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On the uniqueness and stability of an inverse problem in photo-acoustic tomography

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A B S T R A C T

This article deals with the uniqueness and stability of the solution of a problem of optimal control related to the photo-acoustic tomography process. We prove stability results of the optimal solution with respect to the source and to the observation data and we compute the corresponding derivatives.

1. Introduction

In this paper we consider a differential system arising in photo-acoustic tomography. We refer [12] to get a complete description of the model. Let us briefly mention that we deal with two coupled partial differential equations that describes the light intensity (fluence) behavior inside a body that is excited by a laser (pulsed) source and the acoustic pressure wave which is generated by this excitation. The authors of [12] have investigated the model and obtained an optimal control formulation to recover some parameters of interest, namely the absorption and diffusion coefficients (μ, D).

We want to address the optimal solution sensitivity with respect to the source and the observation data that appears in the wave equation. For that purpose, in a first step we assume that the diffusion coefficient is constant (and for sake of simplicity equal to 1).

In this work, we prove uniqueness and stability results provided that the coercivity constant α of the cost functional J, given by (2.4), is large enough.

From this point of view, the result is similar to the one of [10]. Other results of uniqueness and stability, in the context of the photo-acoustic, have been obtained in [2,3,7–9,11]. For example, in [8,9] the authors obtained uniqueness and stability results under the assumption that the function $H(x) := \Gamma(x)\mu(x)I_\mu(x)$

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is known and the absorption and diffusion coefficients are smooth enough. Moreover, they did not consider the whole process which couples lightning and acoustic wave equations.

The stability of optimal controls have also studied in [13,18,20,22,23] in other settings.

The paper is structured as follows. In Section 2, we recall the problem setting and preliminary results. Section 3 is devoted to stability and uniqueness properties. In Section 4, we compute the derivative of the optimal control with respect to the source giving a characterization. We also study the stability of the optimal solution with respect to the observation. We end the paper with conclusions and a few words on future work.

2. Problem setting

2.1. Photo-acoustic modelling

Recall photo-acoustic tomography (PAT) principle: tissues to be imaged are illuminated by a laser (the source). This energy is converted into heat creating a thermally induced pressure jump that propagates as a sound wave, which can be detected. The fluence rate I_μ, that is the average of the luminous intensity in all the directions, satisfies the diffusion equation (see [1,5,12])

$$\begin{cases} \dfrac{1}{c}\dfrac{\partial I_\mu}{\partial t}(t,x) + \mu(x)I_\mu(t,x) - \Delta I_\mu(t,x) = S(t,x) & \text{in } (0,T)\times\Omega \\ I_\mu(t,x) = 0 & \text{on } (0,T)\times\partial\Omega \\ I_\mu(0,x) = 0 & \text{in } \Omega. \end{cases} \tag{2.1}$$

where c is the speed of light, S is the incident light source, μ is the *absorption coefficient*, and $T > 0$ is the duration of the acquisition process.

Here, Ω stands for the part of the body where the diffusion approximation is relevant and the diffusion coefficient has been set to 1 for simplicity. It is an open subset of $\mathbb{R}^d$ ($d \geq 2$) of class C^2. For a fixed $T > 0$, we will often denote $Q := (0,T)\times\Omega$.

The acoustic wave that is generated is described via the pressure p_μ that satisfies (up to the change of variables: $p \mapsto \int_0^t p(s)\,ds$):

$$\begin{cases} \dfrac{\partial^2 p_\mu}{\partial t^2}(t,x) - \operatorname{div}(v_s^2\nabla p_\mu)(t,x) = \mathbb{1}_\Omega(x)\Gamma(x)\mu(x)I_\mu(t,x) & \text{in } (0,T)\times\mathcal{B} \\ p_\mu(t,x) = 0 & \text{on } (0,T)\times\partial\mathcal{B} \\ p_\mu(0,x) = \dfrac{\partial p_\mu}{\partial t}(0,x) = 0 & \text{in } \mathcal{B}. \end{cases} \tag{2.2}$$

Here, the *Grueneisen coefficient* Γ, coupling the energy absorption to the thermal expansion, is assumed to be known. In the sequel we assume that Γ has compact support in Ω so that $\Gamma\mathbb{1}_\Omega = \Gamma$ and that the speed of sound v_s is known and satisfies $v_s \in [v_s^{\min}, v_s^{\max}]$, with $v_s^{\min} > 0$. The ball $\mathcal{B}$ is the domain where the wave propagates. It includes Ω and it has to be bounded in view of numerical simulations. It is large enough to assume that there is no reflected wave before time T.

The absorption coefficient μ is the parameter we want to study. We assume that

$$\mu \in \mathcal{U}_{ad} = \{\mu \in L^\infty(\mathcal{B}) \mid \mu \in [\mu^{\min}, \mu^{\max}] \text{ a.e. in } \mathcal{B}\}, \tag{2.3}$$

where $0 < \mu^{\min} < \mu^{\max}$ are positive real numbers.

The photo-acoustic tomography model is completely described by the coupling of equations (2.1) and (2.2), where I_μ is extended by 0 on $\mathcal{B}\setminus\Omega$. Here S is the incident light source that we assume in $L^2(Q)$.

We first recall the results of [12] (for $D = 1$).

Theorem 2.1. *Let Ω be a bounded connected open set of $\mathbb{R}^d$ with $\mathcal{C}^1$ boundary, $\Gamma \in L^\infty(\mathcal{B})$, $v_s \in L^\infty(\mathcal{B})$ with $v_s \in [v_s^{\min}, v_s^{\max}]$ a.e. in $\mathcal{B}$. Assume that the assumption (2.3) holds. Then,*

1. *Equation (2.1) has a unique solution I_μ such that*

$$I_\mu \in \mathcal{C}^0(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)),$$
$$\frac{\partial I_\mu}{\partial t} \in L^2(0,T;H^{-1}(\Omega)).$$

2. *Equation (2.2) has a unique solution p_μ such that*

$$p_\mu \in \mathcal{C}(0,T;H_0^1(\mathcal{B})) \cap \mathcal{C}^1(0,T;L^2(\mathcal{B})).$$

Using Theorem 2.1, we define the maps

$$I\colon \quad \mathcal{U}_{ad} \longrightarrow \mathcal{C}^0(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$$
$$\mu \longmapsto I_\mu$$

where I_μ satisfies (2.1) and

$$p\colon \quad \mathcal{U}_{ad} \longrightarrow \mathcal{C}^0(0,T;H_0^1(\mathcal{B}))$$
$$\mu \longmapsto p_\mu$$

where p_μ is the solution to (2.2).

