Numerical reconstruction based on Carleman estimates of a source term in a reaction-diffusion equation. *

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Abstract

In this article, we consider a reaction-diffusion equation where the reaction term is given by a cubic function and we are interested in the numerical reconstruction of the time-independent part of the source term from measurements of the solution. For this identification problem, we present an iterative algorithm based on Carleman estimates which consists of minimizing at each iteration cost functionals which are strongly convex on bounded sets. Despite the nonlinear nature of the problem, we prove that our method globally converges and the convergence speed evaluated in weighted norm is linear. In the last part of the paper, we illustrate the effectiveness of our method with several numerical reconstructions in dimension one or two.

Keywords: inverse problems, nonlinear parabolic equations, Carleman estimates, numerical reconstruction.

AMS subject classifications: 35R30, 35K55, 35K57, 93B07.

1 Introduction

Let Ω be a C^2 bounded domain of $\mathbb{R}^d$ for $d = 1, 2$ or 3 and $T > 0$. We consider the following reaction-diffusion equation

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) + u^3(t,x) = \sigma(x)h(t,x), & (t,x) \in (0,T) \times \Omega, \\ u(t,x) = g(t,x), & (t,x) \in (0,T) \times \partial\Omega, \\ u(0,x) = u_\circ(x), & x \in \Omega, \end{cases} \tag{1.1}$$

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where g is the Dirichlet boundary data and $u_\circ$ is the initial condition. In the right hand side of the first equation, we assume that the time-dependent function h is known and we focus on the reconstruction of σ which is assumed to depend only on the spatial variable. To identify this unknown, we have two kinds of measurements, the flux of the solution on a part of a boundary and the solution in the whole domain at a given time:

$$\begin{cases} m(t,x) := \nabla u(t,x) \cdot n(x), & (t,x) \in (0,T) \times \Gamma, \\ r(x) := u(T_0, x), & x \in \Omega, \end{cases} \tag{1.2}$$

where $\Gamma \subset \partial\Omega$, $T_0 \in (0,T)$ and n is the outward-pointing unit normal vector defined on $\partial\Omega$. Regarding the applications, this model can represent the evolution of a pollutant in the atmosphere. The source in the right hand side corresponds to a spill of pollutant and we want to localise it. This model can also be viewed as a simplified model to represent the evolution of the electrical potential in the heart (we refer to [8] for a detailed presentation of this application domain and more precisely to [8, Subsection 2.9.7] for cubic-like reaction models). In our model, the natural propagation of the potential is initiated by the initial condition and the source in the right hand side may correspond to a secondary undesirable source that we want to identify.

Let $\sigma_{max} > 0$ be a fixed constant. We assume that the source term σ that we want to reconstruct belongs to $L^\infty(\Omega)$ and satisfies the following *a priori* bound:

$$\|\sigma\|_{L^\infty(\Omega)} \leq \sigma_{max}. \tag{1.3}$$

For this problem, according to Bukhgeim-Klibanov method, σ is uniquely determined by the measurements and a Lipschitz stability estimate holds under appropriate assumptions on the data (the precise result is stated in Proposition 2.6). Bukhgeim-Klibanov method [5] is a classical theoretical method to prove the uniqueness and stability for parameter identification problems. For a presentation of this method which relies on Carleman estimates and for a survey on its applications, we refer to [18] and in particular to section 3.3 on parabolic equations. For the inverse problem of coefficients identification in nonlinear parabolic equations, let us in particular mention that [3] and [9] deal with the theoretical stability of the reaction term in a semi-linear PDE.

In this paper, our aim is to tackle the numerical reconstruction of σ and to propose for this nonlinear problem a globally convergent method. Our work is drawn from a numerical method presented in [1] for the identification of a potential in a wave equation. The method strongly relies on Carleman inequalities and it consists of an iterative algorithm minimizing at each iteration a cost functional involving Carleman weights. The main strength of this numerical method is that it globally converges to the exact solution *i.e.* it converges independently of the initialization. In particular, contrary to classical minimization techniques like Tikhonov methods [16], it is not necessary to add a priori knowledge on the source term through the data of a background state to convexify the cost functional.

As pointed out in the introduction of [2], this method induces several numerical challenges. In particular, the classical Carleman weights have very strong variations due to the presence of a double exponential involving large coefficients. That is why, as in [2] for the wave equations, we need to construct new Carleman weights for the heat equation which involve single exponentials (these weights are given by (2.3) and (2.4)).

The presence of a nonlinearity in our PDE leads to additional difficulties in the study of the numerical methods. In particular, the strong convexity properties of the cost functional are restricted to bounded spaces. Moreover, the operator appearing in the cost functional has to be modified by adding truncation operators in the nonlinear terms to tackle these terms in the proof of the convergence of the numerical method. At last, contrary to [2] where the PDE is linear, introducing a conjugate variable $e^{s\varphi}z$ does not allow to overcome the fact that, even with single exponentials, the minimization of the cost functional is challenging.

Let us mention that we could have considered general cubic functions of the form

$$a_3u^3 + a_2u^2 + a_1u$$

with $a_3 > 0$ instead of a simple cubic monomial in this semi-linear parabolic partial differential equations (1.1). By this way, the study includes the bistable equation or Allen-Cahn equation. Moreover, we refer to Remark 2.4 for some remarks on the case of other boundary data (Neumann boundary conditions instead of the second equation in (1.1) and boundary measurements on the solution itself).

Using Carleman estimates to solve numerically inverse problems has been first considered in the paper [17] by Klibanov. This method called convexification method has been applied in several papers. In [19], the authors are interested by the reconstruction of a coefficient in a parabolic equation and present a gradient method applied to a strictly convex cost functional involving Carleman weights. In [22], the authors consider the reconstruction of the initial condition in a nonlinear parabolic equation. We also refer to [20] for the most recent paper which applies this convexification method.

For numerical studies applying Carleman estimates to controllability problems, we refer to [7] for the numerical controllability of the wave equation and [12] for the numerical controllability of the heat, Stokes and Navier-Stokes equation.

The paper is organized as follows. Sections 2 give some preliminary results. First, in Section 2.1, we present a Carleman estimate for the heat operator with Dirichlet boundary conditions. In this estimate, we consider two kinds of Carleman weights: the classical weights for the heat equation with a double exponential and new weights involving single exponentials which are introduced for numerical purposes. Then, in Section 2.2, we state a regularity result satisfied by the solution of equation (1.1). The proofs of the Carleman estimate and the regularity result are presented in Appendices A and B respectively. At last, in Section 2.3, we state the stability inequality associated to our inverse problem.

Section 3.1 is the core of the paper and presents the numerical reconstruction method of the source term. The latter is an iterative process which requires at each iteration the minimization of a functional based on the Carleman estimate. This section states the global convergence of the method (Theorem 3.3). In Section 3.3, we establish properties satisfied by the functional to minimize at each step. In particular, the existence of a global minimizer of the functional is stated in Lemma 3.6 and the strong convexity on bounded set is proved in Lemma 3.7. In the last section (Section 3.4), we prove the global convergence property. Finally, Section 4 is devoted to the implementation of the algorithm and the numerical results obtained for several 1D and 2D test cases.

2 Preliminary results

2.1 Carleman inequality for the heat equation

Without loss of generality, from now on, we assume that $T_0 = \dfrac{T}{2}$.
In this section, we state a Carleman inequality for the heat equation in two cases. The first case corresponds to the classical weights with a double exponential while, in the second case, the weights only involve single exponentials as in [27, Section 3]. Let us specify these two cases :

- **Case 1:** For $\lambda > 0$, we define θ and φ by: for all $(t,x) \in (0,T) \times \Omega$

$$\theta(t,x) = \frac{e^{\lambda(2\|\eta_0\|_\infty + \eta_0(x))}}{t(T-t)} \quad \varphi(t,x) = \frac{e^{\lambda(2\|\eta_0\|_\infty + \eta_0(x))} - e^{4\lambda\|\eta_0\|_\infty}}{t(T-t)} \tag{2.1}$$

  where η_0 satisfies the following properties:

$$\eta_0 > 0 \text{ in } \Omega, \quad |\nabla \eta_0| \geq C > 0 \text{ in } \overline{\Omega} \quad \text{and} \quad \eta_0 = 0 \text{ on } \partial\Omega \setminus \Gamma. \tag{2.2}$$

- **Case 2:** For all $(t,x) \in (0,T) \times \Omega$, we define

$$\theta(t) = \frac{1}{t(T-t)} - \frac{1-\rho}{T_0^2} \tag{2.3}$$

  and

$$\varphi(t,x) = \psi(x)\theta(t) \quad \text{with} \quad \psi(x) = |x - x_0|^2 - 2\sup_{x\in\overline{\Omega}} |x - x_0|^2, \tag{2.4}$$

  where x_0 is an arbitrary point in $\mathbb{R}^d \setminus \overline{\Omega}$ and ρ is a constant satisfying $0 < \rho < 1$. We notice that $\theta > 0$ and $\psi < 0$. In this case, we assume in addition that x_0 and Γ are such that

$$\{x \in \partial\Omega \,|\, (x - x_0) \cdot n(x) > 0\} \subset \Gamma. \tag{2.5}$$

Let us mention that the spatial part ψ of the Carleman weight in **Case 2** resembles the one proposed in [1] for the wave equation. Moreover, the geometric condition (2.5) which is classical for the wave equation (see [14, 23]) is unusual for the heat equation and is linked to this new choice of weights. With these weights, we have less flexibility in the computations and we need an extra condition on the measurement domain compared to the classical weights corresponding to **Case 1**. On the other hand, if we take the classical weights, the presence of a double exponential in the functional to minimize (see (3.11)) is prohibitive to address numerical applications (we refer to Remark 2.3 for additional comments). In all our numerical tests presented in Section 4.2, we have considered the weights given by **Case 2**.

Let us now formulate the Carleman inequality in **Case 1** and **Case 2**.

Theorem 2.1. *We assume that θ and φ are given by (2.1) where λ is fixed and large enough or by (2.4). In this last case, we assume that Γ is such that (2.5) holds. Then, there exists $s_0 > 0$ and $C > 0$*

such that, for all $s \geq s_0$:

$$\int_0^T \int_\Omega e^{2s\varphi} \left(\frac{1}{s\theta} |\partial_t z|^2 + \frac{1}{s\theta} |\Delta z|^2 + s\theta |\nabla z|^2 + s^3\theta^3 |z|^2 \right) dxdt$$
$$\leq C \int_0^T \int_\Omega e^{2s\varphi} |\partial_t z - \Delta z|^2 \, dxdt + Cs \int_0^T \int_\Gamma e^{2s\varphi} \theta |\nabla z \cdot n|^2 \, d\gamma dt, \quad (2.6)$$

for all $z \in H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$.

Here and in all the paper, we denote by C a positive constant which depends on T and Ω, λ in **Case 1** and ρ in **Case 2**, unless specified otherwise where appropriated. The proof of this theorem is given in Appendix A. A consequence of Theorem 2.1 is the following lemma:

Lemma 2.2. *Under the same assumptions as Theorem 2.1, there exist $s_0 > 0$ and $C > 0$ such that, for all $s \geq s_0$:*

$$s \int_\Omega e^{2s\varphi(T_0)} |z(T_0)|^2 \, dx \leq C \int_0^T \int_\Omega e^{2s\varphi} |\partial_t z - \Delta z|^2 \, dxdt + Cs \int_0^T \int_\Gamma e^{2s\varphi} \theta |\nabla z \cdot n|^2 \, d\gamma dt, \quad (2.7)$$

for all $z \in H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$.

Proof. We have

$$\begin{aligned}
\int_\Omega e^{2s\varphi(T_0)} |z(T_0)|^2 \, dx &= \int_0^{T_0} \frac{d}{dt} \left(\int_\Omega e^{2s\varphi} |z|^2 \, dx \right) dt = \int_0^{T_0} \int_\Omega \partial_t (e^{2s\varphi} |z|^2) \, dxdt \\
&= \int_0^{T_0} \int_\Omega e^{2s\varphi} \left(2z \left(s\theta^{1/2} \frac{1}{s\theta^{1/2}} \right) \partial_t z + 2s \partial_t \varphi |z|^2 \right) dxdt \\
&\leq \int_0^{T_0} \int_\Omega e^{2s\varphi} \frac{1}{s^2\theta} |\partial_t z|^2 \, dxdt + \int_0^{T_0} \int_\Omega e^{2s\varphi} (s^2\theta + 2s|\partial_t \varphi|) |z|^2 \, dxdt \\
&\leq \frac{C}{s} \left[\int_0^{T_0} \int_\Omega e^{2s\varphi} \frac{1}{s\theta} |\partial_t z|^2 \, dxdt + \int_0^{T_0} \int_\Omega e^{2s\varphi} s^3\theta^2 |z|^2 \, dxdt \right]
\end{aligned}$$

where we have used that $|\partial_t \varphi| \leq C\theta^2$. Thus, the result follows from (2.6). ☐

Remark 2.3. *To better design Carleman weights for numerical purposes, it would be interesting to make a comprehensive comparison between different possible choices of Carleman weights for the heat equation. In particular, in such a study which is beyond the scope of our paper, it would be necessary to spell the lower bound on s in the associated Carleman inequality.*

Remark 2.4. *We could have considered other kinds of boundary data by completing the first equation of (1.1) with Neumann conditions instead of Dirichlet conditions and by replacing the first measurement in (1.2) by a measurement on a part of the boundary of u itself. In this case, following [13] and [11], we still have a Carleman inequality with the classical weights (2.1) and we can still prove the global convergence of the numerical method. For the numerical tests, it would be interesting to see if we can get a Carleman inequality with weights similar to the ones of* **Case 2**.

2.2 Regularity result

Let us give a regularity result for problem (1.1). The proof of this result is presented in Appendix B.

Proposition 2.5. *Assume that* $u_\circ \in H^3(\Omega)$, $\sigma \in L^\infty(\Omega)$, $h \in H^1(0,T;L^2(\Omega))$ *and* $g \in H^1(0,T;H^{3/2}(\partial\Omega)) \cap H^2(0,T;H^{1/2}(\partial\Omega))$. *Moreover, we assume that* $h(0,\cdot) = 0$ *in* Ω.

