34)–(1.42) with (2.13) verifies:*
(1) *If $d < d(\boldsymbol{x}_0, \mathcal{S}(t))$, then $F_\alpha \leqslant CA^\alpha$ for some constants $C > 0$ and $A \in (0, 1)$.*
(2) *If $d > d(\boldsymbol{x}_0, \mathcal{S}(t))$, then $F_\alpha \geqslant CB^\alpha$ for some constants $C > 0$ and $B > 1$ and for $\alpha > 1$.*

Remark 2.7. The above result is based on the construction of spherical geometrical optics solutions. In the case of Stokes-type system, such a construction has been done in [16]. Let us point out that in their method use the Hahn–Banach theorem. In the case of the Calderon problem, another construction that is not using the Hahn–Banach theorem is done in [20].

For systems (1.22)–(1.33), and (1.17)–(1.21), we slightly modify the boundary condition by using $(\boldsymbol{v}_\alpha, q_\alpha)$ depending on time and satisfying (2.10)–(2.11) for all t and we consider the following measurement:

$$K_\alpha := \int_0^T \int_{\partial\Omega} (\boldsymbol{v}_\alpha \cdot \boldsymbol{\sigma}(\boldsymbol{u}_\alpha, p_\alpha)\boldsymbol{n} - \boldsymbol{u}_\alpha \cdot \boldsymbol{\sigma}(\boldsymbol{v}_\alpha, q_\alpha)\boldsymbol{n})\, dt\, d\gamma - \int_0^T \int_{\partial\Omega} (\boldsymbol{f}_\alpha \cdot \boldsymbol{n}) \frac{|\boldsymbol{f}_\alpha|^2}{2}\, d\gamma. \tag{2.16}$$

In the case of system (1.22)–(1.33), we need to assume again (2.15) to prevent possible contacts. Again this condition is satisfied for instance in the case where $\mathcal{S}_0$ and Ω are balls (see [17–19]). It is probably true for other geometries but up to now this has not been proven.

For both systems (1.17)–(1.21) and (1.22)–(1.33), we also impose that $N = 2$ since we are working with regular solutions and for $N = 3$ the existence of global (in time) regular solutions is an open problem. In particular, in the case $N = 3$, one should need to show that the times T_α of existence of the family of solutions $(\boldsymbol{v}_\alpha, q_\alpha)$ can be chosen uniformly with respect to α.

Theorem 2.8. *Suppose $N = 2$. Assume $\partial\mathcal{S}$ is of class C^2, $d > 0$ and $\boldsymbol{x}_0 \notin \operatorname{ch}(\Omega)$. There exists a family of solutions $(\boldsymbol{v}_\alpha, q_\alpha)$ satisfying (2.10)–(2.11) such that:*
(1) *The solution $(\boldsymbol{u}_\alpha, p_\alpha)$ of (1.17)–(1.21) with (2.13) verifies the following implication: if $(K_\alpha)_{\alpha>\alpha_0}$ is bounded, then $d < d(\boldsymbol{x}_0, \mathcal{S})$.*
(2) *The solution $(\boldsymbol{u}_\alpha, p_\alpha, \boldsymbol{\ell}_\alpha, \omega_\alpha)$ of (1.22)–(1.33) with (2.13) verifies the following implication: if $(K_\alpha)_{\alpha>\alpha_0}$ is bounded, then $d < d(\boldsymbol{x}_0, \mathcal{S}(t))$ for all $t \in [0, T]$.*

As explained in the introduction, the above result is only partial since with the other case (as in Theorem 2.6) is not present here. As it appear in the proof, it would imply to prove an estimate on the solutions $(\boldsymbol{u}_\alpha, p_\alpha, \boldsymbol{\ell}_\alpha, \omega_\alpha)$ for system (1.22)–(1.33).

For simplicity, we suppress in the proofs below the explicit dependence on α in the notation. For example, we write $\hat{\boldsymbol{v}}$ instead of $\hat{\boldsymbol{v}}_\alpha$.

3 Preliminaries

Lemma 3.1. *Assume $\hat{\boldsymbol{v}} \in H^1(\Omega)$ such that $\operatorname{div} \hat{\boldsymbol{v}} = 0$ in Ω. Consider a pair $(\boldsymbol{w}, \pi) \in H^1(\mathcal{F}) \times L^2(\mathcal{F})$ such that $\mathbf{div}\, \boldsymbol{\sigma}(\boldsymbol{w}, \pi) \in L^2(\mathcal{F})$. Then there exists a constant $C = C(\Omega, \mathcal{S})$ such that*

$$\left| \int_{\partial\Omega} \hat{\boldsymbol{v}} \cdot \boldsymbol{\sigma}(\boldsymbol{w}, \pi)\boldsymbol{n}\, d\gamma \right| \leqslant C\|\hat{\boldsymbol{v}}\|_{H^1(\Omega)}\big(\|\boldsymbol{D}(\boldsymbol{w})\|_{L^2(\mathcal{F})} + \|\mathbf{div}\, \boldsymbol{\sigma}(\boldsymbol{w}, \pi)\|_{L^2(\mathcal{F})}\big), \tag{3.1}$$

$$\left| \int_{\partial\mathcal{S}} \hat{\boldsymbol{v}} \cdot \boldsymbol{\sigma}(\boldsymbol{w}, \pi)\boldsymbol{n}\, d\gamma \right| \leqslant C\|\hat{\boldsymbol{v}}\|_{H^1(\mathcal{S})}\big(\|\boldsymbol{D}(\boldsymbol{w})\|_{L^2(\mathcal{F})} + \|\mathbf{div}\, \boldsymbol{\sigma}(\boldsymbol{w}, \pi)\|_{L^2(\mathcal{F})}\big). \tag{3.2}$$

Proof. We use [14, p. 176, relations (III.3.31) and (III.3.32)]: there exists $\hat{\boldsymbol{V}} \in H^1(\mathcal{F})$ such that

$$\operatorname{div} \hat{\boldsymbol{V}} = 0 \quad \text{in } \mathcal{F}, \qquad \hat{\boldsymbol{V}} = \hat{\boldsymbol{v}} \quad \text{on } \partial\Omega, \qquad \hat{\boldsymbol{V}} = \boldsymbol{0} \quad \text{on } \partial\mathcal{S},$$

with

$$\|\hat{\boldsymbol{V}}\|_{H^1(\mathcal{F})} \leqslant C\|\hat{\boldsymbol{v}}\|_{H^{1/2}(\partial\Omega)} \leqslant C\|\hat{\boldsymbol{v}}\|_{H^1(\Omega)}. \tag{3.3}$$

We then use integration by parts

$$\int_{\partial\Omega} \hat{\boldsymbol{v}} \cdot \boldsymbol{\sigma}(\boldsymbol{w}, \pi)\boldsymbol{n}\, d\gamma = \int_{\partial\mathcal{F}} \hat{\boldsymbol{V}} \cdot \boldsymbol{\sigma}(\boldsymbol{w}, \pi)\boldsymbol{n}\, d\gamma = \int_{\mathcal{F}} \mathbf{div}\, \boldsymbol{\sigma}(\boldsymbol{w}, \pi) \cdot \hat{\boldsymbol{V}}\, d\boldsymbol{x} + \int_{\mathcal{F}} 2\boldsymbol{D}(\boldsymbol{w}) : \boldsymbol{D}(\hat{\boldsymbol{V}})\, d\boldsymbol{x},$$

and (3.1) follows from (3.3).

The proof of (3.2) is similar, we consider (instead of $\hat{\boldsymbol{V}}$) a function $\hat{\boldsymbol{W}} \in H^1(\mathcal{F})$ such that

$$\operatorname{div} \hat{\boldsymbol{W}} = 0 \quad \text{in } \mathcal{F}, \qquad \hat{\boldsymbol{W}} = \boldsymbol{0} \quad \text{on } \partial\Omega, \qquad \hat{\boldsymbol{W}} = \hat{\boldsymbol{v}} \quad \text{on } \partial\mathcal{S},$$

with

$$\|\hat{\boldsymbol{W}}\|_{H^1(\mathcal{F})} \leqslant C\|\hat{\boldsymbol{v}}\|_{H^{1/2}(\partial\mathcal{S})} \leqslant C\|\hat{\boldsymbol{v}}\|_{H^1(\mathcal{S})}.$$

The proof of the lemma is complete. □

Proposition 3.2. *Assume $\boldsymbol{f} = \chi\hat{\boldsymbol{v}}$, with $\chi \in H^1(0, T)$, $\hat{\boldsymbol{v}} \in H^{3/2}(\partial\Omega)$ satisfying*

$$\chi(0) = 0, \qquad \int_{\partial\Omega} \hat{\boldsymbol{v}} \cdot \boldsymbol{n}\, d\gamma = 0.$$

Then:

(1) *There exists a unique solution $(\boldsymbol{u}, p)$ of system* (1.1)–(1.5)

$$\boldsymbol{u} \in L^2(0, T; H^2(\mathcal{F})) \cap C([0, T]; H^1(\mathcal{F})) \cap H^1(0, T; L^2(\mathcal{F})), \tag{3.4}$$

$$p \in L^2(0, T; H^1(\mathcal{F})/\mathbb{R}); \tag{3.5}$$

(2) *There exists a unique solution $(\boldsymbol{u}, p, \boldsymbol{\ell}, \boldsymbol{\omega})$ of system* (1.8)–(1.15) *satisfying* (3.4), (3.5) *and $\boldsymbol{\ell}, \boldsymbol{\omega} \in H^1(0, T)$.*

The above result is quite classical for system (1.1)–(1.5) and is similar for system (1.8)–(1.15). We only give here some ideas of the proof. Note that the particular form of $\boldsymbol{f}$ is not needed to obtain the result and the result remains true for more general boundary conditions.