Next we define, for every $\mu \in \mathcal{U}_{ad}$, the functional J

$$J(\mu) = \frac{1}{2} \int\limits_{[0,T]\times\omega} (p_\mu(t,x) - p^{\text{obs}}(t,x))^2 \mathrm{d}x\,\mathrm{d}t + \alpha \int\limits_{\Omega} \mu^2(x)\mathrm{d}x, \tag{2.4}$$

where $\alpha \geq 0$ and $\omega \subset \mathbb{R}^d$ is the observation subset; we consider the optimization problem:

$$(\mathcal{P}) \qquad \min_{\mu \in \mathcal{U}_{ad}} J(\mu).$$

Theorem 2.2. *(See [12].) Assume that $\alpha \geq 0$. Then, Problem $(\mathcal{P})$ has at least a solution.*

Moreover, for every $\bar{\mu}$ optimal solution to Problem $(\mathcal{P})$, there exists $q_{1\mu}$ and $q_{2\mu}$ such that

- *The state equations (2.1)–(2.2) are satisfied*

$$\begin{cases} \dfrac{\partial^2 p_\mu}{\partial t^2}(t,x) - \operatorname{div}(v_s^2 \nabla p_\mu)(t,x) = \Gamma(x)\mu(x)I_\mu(t,x) & \text{in } (0,T)\times\mathcal{B} \\ p_\mu(t,x) = 0 & \text{on } (0,T)\times\partial\mathcal{B} \\ p_\mu(0,x) = \dfrac{\partial p_\mu}{\partial t}(0,x) = 0 & \text{in } \mathcal{B} \end{cases}$$

and

$$\begin{cases} \dfrac{1}{c}\dfrac{\partial I_\mu}{\partial t}(t,x) + \mu(x)I_\mu(t,x) - \Delta I_\mu(t,x) = S(t,x) & \text{in } (0,T)\times\Omega \\ I_\mu(t,x) = 0, & \text{on } (0,T)\times\partial\Omega \\ I_\mu(t,x) = 0 & \text{in } (0,T)\times\mathcal{B}\backslash\Omega \\ I_\mu(0,x) = 0 & \text{in } \Omega. \end{cases}$$

- *The adjoint state equations are satisfied*

$$\begin{cases} \dfrac{\partial^2 q_{1\mu}}{\partial t^2} - \operatorname{div}(v_s^2 \nabla q_{1\mu}) = (p_\mu - p^{\mathrm{obs}})\mathbb{1}_\omega & \text{in } (0,T)\times\mathcal{B} \\ q_{1\mu} = 0 & \text{on } (0,T)\times\partial\mathcal{B} \\ q_{1\mu}(T,\cdot) = \dfrac{\partial q_{1\mu}}{\partial t}(T,\cdot) = 0 & \text{in } \mathcal{B} \end{cases} \tag{2.5}$$

$$\begin{cases} -\dfrac{1}{c}\dfrac{\partial q_{2\mu}}{\partial t} + \mu q_{2\mu} - \Delta q_{2\mu} = \Gamma \mu q_{1\mu} & \text{in } (0,T)\times\Omega \\ q_{2\mu} = 0 & \text{on } (0,T)\times\partial\Omega \\ q_{2\mu}(T,\cdot) = 0 & \text{in } \Omega. \end{cases} \tag{2.6}$$

- *For all $\mu \in L^\infty(\Omega)$ such that $\mu \in [\mu^{\min}, \mu^{\max}]$,*

$$\left\langle \int_0^T (\mathbb{1}_\Omega \Gamma q_{1\mu} - q_{2\mu}) I_\mu \, dt + 2\alpha\bar{\mu}, \mu - \bar{\mu} \right\rangle_{L^2(\Omega)} \geq 0. \tag{2.7}$$

Furthermore, systems (2.5)–(2.6) respectively have a unique solution

$$q_{1\mu} \in \mathcal{C}(0,T; H_0^1(\mathcal{B})) \cap \mathcal{C}^1(0,T; L^2(\mathcal{B}))$$

and

$$q_{2\mu} \in \mathcal{C}(0,T; L^2(\mathcal{B})) \cap L^2(0,T; H_0^1(\mathcal{B})).$$

Here $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$ denotes the L^2-inner product.

2.2. Regularity results and estimates

In this subsection we give regularity results for two particular problems, of parabolic type and hyperbolic type respectively, and we provide estimates that we will use extensively in the following sections. These problems are representative of the systems we considered in the previous section and the ones to be studied in the sequel. The proofs of these results can be obtained with a slight change of the proofs in [15] because here we consider systems of equations with less smooth coefficients. Therefore, we omit these proofs.

Theorem 2.3 *(Regularity result for parabolic systems). Let Ω be a bounded connected open set of $\mathbb{R}^d$ with $\mathcal{C}^2$ boundary, $f \in L^2(Q)$ and $\beta \in L^\infty(\Omega)$ such that $\beta(x) \in [\beta^{\min}, \beta^{\max}]$ with $0 < \beta^{\min} \leq \beta^{\max}$. Then, the system*

$$\begin{cases} \dfrac{\partial w}{\partial t}(t,x) + \beta(x) w(t,x) - \Delta w(t,x) = f(t,x) & \text{in } (0,T)\times\Omega \\ w(t,x) = 0 & \text{on } (0,T)\times\partial\Omega \\ w(0,x) = 0 & \text{in } \Omega, \end{cases}$$

has a unique solution w such that

$$w \in L^2(0,T; H^2(\Omega)) \cap L^\infty(0,T; H_0^1(\Omega)), \ \frac{\partial w}{\partial t} \in L^2(Q).$$

Moreover we have the following estimate

$$\sup_{0\le t\le T}\|w(t)\|_{H_0^1(\Omega)}+\|w\|_{L^2(0,T;H^2(\Omega))}+\left\|\frac{\partial w}{\partial t}\right\|_{L^2(Q)}\le C\|f\|_{L^2(Q)} \tag{2.8}$$

where C depends on Ω, T and $\|\beta\|_{L^\infty(\Omega)}$.

Theorem 2.4 *(Regularity result for hyperbolic system). Let Ω be a bounded connected open set of $\mathbb{R}^d$ with $\mathcal{C}^2$ boundary, $g\in H^1(0,T;L^2(\Omega))$ and κ be a Lipschitz continuous function in Ω (that only depends on the space variable) such that $\kappa(x)\in[\kappa^{\min},\kappa^{\max}]$ with $0<\kappa^{\min}\le\kappa^{\max}$. Then, the system*

$$\begin{cases}\dfrac{\partial^2 u}{\partial t^2}(t,x)-\operatorname{div}(\kappa(x)\nabla u)(t,x)=g(t,x) & \text{in } (0,T)\times\Omega\\ u(t,x)=0 & \text{on } (0,T)\times\partial\Omega\\ u(0,x)=\dfrac{\partial u}{\partial t}(0,x)=0 & \text{in } \Omega\end{cases}$$

has a unique solution u such that

$$u\in L^\infty(0,T;H^2(\Omega)),\ \frac{\partial u}{\partial t}\in L^\infty(0,T;H_0^1(\Omega)),$$
$$\frac{\partial^2 u}{\partial t^2}\in L^\infty(0,T;L^2(\Omega)),\ \frac{\partial^3 u}{\partial t^3}\in L^2(0,T;H^{-1}(\Omega)),$$

and we have the following estimate

$$\begin{aligned}&\sup_{0\le t\le T}\left(\|u(t)\|_{H^2(\Omega)}+\left\|\frac{\partial u}{\partial t}(t)\right\|_{H_0^1(\Omega)}+\left\|\frac{\partial^2 u}{\partial t^2}(t)\right\|_{L^2(\Omega)}\right)+\left\|\frac{\partial^3 u}{\partial t^3}\right\|_{L^2(0,T;H^{-1}(\Omega))}\\ &\quad\le C\|g\|_{H^1(0,T;L^2(\Omega))}\end{aligned} \tag{2.9}$$

where C depends on Ω, T and $\|\kappa\|_{L^\infty(\Omega)}$.