Then the solution u *of (1.1) belongs to*

$$u \in C^1(0,T;H^1(\Omega)) \cap H^2(0,T;L^2(\Omega)) \cap H^1(0,T;H^2(\Omega))$$

with the estimate

$$\begin{aligned}
&\|u\|_{C^1(0,T;H^1(\Omega))} + \|u\|_{H^2(0,T;L^2(\Omega))} + \|u\|_{H^1(0,T;H^2(\Omega))} \\
&\le C\left(\|\sigma\|_{L^\infty(\Omega)} + \|\sigma\|^p_{L^\infty(\Omega)}\right)\left(\|h\|_{H^1(0,T;L^2(\Omega))} + \|h\|^p_{H^1(0,T;L^2(\Omega))}\right) + C\left(\|u_\circ\|_{H^3(\Omega)} + \|u_\circ\|^p_{H^3(\Omega)}\right) \\
&\quad + C\left(\|g\|_{H^1(0,T;H^{3/2}(\partial\Omega))\cap H^2(0,T;L^2(\partial\Omega))} + \|g\|^p_{H^1(0,T;H^{3/2}(\partial\Omega))\cap H^2(0,T;H^{1/2}(\partial\Omega))}\right)
\end{aligned} \tag{2.8}$$

where the power $p > 1$ *is a fixed integer and* C *only depends on* T *and* Ω.

Let us note that, in the above proposition, the regularity assumed for g is not optimal, it would indeed be sufficient to assume that $g \in H^1(0,T;H^{3/2}(\partial\Omega)) \cap H^2(0,T;H^\kappa(\partial\Omega))$ with $\kappa > 0$ (see [24, Chapter 1, Subsection 9.2]). In this result, if we do not make the assumption that $h(0,\cdot) = 0$ in Ω, it is necessary to assume that σ belongs to $H^1(\Omega)$ (since we need an initial condition in $H^1(\Omega)$ for the problem satisfied by $\partial_t u$). But this additional regularity assumption on σ leads to difficulties in the construction of the iterations in Algorithm 1.

2.3 Stability inequality

In this paragraph, we state a Lipschitz stability inequality for our inverse problem. This result asserts in particular that the unknown σ is identifiable from the measurements given by (1.2). It is obtained thanks to a direct application of Bukhgeim-Klibanov method [5] and relies on the Carleman inequality given by Theorem 2.1 and the regularity result given by Proposition 2.5. We do not give the proof here and refer to [15] for a closely related result.

Proposition 2.6. *We assume that* $u_\circ \in H^3(\Omega)$, $g \in H^1(0,T;H^{3/2}(\partial\Omega)) \cap H^2(0,T;H^{1/2}(\partial\Omega))$ *and* $h \in H^1(0,T;L^\infty(\Omega))$ *is such that* $h(0,\cdot) = 0$ *in* Ω *and* $|h(T_0,\cdot)| \ge \beta > 0$ *in* Ω. *We consider* σ_1 *and* σ_2 *in* $L^\infty(\Omega)$ *which satisfy (1.3). Then, for* $i = 1, 2$, *if we denote by* u_i *the solution of (1.1) associated to* σ_i, *we have the following inequality: there exists* $C > 0$ *such that*

$$\|\sigma_1 - \sigma_2\|_{L^2(\Omega)} \le C\left(\|u_1(T_0) - u_2(T_0)\|_{H^2(\Omega)} + \|\nabla(u_1 - u_2)\cdot n\|_{H^1(0,T;L^2(\Gamma))}\right).$$

3 Numerical reconstruction method and theoretical study

3.1 Presentation of the algorithm and convergence

In this subsection, we construct a sequence $(\sigma_k)_{k\in\mathbb{N}}$ which approximates the unknown σ and we state the convergence of this sequence. We make the following assumptions:

Hypotheses 3.1.

- $u_\circ \in H^3(\Omega)$ and $g \in H^1(0,T;H^{3/2}(\partial\Omega)) \cap H^2(0,T;H^{1/2}(\partial\Omega))$.
- $\sigma \in L^\infty(\Omega)$ satisfies (1.3).
- h satisfies
$$h \in H^1(0,T;L^\infty(\Omega)), \quad h(0,\cdot) = 0 \text{ in } \Omega \tag{3.1}$$
and
$$|h(T_0,\cdot)| \geq \beta > 0 \quad \text{ in } \Omega. \tag{3.2}$$
- The weights θ and φ are given by **Case 1** or **Case 2** described at the beginning of paragraph 2.1. In **Case 1**, the parameter λ is fixed and large enough.

In our paper, we denote by M an arbitrary constant which only depends on T, Ω, σ_{max}, $\|u_\circ\|_{H^3(\Omega)}$, $\|h\|_{H^1(0,T;L^\infty(\Omega))}$ and $\|g\|_{H^1(0,T;H^{3/2}(\partial\Omega))\cap H^2(0,T;H^{1/2}(\partial\Omega))}$.

First, we initialize the sequence with $\sigma_0 = 0$ (or any guess such that $\|\sigma_0\|_{L^\infty(\Omega)} \leq \sigma_{max}$).
Now, let us assume that we are at step k and that we have constructed σ_k which satisfies
$$\|\sigma_k\|_{L^\infty(\Omega)} \leq \sigma_{max}. \tag{3.3}$$
We denote by u_k the solution of (1.1) associated to σ_k and by u_σ the solution of (1.1) associated to the unknown σ. Moreover, we set $v_k = u_\sigma - u_k$.

We then use Proposition 2.5 and we denote by $\overline{M} > 0$ a fixed constant depending on T, Ω, σ_{max}, $\|u_\circ\|_{H^3(\Omega)}$, $\|h\|_{H^1(0,T;L^2(\Omega))}$ and $\|g\|_{H^1(0,T;H^{3/2}(\partial\Omega))\cap H^2(0,T;H^{1/2}(\partial\Omega))}$ such that
$$\|v_k\|_{C([0,T]\times\overline{\Omega})} + \|v_k\|_{C^1(0,T;H^1(\Omega))} + \|v_k\|_{H^2(0,T;L^2(\Omega))} + \|v_k\|_{H^1(0,T;H^2(\Omega))} \leq \overline{M}. \tag{3.4}$$
The function v_k is solution of
$$\begin{cases} \partial_t v_k(t,x) - \Delta v_k(t,x) + v_k(t,x) q[v_k,u_k](t,x) = (\sigma(x) - \sigma_k(x))h(t,x), & (t,x) \in (0,T)\times\Omega, \\ v_k(t,x) = 0, & (t,x) \in (0,T)\times\partial\Omega, \\ v_k(0,x) = 0, & x \in \Omega, \end{cases} \tag{3.5}$$
where we have set $q[v,u] = 3u^2 + 3uv + v^2$. Let us differentiate the equation with respect to time. We introduce $w_k = \partial_t v_k$ which satisfies:
$$\begin{cases} \partial_t w_k(t,x) - \Delta w_k(t,x) + w_k(t,x) q[v_k,u_k](t,x) + v_k(t,x)\partial_t(q[v_k,u_k])(t,x) = f_k(t,x), & (t,x) \in (0,T)\times\Omega, \\ w_k(t,x) = 0, & (t,x) \in (0,T)\times\partial\Omega, \\ w_k(0,x) = 0, & x \in \Omega, \end{cases} \tag{3.6}$$
where, for all $(t,x) \in (0,T)\times\Omega$,
$$f_k(t,x) = (\sigma(x) - \sigma_k(x))\partial_t h(t,x). \tag{3.7}$$
Let us now explain the core idea of the numerical method that we will introduce below. We notice in (3.5) that
$$w_k(T_0,x) = \partial_t v_k(T_0,x) = \Delta v_k(T_0,x) - v_k(T_0,x) q[v_k,u_k](T_0,x) + (\sigma(x) - \sigma_k(x))h(T_0,x), \qquad x \in \Omega. \tag{3.8}$$

Hence, if $w_k(T_0,\cdot)$ was known, then σ could be directly computed, because we assume that $h(T_0,\cdot)$ satisfies (3.2) and the other terms in (3.8) are given observations thanks to (1.2). However, since f_k defined in (3.7) depends on σ, w_k is unknown. Thus, the idea is to use Z_k obtained via the minimization step (Step 2) of Algorithm 1 as a proxy for w_k. In Section 3.4, we will estimate the discrepancy between Z_k and w_k (both seen as minimizers of functionals) with respect to f_k.

For the constant $\overline{M} > 0$ introduced in estimate (3.4), we consider the following function:

$$\begin{aligned} T_{\overline{M}}:\quad & \mathbb{R} \longrightarrow \mathbb{R} \\ & X \longmapsto X\Phi\left(\frac{X}{\overline{M}}\right), \end{aligned} \tag{3.9}$$

where $\Phi \in C_0^2(\mathbb{R})$ is such that $0 \le \Phi \le 1$ and

$$\Phi(X) = \begin{cases} 1, & \text{if } |X| \le 1, \\ 0, & \text{if } |X| \ge 2. \end{cases} \tag{3.10}$$

The properties satisfied by $T_{\overline{M}}$ are given in section 3.2. For any μ in $L^2((0,T)\times\Gamma)$, we introduce the functional $J_{0,k}[\mu]$ by

$$J_{0,k}[\mu](z) = \frac{1}{2}\int_0^T\int_\Omega e^{2s\varphi}|P_k z|^2\,dxdt + \frac{s}{2}\int_0^T\int_\Gamma e^{2s\varphi}\theta|\nabla z\cdot n - \mu|^2\,d\gamma dt, \tag{3.11}$$

with

$$P_k z = \partial_t z - \Delta z + 3(u_k)^2 z + 6\partial_t u_k u_k T_{\overline{M}}(y) + 3\partial_t u_k T_{\overline{M}}(y)^2 + 6u_k z\, T_{\overline{M}}(y) + 3z\, T_{\overline{M}}(y)^2 \tag{3.12}$$

where

$$y(t,x) = v_k(T_0,x) + \int_{T_0}^t z(t',x)dt', \quad (t,x)\in(0,T)\times\Omega.$$

By this way, since v_k satisfies (3.4), $T_{\overline{M}}(v_k) = v_k$ and $P_k(w_k)$ corresponds to the left hand side of the first equation of (3.6).

We consider the functional $J_{0,k}[\mu]$ on the function space

$$\begin{aligned} \widetilde{E} = \Big\{ & z : e^{s\varphi}(\partial_t z - \Delta z) \in L^2((0,T)\times\Omega), e^{s\varphi}\theta^{1/2}\nabla z\cdot n \in L^2((0,T)\times\Gamma), \\ & e^{s\varphi}\theta^{3/2} z \in L^2((0,T)\times\Omega),\ e^{s\varphi}\theta^{-1/2} z \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega))\Big\} \end{aligned} \tag{3.13}$$

endowed with its natural norm.

The next iteration σ_{k+1} is defined by following four steps:

Algorithm 1. ***Iteration: From*** k ***to*** $k+1$

• *Step 1* - We set $\mu_k = \partial_t(m - \nabla u_k\cdot n)$ on $(0,T)\times\Gamma$, where m is the measurement defined in (1.2) and u_k is the solution of (1.1) associated to σ_k.

- *Step 2* - We denote by Z_k a minimizer of $J_{0,k}[\mu_k]$ in $\widetilde{E}$.
- *Step 3* - We set

$$\widetilde{\sigma}_{k+1}(x) = \sigma_k(x) + \frac{Z_k(T_0, x) - \Delta v_k(T_0, x) + v_k(T_0, x) q[v_k, u_k](T_0, x)}{h(T_0, x)}, \quad x \in \Omega. \tag{3.14}$$

- *Step 4* - At last, we define

$$\sigma_{k+1} = \Pi_{\sigma_{max}}(\widetilde{\sigma}_{k+1}),$$

where $\Pi_{\sigma_{max}}$ is given by

$$\Pi_{\sigma_{max}}(\sigma) = \begin{cases} \sigma, & \text{if } |\sigma| \leq \sigma_{max}, \\ \text{sign}(\sigma)\sigma_{max}, & \text{otherwise.} \end{cases}$$

Remark 3.2. *Let us give some comments regarding the different steps of Algorithm 1.*

- *According to Lemma 3.6,* $J_{0,k}[\mu_k]$ *(defined in (3.11)) admits a global minimizer in* $\widetilde{E}$ *if* s *is large enough. Hence,* Z_k *introduced in Step 2 is well-defined. It depends on* s *but we drop this dependence to simplify the notations.*

- *In Step 3,* $\widetilde{\sigma}_{k+1}$ *is well-defined because* h *satisfies the positivity condition (3.2) and* $h(T_0, \cdot)$ *belongs to* $L^2(\Omega)$*. Moreover, in this expression,* $v_k(T_0, \cdot)$ *is known and given by* $v_k(T_0, \cdot) = r - u_k(T_0, \cdot)$ *where* r *is the measurement defined in (1.2). Since* $u_k(T_0)$ *and* $v_k(T_0)$ *belong to* $H^2(\Omega)$ *and* $Z_k(T_0)$ *belongs to* $L^2(\Omega)$*,* $\widetilde{\sigma}_{k+1}$ *belongs to* $L^2(\Omega)$*.*

- *Step 4 is needed to ensure that* σ_{k+1} *satisfies (3.3) at step* $k + 1$*.*

Now we state the main theoretical result which gives the global linear convergence in the weighted L^2-norm of the sequence $(\sigma_k)_{k\in\mathbb{N}}$:

Theorem 3.3. *Under Hypotheses 3.1, there exist* $s_0 > 0$ *and* $M > 0$ *such that for all* $s \geq s_0$*, for all* $k \in \mathbb{N}$

$$\int_\Omega e^{2s\varphi(T_0)} |\sigma_{k+1} - \sigma|^2 dx \leq \frac{M}{s} \int_\Omega e^{2s\varphi(T_0)} |\sigma_k - \sigma|^2 dx. \tag{3.15}$$

Thus, for s *large enough,* $(\sigma_k)_{k\in\mathbb{N}}$ *tends to* σ *when* k *goes to* $+\infty$*.*

This theorem will be proved in Subsection 3.4.