Proof. Using, for instance, [33], there exists $\hat{\boldsymbol{V}} \in H^2(\mathcal{F})$ such that

$$\operatorname{div} \hat{\boldsymbol{V}} = 0 \quad \text{in } \mathcal{F}, \qquad \hat{\boldsymbol{V}} = \hat{\boldsymbol{v}} \quad \text{on } \partial\Omega, \qquad \hat{\boldsymbol{V}} = \boldsymbol{0} \quad \text{on } \partial\mathcal{S}.$$

Using this lifting, we consider the change of variables

$$\boldsymbol{U} = \boldsymbol{u} - \chi\hat{V}$$

and the equations for $(\boldsymbol{U}, p)$ can be written as

$$\frac{\partial \boldsymbol{U}}{\partial t} - \operatorname{\mathbf{div}} \boldsymbol{\sigma}(\boldsymbol{U}, p) = \boldsymbol{F} \quad \text{in } (0, T) \times \mathcal{F}, \tag{3.6}$$

$$\operatorname{div} \boldsymbol{U} = 0 \quad \text{in } (0, T) \times \mathcal{F}, \tag{3.7}$$

$$\boldsymbol{U} = \boldsymbol{0} \quad \text{on } (0, T) \times \partial\mathcal{F}, \tag{3.8}$$

$$\boldsymbol{U}(0, \cdot) = \boldsymbol{0} \quad \text{in } \mathcal{F}, \tag{3.9}$$

with

$$\boldsymbol{F} = \chi\Delta\hat{\boldsymbol{V}} - \chi'\hat{\boldsymbol{V}} \in L^2(0, T; L^2(\mathcal{F})). \tag{3.10}$$

To end the proof, one can write (3.6)–(3.9) with the Stokes operator $\boldsymbol{A} = -\boldsymbol{P}_0\Delta$ as

$$\boldsymbol{U}' + \boldsymbol{A}\boldsymbol{U} = \boldsymbol{P}_0 F,$$

where $\boldsymbol{P}_0 : L^2(\mathcal{F}) \to \mathcal{H}_0$ is the Leray projection on

$$\mathcal{H}_0 := \{\boldsymbol{w} \in L^2(\mathcal{F}) : \operatorname{div} \boldsymbol{w} = 0,\ \boldsymbol{w} \cdot \boldsymbol{n} = 0 \text{ on } \partial\mathcal{F}\}.$$

Using that $\boldsymbol{A}$ is self-adjoint and positive, we obtain the result.

For system (1.8)–(1.15), we can proceed with the same proof. Using the lifting $\hat{V}$, we are reduced to solve

$$\begin{aligned}
\frac{\partial \boldsymbol{U}}{\partial t} - \operatorname{\mathbf{div}} \boldsymbol{\sigma}(\boldsymbol{U}, p) &= \boldsymbol{F} && \text{in } (0, T) \times \mathcal{F},\\
\operatorname{div} \boldsymbol{U} &= 0 && \text{in } (0, T) \times \mathcal{F},\\
\boldsymbol{U} &= \boldsymbol{0} && \text{on } (0, T) \times \partial\Omega,\\
\boldsymbol{U} &= \boldsymbol{\ell} + \boldsymbol{\omega} \times \boldsymbol{y} && \text{on } (0, T) \times \partial\mathcal{S},\\
m\boldsymbol{\ell}' + \int_{\partial\mathcal{S}} \boldsymbol{\sigma}(\boldsymbol{U}, p)\boldsymbol{n}\, d\gamma &= \boldsymbol{\ell}_{\boldsymbol{F}} && \text{in } (0, T),\\
I_0\boldsymbol{\omega}' + \int_{\partial\mathcal{S}} \boldsymbol{y} \times \boldsymbol{\sigma}(\boldsymbol{U}, p)\boldsymbol{n}\, d\gamma &= \boldsymbol{\omega}_{\boldsymbol{F}} && \text{in } (0, T),\\
\boldsymbol{U}(0, \cdot) &= \boldsymbol{0} && \text{in } \mathcal{F},\\
\boldsymbol{\ell}(0) = \boldsymbol{0}, \quad \boldsymbol{\omega}(0) &= \boldsymbol{0},
\end{aligned}$$

with $\boldsymbol{F}$ given by (3.10) and

$$\boldsymbol{\ell}_{\boldsymbol{F}} = -2\chi \int_{\partial\mathcal{S}} \boldsymbol{D}(\hat{V})\boldsymbol{n}\, d\gamma, \quad \boldsymbol{\omega}_{\boldsymbol{F}} = -2\chi \int_{\partial\mathcal{S}} \boldsymbol{y} \times \boldsymbol{D}(\hat{V})\boldsymbol{n}\, d\gamma.$$

We recall that the operator $\boldsymbol{D}$ is defined in (1.6). It is classical that $\boldsymbol{D}\boldsymbol{w} = \boldsymbol{0}$ on $\mathcal{S}$ if and only if there exist $\boldsymbol{\ell}_{\boldsymbol{w}}, \boldsymbol{\omega}_{\boldsymbol{w}} \in \mathbb{R}^3$ such that $\boldsymbol{w}(\boldsymbol{y}) = \boldsymbol{\ell}_{\boldsymbol{w}} + \boldsymbol{\omega}_{\boldsymbol{w}} \times \boldsymbol{y}$ for $\boldsymbol{y} \in \mathcal{S}$ (see, for instance, in [32, p. 51]).

If we extend $\boldsymbol{U}$ and $\boldsymbol{F}$ in $\mathcal{S}$ by

$$\boldsymbol{U}(t, \boldsymbol{y}) = \boldsymbol{\ell}(t) + \boldsymbol{\omega}(t) \times \boldsymbol{y}, \quad \boldsymbol{F}(t, \boldsymbol{y}) = \boldsymbol{\ell}_{\boldsymbol{F}}(t) + \boldsymbol{\omega}_{\boldsymbol{F}}(t) \times \boldsymbol{y},$$

then the above system can be written as

$$\boldsymbol{U}' + \boldsymbol{A}\boldsymbol{U} = \boldsymbol{P}\boldsymbol{F},$$

with

$$\begin{aligned}
\mathcal{H} &:= \{\boldsymbol{w} \in L^2(\Omega) : \operatorname{div} \boldsymbol{w} = 0,\ \boldsymbol{w} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega,\ \boldsymbol{D}\boldsymbol{w} = \boldsymbol{0} \text{ in } \mathcal{S}\},\\
\boldsymbol{D}(\boldsymbol{A}) &:= \{\boldsymbol{w} \in \mathcal{H} \cap H^1(\Omega) : \boldsymbol{w} = \boldsymbol{0} \text{ on } \partial\Omega,\ \boldsymbol{w}_{|\mathcal{F}} \in H^2\},\\
\mathcal{A}\boldsymbol{w} &:= \begin{cases} -\Delta \boldsymbol{w} & \text{in } \mathcal{F},\\ \Big(\int\limits_{\partial\mathcal{S}} \boldsymbol{D}(\boldsymbol{w})\boldsymbol{n}\, d\gamma\Big) + \Big(\int\limits_{\partial\mathcal{S}} \boldsymbol{y} \times \boldsymbol{D}(\boldsymbol{w})\boldsymbol{n}\, d\gamma\Big) \times \boldsymbol{x}, & \boldsymbol{x} \in \mathcal{S}, \end{cases}\\
\boldsymbol{A} &:= \boldsymbol{P}\mathcal{A}
\end{aligned}$$

and $\boldsymbol{P} : L^2(\Omega) \to \mathcal{H}$ is the orthogonal projection. We have $\boldsymbol{P}\boldsymbol{F} \in L^2(0, T; \mathcal{H})$ and it is proved in [37] that $\boldsymbol{A}$ is self-adjoint and positive in $\mathcal{H}$, and this allows us to prove the result by using classical result on parabolic systems. □

Lemma 3.3. *Assume that $(\boldsymbol{u}, p)$ is the solution of (1.1)–(1.5) with $\boldsymbol{f}$ defined by (2.3). Then for α large enough, there exists a constant C (independent of α) such that*

$$\|\boldsymbol{u}\|_{L^\infty(0,T;L^2(\mathcal{F}))} + \|\boldsymbol{D}(\boldsymbol{u})\|_{L^2(0,T;L^2(\mathcal{F}))} \leqslant C\alpha^2 \|\hat{\boldsymbol{v}}\|_{H^1(\Omega)}. \tag{3.11}$$

Proof. Let us multiply (1.1) by $\boldsymbol{u}$ and integrate by parts:

$$\int\limits_{\mathcal{F}} \frac{d}{dt}\Big(\frac{|\boldsymbol{u}|^2}{2}\Big)\, d\boldsymbol{x} + \int\limits_{\mathcal{F}} 2|\boldsymbol{D}(\boldsymbol{u})|^2\, d\boldsymbol{x} = \chi(t) \int\limits_{\partial\Omega} \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n} \cdot \hat{\boldsymbol{v}}\, d\gamma. \tag{3.12}$$