3. Stability and uniqueness results for optimal controls

In this section we first give a stability result for the optimal solution with respect to the source and with respect to the observation data and we provide a uniqueness result.

Theorem 3.1. *Let $S_1,S_2\in L^2(Q)$ two sources and $p_1^{\text{obs}},p_2^{\text{obs}}\in H^1(0,T;L^2(\omega))$ the (corresponding) measured pressure on $\omega\times[0,T]$. For all $\mu_i\in\mathcal{U}_{ad}$ solution of the optimality system (2.2)–(2.1), (2.5)–(2.7) with source S_i and measurement p_i^{obs}, $i=1,2$, we have the following estimation*

$$\|\mu_1-\mu_2\|_{L^2(\Omega)}\le\frac{C}{2\alpha}\left(\|S_1-S_2\|_{L^2(Q)}+\|\mu_1-\mu_2\|_{L^2(\Omega)}+\|p_1^{\text{obs}}-p_2^{\text{obs}}\|_{L^2(Q)}\right) \tag{3.1}$$

where $C=C\left(d,\Omega,T,\mu^{\min},\mu^{\max},v_s^{\min},v_s^{\max},c,\|\Gamma\|_{L^\infty},\|S_i\|_{L^2(Q)}\right)$, $i=1,2$.

Proof. We consider two sources $S_1,S_2\in L^2(Q)$. Let us write I_i, p_i, q_{1i} and q_{2i} the respective solutions of equations (2.1)–(2.2), (2.5)–(2.6). From Theorem 2.2 there exist $\mu_1,\mu_2\in\mathcal{U}_{ad}$ (not necessarily unique) solutions of problems $(\mathcal{P}_1)$, $(\mathcal{P}_2)$ respectively. With these notations, $p:=p_1-p_2$ satisfies the system

$$\begin{cases} \dfrac{\partial^2 p}{\partial t^2} - \operatorname{div}(v_s^2 \nabla p) = \Gamma\left(\mu_1 I_1 - \mu_2 I_2\right) & \text{in } (0,T)\times\mathcal{B} \\ p = 0 & \text{on } (0,T)\times\partial\mathcal{B} \\ p(0,\cdot) = \dfrac{\partial p}{\partial t}(0,\cdot) = 0 & \text{in } \mathcal{B}, \end{cases} \tag{3.2}$$

and $I := I_1 - I_2$ satisfies

$$\begin{cases} \dfrac{1}{c}\dfrac{\partial I}{\partial t} + \mu_1 I - \Delta I = S_1 - S_2 + I_2(\mu_2 - \mu_1) & \text{in } (0,T)\times\Omega \\ I = 0 & \text{on } (0,T)\times\partial\Omega \\ I(0,\cdot) = 0 & \text{in } \Omega \\ I = 0 & \text{in } (0,T)\times\mathcal{B}\setminus\Omega. \end{cases} \tag{3.3}$$

Similarly $q_1 := q_{11} - q_{12}$ satisfies

$$\begin{cases} \dfrac{\partial^2 q_1}{\partial t^2} - \operatorname{div}(v_s^2 \nabla q_1) = \mathbb{1}_\omega(p - p^{\text{obs}}) & \text{in } (0,T)\times\mathcal{B} \\ q_1 = 0 & \text{on } (0,T)\times\partial\mathcal{B} \\ q_1(T,\cdot) = \dfrac{\partial q_1}{\partial t}(T,\cdot) = 0 & \text{in } \mathcal{B}, \end{cases} \tag{3.4}$$

where $p^{\text{obs}} := p_1^{\text{obs}} - p_2^{\text{obs}}$. And $q_2 := q_{21} - q_{22}$ is solution to

$$\begin{cases} -\dfrac{1}{c}\dfrac{\partial q_2}{\partial t} + \mu_1 q_2 - \Delta q_2 = \Gamma\mu_1 q_1 + (\mu_2 - \mu_1)(q_{22} - \Gamma q_{12}) & \text{in } (0,T)\times\Omega \\ q_2 = 0 & \text{on } (0,T)\times\partial\Omega \\ q_2(T,\cdot) = 0 & \text{in } \Omega. \end{cases} \tag{3.5}$$

We note that, after the change of variables $(t,x)\mapsto(T-t,x)$, Theorem 2.4 can be applied to the systems (2.5) and (3.4). Similarly, Theorem 2.3 can be applied to the systems (2.6) and (3.5). In the sequel, we will use this fact many times.

From Theorem 2.2, μ_1, μ_2 satisfy, for every $\xi \in \mathcal{U}_{ad}$

$$\left\langle \int_0^T (\Gamma q_{11} - q_{21}) I_1 \, dt + 2\alpha\mu_1, \xi - \mu_1 \right\rangle_{L^2(\Omega)} \geq 0 \tag{3.6}$$

and

$$\left\langle \int_0^T (\Gamma q_{12} - q_{22}) I_2 \, dt + 2\alpha\mu_2, \xi - \mu_2 \right\rangle_{L^2(\Omega)} \geq 0. \tag{3.7}$$

Taking $\xi = \mu_2$ in (3.6) and $\xi = \mu_1$ in (3.7) and adding the two inequalities give:

$$\begin{aligned} 0 &\leq \int_\Omega \left[\int_0^T (\Gamma q_{11} - q_{21})\cdot I_1 \, dt - \int_0^T (\Gamma q_{12} - q_{22})\cdot I_2 \, dt + 2\alpha(\mu_1 - \mu_2) \right] (\mu_2 - \mu_1) \, dx \\ &= \int_\Omega \left[\int_0^T (\Gamma q_{11} - q_{21})\cdot I_1 \, dt - \int_0^T (\Gamma q_{12} - q_{22})\cdot I_2 \, dt \right] (\mu_2 - \mu_1) \, dx - 2\alpha\|\mu_1 - \mu_2\|^2_{L^2(\Omega)}. \end{aligned}$$