Remark 3.4. *Let us notice that our method may also be applied to the identification of a source in the simpler case of the linear heat equation. In this case, the inverse problem is linear and thus the properties of our method (in particular the global convergence) are much more classical. If we simply consider the least square method, the functional is quadratic and, thanks to the stability estimate, we can prove that this functional is strongly convex. Thus, a classical gradient descent method globally converges and it is not necessary to introduce our algorithm which is more complex.*

3.2 Properties satisfied by the function $T_{\overline{M}}$

Proposition 3.5. *The function $T_{\overline{M}}$ defined by (3.9) belongs to $C_0^2(\mathbb{R})$ satisfies the following properties:*

a) For all $X \in \mathbb{R}$,

$$\left|T_{\overline{M}}(X)\right| \leq 2\overline{M}. \tag{3.16}$$

b) There exists $L > 0$ such that

$$|T'_{\overline{M}}(X)| \leq L\chi_{[-2\overline{M},2\overline{M}]}(X), \qquad \forall X \in \mathbb{R}, \tag{3.17}$$

where χ_A is the characteristic function of a set A.

c) For all $X_1,\ X_2 \in \mathbb{R}$,

$$|T_{\overline{M}}(X_1) - T_{\overline{M}}(X_2)| \leq L|X_1 - X_2|, \tag{3.18}$$

which implies in particular that $T_{\overline{M}}$ is a Lipschitz operator.

d) There exists $C > 0$ such that, for all $X_1,\ X_2 \in \mathbb{R}$,

$$|T'_{\overline{M}}(X_1) - T'_{\overline{M}}(X_2)| \leq C|X_1 - X_2|, \tag{3.19}$$

Proof. a) For $X \in \mathbb{R}$, we have

$$\left|T_{\overline{M}}(X)\right| = \left|X\Phi\left(\frac{X}{\overline{\overline{M}}}\right)\right| = \left\{\begin{array}{ll} \leq |X|, & \text{if } |X| \leq 2\overline{M} \\ = 0, & \text{if } |X| > 2\overline{M} \end{array}\right\} \leq 2\overline{M}.$$

b) By definition (3.9), $T'_{\overline{M}}(X) = 0$ for all $|X| \geq 2\overline{M}$. Moreover, for all $|X| \leq 2\overline{M}$, we have

$$\begin{aligned} |T'_{\overline{M}}(X)| &= \left|\Phi\left(\frac{X}{\overline{\overline{M}}}\right) + \frac{X}{\overline{\overline{M}}}\Phi'\left(\frac{X}{\overline{\overline{M}}}\right)\right| \\ &\leq 1 + 2\|\Phi'\|_{C_0(\mathbb{R})}. \end{aligned}$$

c) This is a direct consequence of (3.17) and the mean value inequality.

d) By the same arguments as in b), we show that

$$|T''_{\overline{M}}(X)| \leq L\chi_{[-2\overline{M},2\overline{M}]}(X), \qquad \forall X \in \mathbb{R}. \tag{3.20}$$

Hence, (3.19) is a direct consequence of (3.20) and the mean value inequality.

□

3.3 Properties satisfied by $J_{0,k}[\mu]$

The following lemma ensures the existence of a minimizer of $J_{0,k}[\mu]$ in $\widetilde{E}$. This result in particular ensures the existence of the function Z_k introduced at *Step 2* in Algorithm 1.

Lemma 3.6. *Let μ be given in $L^2((0,T)\times\Gamma)$ and assume that Hypotheses 3.1 hold. There exists $s_0 > 0$ which depends on T, Ω, σ_{max}, $\|u_\circ\|_{H^3(\Omega)}$, $\|h\|_{H^1(0,T;L^\infty(\Omega))}$, and $\|g\|_{H^1(0,T;H^{3/2}(\partial\Omega))\cap H^2(0,T;H^{1/2}(\partial\Omega))}$ such that for all $s \geq s_0$ and for all $k \in \mathbb{N}$, the functional $J_{0,k}[\mu]$ defined by (3.11) admits a global minimizer in $\widetilde{E}$.*

Proof. According to Proposition 2.5, there exists a constant $M > 0$ such that

$$\|u_k\|_{C^1(0,T;H^1(\Omega))} + \|u_k\|_{H^2(0,T;L^2(\Omega))} + \|u_k\|_{H^1(0,T;H^2(\Omega))} \leq M. \tag{3.21}$$

Using this estimate and (3.18), we have the continuity of $J_{0,k}[\mu]$ in $\widetilde{E}$. Moreover, since $J_{0,k}[\mu]$ is positive, it admits an infimum in $\widetilde{E}$ and we can introduce a sequence $(z_n)_{n\in\mathbb{N}}$ such that

$$J_{0,k}[\mu](z_n) \underset{n\to+\infty}{\longrightarrow} \inf_{z\in\widetilde{E}} J_{0,k}[\mu](z).$$

Let us study the convergence properties of the sequence $(z_n)_{n\in\mathbb{N}}$. First, we notice that we can write $P_k z_n$ under the form

$$P_k z_n = \partial_t z_n - \Delta z_n + r_{k,n} z_n + s_{k,n},$$

where $r_{k,n}$ and $s_{k,n}$ only depend on u_k, $\partial_t u_k$ and $T_{\overline{M}}(y_n)$. Using inequality (3.21) and the property (3.16), we deduce that $r_{k,n}$ is bounded in $L^\infty((0,T)\times\Omega)$ and $s_{k,n}$ is bounded in $L^2((0,T)\times\Omega)$ by some constant M. Hence, writing that

$$|P_k z_n|^2 \geq \frac{1}{2}|\partial_t z_n - \Delta z_n|^2 - 2|r_{k,n} z_n|^2 - 2|s_{k,n}|^2,$$

we get

$$\begin{aligned} J_{0,k}[\mu](z_n) \geq \frac{1}{4}\int_0^T\int_\Omega e^{2s\varphi}|\partial_t z_n - \Delta z_n|^2 dx\,dt - M\int_0^T\int_\Omega e^{2s\varphi}|z_n|^2 dx\,dt - M \\ + \frac{s}{4}\int_0^T\int_\Gamma e^{2s\varphi}\theta|\nabla z_n\cdot n|^2\,d\gamma dt - \frac{s}{2}\int_0^T\int_\Gamma e^{2s\varphi}\theta\mu^2\,d\gamma dt. \end{aligned}$$

According to the Carleman inequality given by (2.6) and using the fact that $se^{2s\varphi}\theta \leq C$ in $(0,T)\times\Omega$ for the third term in the right hand side, we deduce that, for s large enough,

$$\int_0^T\int_\Omega e^{2s\varphi}\left(\frac{1}{s\theta}|\partial_t z_n|^2 + \frac{1}{s\theta}|\Delta z_n|^2 + s\theta|\nabla z_n|^2 + s^3\theta^3|z_n|^2\right) dxdt \leq J_{0,k}[\mu](z_n) + M + C\|\mu\|^2_{L^2((0,T)\times\Gamma)}. \tag{3.22}$$

By construction of $(z_n)_{n\in\mathbb{N}}$, the sequence $(J_{0,k}[\mu](z_n))_{n\in\mathbb{N}}$ is bounded and thus the left hand side of this last inequality is bounded. According to the definitions of θ and φ which are given by (2.1) or by (2.4), we have in $(0,T)\times\Omega$

$$|\partial_t\theta| + |\partial_t\varphi| \leq C\theta^2 \ \text{ and } \ |\nabla\theta| + |\nabla\varphi| + |D^2\theta| + |D^2\varphi| \leq C\theta$$

and thus $(e^{s\varphi}\theta^{-1/2}z_n)_{n\in\mathbb{N}}$ is bounded in $H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^2(\Omega))$. We deduce that, $(e^{s\varphi}\theta^{-1/2}z_n)_{n\in\mathbb{N}}$ weakly converges to some element in $H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^2(\Omega))$ that we denote $e^{s\varphi}\theta^{-1/2}\tilde{z}$ (all the convergence results given in this proof are valid up to a subsequence but we do not specify it in order to lighten the writing). Moreover, since $H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^2(\Omega))$ is compactly embedded in $L^2((0,T)\times\Omega)$,

$$e^{s\varphi}\theta^{-1/2}z_n \to e^{s\varphi}\theta^{-1/2}\tilde{z} \text{ in } L^2((0,T)\times\Omega) \tag{3.23}$$

and, by identification of the limit, since θ^{-1} belongs to $L^\infty((0,T)\times\Omega)$, we also have

$$e^{s\varphi}\theta^{3/2}z_n \rightharpoonup e^{s\varphi}\theta^{3/2}\tilde{z} \text{ weakly in } L^2((0,T)\times\Omega)$$

and

$$e^{s\varphi}\theta^{1/2}\nabla z_n \rightharpoonup e^{s\varphi}\theta^{1/2}\nabla\tilde{z} \text{ weakly in } L^2(0,T;H^1(\Omega)). \tag{3.24}$$

Let us now prove that $\lim_{n\to+\infty} J_{0,k}[\mu](z_n) = J_{0,k}[\mu](\tilde{z})$ which will imply that $\tilde{z}$ minimizes $J_{0,k}[\mu]$. Since $(J_{0,k}[\mu](z_n))_{n\in\mathbb{N}}$ is bounded, $(e^{s\varphi}\theta^{1/2}(\nabla z_n\cdot n))_{n\in\mathbb{N}}$ weakly converges in $L^2((0,T)\times\Gamma)$ and $(e^{s\varphi}P_k z_n)_{n\in\mathbb{N}}$ weakly converges in $L^2((0,T)\times\Omega)$ and it is sufficient to identify their weak limits. The fact that

$$e^{s\varphi}\theta^{1/2}(\nabla z_n\cdot n) \rightharpoonup e^{s\varphi}\theta^{1/2}(\nabla\tilde{z}\cdot n) \text{ weakly in } L^2((0,T)\times\Gamma)$$

directly comes from (3.24). To identify the limit of $(e^{s\varphi}P_k z_n)_{n\in\mathbb{N}}$, we will prove that

$$e^{s\varphi}\theta^{-1/2}P_k z_n \rightharpoonup e^{s\varphi}\theta^{-1/2}P_k\tilde{z} \text{ weakly in } L^2((0,T)\times\Omega). \tag{3.25}$$

We first consider in the definition (3.12) of P_k the three first terms which correspond to the linear part. The weak convergence of $(e^{s\varphi}\theta^{-1/2}z_n)_{n\in\mathbb{N}}$ to $e^{s\varphi}\theta^{-1/2}\tilde{z}$ in $H^1(0,T;L^2(\Omega))\cap L^2(0,T;H^2(\Omega))$ implies that

$$e^{s\varphi}\theta^{-1/2}(\partial_t z_n - \Delta z_n + 3(u_k)^2 z_n) \rightharpoonup e^{s\varphi}\theta^{-1/2}(\partial_t\tilde{z} - \Delta\tilde{z} + 3(u_k)^2\tilde{z}) \quad \text{weakly in } L^2((0,T)\times\Omega). \tag{3.26}$$

Now we define, for all $t\in(0,T)$

$$y_n(t) = v_k(T_0) + \int_{T_0}^t z_n(t')dt' \quad \text{and} \quad \tilde{y}(t) = v_k(T_0) + \int_{T_0}^t \tilde{z}(t')dt'.$$

For the other terms in the operator P_k, let us first prove that $(e^{s\varphi}\theta^{-1/2}T_{\overline{M}}(y_n))_{n\in\mathbb{N}}$ strongly converges to $e^{s\varphi}\theta^{-1/2}T_{\overline{M}}(\tilde{y})$ in $L^\infty(0,T;L^2(\Omega))$. To do so, we observe that

$$\begin{aligned}\int_\Omega e^{2s\varphi}\theta^{-1/2}|y_n-\tilde{y}|^2dx &= \int_\Omega e^{2s\varphi}\theta^{-1/2}\left|\int_{T_0}^t (z_n-\tilde{z})(t',x)dt'\right|^2 dx\\ &\le C\int_\Omega e^{2s\varphi}\theta^{-1/2}\left|\int_{T_0}^t |z_n-\tilde{z}|^2(t',x)dt'\right| dx.\end{aligned}$$

By definition (2.1) or (2.4) of φ and θ and since $T_0=\frac{T}{2}$, we have, for all t' between T_0 and t, for all $x\in\Omega$

$$\varphi(t,x)\le\varphi(t',x) \text{ and } \theta(t,x)\ge\theta(t',x). \tag{3.27}$$

This implies that

$$\|e^{s\varphi}\theta^{-1/2}(y_n-\tilde{y})\|^2_{L^\infty(0,T;L^2(\Omega))} \le C\int_0^T\int_\Omega e^{2s\varphi}\theta^{-1/2}|z_n-\tilde{z}|^2 dx\, dt.$$

Thus, according to (3.23), $(e^{s\varphi}\theta^{-1/2}y_n)_{n\in\mathbb{N}}$ strongly converges to $e^{s\varphi}\theta^{-1/2}\tilde{y}$ in $L^\infty(0,T;L^2(\Omega))$ and since $T_{\overline{M}}$ satisfies (3.18), this implies that

$$e^{s\varphi}\theta^{-1/2}T_{\overline{M}}(y_n) \to e^{s\varphi}\theta^{-1/2}T_{\overline{M}}(\tilde{y}) \quad \text{in } L^\infty(0,T;L^2(\Omega)). \tag{3.28}$$

We can now study the limit of the remaining terms of $e^{s\varphi}P_k z_n$ when n tends to $+\infty$: using (3.16), (3.21) and (3.28), we have

$$e^{s\varphi}\theta^{-1/2}\partial_t u_k u_k T_{\overline{M}}(y_n) \to e^{s\varphi}\theta^{-1/2}\partial_t u_k u_k T_{\overline{M}}(\tilde{y}) \quad \text{in } L^2((0,T)\times\Omega) \tag{3.29}$$

and

$$e^{s\varphi}\theta^{-1/2}\partial_t u_k T_{\overline{M}}(y_n)^2 \to e^{s\varphi}\theta^{-1/2}\partial_t u_k T_{\overline{M}}(\tilde{y})^2 \quad \text{in } L^2((0,T)\times\Omega). \tag{3.30}$$