In the above relation, we have used (2.3). Then we use the same function $\hat{\boldsymbol{V}}$ used in the proof of Lemma 3.1 and integrate by parts:

$$\begin{aligned}
\int\limits_{\partial\Omega} \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n} \cdot \hat{\boldsymbol{v}}\, d\gamma &= \int\limits_{\partial\mathcal{F}} \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n} \cdot \hat{\boldsymbol{V}}\, d\gamma\\
&= \int\limits_{\mathcal{F}} \operatorname{\mathbf{div}} \boldsymbol{\sigma}(\boldsymbol{u}, p) \cdot \hat{\boldsymbol{V}} + 2\boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\hat{\boldsymbol{V}})\, d\boldsymbol{x}\\
&= \frac{d}{dt} \int\limits_{\mathcal{F}} \boldsymbol{u} \cdot \hat{\boldsymbol{V}}\, d\boldsymbol{x} + 2 \int\limits_{\mathcal{F}} \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\hat{\boldsymbol{V}})\, d\boldsymbol{x}.
\end{aligned} \tag{3.13}$$

Combining (3.12) and (3.13) and integrating on $(0, t)$, we obtain

$$\frac{1}{2} \int\limits_{\mathcal{F}} |\boldsymbol{u}(t)|^2\, d\boldsymbol{x} + \int\limits_0^t \int\limits_{\mathcal{F}} 2|\boldsymbol{D}(\boldsymbol{u})|^2\, d\boldsymbol{x}\, ds = \chi(t) \int\limits_{\mathcal{F}} \boldsymbol{u}(t) \cdot \hat{\boldsymbol{V}}\, d\boldsymbol{x} - \int\limits_0^t \int\limits_{\mathcal{F}} \chi' \boldsymbol{u} \cdot \hat{\boldsymbol{V}}\, d\boldsymbol{x}\, ds + \int\limits_0^t \int\limits_{\mathcal{F}} 2\chi \boldsymbol{D}(\boldsymbol{u}) : \boldsymbol{D}(\hat{\boldsymbol{V}})\, d\boldsymbol{x}\, ds.$$

Using Gronwall's lemma, we deduce the existence of constant depending only on T such that

$$\sup_{(0,T)} \int\limits_{\mathcal{F}} |\boldsymbol{u}|^2\, d\boldsymbol{x} + \int\limits_0^T \int\limits_{\mathcal{F}} 2|\boldsymbol{D}(\boldsymbol{u})|^2\, d\boldsymbol{x}\, ds \leqslant C \|\hat{\boldsymbol{v}}\|^2_{H^1(\Omega)} \|\chi\|^2_{H^1(0,T)}.$$

With the choice (2.4), taking α large enough, we conclude that there exists $C = C(T, \Omega, \mathcal{S}) > 0$ such that (3.11) holds. □

We can obtain in a similar way the following lemma.

Lemma 3.4. *Assume that $(\boldsymbol{u}, p, \boldsymbol{\ell}, \boldsymbol{\omega})$ is the solution of (1.8)–(1.15) with $\boldsymbol{f}$ defined by (2.3). Then for α large enough, there exists a constant C (independent of α) such that*

$$\|\boldsymbol{u}\|_{L^\infty(0,T;L^2(\Omega))} + \|\boldsymbol{D}(\boldsymbol{u})\|_{L^2(0,T;L^2(\Omega))} \leqslant C\alpha^2 \|\hat{\boldsymbol{v}}\|_{H^1(\Omega)}.$$

In the above result, we have extended $\boldsymbol{u}$ in $\mathcal{S}$ by setting

$$\boldsymbol{u}(t, \boldsymbol{x}) := \boldsymbol{\ell}(t) + \boldsymbol{\omega}(t) \times \boldsymbol{x}, \quad \boldsymbol{x} \in \mathcal{S}.$$

Finally, we end this section by recalling existence results for systems (1.17)–(1.21) and (1.22)–(1.33), for $N = 2$.

Proposition 3.5. *Assume $N = 2$ and assume $\boldsymbol{f} = \chi\hat{\boldsymbol{v}}$, with $\chi \in H^1(0,T)$, $\hat{\boldsymbol{v}} \in H^{3/2}(\partial\Omega)$ satisfying*

$$\chi(0) = 0, \qquad \int_{\partial\Omega} \hat{\boldsymbol{v}} \cdot \boldsymbol{n}\, d\gamma = 0.$$

Then the following hold:

(1) *There exists a unique solution $(\boldsymbol{u}, p)$ of system* (1.17)–(1.21) *with* (3.4), (3.5).

(2) *Assume* (2.15). *There exists a unique solution $(\boldsymbol{u}, p, \boldsymbol{\ell}, \omega)$ of system* (1.22)–(1.33) *satisfying $\boldsymbol{h}, \theta \in H^2(0,T)$ and*

$$\boldsymbol{u} \in L^2(0,T; H^2(\mathcal{F}(\boldsymbol{h},\theta))) \cap C([0,T]; H^1(\mathcal{F}(\boldsymbol{h},\theta))) \cap H^1(0,T; L^2(\mathcal{F}(\boldsymbol{h},\theta))), \tag{3.14}$$

$$p \in L^2(0,T; H^1(\mathcal{F}(\boldsymbol{h},\theta))/\mathbb{R}). \tag{3.15}$$

The first result is classical and the second result was proved in [36]. It is possible to prove the first result by using a fixed point approach: one can consider the mapping

$$\boldsymbol{F} \mapsto -(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \tag{3.16}$$

where $(\boldsymbol{u}, p)$ is the solution of

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} - \operatorname{\mathbf{div}} \boldsymbol{\sigma}(\boldsymbol{u}, p) &= \boldsymbol{F} && \text{in } (0,T)\times\mathcal{F},\\ \operatorname{div} \boldsymbol{u} &= 0 && \text{in } (0,T)\times\mathcal{F},\\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } (0,T)\times\partial\mathcal{S},\\ \boldsymbol{u} &= \boldsymbol{f} && \text{on } (0,T)\times\partial\Omega,\\ \boldsymbol{u}(0,\cdot) &= \boldsymbol{0} && \text{in } \mathcal{F}. \end{aligned}$$

Using the Banach fixed point theorem and the above mapping, we can obtain the local in time existence of system (1.17)–(1.21). Then, we derive H^1 estimate (that is possible since $N = 2$) to deduce the global in time existence.

For system (1.22)–(1.33), the approach is similar but with several additional difficulties. First since we are working with a moving domain, it is convenient to consider a change of variables $\boldsymbol{X}(t,\cdot) : \mathcal{F}(0) \to \mathcal{F}(t)$ (construct from $\boldsymbol{h}, \theta$) and transform $\boldsymbol{u}$ in $\tilde{\boldsymbol{u}} := \mathbf{Cof}(\nabla\boldsymbol{X})^\top(\boldsymbol{u}\circ\boldsymbol{X})$ (where $\mathbf{Cof}(\nabla\boldsymbol{X})^\top$ is the transpose of the cofactor matrix of $\nabla\boldsymbol{X}$) and p in $\tilde{p} := (\det\nabla\boldsymbol{X})(p\circ\boldsymbol{X})$. In the above proposition, (3.14)–(3.15) means that

$$\begin{aligned} &\tilde{\boldsymbol{u}} \in L^2(0,T; H^2(\mathcal{F}(0))) \cap C([0,T]; H^1(\mathcal{F}(0))) \cap H^1(0,T; L^2(\mathcal{F}(0))),\\ &\tilde{p} \in L^2(0,T; H^1(\mathcal{F}(0))/\mathbb{R}). \end{aligned}$$

Then we can consider a fixed point as above but with using (1.8)–(1.15) instead of (1.1)–(1.5) and where in the application (3.16) we have to add nonlinear terms coming from the change of variables (see [36] for more details).

4 Proof of Theorem 2.1

Let us define for all $\alpha > 0$,

$$\hat{\boldsymbol{u}}(\boldsymbol{x}) := \int_0^T e^{-\alpha t}\boldsymbol{u}(t,\boldsymbol{x})\,dt, \qquad \hat{p}(\boldsymbol{x}) := \int_0^T e^{-\alpha t} p(t,\boldsymbol{x})\,dt.$$

Then, we deduce from (1.1)–(1.5) that

$$\begin{aligned} \alpha\hat{\boldsymbol{u}} - \operatorname{\mathbf{div}} \boldsymbol{\sigma}(\hat{\boldsymbol{u}}, \hat{p}) &= -e^{-\alpha T}\boldsymbol{u}(T) && \text{in } \mathcal{F},\\ \operatorname{div} \hat{\boldsymbol{u}} &= 0 && \text{in } \mathcal{F},\\ \hat{\boldsymbol{u}} &= \hat{\boldsymbol{f}} && \text{on } \partial\Omega,\\ \hat{\boldsymbol{u}} &= \boldsymbol{0} && \text{on } \partial\mathcal{S}, \end{aligned}$$

with $\hat{\boldsymbol{f}}$ defined by (2.5).