Then, applying Cauchy–Schwarz inequality, we have

$$\begin{aligned}\|\mu_1-\mu_2\|_{L^2(\Omega)} &\le \left\| \frac{1}{2\alpha}\int_0^T (\Gamma q_{11}-q_{21})\, I_1\, dt - \frac{1}{2\alpha}\int_0^T (\Gamma q_{12}-q_{22}) I_2\, dt \right\|_{L^2(\Omega)} \\ &= \frac{1}{2\alpha}\left\| \int_0^T (\Gamma q_1 - q_2) I_1\, dt + \int_0^T (\Gamma q_{12}-q_{22}) I\, dt \right\|_{L^2(\Omega)}. \end{aligned} \tag{3.8}$$

Let us estimate the right hand side of (3.8):

$$\begin{aligned} &\left\| \int_0^T (\Gamma q_1 - q_2) I_1\, dt + \int_0^T (\Gamma q_{12}-q_{22}) I\, dt \right\|_{L^2(\Omega)} \\ &\le \left\| \int_0^T \Gamma q_1 I_1\, dt \right\|_{L^2(\Omega)} + \left\| \int_0^T q_2 I_1\, dt \right\|_{L^2(\Omega)} + \left\| \int_0^T (\Gamma q_{12}-q_{22}) I\, dt \right\|_{L^2(\Omega)}. \end{aligned}$$

Using Theorem 2.4 and Theorem 2.3 we get that $q_1 \in L^\infty(0,T;H^2(\Omega))$, $p \in H^1(0,T;L^2(\Omega))$ and $I_1 \in L^2(Q)$. Then, from regularity of Γ, Sobolev inequalities, estimate (2.9) for q_1 and estimate (2.8) for I_1, we obtain

$$\begin{aligned} \left\| \int_0^T \Gamma q_1 I_1\, dt \right\|^2_{L^2(\Omega)} &\le T \int_0^T \int_\Omega |\Gamma q_1|^2 |I_1|^2\, dx\, dt \\ &\le C\|\Gamma\|^2_{L^\infty(\Omega)} \|q_1\|^2_{L^\infty(H^2(\Omega))} \|I_1\|^2_{L^2(Q)} \\ &\le C\|\Gamma\|^2_{L^\infty(\Omega)} \|p-p^{\mathrm{obs}}\|^2_{H^1(L^2(\omega))} \|S_1\|^2_{L^2(Q)}. \end{aligned} \tag{3.9}$$

Applying Theorem 2.3 for I and I_2 we obtain that $I \in L^2(Q)$ and $I_2 \in L^2(0,T;H^2(\Omega))$. Hence, from estimate (2.9) for p, estimate (2.8) for I and I_2 and Sobolev inequalities we get

$$\begin{aligned} \|p-p^{\mathrm{obs}}\|_{H^1(L^2(\omega))} &\le \|\Gamma\left(\mu_1 I + I_2(\mu_1-\mu_2)\right)\|_{L^2(Q)} + \|p^{\mathrm{obs}}\|_{H^1(L^2(\omega))} \\ &\le C\|\Gamma\|_{L^\infty(\Omega)} \big(\|I\|_{L^2(Q)} + \\ &\quad + C\|I_2\|_{L^2(H^2(\Omega))}\|\mu_1-\mu_2\|_{L^2(\Omega)} \big) + \|p^{\mathrm{obs}}\|_{H^1(L^2(\omega))} \\ &\le C\|\Gamma\|_{L^\infty(\Omega)} \big(\|S_1-S_2\|_{L^2(Q)} + \\ &\quad + 2C\|I_2\|_{L^2(H^2(\Omega))}\|\mu_1-\mu_2\|_{L^2(\Omega)} \big) + \|p^{\mathrm{obs}}\|_{H^1(L^2(\omega))} \\ &\le C\|\Gamma\|_{L^\infty(\Omega)} \big(\|S_1-S_2\|_{L^2(Q)} + \\ &\quad + C\|S_2\|_{L^2(Q)}\|\mu_1-\mu_2\|_{L^2(\Omega)} \big) + \|p^{\mathrm{obs}}\|_{H^1(L^2(\omega))}. \end{aligned} \tag{3.10}$$

Replacing this last inequality in (3.9), we obtain

$$\left\| \int_0^T \Gamma q_1 I_1\, dt \right\|^2_{L^2(\Omega)} \le C\left(\|S_1-S_2\|^2_{L^2(Q)} + \|\mu_1-\mu_2\|^2_{L^2(\Omega)} + \|p_1^{\mathrm{obs}} - p_2^{\mathrm{obs}}\|^2_{H^1(L^2(\omega))} \right). \tag{3.11}$$

Next, applying Theorem 2.3 for q_2, q_{22} and I_1 and Theorem 2.4 for q_1 and p give $q_2 \in L^\infty(0,T;L^2(\Omega))$, $q_{22} \in L^2(0,T;L^\infty(\Omega))$, $I_1 \in L^2(0,T;H^2(\Omega))$, $q_1 \in L^2(Q)$ and $p \in L^2(Q)$. So, once again

$$\begin{aligned}
\left\| \int_0^T q_2 I_1 \, dt \right\|^2_{L^2(\Omega)} &\le T \int_0^T \int_\Omega |q_2|^2 |I_1|^2 \, dx \, dt \\
&\le C\|q_2\|^2_{L^\infty(L^2(\Omega))} \|I_1\|^2_{L^2(H^2(\Omega))} \\
&\le C\|q_2\|^2_{L^\infty(L^2(\Omega))} \|S_1\|^2_{L^2(Q)} \\
&\le C\|S_1\|^2_{L^2(Q)} \|\Gamma \mu_1 q_1 + (\mu_2 - \mu_1)(q_{22} - \Gamma q_{12})\|^2_{L^2(Q)} \\
&\le C\|S_1\|^2_{L^2(Q)} \Big(\|\Gamma\|^2_{L^\infty(\Omega)} \|q_1\|^2_{L^2(Q)} + \\
&\quad + \|\mu_2 - \mu_1\|^2_{L^2(\Omega)} \|q_{22} - \Gamma q_{12}\|^2_{L^2(L^\infty(\Omega))} \Big) \\
&\le C\|S_1\|^2_{L^2(Q)} \Big(\|\Gamma\|^2_{L^\infty(\Omega)} \|p - p^{\text{obs}}\|^2_{L^2(L^2(\omega))} + \\
&\quad + \|\mu_2 - \mu_1\|^2_{L^2(\Omega)} \|q_{22} - \Gamma q_{12}\|^2_{L^2(L^\infty(\Omega))} \Big) . \qquad (3.12)
\end{aligned}$$