Let us now prove that

$$e^{s\varphi}\theta^{-1/2}u_k z_n\, T_{\overline{M}}(y_n) \to e^{s\varphi}\theta^{-1/2}u_k\tilde{z}\, T_{\overline{M}}(\tilde{y}) \quad \text{in } L^2((0,T)\times\Omega). \tag{3.31}$$

The strong convergence (3.23) of $(e^{s\varphi}\theta^{-1/2}z_n)_{n\in\mathbb{N}}$ implies the almost everywhere convergence of $(z_n)_{n\in\mathbb{N}}$ to $\tilde{z}$ and the existence of a function z_b in $L^2((0,T)\times\Omega)$ such that, for all $n\in\mathbb{N}$

$$|e^{s\varphi}\theta^{-1/2}z_n| \le z_b.$$

Moreover, the strong convergence of $(e^{s\varphi}\theta^{-1/2}y_n)_{n\in\mathbb{N}}$ in $L^2((0,T)\times\Omega)$ implies the almost everywhere convergence of $(y_n)_{n\in\mathbb{N}}$ to $\tilde{y}$. Thus, we deduce that

$$e^{s\varphi}\theta^{-1/2}u_k z_n\, T_{\overline{M}}(y_n) \to e^{s\varphi}\theta^{-1/2}u_k\tilde{z}\, T_{\overline{M}}(\tilde{y}) \text{ a.e.}$$

and

$$|e^{s\varphi}\theta^{-1/2}u_k z_n\, T_{\overline{M}}(y_n)| \le M z_b.$$

And these two properties imply (3.31) according to Lebesgue's dominated convergence theorem. At last, we use the same arguments to prove that

$$e^{s\varphi}\theta^{-1/2}z_n\, T_{\overline{M}}(y_n)^2 \to e^{s\varphi}\theta^{-1/2}u_k\tilde{z}\, T_{\overline{M}}(\tilde{y}) \quad \text{in } L^2((0,T)\times\Omega). \tag{3.32}$$

Finally, gathering (3.26) and (3.29) to (3.32), we obtain (3.25) and we conclude that $\tilde{z}$ is a minimizer of $J_{0,k}[\mu]$.

□

Due to the nonlinearities in our equation, we can not state the strong convexity of $J_{0,k}[\mu]$ in $\widetilde{E}$. Nevertheless, it is interesting to notice that the strong convexity property holds if we consider a smaller space than $\widetilde{E}$ including some boundedness hypotheses which allow to deal with the nonlinearities (similar results are obtained in [19])). We state this property in the following lemma:

Lemma 3.7. *Let $C > 0$ be fixed. We define*

$$E_C := \left\{ z \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)), \|z\|_{L^2(0,T;H^2(\Omega))} + \|z\|_{H^1(0,T;L^2(\Omega))} \leq C \right\}. \tag{3.33}$$

For s_0 large enough, $J_{0,k}[\mu]$ is strongly convex in E_C for any $s \geq s_0$.

This Lemma is proved in Appendix C. Contrary to the wave equations where the weights stay far from 0 (see [1, Section 4]), our weights, as usual for the heat equation, vanish at 0 and T and it is not clear that a minimizer of $J_{0,k}[\mu]$ in $\widetilde{E}$ will belong to E_C for some $C > 0$. Therefore, it is not possible to deduce the uniqueness in $\widetilde{E}$ from the strong convexity in E_C.
This convexity property is not used in the proof of Theorem 3.3 but it is an important property for the convergence of numerical minimization methods (we refer to Remark 4.1 for a discussion on the numerical methods and the fact that the property shown here does not correspond exactly to the numerical framework).

3.4 Proof of the convergence result stated in Theorem 3.3

For $\mu_k = \partial_t (m - \nabla u_k \cdot n)$ on $(0,T)\times\Gamma$, we define the functional

$$J_k[\mu_k](z) = \frac{1}{2}\int_0^T \int_\Omega e^{2s\varphi} |P_k z - f_k|^2 \, dxdt + \frac{s}{2}\int_0^T \int_\Gamma e^{2s\varphi}\theta |\nabla z \cdot n - \mu_k|^2 \, d\gamma dt,$$

where P_k is given by (3.12) and f_k is defined by (3.7). We notice that w_k, solution of the equation (3.6) minimizes $J_k[\mu_k]$ in $\widetilde{E}$. Indeed, according to (3.4), $T_{\overline{M}}(v_k) = v_k$ and this implies that $J_k[\mu_k](w_k) = 0$.

Let us now compute the Gâteaux derivative of P_k at point w, for any $w \in \widetilde{E}$. Let $z \in \widetilde{E}$,

$$\begin{aligned} DP_k(w)(z) &= \lim_{\epsilon\to 0} \frac{P_k(w+\epsilon z) - P_k(w)}{\epsilon} \\ &= \partial_t z - \Delta z + 3z\left((u_k)^2 + 2u_k T_{\overline{M}}(v) + T_{\overline{M}}(v)^2\right) \\ &\qquad + 6T'_{\overline{M}}(v)\overline{y}\left(\partial_t u_k u_k + \partial_t u_k T_{\overline{M}}(v) + u_k w + w T_{\overline{M}}(v)\right), \end{aligned} \tag{3.34}$$

where $v(t) = v_k(T_0) + \int_{T_0}^t w(t')dt'$, $\overline{y}(t) = \int_{T_0}^t z(t')dt'$.

Then, w_k satisfies the first order optimality condition given by

$$\int_0^T \int_\Omega e^{2s\varphi}(P_k w_k - f_k) DP_k(w_k)(z) \, dxdt + s\int_0^T \int_\Gamma e^{2s\varphi}\theta(\nabla w_k \cdot n - \mu_k)(\nabla z\cdot n) \, d\gamma dt = 0, \qquad \forall z \in \widetilde{E}. \tag{3.35}$$

Similarly, Z_k satisfies the first order optimality condition

$$\int_0^T \int_\Omega e^{2s\varphi}(P_k Z_k) DP_k(Z_k)(z) \, dxdt + s\int_0^T \int_\Gamma e^{2s\varphi}\theta(\nabla Z_k \cdot n - \mu_k)(\nabla z\cdot n) \, d\gamma dt = 0, \qquad \forall z \in \widetilde{E}. \tag{3.36}$$

Let us define $z_k = w_k - Z_k$. We compute the difference between (3.35) and (3.36) and take $z = z_k$. We get

$$\begin{aligned}&\int_0^T\int_\Omega e^{2s\varphi}\big((P_kw_k - P_kZ_k)DP_k(Z_k)(z_k) + P_kw_k(DP_k(w_k)(z_k) - DP_k(Z_k)(z_k))\big)\,dxdt\\&+ s\int_0^T\int_\Gamma e^{2s\varphi}\theta|\nabla z_k\cdot n|^2\,d\gamma dt = \int_0^T\int_\Omega e^{2s\varphi}f_kDP_k(w_k)(z_k)\,dxdt.\end{aligned} \tag{3.37}$$

This implies that

$$\begin{aligned}&\int_0^T\int_\Omega e^{2s\varphi}(P_kw_k - P_kZ_k)DP_k(Z_k)(z_k)\,dxdt + s\int_0^T\int_\Gamma e^{2s\varphi}\theta|\nabla z_k\cdot n|^2\,dxdt \le \int_0^T\int_\Omega e^{2s\varphi}|f_k|^2\,dxdt\\&+\frac{1}{4}\int_0^T\int_\Omega e^{2s\varphi}|DP_k(w_k)(z_k)|^2\,dxdt + \int_0^T\int_\Omega e^{2s\varphi}|P_kw_k||DP_k(w_k)(z_k) - DP_k(Z_k)(z_k)|\,dxdt.\end{aligned} \tag{3.38}$$

For what follows, we define, for all $t\in(0,T)$

$$Y_k(t,\cdot) = v_k(T_0,\cdot) + \int_{T_0}^t Z_k(t',\cdot)dt' \quad\text{and}\quad \overline{y}_k(t,\cdot) = v_k(t,\cdot) - Y_k(t,\cdot) = \int_{T_0}^t z_k(t',\cdot)dt'.$$

We will estimate separately the different terms of this inequality. We divide the computations in several steps.

- *Step 1.* Let us first find a lower bound for the first term in the left-hand side of (3.38).

$$\begin{aligned}P_kw_k - P_kZ_k =&\partial_t z_k - \Delta z_k + 3(u_k)^2z_k + 6\partial_t u_ku_k(T_{\overline{M}}(v_k) - T_{\overline{M}}(Y_k))\\&+3\partial_t u_k(T_{\overline{M}}(v_k)^2 - T_{\overline{M}}(Y_k)^2) + 6u_k(T_{\overline{M}}(v_k)w_k - T_{\overline{M}}(Y_k)Z_k)\\&+3(T_{\overline{M}}(v_k)^2w_k - T_{\overline{M}}(Y_k)^2Z_k)\\=&\partial_t z_k - \Delta z_k + 3(u_k)^2z_k + 6\partial_t u_ku_k(T_{\overline{M}}(v_k) - T_{\overline{M}}(Y_k))\\&+3\partial_t u_k(T_{\overline{M}}(v_k) - T_{\overline{M}}(Y_k))(T_{\overline{M}}(v_k) + T_{\overline{M}}(Y_k)) + 6u_kz_kT_{\overline{M}}(Y_k)\\&+6u_k(T_{\overline{M}}(v_k) - T_{\overline{M}}(Y_k))w_k + 3z_kT_{\overline{M}}(Y_k)^2\\&+3(T_{\overline{M}}(v_k) + T_{\overline{M}}(Y_k))(T_{\overline{M}}(v_k) - T_{\overline{M}}(Y_k))w_k\\:=&\partial_t z_k - \Delta z_k + R_{1,k}.\end{aligned}$$

  Using (3.16), (3.18) and (3.21), we can estimate $R_{1,k}$

$$|R_{1,k}| \le M|z_k| + M|\overline{y}_k|(|\partial_t u_k| + |w_k|).$$

  Moreover, from (3.34), we can write $DP_k(Z_k)(z_k) = \partial_t z_k - \Delta z_k + R_{2,k}$ where, according to (3.16) and (3.21)

$$|R_{2,k}| \le M|z_k| + M|T'_{\overline{M}}(Y_k)||\overline{y}_k|(|\partial_t u_k| + |Z_k|).$$

We deduce from these inequalities that

$$\begin{aligned}&\int_0^T\int_\Omega e^{2s\varphi}(P_kw_k-P_kZ_k)DP_k(Z_k)(z_k)dxdt\geq\frac{3}{4}\int_0^T\int_\Omega e^{2s\varphi}|\partial_tz_k-\Delta z_k|^2dxdt\\&-M\int_0^T\int_\Omega e^{2s\varphi}|z_k|^2dxdt-M\int_0^T\int_\Omega e^{2s\varphi}|\overline{y}_k|^2(|\partial_tu_k|^2+|w_k|^2+|T'_{\overline{M}}(Y_k)|^2|\partial_tu_k|^2)dxdt\\&-M\int_0^T\int_\Omega e^{2s\varphi}|\overline{y}_k|^2|T'_{\overline{M}}(Y_k)|^2|Z_k|^2dxdt\end{aligned}\tag{3.39}$$

According to (3.17) and using that $Z_k=w_k-z_k$ and that $\chi_{[-2\overline{M},2\overline{M}]}(Y_k)\leq\chi_{[-3\overline{M},3\overline{M}]}(\overline{y}_k)$ thanks to (3.4), we get for the last two terms that

$$\begin{aligned}&\int_0^T\int_\Omega e^{2s\varphi}|\overline{y}_k|^2(|\partial_tu_k|^2+|w_k|^2+|T'_{\overline{M}}(Y_k)|^2|\partial_tu_k|^2+|T'_{\overline{M}}(Y_k)|^2|Z_k|^2)dxdt\\&\leq C\int_0^T\int_\Omega e^{2s\varphi}|\overline{y}_k|^2(|\partial_tu_k|^2+|w_k|^2+\chi_{[-3\overline{M},3\overline{M}]}(\overline{y}_k)|z_k|^2)dxdt.\end{aligned}\tag{3.40}$$

Let us estimate this last integral. We first notice that

$$\int_0^T\int_\Omega e^{2s\varphi}|\overline{y}_k|^2\chi_{[-3\overline{M},3\overline{M}]}(\overline{y}_k)|z_k|^2dxdt\leq M\int_0^T\int_\Omega e^{2s\varphi}|z_k|^2dxdt.\tag{3.41}$$

Next, according to (3.21) and (3.4), we have

$$\begin{aligned}\int_0^T\int_\Omega e^{2s\varphi}|\overline{y}_k|^2(|\partial_tu_k|^2+|w_k|^2)\,dxdt&\leq\|e^{s\varphi}\overline{y}_k\|^2_{L^\infty(0,T;L^2(\Omega))}(\|\partial_tu_k\|^2_{L^2(0,T;L^\infty(\Omega))}+\|w_k\|^2_{L^2(0,T;L^\infty(\Omega))})\\&\leq M\|e^{s\varphi}\overline{y}_k\|^2_{L^\infty(0,T;L^2(\Omega))}.\end{aligned}$$

We have, for all $t\in(0,T)$

$$\begin{aligned}\int_\Omega e^{2s\varphi(t,x)}|\overline{y}_k(t,x)|^2dx&=\int_\Omega e^{2s\varphi(t,x)}\left|\int_{T_0}^t z_k(t',x)dt'\right|^2dx\\&\leq C\int_\Omega e^{2s\varphi(t,x)}\left|\int_{T_0}^t|z_k(t',x)|^2dt'\right|dx\leq C\int_0^T\int_\Omega e^{2s\varphi}|z_k|^2dx\,dt'\end{aligned}\tag{3.42}$$

using inequality (C.5) for φ. By this way, we deduce that

$$\int_0^T\int_\Omega e^{2s\varphi}|\overline{y}_k|^2(|\partial_tu_k|^2+|w_k|^2)\,dxdt\leq M\int_0^T\int_\Omega e^{2s\varphi}|z_k|^2dx\,dt.\tag{3.43}$$