We consider the solution $(\hat{\boldsymbol{w}}, \hat{r})$ of the problem

$$\begin{aligned} \alpha\hat{\boldsymbol{w}} - \mathbf{div}\,\boldsymbol{\sigma}(\hat{\boldsymbol{w}}, \hat{r}) &= \mathbf{0} && \text{in } \mathcal{F},\\ \operatorname{div}\hat{\boldsymbol{w}} &= 0 && \text{in } \mathcal{F},\\ \hat{\boldsymbol{w}} &= \hat{\boldsymbol{f}} && \text{on } \partial\Omega,\\ \hat{\boldsymbol{w}} &= \mathbf{0} && \text{on } \partial\mathcal{S}. \end{aligned}$$

The couple $(\hat{\boldsymbol{u}} - \hat{\boldsymbol{w}}, \hat{p} - \hat{r})$ satisfies the system

$$\begin{aligned} \alpha(\hat{\boldsymbol{u}} - \hat{\boldsymbol{w}}) - \mathbf{div}\,\boldsymbol{\sigma}((\hat{\boldsymbol{u}} - \hat{\boldsymbol{w}}), (\hat{p} - \hat{r})) &= -e^{-\alpha T}\boldsymbol{u}(T) && \text{in } \mathcal{F},\\ \operatorname{div}(\hat{\boldsymbol{u}} - \hat{\boldsymbol{w}}) &= 0 && \text{in } \mathcal{F},\\ (\hat{\boldsymbol{u}} - \hat{\boldsymbol{w}}) &= \mathbf{0} && \text{on } \partial\mathcal{F}. \end{aligned} \tag{4.1}$$

Taking the inner product of (4.1) with $\hat{\boldsymbol{u}} - \hat{\boldsymbol{w}}$ and integrating by parts, we obtain

$$\alpha\|\hat{\boldsymbol{u}} - \hat{\boldsymbol{w}}\|^2_{L^2(\mathcal{F})} + 4\|\boldsymbol{D}(\hat{\boldsymbol{u}} - \hat{\boldsymbol{w}})\|^2_{L^2(\mathcal{F})} \leqslant \frac{1}{\alpha}e^{-2\alpha T}\|\boldsymbol{u}(T)\|^2_{L^2(\mathcal{F})}. \tag{4.2}$$

Since $(\hat{\boldsymbol{v}}, \hat{q})$ satisfies (2.1)–(2.2), the couple $(\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}}, \hat{q} - \hat{r})$ is solution of the system

$$\begin{aligned} \alpha(\hat{v} - \hat{\boldsymbol{w}}) - \mathbf{div}\,\boldsymbol{\sigma}((\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}}), (\hat{q} - \hat{r})) &= \mathbf{0} && \text{in } \mathcal{F},\\ \operatorname{div}(\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}}) &= 0 && \text{in } \mathcal{F},\\ (\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}}) &= \hat{\boldsymbol{v}} && \text{on } \partial\mathcal{S},\\ (\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}}) &= \mathbf{0} && \text{on } \partial\Omega. \end{aligned} \tag{4.3}$$

Taking the inner product of (4.3) with $(\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}})$ and integrating by parts, it follows

$$0 = \int_{\mathcal{F}} \alpha|\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}}|^2 + 2|\boldsymbol{D}(\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}})|^2\,d\boldsymbol{x} - \int_{\partial\mathcal{S}} \boldsymbol{\sigma}((\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}}), (\hat{q} - \hat{r}))n \cdot \hat{\boldsymbol{v}}\,d\boldsymbol{\gamma}. \tag{4.4}$$

Taking the inner product of (4.3) with $\hat{\boldsymbol{v}}$, taking the inner product of (2.1) with $(\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}})$ and integrating by parts yield

$$0 = -\int_{\partial\mathcal{F}} \boldsymbol{\sigma}((\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}}), (\hat{q} - \hat{r}))\boldsymbol{n} \cdot \hat{\boldsymbol{v}}\,d\boldsymbol{\gamma} + \int_{\partial\mathcal{F}} \boldsymbol{\sigma}(\hat{\boldsymbol{v}}, \hat{q})\boldsymbol{n} \cdot (\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}})\,d\boldsymbol{\gamma}.$$

The above relation implies

$$0 = -\int_{\partial\Omega} \boldsymbol{\sigma}((\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}}), (\hat{q} - \hat{r}))\boldsymbol{n} \cdot \hat{\boldsymbol{v}}\,d\boldsymbol{\gamma} - \int_{\partial\mathcal{S}} \boldsymbol{\sigma}((\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}}), (\hat{q} - \hat{r}))\boldsymbol{n} \cdot \hat{\boldsymbol{v}}\,d\boldsymbol{\gamma} + \int_{\partial\mathcal{S}} \boldsymbol{\sigma}(\hat{\boldsymbol{v}}, \hat{q})\boldsymbol{n} \cdot \hat{\boldsymbol{v}}\,d\boldsymbol{\gamma}. \tag{4.5}$$

Taking the inner product of (2.1) with $\hat{\boldsymbol{v}}$ and integrating by parts on $\mathcal{S}$, we obtain

$$\int_{\mathcal{S}} \alpha|\hat{\boldsymbol{v}}|^2 + 2|\boldsymbol{D}(\hat{\boldsymbol{v}})|^2\,d\boldsymbol{x} + \int_{\partial\mathcal{S}} \boldsymbol{\sigma}(\hat{\boldsymbol{v}}, \hat{q})\boldsymbol{n} \cdot \hat{\boldsymbol{v}}\,d\boldsymbol{\gamma} = 0. \tag{4.6}$$

Combining (4.4), (4.5) and (4.6), we deduce

$$-\int_{\partial\Omega} \boldsymbol{\sigma}((\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}}), (\hat{q} - \hat{r}))\boldsymbol{n} \cdot \hat{\boldsymbol{v}}\,d\boldsymbol{\gamma} = \int_{\mathcal{F}} \alpha|\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}}|^2 + 2|\boldsymbol{D}(\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}})|^2\,d\boldsymbol{x} + \int_{\mathcal{S}} \alpha|\hat{\boldsymbol{v}}|^2 + 2|\boldsymbol{D}(\hat{\boldsymbol{v}})|^2\,d\boldsymbol{x}. \tag{4.7}$$

We are now in a position to deal with E_α defined by (2.7). First we rewrite it as

$$E_\alpha = \int_{\partial\Omega} (\hat{\boldsymbol{v}} \cdot \boldsymbol{\sigma}(\hat{\boldsymbol{u}}, \hat{p})\boldsymbol{n} - \hat{\boldsymbol{u}} \cdot \boldsymbol{\sigma}(\hat{\boldsymbol{v}}, \hat{q})\boldsymbol{n})\,d\boldsymbol{\gamma} = \int_{\partial\Omega} \hat{\boldsymbol{f}} \cdot (\boldsymbol{\sigma}(\hat{\boldsymbol{u}}, \hat{p})\boldsymbol{n} - \boldsymbol{\sigma}(\hat{\boldsymbol{v}}, \hat{q})\boldsymbol{n})\,d\boldsymbol{\gamma}.$$

We can split E_α into two parts:

$$E_\alpha = \int_{\partial\Omega} \hat{\boldsymbol{f}} \cdot (\boldsymbol{\sigma}(\hat{\boldsymbol{w}}, \hat{r})\boldsymbol{n} - \boldsymbol{\sigma}(\hat{\boldsymbol{v}}, \hat{q})\boldsymbol{n})\,d\boldsymbol{\gamma} + \int_{\partial\Omega} \hat{\boldsymbol{f}} \cdot (\boldsymbol{\sigma}(\hat{\boldsymbol{u}}, \hat{p})\boldsymbol{n} - \boldsymbol{\sigma}(\hat{\boldsymbol{w}}, \hat{r})\boldsymbol{n})\,d\boldsymbol{\gamma}. \tag{4.8}$$

The second term on the right-hand side of the above relation can be estimated by using (3.1):

$$\Big|\int_{\partial\Omega} \hat{\boldsymbol{f}} \cdot (\boldsymbol{\sigma}(\hat{\boldsymbol{u}}, \hat{p})\boldsymbol{n} - \boldsymbol{\sigma}(\hat{\boldsymbol{w}}, \hat{r})\boldsymbol{n})\, d\gamma\Big| \leqslant C\|\hat{\boldsymbol{v}}\|_{H^1(\Omega)}(\|\boldsymbol{D}(\hat{\boldsymbol{u}} - \hat{\boldsymbol{w}})\|_{L^2(\mathcal{F})} + \|\mathbf{div}\,\boldsymbol{\sigma}(\hat{\boldsymbol{u}} - \hat{\boldsymbol{w}}, \hat{p} - \hat{r})\|_{L^2(\mathcal{F})})$$

and combining the above estimate with (4.1), we obtain

$$\Big|\int_{\partial\Omega} \hat{\boldsymbol{f}} \cdot (\boldsymbol{\sigma}(\hat{\boldsymbol{u}}, \hat{p})\boldsymbol{n} - \boldsymbol{\sigma}(\hat{\boldsymbol{w}}, \hat{r})\boldsymbol{n})\, d\gamma\Big| \leqslant C\|\hat{\boldsymbol{v}}\|_{H^1(\Omega)}\big(\|\boldsymbol{D}(\hat{\boldsymbol{u}} - \hat{\boldsymbol{w}})\|_{L^2(\mathcal{F})} + \alpha\|\hat{\boldsymbol{u}} - \hat{\boldsymbol{w}}\|_{L^2(\mathcal{F})} + \|e^{-\alpha T}\boldsymbol{u}(T)\|_{L^2(\mathcal{F})}\big).$$

Gathering the above inequality, (4.2) and Lemma 3.3, we finally deduce that, for $\alpha \geqslant 1$,

$$\Big|\int_{\partial\Omega} \hat{\boldsymbol{f}} \cdot (\boldsymbol{\sigma}(\hat{\boldsymbol{u}}, \hat{p})\boldsymbol{n} - \boldsymbol{\sigma}(\hat{\boldsymbol{w}}, \hat{r})\boldsymbol{n})\, d\gamma\Big| \leqslant C\alpha^2 e^{-\alpha T}\|\hat{\boldsymbol{v}}\|^2_{H^1(\Omega)}. \tag{4.9}$$

To estimate the first term on the right-hand side of (4.8), we use (4.7), (4.4) and (3.2)

$$\begin{aligned}\int_{\mathcal{F}} \alpha|\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}}|^2 + 2|\boldsymbol{D}(\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}})|^2\, d\boldsymbol{x} &= \int_{\partial\mathcal{S}} \boldsymbol{\sigma}((\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}}), (\hat{q} - \hat{r}))n \cdot \hat{\boldsymbol{v}}\, d\gamma \\ &\leqslant C\|\hat{\boldsymbol{v}}\|_{H^1(\mathcal{S})}\big(\|\boldsymbol{D}(\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}})\|_{L^2(\mathcal{F})} + \|\,\mathbf{div}\,\boldsymbol{\sigma}((\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}}), (\hat{q} - \hat{r}))\|_{L^2(\mathcal{F})}\big).\end{aligned}$$

Therefore, using (4.3) we deduce that, for $\alpha \geqslant 1$,

$$\int_{\mathcal{F}} \alpha|\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}}|^2 + 2|\boldsymbol{D}(\hat{\boldsymbol{v}} - \hat{\boldsymbol{w}})|^2\, d\boldsymbol{x} \leqslant C(\alpha + 1)\Big(\int_{\mathcal{S}} |\hat{\boldsymbol{v}}|^2 + 2|\boldsymbol{D}(\hat{\boldsymbol{v}})|^2\, d\boldsymbol{x}\Big). \tag{4.10}$$

We conclude from (4.8), (4.9), (4.7) and (4.10) the relation (2.8).