Similarly, using again Theorem 2.3 and Theorem 2.4 we get $q_{22} \in L^2(0, T; H^2(\Omega))$, $q_{12} \in L^2(0, T; H^2(\Omega))$ and $p_2 \in H^1(0, T; L^2(\Omega))$ and

$$\begin{aligned}
\|q_{22} - \Gamma q_{12}\|_{L^2(L^\infty(\Omega))} &\le \|q_{22}\|_{L^2(L^\infty(\Omega))} + \|\Gamma\|_{L^\infty(\Omega)} \|q_{12}\|_{L^2(L^\infty(\Omega))} \\
&\le C\|q_{22}\|_{L^2(H^2(\Omega))} + \|\Gamma\|_{L^\infty(\Omega)} \|q_{12}\|_{L^2(H^2(\Omega))} \\
&\le C\|\Gamma \mu_2 q_{12}\|_{L^2(Q)} + \|\Gamma\|_{L^\infty(\Omega)} \|q_{12}\|_{L^2(H^2(\Omega))} \\
&\le C\|\Gamma\|_{L^\infty(\Omega)} \|q_{12}\|_{L^2(H^2(\Omega))} \\
&\le C\|\Gamma\|_{L^\infty(\Omega)} \|p_2 - p_2^{\text{obs}}\|_{H^1(L^2(\omega))} \\
&\le C\|\Gamma\|_{L^\infty(\Omega)} \left(\|\Gamma\|_{L^\infty(\Omega)} \|S_2\|_{L^2(Q)} + \|p_2^{\text{obs}}\|_{H^1(L^2(\omega))} \right) . \qquad (3.13)
\end{aligned}$$

By using (3.10) and (3.13) in (3.12), we deduce

$$\left\| \int_0^T q_2 I_1 \, dt \right\|^2_{L^2(\Omega)} \le C \left(\|S_1 - S_2\|^2_{L^2(Q)} + \|\mu_1 - \mu_2\|^2_{L^2(\Omega)} + \|p_1^{\text{obs}} - p_2^{\text{obs}}\|^2_{L^2(L^2(\omega))} \right) . \qquad (3.14)$$

Eventually, inequality (3.13) and Theorem 2.3 applied to I yield

$$\begin{aligned}
\left\| \int_0^T (\Gamma q_{12} - q_{22}) I \, dt \right\|^2_{L^2(\Omega)} &\le T \int_0^T \int_\Omega |\Gamma q_{12} - q_{22}|^2 |I|^2 \, dx \, dt \\
&\le T \, \|\Gamma q_{12} - q_{22}\|^2_{L^2(L^\infty(\Omega))} \|I\|^2_{L^\infty(L^2(\Omega))} \\
&\le C\|\Gamma\|^2_{L^\infty(\Omega)} \left(\|\Gamma\|^2_{L^\infty(\Omega)} \|S_2\|^2_{L^2(Q)} + \|p_2^{\text{obs}}\|^2_{H^1(L^2(\omega))} \right) \\
&\quad \left(\|S_1 - S_2\|^2_{L^2(Q)} + \|I_2\|^2_{L^2(H^2(\Omega))} \|\mu_1 - \mu_2\|^2_{L^2(\Omega)} \right) \\
&\le C \left(\|S_1 - S_2\|^2_{L^2(Q)} + \|S_2\|^2_{L^2(Q)} \|\mu_1 - \mu_2\|^2_{L^2(\Omega)} \right) \\
&\le C \left(\|S_1 - S_2\|^2_{L^2(Q)} + \|\mu_2 - \mu_1\|^2_{L^2(\Omega)} \right) . \qquad (3.15)
\end{aligned}$$

Using estimates (3.11), (3.14) and (3.15) in (3.8) give

$$\|\mu_1 - \mu_2\|_{L^2(\Omega)} \leq \frac{C}{2\alpha}\left(\|S_1 - S_2\|_{L^2(Q)} + \|\mu_1 - \mu_2\|_{L^2(\Omega)} + \|p_1^{\text{obs}} - p_2^{\text{obs}}\|_{L^2(L^2(\omega))}\right).$$

In the previous estimates the generic constant C depends on $d, \Omega, T, \mu^{\min}, \mu^{\max}, v_s^{\min}, v_s^{\max}, c, \|\Gamma\|_{L^\infty}$, $\|S_1\|_{L^2(Q)}, \|S_2\|_{L^2(Q)}$ and $\|p_2^{\text{obs}}\|_{H^1(L^2(\omega))}$. □

As a consequence of Theorem 3.1 we deduce the uniqueness of the optimal solution that satisfies the optimality system (2.2)–(2.1), (2.5)–(2.7).

Corollary 3.1 *(Uniqueness). Let $\alpha > C/2$ with C as in Theorem 3.1. If the sources are the same $S_1 = S_2$ then the optimal control given by Theorem 2.2 is unique.*

Proof. As $S_1 = S_2$ it follows that the measurements $p_1^{\text{obs}} = p_2^{\text{obs}}$ and the optimality systems are the same. Let μ_1, μ_2 be two solutions of (2.2)–(2.1), (2.5)–(2.7), then from inequality (3.1) we deduce

$$\|\mu_1 - \mu_2\|_{L^2(\Omega)} \leq 0,$$

which concludes the proof. □

4. Computation of the derivative of the optimal control with respect to the source and the observation

In this section, we investigate two particular cases corresponding to practical issues. In the first one, we assume that we have an object to image, for instance a biological tissue (in the case of breast tumors) at a fixed date. The reconstruction process is sensitive to the sources and we can control the process as shown in the previous section. In this case, we assume that the measured pressure variation $p_1^{\text{obs}} - p_2^{\text{obs}}$ is controlled by the source variation $S_1 - S_2$. We are going to make this precise in next subsection.

In the second case, we decide to illuminate two different objects with the same source: it is the case, for example in a calibration process. Usually, physicists perform acquisitions by difference when the object is hard to recover. They image the background without the object and the background with the object. Of course, the measurements are different but for many situations quite close (when the object is difficult to locate). In addition to such calibration processes, consider a biological tissue we want to image in a large time scale, to check micro-tumors that could appear for example. In this case, the objects to image are close (if the acquisition dates are close enough, and the disease not too severe) and we want to estimate the difference between the two objects, namely the new tumors or those that have disappeared.

In both case, the goal can be achieved by characterizing the derivative of μ with respect to the source S and/or the observation p^{obs}.

4.1. Derivative with respect to the source S

In this section we are interested in characterizing the derivative of μ with respect to the source S. We slightly change the notations in what follows: we fix $S \in L^2(Q)$ and write $S_0 = S$. Then we consider $\tilde{S} \in L^2(Q)$ such that $\|\tilde{S}\|_{L^2(Q)} \leq 1$. For $\lambda > 0$, we set $S_\lambda = S_0 + \lambda\tilde{S}$. As the previous section we write I_i, p_i, q_{1i} and q_{2i} the respective solutions of Equations (2.1)–(2.2), (2.5)–(2.6), and p_i^{obs} the measured pressure on $\omega \times [0, T]$ when the source signal is S_i $(i = 0, \lambda)$, we will assume that $p_i^{\text{obs}} \in H^1(0, T; L^2(\omega))$. As the object is unchanged, we also suppose that

$$\|p_0^{\text{obs}} - p_\lambda^{\text{obs}}\|_{L^2(L^2(\omega))} \leq C\|S_0 - S_\lambda\|_{L^2(Q)} \qquad (\mathcal{H})$$