Using, (3.40), (3.41) and this last inequality, (3.39) becomes

$$\begin{aligned}\int_0^T\int_\Omega e^{2s\varphi}(P_kw_k-P_kZ_k)DP_k(Z_k)(z_k)dxdt\geq\frac{3}{4}&\int_0^T\int_\Omega e^{2s\varphi}|\partial_tz_k-\Delta z_k|^2dxdt\\&-M\int_0^T\int_\Omega e^{2s\varphi}|z_k|^2dxdt.\end{aligned}\tag{3.44}$$

- *Step 2.* To bound the second term in the right-hand side of (3.38), by definition (3.34) of DP_k, we have, according to (3.17) and (3.21)

$$|DP_k(w_k)(z_k)|^2 \leq 2|\partial_t z_k - \Delta z_k|^2 + M|z_k|^2 + M|\overline{y}_k|^2(|\partial_t u_k|^2 + |w_k|^2).$$

Thus, using again (3.43), we get

$$\frac{1}{4}\int_0^T \int_\Omega e^{2s\varphi}|DP_k(w_k)(z_k)|^2 dxdt \leq \frac{1}{2}\int_0^T \int_\Omega e^{2s\varphi}|\partial_t z_k - \Delta z_k|^2 dxdt + M\int_0^T \int_\Omega e^{2s\varphi}|z_k|^2 dxdt. \tag{3.45}$$

- *Step 3.* To bound the last term of (3.38), we notice that

$$\begin{aligned}
&DP_k(w_k)(z_k) - DP_k(Z_k)(z_k) = 3z_k\left(2u_k T_{\overline{M}}(v_k) + T_{\overline{M}}(v_k)^2\right) - 3z_k\left(2u_k T_{\overline{M}}(Y_k) + T_{\overline{M}}(Y_k)^2\right)\\
&\quad + 6T'_{\overline{M}}(v_k)\overline{y}_k\left(\partial_t u_k u_k + \partial_t u_k T_{\overline{M}}(v_k) + u_k w_k + w_k T_{\overline{M}}(v_k)\right)\\
&\quad - 6T'_{\overline{M}}(Y_k)\overline{y}_k\left(\partial_t u_k u_k + \partial_t u_k T_{\overline{M}}(Y_k) + u_k Z_k + Z_k T_{\overline{M}}(Y_k)\right)\\
&= 6z_k u_k\left(T_{\overline{M}}(v_k) - T_{\overline{M}}(Y_k)\right) + 3z_k\left(T_{\overline{M}}(v_k)^2 - T_{\overline{M}}(Y_k)^2\right) + 6\partial_t u_k u_k \overline{y}_k\left(T'_{\overline{M}}(v_k) - T'_{\overline{M}}(Y_k)\right)\\
&\quad + 6\partial_t u_k \overline{y}_k\left(T'_{\overline{M}}(v_k) - T'_{\overline{M}}(Y_k)\right)T_{\overline{M}}(v_k) + 6\partial_t u_k T'_{\overline{M}}(Y_k)\overline{y}_k(T_{\overline{M}}(v_k) - T_{\overline{M}}(Y_k))\\
&\quad + 6u_k\overline{y}_k\left(T'_{\overline{M}}(v_k) - T'_{\overline{M}}(Y_k)\right)w_k + 6u_k T'_{\overline{M}}(Y_k)\overline{y}_k z_k + 6\overline{y}_k\left(T'_{\overline{M}}(v_k) - T'_{\overline{M}}(Y_k)\right)w_k T_{\overline{M}}(v_k)\\
&\quad + 6T'_{\overline{M}}(Y_k)\overline{y}_k w_k(T_{\overline{M}}(v_k) - T_{\overline{M}}(Y_k)) + 6T'_{\overline{M}}(Y_k)\overline{y}_k z_k T_{\overline{M}}(Y_k).
\end{aligned}$$

Hence, using that $|T'_{\overline{M}}(Y_k)| \leq L\chi_{[-2\overline{M},2\overline{M}]}(Y_k) \leq L\chi_{[-3\overline{M},3\overline{M}]}(\overline{y}_k)$

$$|DP_k(w_k)(z_k) - DP_k(Z_k)(z_k)| \leq M|z_k| + M|\overline{y}_k|(|\partial_t u_k| + |w_k| + |z_k|\chi_{[-3\overline{M},3\overline{M}]}(\overline{y}_k)).$$

This implies that

$$\begin{aligned}
&\int_0^T \int_\Omega e^{2s\varphi}|P_k w_k||DP_k(w_k)(z_k) - DP_k(Z_k)(z_k)|dxdt\\
&\leq \frac{1}{2}\int_0^T \int_\Omega e^{2s\varphi}|P_k w_k|^2 dxdt + \frac{1}{2}\int_0^T \int_\Omega e^{2s\varphi}|DP_k(w_k)(z_k) - DP_k(Z_k)(z_k)|^2 dxdt\\
&\leq \frac{1}{2}\int_0^T \int_\Omega e^{2s\varphi}|f_k|^2\, dxdt + M\int_0^T \int_\Omega e^{2s\varphi}|z_k|^2 dxdt
\end{aligned} \tag{3.46}$$

according to (3.41) and (3.43).

Using (3.44), (3.45) and (3.46), inequality (3.38) becomes:

$$\begin{aligned}
\frac{1}{4}\int_0^T \int_\Omega e^{2s\varphi}|\partial_t z_k - \Delta z_k|^2\, dxdt + s\int_0^T \int_\Gamma e^{2s\varphi}\theta|\nabla z_k \cdot n|^2\, d\gamma dt&\\
\leq \frac{3}{2}\int_0^T \int_\Omega e^{2s\varphi}|f_k|^2\, dxdt + M\int_0^T \int_\Omega e^{2s\varphi}|z_k|^2\, dxdt.&
\end{aligned} \tag{3.47}$$

Using Theorem 2.1, we can eliminate the last term in the right hand-side of (3.47) for s larger than some constant s_0. Thus, using inequality (2.7), we get the following bound on $z_k(T_0)$:

$$s\int_\Omega e^{2s\varphi(T_0)}|z_k(T_0)|^2\, dx \leq M\int_0^T \int_\Omega e^{2s\varphi}|f_k|^2\, dxdt. \tag{3.48}$$

In the left hand-side of this inequality, we have $z_k(T_0,x) = w_k(T_0,x) - Z_k(T_0,x)$ for $x \in \Omega$ and, using (3.8) and (3.14), we get that

$$z_k(T_0,x) = -h(T_0,x)(\widetilde{\sigma}_{k+1}(x) - \sigma(x)), \quad x \in \Omega.$$

In the right hand-side of (3.48), since $f_k = (\sigma_k - \sigma)\partial_t h$ and h is assumed to be bounded in $H^1(0,T;L^\infty(\Omega))$, we have

$$\int_0^T \int_\Omega e^{2s\varphi}|f_k|^2\,dxdt \leq M \int_\Omega e^{2s\varphi(T_0)}|\sigma_k - \sigma|^2\,dx.$$

Using (3.2), we get that

$$s\int_\Omega e^{2s\varphi(T_0)}|\widetilde{\sigma}_{k+1} - \sigma|^2\,dx \leq M \int_\Omega e^{2s\varphi(T_0)}|\sigma_k - \sigma|^2\,dx.$$

Now, to estimate $\sigma_{k+1} = \Pi_{\sigma_{max}}(\widetilde{\sigma}_{k+1})$, we notice that, since σ satisfies (1.3), we have

$$|\sigma_{k+1} - \sigma| \leq |\widetilde{\sigma}_{k+1} - \sigma| \text{ in } \Omega. \tag{3.49}$$

Thus, we get (3.15) and, applying iteratively this estimate, we obtain that

$$\int_\Omega e^{2s\varphi(T_0)}|\sigma_{k+1} - \sigma|^2\,dx \leq \left(\frac{M}{s}\right)^{k+1} \int_\Omega e^{2s\varphi(T_0)}|\sigma_0 - \sigma|^2\,dx.$$

Thus, for s large enough we deduce from this inequality the convergence of the sequence $(\sigma_k)_{k\in\mathbb{N}}$ to σ.

This concludes the proof of Theorem 3.3.

4 Numerical issues

4.1 Numerical methods

In this subsection, we present the discretization procedure and the numerical methods used in our numerical simulations. To simplify the presentation, we explain the discretization scheme in the one-dimensional case and assume that $\Omega = (0,L)$ for $L > 0$ and $\Gamma = \{x = L\}$.

Generation of the data

In this article, we work with synthetic data. To discretize the reaction-diffusion equation (1.1) for the exact source σ, we use a finite differences scheme based on the three-point backward Euler scheme and a linearization of the cubic term. We denote by $N_x \in \mathbb{N}$ the number of discretization points in the interior of $[0,L]$ and by $N_t \in \mathbb{N}$ the number of discretization points in the interior of $[0,T]$. The space and time steps are denoted by $\Delta x = \dfrac{L}{N_x+1}$ and $\Delta t = \dfrac{T}{N_t+1}$ respectively and we define, for $0 \leq j \leq N_x + 1$ and $0 \leq n \leq N_t + 1$, u_j^n a numerical approximation of the solution $u(t^n, x_j)$ with $t^n = n\Delta t$ and $x_j = j\Delta x$. The approximated solution is computed in the following way:

Initialize: $u_j^0 = u_\circ(x_j), \qquad 0 \leq j \leq N_x + 1.$

For $0 \leq n \leq N_t$, knowing u^n, compute u^{n+1} as the solution of the linear system:

$$\begin{cases} \dfrac{u_j^{n+1} - u_j^n}{\Delta t} - \dfrac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} + (u_j^n)^3 + 3(u_j^n)^2(u_j^{n+1} - u_j^n) = \sigma(x_j)h(t^n, x_j), \\ u_0^{n+1} = g(t^{n+1}, 0) \quad \text{and} \quad u_{N_x+1}^{n+1} = g(t^{n+1}, L), \qquad 1 \leq j \leq N_x, \end{cases} \tag{4.1}$$

where the time implicit cubic term $(u_j^{n+1})^3$ has been approximated by its first order Taylor expansion $(u_j^n)^3 + 3(u_j^n)^2(u_j^{n+1} - u_j^n)$. Then, we compute the counterpart of the continuous measurements r and m given in (1.2) as follows:

$$m^n = \frac{u^n_{N_x+1} - u^n_{N_x}}{\Delta x}, \quad 0 \le n \le N_t + 1 \quad \text{and} \quad r_j = u_j^{n_0}, \quad 0 \le j \le N_x + 1,$$

with n_0 is the integer part of $N_t/2 + 1$.

On the computed data, we may add a Gaussian noise:

$$\begin{aligned} m^n &\longleftarrow m^n + \alpha(\max_n m^n)\mathcal{N}(0,1), && 0 \le n \le N_t + 1, \\ r_j &\longleftarrow r_j + \alpha(\max_j r_j)\mathcal{N}(0,1), && 0 \le j \le N_x + 1, \end{aligned} \tag{4.2}$$

where $\mathcal{N}(0,1)$ satisfies a centered normal law with deviation 1 and α is the level of noise (*i.e.* $\alpha = 0.01$ corresponds to a noise of 1%).

Discrete algorithm

We present in this subsection the discrete version of Algorithm 1.

Algorithm 2. ***Initialisation :*** Start with $\bar\sigma = 0$.
Iteration : Until the convergence criteria is reached, do
• *Step 1* - Knowing $\bar\sigma \in \mathbb{R}^{N_x}$, solve

$$\begin{cases} \dfrac{\bar u_j^{n+1} - \bar u_j^n}{\Delta t} - \dfrac{\bar u_{j+1}^{n+1} - 2\bar u_j^{n+1} + \bar u_{j-1}^{n+1}}{\Delta x^2} + (\bar u_j^n)^3 = \bar\sigma_j h(t^n, x_j), \\ \bar u_0^{n+1} = g(t^{n+1}, 0) \quad \text{and} \quad \bar u_{N_x+1}^{n+1} = g(t^{n+1}, L), \qquad 0 \le n \le N_t, \\ \bar u_j^0 = u_\circ(x_j), \qquad 1 \le j \le N_x, \end{cases} \tag{4.3}$$

and set $v_j = r_j - \bar u_j^{n_0}$.
• *Step 2* - Define for $1 \le n \le N_t$,

$$\mu^n = \frac{\left(m - \dfrac{\bar u_{N_x+1} - \bar u_{N_x}}{\Delta x}\right)^{n+1} - \left(m - \dfrac{\bar u_{N_x+1} - \bar u_{N_x}}{\Delta x}\right)^{n-1}}{2\Delta t} \tag{4.4}$$

and discretize the functional (3.11) as follows:

$$J_{0,k}[\mu](z) = \frac{1}{2}\Delta t \Delta x \sum_{n=1}^{N_t}\sum_{j=1}^{N_x} e^{2s\varphi(t^n, x_j)} |(P_k z)_j^n|^2 + \frac{s}{2}\Delta t \sum_{n=1}^{N_t} e^{2s\varphi(t^n, L)}\theta(t^n)\left|\frac{-z^n_{N_x}}{\Delta x} - \mu^n\right|^2, \tag{4.5}$$

where

$$\begin{aligned} (P_k z)_j^n = {}& \frac{z_j^{n+1} - z_j^{n-1}}{2\Delta t} - \frac{z_{j+1}^n - 2z_j^n + z_{j-1}^n}{\Delta x^2} + 3(\bar u_j^n)^2 z_j^n + 6\frac{\bar u_j^{n+1} - \bar u_j^{n-1}}{2\Delta t}\bar u_j^n T_{\overline{M}}(y_j^n) \\ & + 3\frac{\bar u_j^{n+1} - \bar u_j^{n-1}}{2\Delta t} T_{\overline{M}}(y_j^n)^2 + 6\bar u_j^n z_j^n T_{\overline{M}}(y_j^n) + 3 z_j^n T_{\overline{M}}(y_j^n)^2, \end{aligned} \tag{4.6}$$

with $z_0^n = z_{N_x+1}^n = 0$ for all $0 \leq n \leq N_t + 1$ and

$$\begin{cases} y_j^{n_0} = v_j, & 1 \leq j \leq N_x, \\ y_j^n = y_j^{n-1} + \Delta t z_j^n, & \text{if } n > n_0, \\ y_j^n = y_j^{n+1} - \Delta t z_j^n, & \text{if } n < n_0. \end{cases}$$

Minimize $J_{0,k}[\mu]$ and denote by $Z = (Z_j^n)_{1\leq j\leq N_x, 0\leq n\leq N_t+1}$ the minimizer.