5 Proof of Corollary 2.2

The aim of this section is to prove Corollary 2.2, and in particular to construct a family $(\hat{\boldsymbol{v}}, \hat{q})$ depending on $\alpha > 0$ allowing to recover the support function h_S defined by (2.9).

The proof is similar to the one in [22] or in [25], but we include here the proof for completeness.

We set

$$\hat{\boldsymbol{v}}(\boldsymbol{x}) := \boldsymbol{\ell} e^{\sqrt{\alpha}\boldsymbol{\kappa}\cdot\boldsymbol{x}}, \quad \hat{q}(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \mathbb{R}^3, \tag{5.1}$$

with

$$\boldsymbol{\ell}, \boldsymbol{\kappa} \in \mathbb{S}^2, \quad \boldsymbol{\ell} \cdot \boldsymbol{\kappa} = 0.$$

We can check that

$$\Delta\hat{\boldsymbol{v}}(\boldsymbol{x}) = \alpha\hat{\boldsymbol{v}}(\boldsymbol{x}), \quad \operatorname{div}\hat{\boldsymbol{v}} = 0,$$

so that $(\hat{\boldsymbol{v}}, \hat{q})$ is a solution of (2.1), (2.2).

In order to estimate E_α, we first recall the following proposition (see [22, Proposition 3.2]).

Proposition 5.1. *Assume G is an open subset of $\mathbb{R}^3$. If ∂G is of class C^2, then for any $\boldsymbol{\kappa} \in \mathbb{S}^2$, there exist constants $M = M_{\boldsymbol{\kappa}} > 0$, $\varepsilon = \varepsilon_{\boldsymbol{\kappa}} > 0$ and $p = p_{\boldsymbol{\kappa}} \in [0, 1]$ such that*

$$\mu_2\big(\{\boldsymbol{x} \in G : \boldsymbol{x} \cdot \boldsymbol{\kappa} = h_G(\boldsymbol{\kappa}) - r\}\big) \geqslant Mr^p \quad \textit{for all } r \in (0, \varepsilon), \tag{5.2}$$

where μ_2 denotes the Lebesgue measure of $\mathbb{R}^2$.

As can be seen in the remaining part of the proof, we only need relation (5.2), and thus the corollary is valid for "regular sets" in this sense (see [22] for more details about this notion). Let us introduce the following notation:

$$G_{\boldsymbol{\kappa}}(\delta) := \big\{\boldsymbol{x} \in G : h_G(\boldsymbol{\kappa}) - \delta < \boldsymbol{x} \cdot \boldsymbol{\kappa} \leqslant h_G(\boldsymbol{\kappa})\big\}.$$

Now we are in a position to prove Corollary 2.2. First, it is straightforward from the definition of the support function (recalled in (2.9)) that

$$\int_{\mathcal{S}} e^{2\sqrt{\alpha}\boldsymbol{\kappa}\cdot\boldsymbol{x}}\,d\boldsymbol{x} \leqslant \mu_3(\mathcal{S})e^{2\sqrt{\alpha}h_{\mathcal{S}}(\boldsymbol{\kappa})},$$

where μ_3 is the Lebesgue measure in $\mathbb{R}^3$. Second,

$$\begin{aligned}
\int_{\mathcal{S}} e^{2\sqrt{\alpha}(\boldsymbol{\kappa}\cdot\boldsymbol{x}-h_{\mathcal{S}}(\boldsymbol{\kappa}))}\,d\boldsymbol{x} &\geqslant \int_{\mathcal{S}_{\boldsymbol{\kappa}}(\delta)} e^{2\sqrt{\alpha}(\boldsymbol{\kappa}\cdot\boldsymbol{x}-h_{\mathcal{S}}(\boldsymbol{\kappa}))}\,d\boldsymbol{x}\\
&= \int_0^\delta \int_{\{\boldsymbol{x}\in\mathcal{S}\,:\,\boldsymbol{\kappa}\cdot\boldsymbol{x}-h_{\mathcal{S}}(\boldsymbol{\kappa})=-r\}} e^{2\sqrt{\alpha}(\boldsymbol{\kappa}\cdot\boldsymbol{x}-h_{\mathcal{S}}(\boldsymbol{\kappa}))}\,d\boldsymbol{x}\,dr\\
&= \int_0^\delta \mu_2(\{\boldsymbol{x}\in\mathcal{S} : \boldsymbol{\kappa}\cdot\boldsymbol{x}-h_{\mathcal{S}}(\boldsymbol{\kappa})=-r\})e^{-2\sqrt{\alpha}r}\,dr\\
&\geqslant M\int_0^\delta r^p e^{-2\sqrt{\alpha}r}\,dr\\
&\geqslant Me^{-2\sqrt{\alpha}\delta}\frac{\delta^{p+1}}{p+1}.
\end{aligned}$$

Then, if we take $\delta = \alpha^{-1/2}$, we obtain

$$\int_{\mathcal{S}} e^{2\sqrt{\alpha}\boldsymbol{\kappa}\cdot\boldsymbol{x}}\,d\boldsymbol{x} \geqslant C_2(\mathcal{S},\boldsymbol{\kappa})e^{2\sqrt{\alpha}h_{\mathcal{S}}(\boldsymbol{\kappa})}\frac{1}{\alpha^{(p+1)/2}}.$$

Setting $\beta = \frac{p+1}{2} \in [0,1]$, we deduce

$$C_2\frac{1}{\alpha^\beta}e^{2\sqrt{\alpha}h_{\mathcal{S}}(\boldsymbol{\kappa})} \leqslant \int_{\mathcal{S}} e^{2\sqrt{\alpha}\boldsymbol{\kappa}\cdot\boldsymbol{x}}\,d\boldsymbol{x} \leqslant C_1e^{2\sqrt{\alpha}h_{\mathcal{S}}(\boldsymbol{\kappa})}. \tag{5.3}$$

Using (5.1), we can check that

$$\int_{\mathcal{S}} |\hat{\boldsymbol{v}}|^2\,d\boldsymbol{x} = \int_{\mathcal{S}} e^{2\sqrt{\alpha}\boldsymbol{\kappa}\cdot\boldsymbol{x}}\,d\boldsymbol{x} \quad\text{and}\quad \int_{\mathcal{S}} |\boldsymbol{D}(\hat{\boldsymbol{v}})|^2\,d\boldsymbol{x} = \alpha\left|\frac{\boldsymbol{\ell}\otimes\boldsymbol{\kappa}+\boldsymbol{\kappa}\otimes\boldsymbol{\ell}}{2}\right|^2\int_{\mathcal{S}} e^{2\sqrt{\alpha}\boldsymbol{\kappa}\cdot\boldsymbol{x}}\,d\boldsymbol{x}. \tag{5.4}$$

We can also see that

$$\|\hat{\boldsymbol{v}}\|^2_{H^1(\Omega)} \leqslant C(1+\alpha)e^{2\sqrt{\alpha}h_\Omega(\boldsymbol{\kappa})}, \tag{5.5}$$

where $C = C(\Omega)$ is a positive constant. Therefore, from (5.3), (5.4) and (5.5), (2.8) we obtain

$$C\alpha^{(1-\beta)}e^{2\sqrt{\alpha}h_{\mathcal{S}}(\boldsymbol{\kappa})} - C\alpha^2e^{-\alpha T}(\alpha+1)e^{2\sqrt{\alpha}h_\Omega(\boldsymbol{\kappa})} \leqslant E_\alpha \leqslant C(\alpha+1)^2e^{2\sqrt{\alpha}h_{\mathcal{S}}(\boldsymbol{\kappa})} + C\alpha^2e^{-\alpha T}(\alpha+1)e^{2\sqrt{\alpha}h_\Omega(\boldsymbol{\kappa})}. \tag{5.6}$$

Since

$$\alpha^{(\beta+1)}(\alpha+1)e^{-\alpha T}e^{2\sqrt{\alpha}h_\Omega(\boldsymbol{\kappa})}e^{-2\sqrt{\alpha}h_{\mathcal{S}}(\boldsymbol{\kappa})} \to 0$$

and

$$\alpha^2(\alpha+1)^{-1}e^{-\alpha T}e^{2\sqrt{\alpha}h_\Omega(\boldsymbol{\kappa})}e^{-2\sqrt{\alpha}h_{\mathcal{S}}(\boldsymbol{\kappa})} \to 0$$

as $\alpha \to +\infty$, (5.6) implies

$$\frac{\log C}{2\sqrt{\alpha}} + \frac{(1-\beta)\log(\alpha)}{2\sqrt{\alpha}} + h_{\mathcal{S}}(\boldsymbol{\kappa}) + o(1) \leqslant \frac{1}{2\sqrt{\alpha}}\log(E_\alpha) \leqslant \frac{\log C}{2\sqrt{\alpha}} + \frac{\log((\alpha+1)\alpha)}{2\sqrt{\alpha}} + h_{\mathcal{S}}(\boldsymbol{\kappa}) + o(1)$$

for $\alpha \to +\infty$. This allows us to conclude the proof of Corollary 2.2.