for some constant C and λ small enough (say $|\lambda| \le \lambda_{\max}$ for example). This is realistic form a practical point of view. From Theorem 2.2 and Theorem 3.1 there exist unique $\mu_0, \mu_\lambda \in \mathcal{U}_{ad}$ solutions of Problems $(\mathcal{P}_0)$, $(\mathcal{P}_\lambda)$ respectively. From inequality (3.1), with α large enough, we can conclude that the map

$$T : L^2(0,T;L^2(\Omega)) \to L^2(\Omega)$$
$$S \mapsto \mu$$

is locally Lipschitz continuous. There exists an extensively literature devoted to the study of the differentiability of Lipschitz continuous maps between Banach spaces, more precisely in obtaining an extension of Rademacher's Theorem, and we refer to [4,17,14,6,16,19,21,24] for this purpose. Following the results obtained in the previous papers we can deduce that T is Gâteaux differentiable at S for all $S \in L^2(Q) \setminus \mathfrak{A}$, where $\mathfrak{A}$ is the class of *exceptional sets*, this sets take the place of sets of Lebesgue measure 0 in finite dimensional spaces. We refer to [4, Chapter 1] for the definition and properties of sets $\mathfrak{A}$.

The derivative is the map $\dot{\mu} : L^2(Q) \to \mathcal{T}(L^2(Q); L^2(\Omega))$ given by

$$\dot{\mu}(S;\tilde{S}) = \lim_{\lambda\to 0} \frac{T(S+\lambda\tilde{S}) - T(S)}{\lambda}, \tag{4.1}$$

where $\mathcal{T}(X;Y)$ denotes the set of all maps on X in Y. We note that the previous limit is uniform in $\tilde{S}$ on each compact set. As the domain space and the image space of T are separable Hilbert spaces the limit holds in the sense of the strong topology on $L^2(\Omega)$ (see for example [4, Theorem 1]). Then from (4.1) we have

$$\mu_\lambda = \mu_0 + \lambda\dot{\mu} + o(\lambda) \tag{4.2}$$

with $\|o(\lambda)\|_{L^2(\Omega)}/\lambda \to 0$ as $\lambda \to 0$.

In the same way, from estimates given by Theorem 2.3 and Theorem 2.4, we can deduce that the maps: $S \mapsto I$, $S \mapsto p$, $S \mapsto q_1$ et $S \mapsto q_2$ are Lipschitz continuous. As before, we deduce that these maps are Gâteaux differentiable and then there exist $\dot{I}$, $\dot{p}$, $\dot{q}_1$ and $\dot{q}_2$ such that

$$I_\lambda = I_0 + \lambda\dot{I} + o(\lambda), \quad p_\lambda = p_0 + \lambda\dot{p} + o(\lambda),$$
$$q_{1\lambda} = q_{10} + \lambda\dot{q}_1 + o(\lambda), \quad q_{2\lambda} = q_{20} + \lambda\dot{q}_2 + o(\lambda),$$

with $\|o(\lambda)\|_{L^2(Q)}/\lambda \to 0$ as $\lambda \to 0$. In addition, from the hypothesis made on $p^{\rm obs}$ we can conclude the existence of $\dot{p}^{\rm obs}$ such that

$$p^{\rm obs}_\lambda = p^{\rm obs}_0 + \lambda\dot{p}^{\rm obs} + o(\lambda),$$

with $\|o(\lambda)\|_{L^2(Q)}/\lambda \to 0$ as $\lambda \to 0$.

Furthermore $\dot{I}$, $\dot{p}$, $\dot{q}_1$ and $\dot{q}_2$ satisfy:

$$\begin{cases} \dfrac{1}{c}\dfrac{\partial \dot{I}}{\partial t} + \dot{\mu}I_0 + \mu_0\dot{I} - \Delta\dot{I} = \tilde{S} & \text{in } (0,T)\times\Omega \\ \dot{I} = 0 & \text{on } (0,T)\times\partial\Omega \\ \dot{I} = 0 & \text{in } (0,T)\times\mathcal{B}\setminus\Omega \\ \dot{I}(0,\cdot) = 0 & \text{in } \Omega \end{cases} \tag{4.3}$$

$$\begin{cases} \dfrac{\partial^2 \dot{p}}{\partial t^2} - \operatorname{div}(v_s^2 \nabla \dot{p}) = \mathbb{1}_\Omega \Gamma \left(\dot{\mu} I_0 + \mu_0 \dot{I} \right) & \text{in } (0,T) \times \mathcal{B} \\ \dot{p} = 0 & \text{on } (0,T) \times \partial\mathcal{B} \\ \dot{p}(0,\cdot) = \dfrac{\partial \dot{p}}{\partial t}(0,\cdot) = 0 & \text{in } \mathcal{B}, \end{cases} \tag{4.4}$$

$$\begin{cases} \dfrac{\partial^2 \dot{q}_1}{\partial t^2} - \operatorname{div}(v_s^2 \nabla \dot{q}_1) = \mathbb{1}_\omega (\dot{p} - \dot{p}^{\text{obs}}) & \text{in } (0,T) \times \mathcal{B} \\ \dot{q}_1 = 0 & \text{on } (0,T) \times \partial\mathcal{B} \\ \dot{q}_1(T,\cdot) = \dfrac{\partial \dot{q}_1}{\partial t}(T,\cdot) = 0 & \text{in } \mathcal{B}, \end{cases} \tag{4.5}$$

and

$$\begin{cases} -\dfrac{1}{c}\dfrac{\partial \dot{q}_2}{\partial t} + \mu_0 \dot{q}_2 - \Delta \dot{q}_2 = \Gamma(\dot{\mu} q_{10} + \mu_0 \dot{q}_1) - \dot{\mu} q_{20} & \text{in } (0,T) \times \Omega \\ \dot{q}_2 = 0 & \text{on } (0,T) \times \partial\Omega \\ \dot{q}_2(T,\cdot) = 0 & \text{in } \Omega. \end{cases} \tag{4.6}$$