• *Step 3* - Update

$$\bar{\sigma}_j \longleftarrow \bar{\sigma}_j + \frac{Z_j^{n_0} - \dfrac{v_{j+1} - 2v_j + v_{j-1}}{\Delta x^2} + v_j q[v_j, \bar{u}_j^{n_0}]}{h(t^{n_0}, x_j)}, \quad 1 \leq j \leq N_x. \tag{4.7}$$

• *Step 4* - At last, define

$$\bar{\sigma}_j \longleftarrow \operatorname{sign}(\bar{\sigma}_j) \min(\sigma_{max}, |\bar{\sigma}_j|).$$

The iterative loop is stopped when two consecutive $\bar{\sigma}$ are closer than a fixed relative tolerance ε or when the maximal number of iterations is reached. In the absence of knowledge of the exact solution σ, the quality of the converged solution is measured thanks to the following criteria

$$\text{err}_r = \frac{\|r - \bar{u}^{n_0}\|_2}{\|r\|_2} \qquad \text{and} \qquad \text{err}_m = \frac{\left\| m - \dfrac{\bar{u}_{N_x+1} - \bar{u}_{N_x}}{\Delta x} \right\|_2}{\|m\|_2}, \tag{4.8}$$

that should be of the order of the noise level on the observations. If the exact solution σ is known, we can also compute the relative error

$$\text{err}_\sigma = \frac{\|\sigma - \bar{\sigma}\|_2}{\|\sigma\|_2}.$$

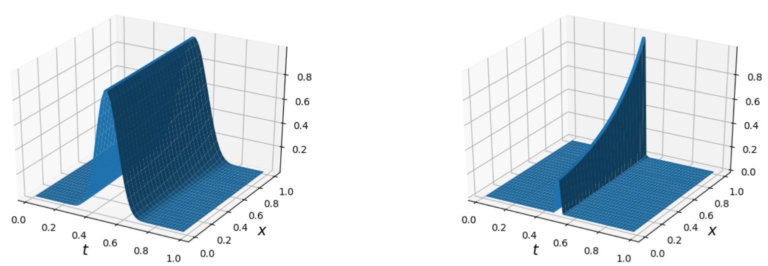
Remark 4.1. *In Step 2 of Algorithm 2, the minimization of $J_{0,k}[\mu]$ is achieved thanks to the Newton method or Newton-Krylov method [4, 21]. This last method belongs to the family of inexact Newton methods and consists of solving at each step the linear system in a Krylov subspace. The Newton minimization method is globally convergent if the functional is strongly convex. To be in this framework, it would be necessary to prove that the discrete functional $J_{0,k}$ (4.5) is strongly convex under boundedness assumptions (like in Lemma 3.7 for the functional in the continuous setting) and to prove that the discrete minimization sequence satisfies these bounds. A complete study of the discretized algorithm could be tackled in the future and would involve in particular a Carleman inequality for the discretized heat equation. In our numerical simulations, we have taken the initial guess of the iterative Newton-Krylov method equal to* 0 *and checked that the convergence does not depend on this initialization. Thus, we observe numerically global convergence properties for the minimization of $J_{0,k}[\mu]$.*

Remark 4.2. *In order to avoid the inverse crime, we introduce a bias by taking different schemes for the direct and the inverse problems. Hence, we solve* (4.1) *associated to σ thanks to a linearized implicit scheme and we use an explicit scheme for the nonlinear term in equation* (4.3) *with $\bar{\sigma} = \sigma_k$.*

Numerical challenges

One of the main drawbacks of the numerical method presented in Algorithm 1 is that we have to differentiate in time the observation m in (4.4) and to take the Laplacian of the observation r in (4.7). Thus, even a small perturbation (noise) on the observations may induce a large perturbation on its derivatives. In order to partially remedy this problem in the presence of noise, we first regularize the data (m, r) thanks to a 3-order low-pass Butterworth filter [6] associated to a cutoff frequency ω. We also replace the classical finite difference formulae in (4.4) and (4.7) that generate instabilities by a Savitzki-Golay formula [25] associated with a cubic polynomial and a window size of 5 points.

As already mentioned previously, another difficulty is the presence of the exponential weights in the functional that leads to severe numerical difficulties when performing the minimization for s large. Indeed, to ensure the strong convexity of the functional $J_{0,k}$ (see Lemma 3.7) and the convergence of Algorithm 1 (see Theorem 3.3), s has to be large. In [2], this difficulty was solved by choosing a functional that only depended on the conjugate variable $e^{s\varphi}z$ and the corresponding conjugate operator. But this was possible because the considered operator was linear. Here, we managed to deal with this difficulty by introducing the new weight functions (2.4). In Figure 1, we plot $e^{s\varphi}$ in $(0, T) \times (0, L)$ for $s = 1$ and $s = 100$. Notice that even for s large, the function does not vanish at the observation time $T_0 = 0.5$ what allows a good reconstruction of the source term in the whole domain Ω. Numerically, we observe that for $s = 1$, the minimisation step is slow (5202 seconds for $s = 1$ versus 17 seconds for $s = 100$ for the test case of Figure 3 (a)) and in some cases the convergence of the algorithm is not achieved (for example in the test case of Figure 3 (b)).



(a) $s = 1$ (b) $s = 100$

Figure 1: Carleman weight function $e^{s\varphi}$ defined in (2.4) for different values of s.

4.2 Numerical results

This subsection is devoted to the presentation of some numerical examples to illustrate the properties of the numerical reconstruction method and its efficiency. All simulations are executed with Python. The source codes are available on request. Table 1 gathers the numerical values used for all the following examples, unless specified otherwise where appropriate. Moreover, we construct the function

Φ introduced in (3.10) in the form:

$$\Phi(X) = \begin{cases} 1, & \text{if } |X| \leq 1, \\ \dfrac{\int_1^{|X|} \exp\left(\frac{-1}{(x-1)(2-x)}\right) dx}{\int_1^2 \exp\left(\frac{-1}{(x-1)(2-x)}\right) dx}, & \text{if } 1 < |X| < 2, \\ 0, & \text{if } |X| \geq 2. \end{cases}$$

Figure 2 presents some examples of data generated by the direct problem. In all the figures presenting the numerical results, the exact source that we want to recover is plotted by a red line, whereas the numerical source recovered by our method is represented by a dotted black line. The convergence informations (number of iterations, running time, convergence errors) are reported in Table 2.

L	T	N_x	N_t	g	$u_\circ$	σ_{max}
1	1	25	50	0	0	2
α	x_0	s	$\overline{M}$	ρ	ε	ω
0	-0.3	100	10	10^{-3}	10^{-3}	0.15

Table 1: Numerical values for the variables.

Example	Number of iterations	Running time in seconds	err_m	err_r	err_σ
Figure 3 (a)	3	117	0.1%	0.2%	0.02%
Figure 3 (b)	16	554	0.7%	0.1%	0.8%
Figure 5 (a)	3	87	1%	0.3%	2%
Figure 5 (b)	3	91	1%	0.3%	4%
Figure 5 (c)	3	97	3%	0.5%	9%
Figure 6 (b)	4	497	0.05%	0.1%	0.05%
Figure 6 (d)	6	802	0.1%	0.1%	0.01%

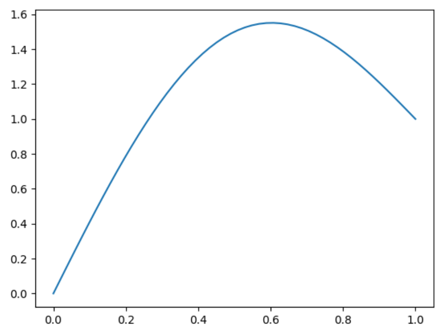
Table 2: Convergence results of the test cases.

Simulations from data without noise

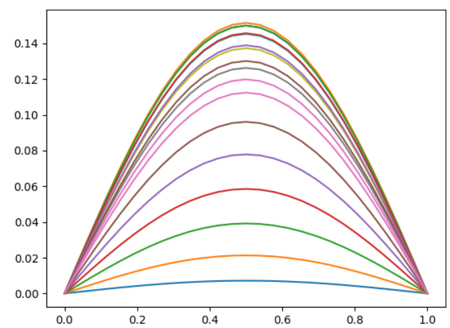
In Figure 3, we present the successive results obtained at each iteration of Algorithm 1 in the case of the reconstruction of the source $\sigma(x) = \sin(\pi x)$ for two different choices of h. One can observe that in both cases the convergence criteria (4.8) for ε is achieved in less than 20 iterations. In Figure 4, several results of reconstruction of sources obtained using Algorithm 1 in the absence of noise are given.

Simulations with several levels of noise

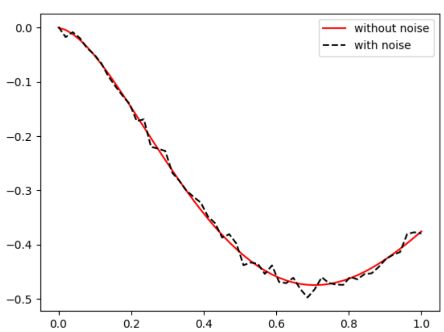
Figure 5 shows the results for $\sigma(x) = \sin(\pi x)$ with different levels of noise in the measurements ($\alpha = 1\%$, 2% and 5%). In Table 2, we report the corresponding errors on the reconstructed source. In



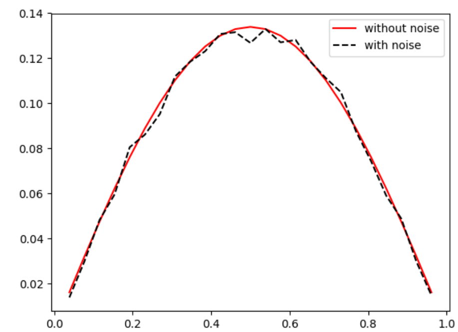
(a) $h(t) = t + \sin(\pi t)$



(b) $u(t, \cdot)$ for different times



(c) $m(t)$ the measurement of the flux at $x = L$



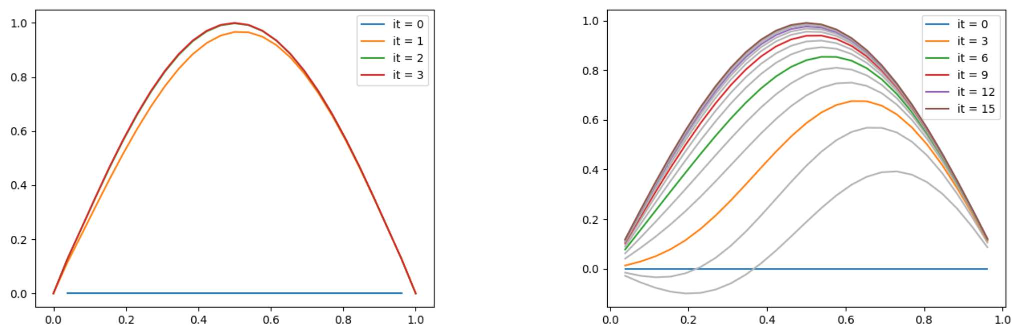
(d) $r(x)$ the measurement of u at $t = T_0$

Figure 2: Examples of data used in the numerical examples for $\sigma(x) = \sin(\pi x)$ and $\alpha = 2\%$.

fact, we observe that a noise of level α in the measurements gives rise to an error of order 2α in the recovered source.

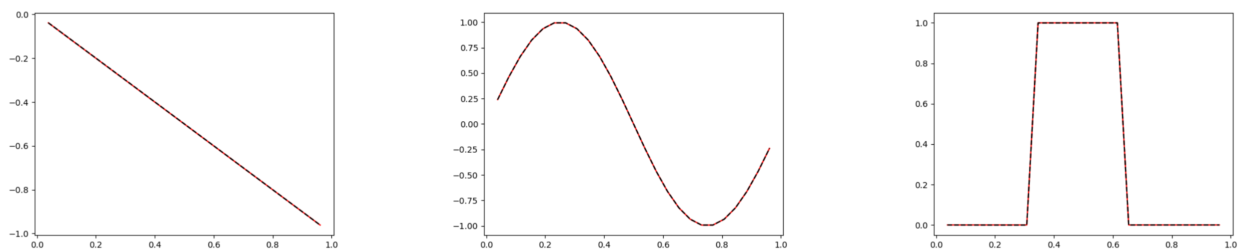
Simulations in two dimensions

We also performed some reconstructions in two dimensions where $\Omega = (0,1)^2$, $x_0 = (-0.3, -0.3)$ and $\Gamma = \big(\{0\} \times [0,1]\big) \cup \big([0,1] \times \{0\}\big)$. By this way, assumption (2.5) is satisfied. Figure 6 presents the results obtained for two different sources in the absence of additional noise. The gray scales are identical for the exact and the recovered graphics. The final error (reported in Table 2) is less than 0.1% what shows the effectiveness of the reconstruction obtained in a few minutes on a personal laptop.