6 Proof of Corollary 2.3

In this section, we prove Corollary 2.3. In order to do this, we construct a family $(\hat{\boldsymbol{v}}, \hat{q})$ depending on $\alpha > 0$ allowing to recover the distance $d(\boldsymbol{x}_0, \mathcal{S})$ of $\mathcal{S}$ to a point $\boldsymbol{x}_0 \notin \mathrm{ch}(\Omega)$.

In order to construct $(\hat{\boldsymbol{v}}, \hat{q})$, we use spherical coordinates for a frame centered in $\boldsymbol{x}_0$ and such that the e_3 direction is parallel to a plane separating $\boldsymbol{x}_0$ and Ω. More precisely, every point of the space is defined by its spherical coordinates $(r, \theta, \varphi) \in \mathbb{R}^+ \times [0, \pi] \times [0, 2\pi]$ through the formula

$$\begin{aligned} x_1 &= r \sin\theta \cos\varphi, \\ x_2 &= r \sin\theta \sin\varphi, \\ x_3 &= r\cos\theta. \end{aligned}$$

Since $\boldsymbol{x}_0 \notin \mathrm{ch}(\Omega)$, we can assume that Ω is contained in a region of the form $\{(r, \theta, \varphi) : r > 0,\ \theta_1 < \theta < \theta_2\}$, where $0 < \theta_1 < \theta_2 < \pi$.

With the customary abuse of notation, the same symbol is used for the function of $\boldsymbol{x} = (x_1, x_2, x_3)$ and of (r, θ, φ). In the orthonormal basis $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\varphi)$ associated to the spherical coordinates, we take

$$\hat{\boldsymbol{v}}(r, \theta, \varphi) := \frac{e^{-\sqrt{\alpha} r}}{r \sin\theta} \boldsymbol{e}_\varphi, \quad \hat{q}(r, \theta, \varphi) = 0, \quad r > 0,\ \theta_1 < \theta < \theta_2. \tag{6.1}$$

In what follows, we write

$$g(r, \theta) := \frac{e^{-\sqrt{\alpha} r}}{r \sin\theta}.$$

We are going now to use several classical formulas of operators in spherical coordinates (see, for instance, [11, pp. 285–287]). First, for the divergence, we have

$$\operatorname{div} \hat{\boldsymbol{v}} = \frac{1}{r\sin\theta} \frac{\partial g}{\partial \varphi} = 0.$$

We also have the Laplacian operator in spherical coordinates:

$$\Delta \hat{\boldsymbol{v}} = (\Delta \hat{\boldsymbol{v}})_r \boldsymbol{e}_r + (\Delta \hat{\boldsymbol{v}})_\theta \boldsymbol{e}_\theta + (\Delta \hat{\boldsymbol{v}})_\varphi \boldsymbol{e}_\varphi, \tag{6.2}$$

with

$$(\Delta \hat{\boldsymbol{v}})_r = -\frac{2}{r^2 \sin\theta} \frac{\partial g}{\partial \varphi} = 0, \quad (\Delta \hat{\boldsymbol{v}})_\theta = -\frac{2\cos\theta}{r^2 \sin^2\theta} \frac{\partial g}{\partial \varphi} = 0, \tag{6.3}$$

$$(\Delta \hat{\boldsymbol{v}})_\varphi = \frac{\partial^2 g}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 g}{\partial \theta^2} + \frac{2}{r} \frac{\partial g}{\partial r} + \frac{\cos\theta}{r^2 \sin\theta} \frac{\partial g}{\partial \theta} - \frac{1}{r^2 \sin^2\theta} g. \tag{6.4}$$

Some calculation gives

$$\frac{\partial g}{\partial r} = -\Big(\sqrt{\alpha} + \frac{1}{r}\Big) g, \tag{6.5}$$

$$\frac{\partial^2 g}{\partial r^2} = \Big(\alpha + \frac{2\sqrt{\alpha}}{r} + \frac{2}{r^2}\Big) g, \tag{6.6}$$

$$\frac{\partial g}{\partial \theta} = -\frac{\cos\theta}{\sin\theta} g, \tag{6.7}$$

$$\frac{\partial^2 g}{\partial \theta^2} = \Big(-1 + \frac{2}{\sin^2\theta}\Big) g. \tag{6.8}$$

Inserting (6.5)–(6.8) in (6.4) yields

$$(\Delta \hat{\boldsymbol{v}})_\varphi = \Big(\alpha + \frac{2\sqrt{\alpha}}{r} + \frac{2}{r^2} - \frac{1}{r^2} + \frac{2}{r^2 \sin^2\theta} - \frac{2\sqrt{\alpha}}{r} - \frac{2}{r^2} - \frac{\cos^2\theta}{r^2\sin^2\theta} - \frac{1}{r^2 \sin^2\theta}\Big) g = \alpha g.$$

The above relation, (6.2), (6.4) and (6.3) imply

$$\Delta \hat{\boldsymbol{v}} = \alpha \hat{\boldsymbol{v}},$$

so that $(\hat{\boldsymbol{v}}, \hat{q})$ defined by (6.1) is a solution of (2.1), (2.2).

We can thus use this family and apply Theorem 2.1 to prove Corollary 2.3. More precisely, this corollary will be proved if we can estimate the integrals of $\hat{\boldsymbol{v}}$, $\boldsymbol{D}(\hat{\boldsymbol{v}})$ and $\nabla\hat{\boldsymbol{v}}$. We use again classical formula for differential operators in spherical coordinates (see, for instance, [11, pp. 285–287]): setting

$$M_{ij} = \boldsymbol{M}\boldsymbol{e}_i \cdot \boldsymbol{e}_j \quad i, j \in \{r, \theta, \varphi\},$$

we have

$$(\nabla\hat{\boldsymbol{v}})_{rr} = (\nabla\hat{\boldsymbol{v}})_{\theta\theta} = (\nabla\hat{\boldsymbol{v}})_{r\theta} = (\nabla\hat{\boldsymbol{v}})_{\theta r} = 0, \tag{6.9}$$

$$(\nabla\hat{\boldsymbol{v}})_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial g}{\partial\varphi} = 0, \tag{6.10}$$

$$(\nabla\hat{\boldsymbol{v}})_{\theta\varphi} = -\frac{g\cos\theta}{r\sin\theta}, \quad (\nabla\hat{\boldsymbol{v}})_{\varphi\theta} = \frac{1}{r}\frac{\partial g}{\partial\theta}, \tag{6.11}$$

$$(\nabla\hat{\boldsymbol{v}})_{\varphi r} = \frac{\partial g}{\partial r}, \quad (\nabla\hat{\boldsymbol{v}})_{r\varphi} = -\frac{g}{r}, \tag{6.12}$$

and

$$\begin{aligned}
\boldsymbol{D}(\hat{\boldsymbol{v}})_{rr} &= \boldsymbol{D}(\hat{\boldsymbol{v}})_{\theta\theta} = \boldsymbol{D}(\hat{\boldsymbol{v}})_{r\theta} = \boldsymbol{D}(\hat{\boldsymbol{v}})_{\theta r} = 0,\\
\boldsymbol{D}(\hat{\boldsymbol{v}})_{\varphi\varphi} &= \frac{1}{r\sin\theta}\frac{\partial g}{\partial\varphi} = 0,\\
\boldsymbol{D}(\hat{\boldsymbol{v}})_{\theta\varphi} &= \boldsymbol{D}(\hat{\boldsymbol{v}})_{\varphi\theta} = \frac{1}{2}\Big(\frac{1}{r}\frac{\partial g}{\partial\theta} - \frac{g\cos\theta}{r\sin\theta}\Big),\\
\boldsymbol{D}(\hat{\boldsymbol{v}})_{r\varphi} &= \boldsymbol{D}(\hat{\boldsymbol{v}})_{\varphi r} = \frac{1}{2}\Big(\frac{\partial g}{\partial r} - \frac{g}{r}\Big).
\end{aligned}$$

Using (6.5) and (6.7), we deduce

$$\boldsymbol{D}(\hat{\boldsymbol{v}})_{r\varphi} = -\Big(\frac{\sqrt{\alpha}}{2} + \frac{1}{r}\Big)g \quad \text{and} \quad \boldsymbol{D}(\hat{\boldsymbol{v}})_{\theta\varphi} = -\frac{\cos\theta}{r\sin\theta}g.$$

The above relation implies

$$I := \int_{\mathcal{S}} \alpha|\hat{\boldsymbol{v}}|^2 + 2|\boldsymbol{D}(\hat{\boldsymbol{v}})|^2 \, d\boldsymbol{x} = \int_{\mathcal{S}} \Big(2\alpha + 4\frac{\sqrt{\alpha}}{r} + 4\frac{1}{r^2\sin^2\theta}\Big)|g|^2 r^2 \sin\theta \, dr \, d\theta \, d\varphi. \tag{6.13}$$

Using the hypothesis on $\boldsymbol{x}_0$ and Ω, we can assume that

$$\mathcal{S} \subseteq \{(r, \theta, \varphi) : 0 < r_1 \leqslant r < r_2,\ 0 < \theta_1 < \theta < \theta_2 < \pi\}. \tag{6.14}$$

We can take r_1 such that

$$r_1 = \min_{\mathcal{S}} r = \min_{\boldsymbol{x}\in\mathcal{S}} |\boldsymbol{x} - \boldsymbol{x}_0| = d(\boldsymbol{x}_0, \mathcal{S}). \tag{6.15}$$

From (6.14), we can assume that

$$\sin\theta > s^* > 0 \quad \text{in } \mathcal{S}. \tag{6.16}$$

In what follows, α is taken large enough (for instance, $\alpha > 1$). Using (6.14), (6.15) and (6.16), we can estimate I defined by (6.13) as

$$I \leqslant \mu_3(\mathcal{S})\Big(2\alpha + 4\frac{\sqrt{\alpha}}{r_1} + 4\frac{1}{r_1^2(s^*)^2}\Big)\frac{e^{-2\sqrt{\alpha}r_1}}{r_1^2(s^*)^2} \leqslant C_1(\mathcal{S})(\alpha + 1)e^{-2\sqrt{\alpha}d(\boldsymbol{x}_0,\mathcal{S})}.$$

The lower bound on the integral is obtained from the following result that is proved, for instance, in [27, Proposition 3.2].