Now, we define

$$m_\lambda := -\frac{1}{2\alpha} \int_0^T (\mathbb{1}_\Omega \Gamma q_{1\lambda} - q_{2\lambda}) \cdot I_\lambda \, dt; \tag{4.7}$$

so from (2.7), the optimal control μ_λ is equal to $\mathbb{P}_{\mathcal{U}_{ad}}(m_\lambda)$, where $\mathbb{P}_{\mathcal{U}_{ad}}$ denotes the projection in $L^2(\Omega)$ onto $\mathcal{U}_{ad}$. Moreover, the calculations made in the previous section show that the map

$$L^2(Q) \to L^2(\Omega)$$
$$S \mapsto m$$

is Lipschitz continuous. Then, repeating the above argument, we deduce that there exists $\dot{m} \in L^2(\Omega)$ such that

$$m_\lambda = m_0 + \lambda \dot{m} + o(\lambda) \tag{4.8}$$

with $\|o(\lambda)\|_{L^2(\Omega)}/\lambda \to 0$ as $\lambda \to 0$. It is easy to check that

$$\dot{m} = -\frac{1}{2\alpha} \left(\int_0^T (\mathbb{1}_\Omega \Gamma \dot{q}_1 - \dot{q}_2) I_0 \, dt + \int_0^T (\mathbb{1}_\Omega \Gamma q_{10} - q_{20}) \dot{I} \, dt \right) \tag{4.9}$$

with $\dot{I}$, $\dot{p}$, $\dot{q}_1$ and $\dot{q}_2$ satisfying (4.3)–(4.6) respectively. From [23, Lemma 2.1] it follows that there exists $\nu \in L^2(\Omega)$ such that

$$\mathbb{P}_{\mathcal{U}_{ad}}(m_\lambda) = \mathbb{P}_{\mathcal{U}_{ad}}(m_0) + \lambda \nu + o(\lambda)$$

with $\|o(\lambda)\|_{L^2(\Omega)}/\lambda \to 0$ as $\lambda \to 0$. From (4.2) and the above equality, we deduce

$$\dot{\mu} = \nu.$$

Definition 4.1. *Let H be a Hilbert space, $K \subset H$ a closed convex subset. For every $\zeta \in K$, we set*

$$C_K(\zeta) = \bigcup_{\xi > 0} \xi(K - \zeta).$$

$C_K(\zeta)$ is called the tangent cone at point ζ.

Theorem 4.1. *The derivative of the optimal control μ_0 at point S_0 in the direction of $\tilde{S}$ given by (4.1) satisfies the following properties:*

1. *$\dot{\mu} \in \overline{C_{\mathcal{U}_{ad}}(\mu_0)}$, where $\overline{A}$ denotes the closure in $L^2(\Omega)$ of a set A.*
2. *$\dot{\mu} \in \{\mu_0 - m_0\}^{\perp}$, where $A^{\perp}$ denotes the orthogonal set to A.*
3. *$\langle \dot{\mu} - \dot{m}, \dot{\mu} \rangle_{L^2(\Omega)} \leq 0$.*
4. *For all $w \in \overline{C_{\mathcal{U}_{ad}}(\mu_0) \cap \{\mu_0 - m_0\}^{\perp}}$ we have*

$$\langle \dot{\mu} - \dot{m}, w \rangle_{L^2(\Omega)} \geq 0.$$

Proof. Item (1) follows directly from (4.2): $\dot{\mu} = \lim_{\lambda \to 0} \frac{1}{\lambda}(\mu_\lambda - \mu_0)$ and the fact that $\mu_\lambda \in \mathcal{U}_{ad}$ for all $\lambda \geq 0$.

From Theorem 2.2 and equations (2.7), (4.7), we get for every $\lambda > 0$ and $\xi \in \mathcal{U}_{ad}$

$$\langle \mu_\lambda - m_\lambda, \xi - \mu_\lambda \rangle_{L^2(\Omega)} \geq 0 \tag{4.10}$$

and

$$\langle \mu_0 - m_0, \xi - \mu_0 \rangle_{L^2(\Omega)} \geq 0. \tag{4.11}$$

Taking $\xi = \mu_0$ in (4.10), dividing by λ and taking limit as $\lambda \to 0$ we obtain

$$\langle \mu_0 - m_0, \dot{\mu} \rangle_{L^2(\Omega)} \leq 0.$$

Similarly, taking $\xi = \mu_\lambda$ in (4.11), dividing by λ and taking limit as $\lambda \to 0$ we obtain

$$\langle \mu_0 - m_0, \dot{\mu} \rangle_{L^2(\Omega)} \geq 0.$$

Hence, we have item (2). Similarly, taking $\xi = \mu_0$ in (4.10) and $\xi = \mu_\lambda$ in (4.11), adding the two inequalities, dividing by λ^2 and passing to the limit as $\lambda \to 0$ gives $\langle \dot{\mu} - \dot{m}, \dot{\mu} \rangle_{L^2(\Omega)} \leq 0$.

In order to prove item (4) let us choose $\lambda_n \to 0$ as $n \to +\infty$. First, we check that

$$\begin{aligned} 0 &\geq \langle m_\lambda - \mu_\lambda, \xi - \mu_\lambda \rangle_{L^2(\Omega)} \\ &= \langle m_0 + \lambda_n \dot{m} - \mu_0 - \lambda_n \dot{\mu} + o(\lambda_n), \xi - \mu_0 - \lambda_n \dot{\mu} + o(\lambda_n) \rangle_{L^2(\Omega)} \\ &= \langle m_0 - \mu_0, \xi - \mu_0 \rangle_{L^2(\Omega)} - \langle m_0 - \mu_0, \lambda_n \dot{\mu} \rangle_{L^2(\Omega)} + \\ &\quad + \langle \lambda_n(\dot{m} - \dot{\mu}), \xi - \mu_0 \rangle_{L^2(\Omega)} - \langle \lambda_n(\dot{m} - \dot{\mu}), \lambda_n \dot{\mu} \rangle_{L^2(\Omega)} + o(\lambda_n). \end{aligned} \tag{4.12}$$

Here we used (4.10), (4.2), (4.8) and that $\lambda_n \to 0$. Now, we consider $\xi \in \mathcal{U}_{ad}$ such that $(\xi - \mu_0) \in \{\mu_0 - m_0\}^{\perp}$, the inequality (4.12) yields

$$\lambda_n \langle \dot{m} - \dot{\mu}, \xi - \mu_0 \rangle_{L^2(\Omega)} \leq \lambda_n \langle m_0 - \mu_0, \dot{\mu} \rangle_{L^2(\Omega)} + \lambda_n^2 \langle \dot{m} - \dot{\mu}, \dot{\mu} \rangle_{L^2(\Omega)} + o(\lambda_n).$$

Dividing by $\lambda_n > 0$ taking limit as $n \to +\infty$ and using item (2), we get

$$\langle \dot m - \dot\mu, \xi - \mu_0\rangle_{L^2(\Omega)} \le 0.$$

Now, choose $w \in C_{\mathcal{U}_{ad}}(\mu_0) \cap \{\mu_0 - m_0\}^\perp$: there exist $\tau > 0$ and $\xi \in \mathcal{U}_{ad}$ such that $w = \tau(\xi - \mu_0)$. So, we have

$$0 = \langle m_0 - \mu_0, w\rangle_{L^2(\Omega)} = \tau \langle m_0 - \mu_0, \xi - \mu_0\rangle_{L^2(\Omega)}.$$

As $\xi \in \mathcal{U}_{ad}$ and $(\xi - \mu_0) \in \{\mu_0 - m_0\}^\perp$, the previous computation shows that $\langle \dot\mu - \dot m, \xi - \mu_0\rangle_{L^2(\Omega)} \le 0$. Since $\tau > 0$, we obtain the inequality of item (4) for all $w \in C_{\mathcal{U}_{ad}}(\mu_0) \cap \{\mu_0 - m_0\}^\perp$.