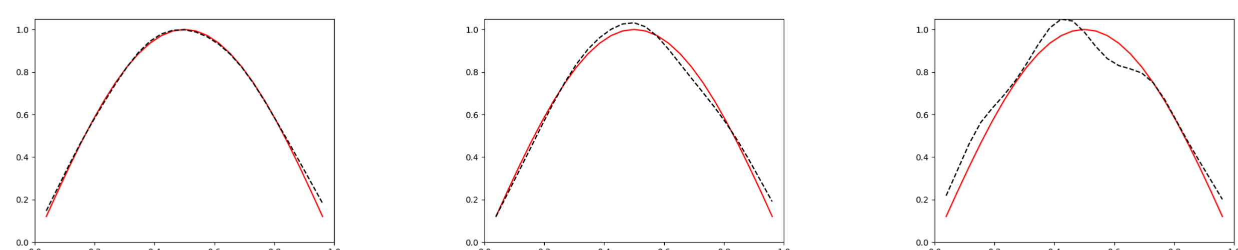
(a) $h(t) = t + \sin(\pi t)$ (b) $h(t) = t + \sin(7\pi t)$

Figure 3: Reconstruction of $\sigma(x) = \sin(\pi x)$. Different choices for h and the corresponding convergence history.



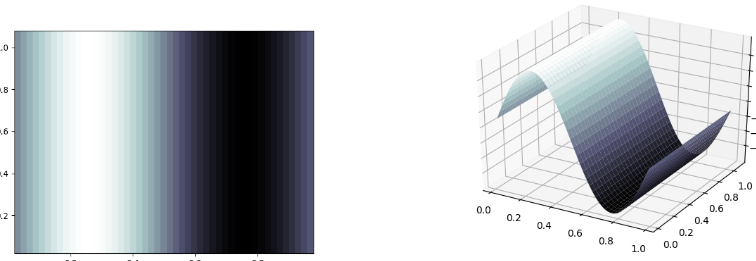
(a) $\sigma(x) = -x$ (b) $\sigma(x) = \sin(2\pi x)$ (c) σ rectangular

Figure 4: Different examples of reconstruction for $h(t) = t + \sin(\pi t)$.

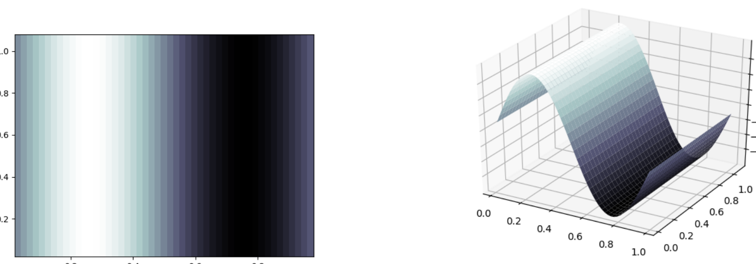


(a) $\alpha = 1\%$ (b) $\alpha = 2\%$ (c) $\alpha = 5\%$

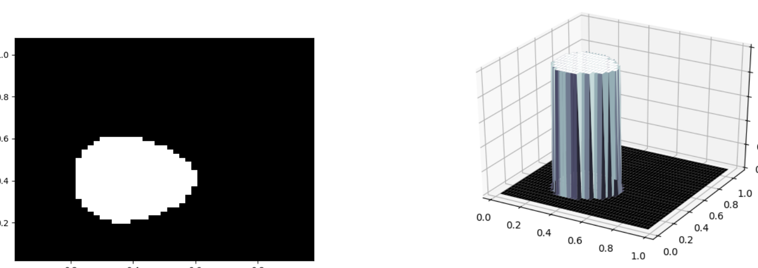
Figure 5: Reconstruction of the source $\sigma(x) = \sin(\pi x)$ for $h(t) = t + \sin(\pi t)$ in presence of noise in the data. The level of noise is denoted by α.



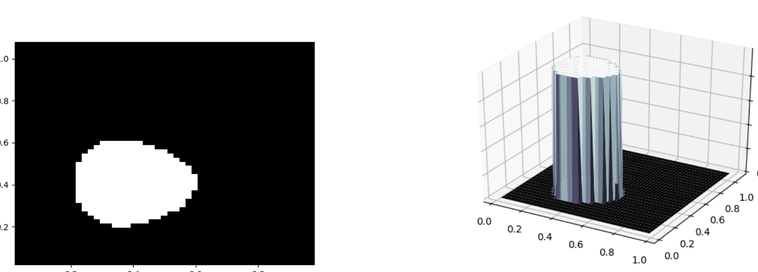
(a) Exact sources.



(b) Sources recovered numerically.



(c) Exact sources.



(d) Sources recovered numerically.

Figure 6: Different examples of reconstruction in the 2d case.

Appendix

A Proof of the Carleman inequality given by Theorem 2.1

If we are in **Case 1** with the first choice of weight (2.1), this result is proved in an identical way as Lemma 1.2 in [13] which considers the case of internal measurements. Assume now that we are in **Case 2** where θ and ψ are given by (2.4).

Let us give some properties on φ which will be useful in what follows:

$$\varphi(t,x) \leq \varphi(T_0,x),\ \forall\,(t,x) \in (0,T)\times\Omega, \quad \nabla^2\varphi = 2\theta I_d, \tag{A.1}$$

$$|\nabla\varphi| \leq C\theta, \quad |\partial_t\varphi| \leq C\theta^2, \quad |\partial_t\nabla\varphi| \leq C\theta^2, \quad |\partial_{tt}\varphi| \leq C\theta^3. \tag{A.2}$$

In the proof, we assume that z belongs to $C^2([0,T]\times\overline{\Omega})$ and satisfies $z=0$ on $(0,T)\times\partial\Omega$. A density argument allows to come back to the regularity hypotheses of the theorem.

For all $s>0$, we set $w=e^{s\varphi}z$ and we introduce the conjugate operator Q defined by

$$Qw = e^{s\varphi}(\partial_t - \Delta)(e^{-s\varphi}w). \tag{A.3}$$

If we set $f=\partial_t z - \Delta z$, we have

$$Qw = e^{s\varphi}f.$$

Some computations give

$$Qw \;=\; \partial_t w + 2s\nabla\varphi\cdot\nabla w + s\Delta\varphi w - \Delta w - (s^2|\nabla\varphi|^2 + s\partial_t\varphi)w = Q_+w + Q_-w,$$

where the operators Q_+ and Q_- are defined by

$$Q_+w \;=\; -\Delta w - (s^2|\nabla\varphi|^2 + s\partial_t\varphi)w, \tag{A.4}$$

$$Q_-w \;=\; \partial_t w + 2s\nabla\varphi\cdot\nabla w + s\Delta\varphi w. \tag{A.5}$$

In a classical way, we write that

$$\int_0^T\int_\Omega e^{2s\varphi}|f|^2\,dxdt = \int_0^T\int_\Omega |Q_+w|^2\,dxdt + \int_0^T\int_\Omega |Q_-w|^2\,dxdt + 2\int_0^T\int_\Omega Q_+w\,Q_-w\,dxdt. \tag{A.6}$$

The main part of the proof consists of bounding from below the terms in the right hand side by positive and dominant terms and a negative observation term located in $(0,T)\times\Gamma$. For the sake of clarity, we divide the proof in several steps.

- *Step 1* - Explicit calculation of the cross-term.

  We set

$$\int_0^T\int_\Omega Q_+w\,Q_-w\,dxdt = \sum_{1\leq i\leq 2,\,1\leq k\leq 3} I_{i,k},$$

where $I_{i,k}$ is the integral of the product of the ith-term in Q_+w and the kth-term in Q_-w. Integrations by parts in time give easily

$$\begin{aligned} I_{11} &= \int_0^T \int_\Omega (-\Delta w)\partial_t w \, dxdt = \int_0^T \int_\Omega \nabla w \cdot \nabla \partial_t w \, dxdt - \int_0^T \int_{\partial\Omega} \nabla w \cdot n \; \partial_t w \, d\gamma dt \\ &= \frac{1}{2}\left[\int_\Omega |\nabla w|^2 \, dx\right]_0^T - \int_0^T \int_{\partial\Omega} \nabla w \cdot n \; \partial_t w \, d\gamma dt = 0 \end{aligned}$$

since $w(0) = w(T) = 0$ in Ω and $w = 0$ on $(0,T) \times \partial\Omega$. An integration by parts in time gives for I_{21}

$$I_{21} = -\int_0^T \int_\Omega (s^2|\nabla\varphi|^2 + s\partial_t\varphi) w \partial_t w \, dxdt = \frac{1}{2}\int_0^T \int_\Omega \partial_t(s^2|\nabla\varphi|^2 + s\partial_t\varphi)|w|^2 \, dxdt.$$

We compute in the same way, by integrating by parts in space

$$\begin{aligned} I_{12} &= -\int_0^T \int_\Omega \Delta w(2s\nabla\varphi \cdot \nabla w) \, dxdt \\ &= 2s\int_0^T \int_\Omega \nabla w \cdot \nabla(\nabla\varphi \cdot \nabla w) \, dxdt - 2s\int_0^T \int_{\partial\Omega} \nabla w \cdot n(\nabla\varphi \cdot \nabla w) \, d\gamma dt \\ &= 2s\int_0^T \int_\Omega (\nabla^2\varphi)\nabla w \cdot \nabla w \, dxdt + 2s\int_0^T \int_\Omega (\nabla^2 w)\nabla w \cdot \nabla\varphi \, dxdt - 2s\int_0^T \int_{\partial\Omega} \nabla w \cdot n(\nabla\varphi \cdot \nabla w) \, d\gamma dt \\ &= 2s\int_0^T \int_\Omega (\nabla^2\varphi)\nabla w \cdot \nabla w \, dxdt - s\int_0^T \int_\Omega |\nabla w|^2 \Delta\varphi \, dxdt + s\int_0^T \int_{\partial\Omega} |\nabla w|^2 \nabla\varphi \cdot n \, d\gamma dt \\ &\quad -2s\int_0^T \int_{\partial\Omega} \nabla w \cdot n(\nabla\varphi \cdot \nabla w) \, d\gamma dt \end{aligned}$$

and

$$I_{22} = -\int_0^T \int_\Omega (s^2|\nabla\varphi|^2 + s\partial_t\varphi) w(2s\nabla\varphi \cdot \nabla w) \, dxdt = \int_0^T \int_\Omega \nabla \cdot \left[(s^3|\nabla\varphi|^2 + s^2\partial_t\varphi)\nabla\varphi\right] |w|^2 \, dxdt.$$

Since $\Delta\varphi$ is independent of x, by integration by parts in space, we have

$$I_{13} = -\int_0^T \int_\Omega \Delta w(s\Delta\varphi w) \, dxdt = s\int_0^T \int_\Omega |\nabla w|^2 \Delta\varphi \, dxdt.$$

At last, we have

$$I_{23} = -\int_0^T \int_\Omega (s^2|\nabla\varphi|^2 + s\partial_t\varphi) w(s\Delta\varphi w) \, dxdt = -\int_0^T \int_\Omega (s^3|\nabla\varphi|^2 + s^2\partial_t\varphi)\Delta\varphi |w|^2 \, dxdt.$$

Gathering all these computations and using the second property in (A.1), we get

$$\int_0^T\int_\Omega Q_+w\,Q_-w\,dxdt = 4s\int_0^T\int_\Omega \theta|\nabla w|^2\,dxdt$$
$$+\int_0^T\int_\Omega\left(\frac{1}{2}\partial_t(s^2|\nabla\varphi|^2+s\partial_t\varphi)+\nabla\cdot\left[(s^3|\nabla\varphi|^2+s^2\partial_t\varphi)\nabla\varphi\right]-(s^3|\nabla\varphi|^2+s^2\partial_t\varphi)\Delta\varphi\right)|w|^2\,dxdt$$
$$+\,s\int_0^T\int_{\partial\Omega}|\nabla w|^2\nabla\varphi\cdot n\,d\gamma dt-2s\int_0^T\int_{\partial\Omega}\nabla w\cdot n(\nabla\varphi\cdot\nabla w)\,d\gamma dt. \tag{A.7}$$

For the second term in the right hand side, we notice that the main part in s^3 is given by

$$s^3(\nabla\cdot(|\nabla\varphi|^2\nabla\varphi)-|\nabla\varphi|^2\Delta\varphi)=s^3\nabla(|\nabla\varphi|^2)\cdot\nabla\varphi=8s^3\theta^3|x-x_0|^2\geq Cs^3\theta^3.$$

For the boundary terms in (A.7), we notice that, since $z=0$ on $(0,T)\times\partial\Omega$, $\nabla w=e^{s\varphi}\nabla z$. In particular, $\nabla w\cdot\tau=0$ on $(0,T)\times\partial\Omega$. Thus, we get

$$s\int_0^T\int_{\partial\Omega}|\nabla w|^2\nabla\varphi\cdot n\,d\gamma dt-2s\int_0^T\int_{\partial\Omega}\nabla w\cdot n(\nabla\varphi\cdot\nabla w)\,d\gamma dt=-s\int_0^T\int_{\partial\Omega}|\nabla w\cdot n|^2\nabla\varphi\cdot n\,d\gamma dt.$$

We divide this last integral as follows

$$-s\int_0^T\int_{\partial\Omega}|\nabla w\cdot n|^2\nabla\varphi\cdot n\,d\gamma dt=-s\int_0^T\int_\Gamma|\nabla w\cdot n|^2\nabla\varphi\cdot n\,d\gamma dt-s\int_0^T\int_{\partial\Omega\setminus\Gamma}|\nabla w\cdot n|^2\nabla\varphi\cdot n\,d\gamma dt.$$

According to (2.5), the second integral is positive and the first integral corresponds to an observation integral.

Gathering these estimates, (A.7) becomes, for s large enough

$$\begin{aligned}\int_0^T\int_\Omega Q_+w\,Q_-w\,dxdt &\geq 4s\int_0^T\int_\Omega\theta|\nabla w|^2\,dxdt+Cs^3\int_0^T\int_\Omega\theta^3|w|^2\,dxdt\\ &\quad -s\int_0^T\int_\Gamma|\nabla w\cdot n|^2\nabla\varphi\cdot n\,d\gamma dt.\end{aligned}\tag{A.8}$$

- *Step 2* - Bounds on Δw and $\partial_t w$.