Proposition 6.1. *Assume $\partial\mathcal{S}$ is of class C^2. There exists $\gamma \in \mathbb{R}$ such that*

$$\liminf_{\alpha\to\infty} \alpha^{\gamma} e^{2\sqrt{\alpha}d(\boldsymbol{x}_0,\mathcal{S})} \int_{\mathcal{S}} e^{-2\sqrt{\alpha}|\boldsymbol{x}-\boldsymbol{x}_0|} \, d\boldsymbol{x} > 0.$$

Using the above proposition and (6.14), we deduce that

$$I \geqslant C(\mathcal{S})\Big(2\alpha + 4\frac{\sqrt{\alpha}}{r_2} + 4\frac{1}{r_2^2}\Big)\frac{1}{r_2^2}\alpha^{-\gamma}e^{-2\sqrt{\alpha}d(\boldsymbol{x}_0,\mathcal{S})} \geqslant C_2(\mathcal{S})\alpha^{1-\gamma}e^{-2\sqrt{\alpha}d(\boldsymbol{x}_0,\mathcal{S})}.$$

On the other hand, using (6.9)–(6.12), (6.5), (6.6), we can check (as in (6.13))

$$\|\hat{\boldsymbol{v}}\|^2_{H^1(\Omega)} \leqslant C(\Omega) \int_\Omega \Big(1 + \alpha + \frac{1}{r^2}\Big) |g|^2 r^2 \sin\theta \, dr \, d\theta \, d\varphi \leqslant C(\Omega)(\alpha + 1) e^{-2\sqrt{\alpha} d(\boldsymbol{x}_0, \Omega)}.$$

Therefore, by the same kind of reasoning as in the end of Section 5, we conclude the proof of Corollary 2.3.

7 Spherical geometrical optics solutions

In this section, we prove Theorem 2.6 and Theorem 2.8 by using the spherical geometrical optics solutions. Let us first recall the following result proved in [16]:

Theorem 7.1. *For all $\boldsymbol{x}_0 \notin \mathrm{ch}(\Omega)$ (the convex hull of Ω) and $d > 0$, there exists a family $(\boldsymbol{v}_\alpha, q_\alpha) \in C^2(\overline{\Omega}) \times C^1(\overline{\Omega})$ such that*

$$-\operatorname{\mathbf{div}} \boldsymbol{\sigma}(\boldsymbol{v}_\alpha, q_\alpha) = \mathbf{0} \quad \text{in } \widetilde{\Omega}, \tag{7.1}$$

$$\operatorname{div} \boldsymbol{v}_\alpha = 0 \quad \text{in } \widetilde{\Omega}, \tag{7.2}$$

for some domain $\widetilde{\Omega} \supseteq \Omega$ and for $\alpha > 0$ and such that for $\alpha > \alpha_0$,

$$c\alpha^2 \Big(\frac{d}{d(\boldsymbol{x}_0, \mathcal{S})}\Big)^{2\alpha} \leqslant \int_{\mathcal{S}} |\boldsymbol{v}_\alpha|^2 \, d\boldsymbol{x} \leqslant C\alpha^2 \Big(\frac{d}{d(\boldsymbol{x}_0, \mathcal{S})}\Big)^{2\alpha}$$

and

$$c\alpha^4 \Big(\frac{d}{d(\boldsymbol{x}_0, \mathcal{S})}\Big)^{2\alpha} \leqslant \int_{\mathcal{S}} |\boldsymbol{D}(\boldsymbol{v}_\alpha)|^2 \, d\boldsymbol{x} \leqslant C\alpha^4 \Big(\frac{d}{d(\boldsymbol{x}_0, \mathcal{S})}\Big)^{2\alpha}.$$

Here c and C are constants that may depend on $\mathcal{S}$.

7.1 Proof of Theorem 2.6

For simplicity, we suppress in the proofs below the explicit dependence on α in the notation. For example, we write $\boldsymbol{v}$ instead of $\boldsymbol{v}_\alpha$.

Multiplying (1.34) by $\boldsymbol{u}$, integrating by part and using (1.35)–(1.39), we obtain

$$\int_{\partial\Omega} \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n} \cdot \boldsymbol{f} \, d\gamma = \int_{\mathcal{F}(t)} 2|\boldsymbol{D}(\boldsymbol{u})|^2 \, d\boldsymbol{x}. \tag{7.3}$$

Multiplying (7.1) by $\boldsymbol{v}$, integrating by part and using (2.12) we deduce

$$\int_{\partial\Omega} \boldsymbol{\sigma}(\boldsymbol{v}, q)\boldsymbol{n} \cdot \boldsymbol{f} \, d\gamma = \int_{\Omega} 2|\boldsymbol{D}(\boldsymbol{v})|^2 \, d\boldsymbol{x}. \tag{7.4}$$

Multiplying (7.1) by a smooth divergence free map w and integrating on $\mathcal{S}(t)$, we obtain

$$\int_{\partial\mathcal{S}(t)} \boldsymbol{\sigma}(\boldsymbol{v}, q)\boldsymbol{n} \cdot w \, d\gamma + 2 \int_{\mathcal{S}(t)} \boldsymbol{D}(\boldsymbol{v}) : \boldsymbol{D}(w) \, d\boldsymbol{x} = 0.$$

Consequently, taking particular choices of w, we have

$$\int_{\partial\mathcal{S}(t)} \boldsymbol{\sigma}(\boldsymbol{v}, q)\boldsymbol{n} \, d\gamma = \int_{\partial\mathcal{S}(t)} \boldsymbol{x} \times \boldsymbol{\sigma}(\boldsymbol{v}, q)\boldsymbol{n} \, d\gamma = 0. \tag{7.5}$$

Then multiplying (7.1) by $\boldsymbol{u}$, integrating on $\mathcal{F}(t)$, integrating by parts and using (7.5) implies

$$\int_{\partial\Omega} \boldsymbol{\sigma}(\boldsymbol{v}, q)\boldsymbol{n} \cdot \boldsymbol{f} \, d\gamma = \int_{\mathcal{F}(t)} 2\boldsymbol{D}(\boldsymbol{v}) : \boldsymbol{D}(\boldsymbol{u}) \, d\boldsymbol{x}. \tag{7.6}$$

Combining (7.3), (7.6) and (7.4),

$$\int_{\partial\Omega} \boldsymbol{\sigma}(\boldsymbol{u}-\boldsymbol{v}, p-q)\boldsymbol{n}\cdot\boldsymbol{f}\,d\gamma = 2\int_{\mathcal{S}(t)} |\boldsymbol{D}(\boldsymbol{v})|^2\,d\boldsymbol{x} + 2\int_{\mathcal{F}(t)} |\boldsymbol{D}(\boldsymbol{v}-\boldsymbol{u})|^2\,d\boldsymbol{x}.$$

On the other hand, combining (1.34)–(1.39), (7.1)–(7.2) and (7.5), we deduce

$$-\operatorname{\mathbf{div}}\boldsymbol{\sigma}(\boldsymbol{u}-\boldsymbol{v}, p-q) = \mathbf{0} \quad \text{in } \mathcal{F}(t),\ t\in(0,T), \tag{7.7}$$

$$\operatorname{div}(\boldsymbol{u}-\boldsymbol{v}) = 0 \quad \text{in } \mathcal{F}(t),\ t\in(0,T), \tag{7.8}$$

$$(\boldsymbol{u}-\boldsymbol{v}) = \mathbf{0} \quad \text{on } \partial\Omega,\ t\in(0,T), \tag{7.9}$$

$$(\boldsymbol{u}-\boldsymbol{v}) = \boldsymbol{\ell} + \boldsymbol{\omega}\times(\boldsymbol{x}-\boldsymbol{h}) - \boldsymbol{v} \quad \text{on } \partial\mathcal{S}(t),\ t\in(0,T), \tag{7.10}$$

$$\int_{\partial\mathcal{S}(t)} \boldsymbol{\sigma}(\boldsymbol{u}-\boldsymbol{v}, p-q)\boldsymbol{n}\,d\gamma = \mathbf{0}, \quad t\in(0,T), \tag{7.11}$$

$$\int_{\partial\mathcal{S}(t)} (\boldsymbol{x}-\boldsymbol{h})\times\boldsymbol{\sigma}(\boldsymbol{u}-\boldsymbol{v}, p-q)\boldsymbol{n}\,d\gamma = \mathbf{0}, \quad t\in(0,T). \tag{7.12}$$