Passing to the limit for appropriate sequences, we obtain the last inequality for all w in the closure of $C_{\mathcal{U}_{ad}}(\mu_0) \cap \{\mu_0 - m_0\}^\perp$. This concludes the proof of the theorem. □

Corollary 4.1. *The Gâteaux-derivative $\dot\mu$ of the optimal control μ at S_0 with respect to the source, can be characterized as the unique function in $L^2(\Omega)$ that verifies the optimality system (4.3)–(4.6) and*

$$\forall w \in \overline{C_{\mathcal{U}_{ad}}(\mu_0) \cap \{\mu_0 - m_0\}^\perp}, \qquad \langle \dot\mu - \dot m, w - \dot\mu\rangle_{L^2(\Omega)} \ge 0 \tag{4.13}$$

with $\dot m$ given by (4.9).

Proof. The demonstration follows immediately from Theorem 4.1, items (3) and (4). The inequality (4.13) says that $\dot\mu$ is the projection of $\dot m$ in $L^2(\Omega)$ onto $\overline{C_{\mathcal{U}_{ad}}(\mu_0) \cap \{\mu_0 - m_0\}^\perp}$. □

This corollary will be used to perform the numerical computation.

4.2. Derivative with respect to the observation

In this subsection, we consider the case where the source is unchanged and discuss the stability of the optimal solution of Problem $(\mathcal{P})$ with respect to the observation p^{obs}.

Let $S \in L^2(Q)$ be fixed. For any fixed $\tilde p \in L^2(0,T;L^2(\omega))$, we consider p_0^{obs} and $p_\lambda^{\text{obs}} = p_0^{\text{obs}} + \lambda\tilde p$ two observations on $[0,T]\times\omega$ and call μ_0, μ_λ the unique corresponding solutions of optimality systems (2.1)–(2.2), (2.5)–(2.7). For example, $\tilde p$ can be equal to $(p_1^{\text{obs}} - p_0^{\text{obs}})/\|p_1^{\text{obs}} - p_0^{\text{obs}}\|_{L^2}$ where p_1^{obs} is a measurement of the pressure corresponding to another abject, and $\lambda = \|p_1^{\text{obs}} - p_0^{\text{obs}}\|_{L^2}$. We will denote I_i, p_i, q_{1i} and q_{2i} the respective solutions of Equations (2.1)–(2.2), (2.5)–(2.6), when the observation data is p_i^{obs} with $i = 0, \lambda$.

Relation (3.1) of Theorem 3.1 yields that the map $T : p^{\text{obs}} \to \mu$ is Lipschitz continuous (if α is large enough). With the same arguments as in Subsection 4.1 we deduce there exists $\dot\mu(\tilde p) \in L^2(\Omega)$ such that

$$\dot\mu(p_0^{\text{obs}};\tilde p) = \lim_{\lambda\to 0} \frac{T(p_0^{\text{obs}} + \lambda\tilde p) - T(p_0^{\text{obs}})}{\lambda},$$

and

$$\mu_\lambda = \mu_0 + \lambda\dot\mu + o(\lambda)$$

with $\|o(\lambda)\|_{L^2(\Omega)}/\lambda \to 0$ as $\lambda \to 0$. We also get the existence of $\dot I$, $\dot p$, $\dot q_1$ and $\dot q_2$ such that

$$I_\lambda = I_0 + \lambda\dot I + o(\lambda), \quad p_\lambda = p_0 + \lambda\dot p + o(\lambda),$$
$$q_{1\lambda} = q_{10} + \lambda\dot q_1 + o(\lambda), \quad q_{2\lambda} = q_{20} + \lambda\dot q_2 + o(\lambda),$$

with $\|o(\lambda)\|_{L^2(Q)}/\lambda \to 0$ as $\lambda \to 0$. Then we can prove similarly

Theorem 4.2. *The Gâteaux-derivative $\dot{\mu}$ of the optimal control μ at p_0^{obs} with respect to the observation, can be characterized as the unique function in $L^2(\Omega)$ that verifies the following optimality system:*

$$\begin{cases} \dfrac{1}{c}\dfrac{\partial \dot{I}}{\partial t} + \mu_0 \dot{I} - \Delta \dot{I} = -\dot{\mu} I_0 & \text{in } (0,T)\times\Omega \\ \dot{I} = 0 & \text{on } (0,T)\times\partial\Omega \\ \dot{I} = 0 & \text{in } (0,T)\times\mathcal{B}\setminus\Omega \\ \dot{I}(0,\cdot) = 0 & \text{in } \Omega, \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 \dot{p}}{\partial t^2} - \operatorname{div}(v_s^2 \nabla \dot{p}) = \mathbb{1}_\Omega \Gamma\left(\dot{\mu} I_0 + \mu_0 \dot{I}\right) & \text{in } (0,T)\times\mathcal{B} \\ \dot{p} = 0 & \text{on } (0,T)\times\partial\mathcal{B} \\ \dot{p}(0,\cdot) = \dfrac{\partial \dot{p}}{\partial t}(0,\cdot) = 0 & \text{in } \mathcal{B}, \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 \dot{q}_1}{\partial t^2} - \operatorname{div}(v_s^2 \nabla \dot{q}_1) = \mathbb{1}_\omega (\dot{p} - \tilde{p}) & \text{in } (0,T)\times\mathcal{B} \\ \dot{q}_1 = 0 & \text{on } (0,T)\times\partial\mathcal{B} \\ \dot{q}_1(T,\cdot) = \dfrac{\partial \dot{q}_1}{\partial t}(T,\cdot) = 0 & \text{in } \mathcal{B}, \end{cases}$$

$$\begin{cases} -\dfrac{1}{c}\dfrac{\partial \dot{q}_2}{\partial t} + \mu_0 \dot{q}_2 - \Delta \dot{q}_2 = \Gamma(\dot{\mu} q_{10} + \mu_0 \dot{q}_1) - \dot{\mu} q_{20} & \text{in } (0,T)\times\Omega \\ \dot{q}_2 = 0 & \text{on } (0,T)\times\partial\Omega \\ \dot{q}_2(T,\cdot) = 0 & \text{in } \Omega, \end{cases}$$

and

$$\forall w \in \overline{C_{\mathcal{U}_{ad}}(\mu_0) \cap \{\mu_0 - m_0\}^\perp}, \qquad \langle \dot{\mu} - \dot{m}, w - \dot{\mu} \rangle_{L^2(\Omega)} \geq 0$$

where $\dot{m}$ is given by (4.9).

5. Conclusions

We have proved the uniqueness of the solution of the optimal control problem studied in [12] and given a stability result. We also provide a characterization of the derivative of the optimal control with respect both to the source and the observation. Furthermore, this characterization leads to a numerical scheme for the computation of these derivatives. This numerical aspect will be discussed in a forthcoming paper.

Open questions remain, as the stability of the optimal control with respect to the sound speed. The study of the uniqueness and stability in the case when the diffusion coefficient D is not constant will be studied in future work.

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