Therefore, multiplying (7.7) by $\boldsymbol{u}-\boldsymbol{v}$, using (7.8)–(7.12), and applying Lemma 3.1 and the Korn inequality, we deduce

$$\|(\boldsymbol{u}-\boldsymbol{v})(t)\|_{H^1(\mathcal{F}(t))} \leqslant C\|\boldsymbol{v}\|_{H^1(\mathcal{S}(t))}.$$

Consequently, we obtain

$$2\int_{\mathcal{S}(t)} |\boldsymbol{D}(\boldsymbol{v})|^2\,d\boldsymbol{x} \leqslant \int_{\partial\Omega} \boldsymbol{\sigma}(\boldsymbol{u}-\boldsymbol{v}, p-q)\boldsymbol{n}\cdot\boldsymbol{f}\,d\gamma \leqslant C\Big(\int_{\mathcal{S}(t)} 2|\boldsymbol{D}(\boldsymbol{v})|^2 + |\boldsymbol{v}|^2\,d\boldsymbol{x}\Big).$$

Using Theorem 7.1, we obtain

$$c\alpha^4\Big(\frac{d}{d(\boldsymbol{x}_0, \mathcal{S}(t))}\Big)^{2\alpha} \leqslant \int_{\partial\Omega} \boldsymbol{\sigma}(\boldsymbol{u}-\boldsymbol{v}, p-q)\boldsymbol{n}\cdot\boldsymbol{f}\,d\gamma \leqslant C(\alpha^2+\alpha^4)\Big(\frac{d}{d(\boldsymbol{x}_0, \mathcal{S}(t))}\Big)^{2\alpha}.$$

If $d < d(\boldsymbol{x}_0, \mathcal{S}(t))$, then the above estimate yields

$$F_\alpha \leqslant C\Big(\frac{d}{d(\boldsymbol{x}_0, \mathcal{S}(t))}\Big)^{\alpha}.$$

If $d > d(\boldsymbol{x}_0, \mathcal{S}(t))$, then we deduce

$$F_\alpha \geqslant c\Big(\frac{d}{d(\boldsymbol{x}_0, \mathcal{S}(t))}\Big)^{2\alpha}.$$

We conclude the proof of Theorem 2.6.

7.2 Proof of Theorem 2.8

We only prove the result for system (1.22)–(1.33). A similar and simpler proof can be done for the Navier–Stokes system (1.17)–(1.21).

We modify the function $(\boldsymbol{v}_\alpha, q_\alpha)$ of Theorem 7.1 by multiplying it by a function $\chi \in C^\infty([0,T])$ such that $\chi(0)=0$, $\chi>0$ in $(0,T]$. This modification allows us to have regular solutions for system (1.22)–(1.33) or for the Navier–Stokes system (1.17)–(1.21) if $N=2$ (see Proposition 3.5).

First, the Reynolds formula implies

$$\frac{d}{dt}\int_{\mathcal{F}(t)} \frac{|\boldsymbol{u}|^2}{2}\,d\boldsymbol{x} = \int_{\mathcal{F}(t)} \frac{\partial\boldsymbol{u}}{\partial t}\cdot\boldsymbol{u}\,d\boldsymbol{x} + \int_{\partial\mathcal{S}(t)} \boldsymbol{u}\cdot\boldsymbol{n}\frac{|\boldsymbol{u}|^2}{2}\,d\gamma. \tag{7.13}$$

On the other hand, an integration by parts gives

$$\int_{\mathcal{F}(t)} (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{u}\,d\boldsymbol{x} = \int_{\partial\mathcal{F}(t)} \boldsymbol{u}\cdot\boldsymbol{n}\frac{|\boldsymbol{u}|^2}{2}\,d\gamma = \int_{\partial\mathcal{S}(t)} \boldsymbol{u}\cdot\boldsymbol{n}\frac{|\boldsymbol{u}|^2}{2}\,d\gamma + \int_{\partial\Omega} \boldsymbol{f}\cdot\boldsymbol{n}\frac{|\boldsymbol{f}|^2}{2}\,d\gamma. \tag{7.14}$$

Multiplying (1.22) by $\boldsymbol{u}$, using (7.13)–(7.14) and integrating by parts yields

$$0 = \frac{d}{dt} \int_{\mathcal{F}(t)} \frac{|\boldsymbol{u}|^2}{2} \, d\boldsymbol{x} + \int_{\mathcal{F}(t)} 2|\boldsymbol{D}(\boldsymbol{u})|^2 \, d\boldsymbol{x} + \int_{\partial\Omega} (\boldsymbol{f} \cdot \boldsymbol{n}) \frac{|\boldsymbol{f}|^2}{2} \, d\gamma + m\boldsymbol{h}'' \cdot \boldsymbol{h}' + I_0 \boldsymbol{\omega}' \boldsymbol{\omega} - \int_{\partial\Omega} \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n} \cdot \boldsymbol{f} \, d\gamma. \tag{7.15}$$

Let us extend $\boldsymbol{u}$ in $\mathcal{S}(t)$ by

$$\boldsymbol{u}(t, \boldsymbol{x}) = \boldsymbol{\ell}(t) + \omega(t)(\boldsymbol{x} - \boldsymbol{h}(t))^{\perp} \quad \text{in } \mathcal{S}(t). \tag{7.16}$$

We also define a global density function ρ as

$$\rho(t, \boldsymbol{x}) := \begin{cases} 1 & \text{if } \boldsymbol{x} \in \mathcal{F}(t), \\ \rho^{\mathcal{S}} & \text{if } \boldsymbol{x} \in \mathcal{S}(t). \end{cases} \tag{7.17}$$

Using (1.16), we can prove that

$$\frac{d}{dt} \int_{\mathcal{S}(t)} \rho^{\mathcal{S}} \frac{|\boldsymbol{u}|^2}{2} \, d\boldsymbol{x} = m\boldsymbol{h}''(t) \cdot \boldsymbol{h}'(t) + I_0 \omega'(t)\omega(t).$$

Combining the above equation with (7.15) and using the notation (7.16)–(7.17), we deduce

$$\int_{\partial\Omega} \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n} \cdot \boldsymbol{f} \, d\gamma = \frac{d}{dt} \int_{\Omega} \rho \frac{|\boldsymbol{u}|^2}{2} \, d\boldsymbol{x} + \int_{\mathcal{F}(t)} 2|\boldsymbol{D}(\boldsymbol{u})|^2 \, d\boldsymbol{x} + \int_{\partial\Omega} (\boldsymbol{f} \cdot \boldsymbol{n}) \frac{|\boldsymbol{f}|^2}{2} \, d\gamma. \tag{7.18}$$

Multiplying (7.1) by $\boldsymbol{v}$, integrating by part and using (2.12), it follows

$$\int_{\partial\Omega} \boldsymbol{\sigma}(\boldsymbol{v}, q)\boldsymbol{n} \cdot \boldsymbol{f} \, d\gamma = \int_{\Omega} 2|\boldsymbol{D}(\boldsymbol{v})|^2 \, d\boldsymbol{x}. \tag{7.19}$$

Using (7.5) and using (2.12) and multiplying (7.1) by $\boldsymbol{u}$, we obtain

$$\int_{\partial\Omega} \boldsymbol{\sigma}(\boldsymbol{v}, q)\boldsymbol{n} \cdot \boldsymbol{f} \, d\gamma = \int_{\mathcal{F}(t)} 2\boldsymbol{D}(\boldsymbol{v}) : \boldsymbol{D}(\boldsymbol{u}) \, d\boldsymbol{x}. \tag{7.20}$$

By combining (7.18), (7.19) and (7.20), we deduce

$$\int_{\partial\Omega} \boldsymbol{\sigma}(\boldsymbol{u} - \boldsymbol{v}, p - q)\boldsymbol{n} \cdot \boldsymbol{f} \, d\gamma = \int_{\mathcal{S}(t)} 2|\boldsymbol{D}(\boldsymbol{v})|^2 \, d\boldsymbol{x} + \int_{\mathcal{F}(t)} 2|\boldsymbol{D}(\boldsymbol{v} - \boldsymbol{u})|^2 \, d\boldsymbol{x} + \frac{d}{dt} \int_{\Omega} \rho \frac{|\boldsymbol{u}|^2}{2} \, d\boldsymbol{x} + \int_{\partial\Omega} (\boldsymbol{f} \cdot \boldsymbol{n}) \frac{|\boldsymbol{f}|^2}{2} \, d\gamma.$$

We deduce that

$$\int_0^T \int_{\partial\Omega} \boldsymbol{\sigma}(\boldsymbol{u} - \boldsymbol{v}, p - q)\boldsymbol{n} \cdot \boldsymbol{f} \, d\gamma \, dt - \int_0^T \int_{\partial\Omega} (\boldsymbol{f} \cdot \boldsymbol{n}) \frac{|\boldsymbol{f}|^2}{2} \, d\gamma \, dt \geqslant \int_0^T \int_{\mathcal{S}(t)} 2|\boldsymbol{D}(\boldsymbol{v})|^2 \, d\boldsymbol{x} \, dt.$$

As a consequence, if the observation K_α defined by (2.16) remains bounded as $\alpha \to \infty$, then it implies that

$$\int_0^T \int_{\mathcal{S}(t)} |\boldsymbol{D}(\boldsymbol{v})|^2 \, d\boldsymbol{x} \, dt$$

is also bounded as $\alpha \to \infty$. From Theorem 7.1, this yields that for almost all $t \in [0, T]$, $d < d(\boldsymbol{x}_0, \mathcal{S}(t))$. Since $\boldsymbol{h}$ and $\boldsymbol{Q}$ are continuous, it implies that

$$\mathcal{S}(t) \cap B(\boldsymbol{x}_0, d) = \emptyset \quad \text{for all } t \in [0, T].$$

This ends the proof of Theorem 2.8.

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