  From the definition of Q_- (A.5), we have

$$\begin{aligned}\frac{1}{2}\int_0^T\int_\Omega\frac{1}{s\theta}|\partial_t w|^2\,dxdt &\leq \int_0^T\int_\Omega\frac{1}{s\theta}|Q_-w|^2\,dxdt+\int_0^T\int_\Omega\frac{1}{s\theta}|2s\nabla\varphi\cdot\nabla w+s\Delta\varphi w|^2\,dxdt\\ &\leq\int_0^T\int_\Omega|Q_-w|^2\,dxdt+C\left(\int_0^T\int_\Omega s\theta|\nabla w|^2\,dxdt+\int_0^T\int_\Omega s\theta|w|^2\,dxdt\right).\end{aligned}$$

  In the same way,

$$\begin{aligned}\frac{1}{2}\int_0^T\int_\Omega\frac{1}{s\theta}|\Delta w|^2\,dxdt &\leq\int_0^T\int_\Omega\frac{1}{s\theta}|Q_+w|^2\,dxdt+\int_0^T\int_\Omega\frac{1}{s\theta}(s^2|\nabla\varphi|^2+s\partial_t\varphi)^2|w|^2\,dxdt\\ &\leq\int_0^T\int_\Omega|Q_+w|^2\,dxdt+C\int_0^T\int_\Omega s^3\theta^3|w|^2\,dxdt.\end{aligned}$$

Thus, coming back to (A.6) and, gathering (A.8) and these last two estimates, we get, for s large enough

$$\int_0^T \int_\Omega \left(\frac{1}{s\theta} |\partial_t w|^2 + \frac{1}{s\theta} |\Delta w|^2 + s\theta |\nabla w|^2 + s^3\theta^3 |w|^2 \right) dxdt \\ \leq C \int_0^T \int_\Omega e^{2s\varphi} |f|^2 \, dxdt + Cs \int_0^T \int_\Gamma |\nabla w \cdot n|^2 \nabla\varphi \cdot n \, d\gamma dt. \tag{A.9}$$

- *Step 3* - Back to the variable z.

  Since $z = e^{-s\varphi} w$ and according to (A.2), we have, in $\Omega \times (0,T)$

$$|\partial_t z|^2 \leq Ce^{-2s\varphi}(|\partial_t w|^2 + s^2\theta^4 |w|^2), \quad |\nabla z|^2 \leq Ce^{-2s\varphi}(|\nabla w|^2 + s^2\theta^2 |w|^2),$$
$$|\Delta z|^2 \leq Ce^{-2s\varphi}(|\Delta w|^2 + s^2\theta^2 |\nabla w|^2 + s^4\theta^4 |w|^2).$$

  Thus, (A.9) gives inequality (2.6) for s large enough.

B Proof of the regularity result given by Proposition 2.5

We split the proof in several steps.

- *Step 1* - A lifting of the boundary condition of (1.1).

  First, we will use a lifting for the boundary condition. Since $g \in H^1(0,T;H^{3/2}(\partial\Omega)) \cap H^2(0,T;H^{1/2}(\partial\Omega))$, from trace theorem, we deduce that there exists a function $\tilde{u} \in H^1(0,T;H^2(\Omega)) \cap H^2(0,T;H^1(\Omega))$ such that $\tilde{u} = g$ on $(0,T) \times \partial\Omega$ and

$$\|\tilde{u}\|_{H^1(0,T;H^2(\Omega))} \leq C\|g\|_{H^1(0,T;H^{3/2}(\partial\Omega))}, \quad \|\tilde{u}\|_{H^2(0,T;H^1(\Omega))} \leq C\|g\|_{H^2(0,T;H^{1/2}(\partial\Omega))}. \tag{B.1}$$

  The function $\overline{u} = u - \tilde{u}$ satisfies

$$\begin{cases} \partial_t \overline{u} - \Delta \overline{u} + \overline{u}^3 + 3\tilde{u}\overline{u}^2 + 3\tilde{u}^2\overline{u} = F, & \text{in } (0,T) \times \Omega, \\ \overline{u} = 0, & \text{on } (0,T) \times \partial\Omega, \\ \overline{u}(0,\cdot) = u_\circ - \tilde{u}(0,\cdot), & \text{in } \Omega, \end{cases} \tag{B.2}$$

  with F defined by $F = \sigma h - \partial_t \tilde{u} + \Delta \tilde{u} - \tilde{u}^3$.

  Multiplying the main equation of (B.2) by $\phi \in H_0^1(\Omega)$ and integrating by parts, we obtain

$$\int_\Omega \partial_t \overline{u}(t,x)\phi(x) \; dx + \int_\Omega \nabla \overline{u}(t,x) \cdot \nabla \phi(x) \; dx + \int_\Omega \overline{u}^3(t,x)\phi(x) \; dx + 3\int_\Omega \tilde{u}\overline{u}^2(t,x)\phi(x) \; dx \\ + 3\int_\Omega \tilde{u}^2\overline{u}(t,x)\phi(x) \; dx = \int_\Omega F(t,x)\phi(x) \; dx, \tag{B.3}$$

  a.e. $t \in (0,T)$.

- *Step 2* - Finite-dimensional approximated solutions.

  At this step, we use the Faedo-Galerkin method and introduce a family of functions $\{\phi_m\}_{m\geq 1}$ in $H_0^1(\Omega)$ which is an orthogonal basis in $H_0^1(\Omega)$ and an orthonormal basis in $L^2(\Omega)$.

  A positive integer m being fixed, we look for an approximated solution of (B.3) $\overline{u}_m : [0,T] \to H_0^1(\Omega)$ under the form

$$\overline{u}_m(t) = \sum_{i=1}^{m} \alpha_{im}(t)\phi_i, \tag{B.4}$$

  where the coefficients $(\alpha_{im})_{1\leq i\leq m}$ being to be determined by the conditions:

$$\begin{aligned}\int_\Omega \partial_t \overline{u}_m(t,x)\phi_i(x)\ dx + \int_\Omega \nabla \overline{u}_m(t,x)\cdot\nabla\phi_i(x)\ dx + \int_\Omega \overline{u}_m^3(t,x)\phi_i(x)\ dx + 3\int_\Omega \tilde{u}\overline{u}_m^2(t,x)\phi_i(x)\ dx \\ + 3\int_\Omega \tilde{u}^2\overline{u}_m(t,x)\phi_i(x)\ dx = \int_\Omega F(t,x)\phi_i(x)\ dx, \quad \forall i=1,\dots,m\end{aligned} \tag{B.5}$$

  along with

$$\alpha_{im}(0) = \int_\Omega \overline{u}(0,x)\phi_i(x)\ dx, \quad \forall i = 1,\dots,m. \tag{B.6}$$

  From Picard-Lindelöf theorem (see, for example [26]), the system (B.5)-(B.6) of nonlinear ordinary differential equations, admits a unique local in time solution $(\alpha_{im})_{1\leq i\leq m}$ in C^1 defined on a maximal interval $(0,T_m)$.

- *Step 3* - A priori estimates.

  Multiplying the equation (B.5) by α_{im}, summing over i and integrating on $(0,t)$, we deduce that

$$\begin{aligned}\|\overline{u}_m\|_{C^0(0,t;L^2(\Omega))} + \|\overline{u}_m\|_{L^2(0,t;H^1(\Omega))} + \|\overline{u}_m\|^2_{L^4((0,t)\times\Omega)} \\ \leq C\left(\|F\|_{L^2((0,T)\times\Omega)} + \|\tilde{u}\|^2_{L^4((0,T)\times\Omega)} + \|\overline{u}(0,\cdot)\|_{L^2(\Omega)}\right).\end{aligned}$$

  Thus, the coefficients $(\alpha_{im})_{1\leq i\leq m}$ stay bounded in $C^0(0,T_m)$ and this ensures that they are defined on the global interval $(0,T)$.

- *Step 4* - Passage to the limit $m\to\infty$.

  Now, we multiply the equation (B.5) by α'_{im}, sum over i and integrate on $(0,T)$. We get that

$$\begin{aligned}\|\overline{u}_m\|_{H^1(0,T;L^2(\Omega))} + \|\overline{u}_m\|_{L^\infty(0,T;H^1(\Omega))} \\ \leq C\left(1+\|\tilde{u}\|^2_{L^\infty((0,T)\times\Omega)}\right)\left(\|F\|_{L^2((0,T)\times\Omega)} + \|\tilde{u}\|^2_{L^4((0,T)\times\Omega)} + \|\overline{u}(0,\cdot)\|_{H^1(\Omega)}\right).\end{aligned}$$

  Thus we deduce that, up to a subsequence, $(\overline{u}_m)_m$ weakly converges in $H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;H_0^1(\Omega))$ and strongly converges in $L^2((0,T)\times\Omega)$. This convergence properties allow to deduce that the limit $\overline{u}$ satisfies the weak formulation (B.3) and the estimate

$$\begin{aligned}\|\overline{u}\|_{H^1(0,T;L^2(\Omega))} + \|\overline{u}\|_{L^\infty(0,T;H^1(\Omega))} + \|\overline{u}\|^2_{L^4((0,T)\times\Omega)} \\ \leq C\left(1+\|\tilde{u}\|^2_{L^\infty((0,T)\times\Omega)}\right)\left(\|F\|_{L^2((0,T)\times\Omega)} + \|\tilde{u}\|^2_{L^4((0,T)\times\Omega)} + \|\overline{u}(0,\cdot)\|_{H^1(\Omega)}\right).\end{aligned} \tag{B.7}$$

- *Step 5* - Higher regularity.

  Looking at (B.2) as an elliptic problem, the elliptic regularity implies that $\overline{u}$ belongs to $L^2(0,T;H^2(\Omega))$ and

$$\|\overline{u}\|_{L^2(0,T;H^2(\Omega))} \leq C\left(1+\|\tilde{u}\|^2_{L^\infty((0,T)\times\Omega)}\right)\left(\|F\|_{L^2((0,T)\times\Omega)}+\|\tilde{u}\|^2_{L^4((0,T)\times\Omega)}+\|\overline{u}(0,\cdot)\|_{H^1(\Omega)}\right) \\ + C\left(1+\|\tilde{u}\|^2_{L^\infty((0,T)\times\Omega)}\right)^3\left(\|F\|_{L^2((0,T)\times\Omega)}+\|\tilde{u}\|^2_{L^4((0,T)\times\Omega)}+\|\overline{u}(0,\cdot)\|_{H^1(\Omega)}\right)^3.$$

  Moreover, since $\overline{u}$ belongs to $H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$, according to [10, Section 5.9, Theorem 4], we deduce that $\overline{u}$ belongs to $C^0(0,T;H^1(\Omega))$.

- *Step 6* - Return to the variable u.

  Coming back to $u = \overline{u} + \tilde{u}$ and using (B.1), we conclude that

$$u \in L^2(0,T;H^2(\Omega)) \cap C^0(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$$

  with the following estimate

$$\|u\|_{L^2(0,T;H^2(\Omega))} + \|u\|_{C^0(0,T;H^1(\Omega))} + \|u\|_{H^1(0,T;L^2(\Omega))} \leq C\left(\|\sigma h\|_{L^2(0,T;L^2(\Omega))} + \|\sigma h\|^p_{L^2(0,T;L^2(\Omega))} \right. \\ \left. + \|g\|_{H^1(0,T;H^{3/2}(\partial\Omega))} + \|g\|^p_{H^1(0,T;H^{3/2}(\partial\Omega))} + \|u_\circ\|_{H^1(\Omega)} + \|u_\circ\|^p_{H^1(\Omega)}\right), \tag{B.8}$$

  where the power p is a positive integer that can change from line to line.

- *Step 7* - Improved regularity.

  Next, let us consider $w = \partial_t u$ which is, according to (1.1) and (3.1), formally the solution of

$$\begin{cases} \partial_t w - \Delta w + 3u^2 w = \sigma \partial_t h, & \text{in } (0,T)\times\Omega, \\ w = \partial_t g, & \text{on } (0,T)\times\partial\Omega, \\ w(0,\cdot) = \Delta u_\circ - (u_\circ)^3, & \text{in } \Omega. \end{cases} \tag{B.9}$$

  We use the same lifting as in *Step 1* and define the function $\overline{w} = w - \partial_t \tilde{u}$, which satisfies

$$\begin{cases} \partial_t \overline{w} - \Delta \overline{w} + 3u^2 \overline{w} = G, & \text{in } (0,T)\times\Omega, \\ \overline{w} = 0, & \text{on } (0,T)\times\partial\Omega, \\ \overline{w}(0,\cdot) = w(0,\cdot) - \partial_t\tilde{u}(0,\cdot), & \text{in } \Omega, \end{cases} \tag{B.10}$$

  with G defined by $G = \sigma\partial_t h - \partial_{tt}\tilde{u} + \Delta\partial_t\tilde{u} - 3u^2\partial_t\tilde{u}$. For this system, we have a unique solution

$$\overline{w} \in C^0(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$$

  which satisfies

$$\|\overline{w}\|_{C^0(0,T;L^2(\Omega))} + \|\overline{w}\|_{L^2(0,T;H^1(\Omega))} \leq C\left(\|G\|_{L^2((0,T)\times\Omega)} + \|\overline{w}(0,\cdot)\|_{L^2(\Omega)}\right).$$

For the first term in the right hand side, we have

$$\|G\|_{L^2((0,T)\times\Omega)} \leq \|\sigma h\|_{H^1(0,T;L^2(\Omega))} + \|\tilde{u}\|_{H^2(0,T;L^2(\Omega))\cap H^1(0,T;H^2(\Omega))} + C\left(\|u\|^2_{C^0(0,T;H^1(\Omega))} + \|\tilde{u}\|_{H^1(0,T;H^2(\Omega))}\right).$$

Taking account (B.1) and (B.8), and going back to $w = \overline{w} + \partial_t \tilde{u}$, we conclude that

$$u \in C^1(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)),$$

with the following estimate

$$\|u\|_{C^1(0,T;L^2(\Omega))} + \|u\|_{H^1(0,T;H^1(\Omega))} \leq C\left(\|\sigma h\|_{H^1(0,T;L^2(\Omega))} + \|\sigma h\|^p_{H^1(0,T;L^2(\Omega))} + \|g\|_{H^1(0,T;H^{3/2}(\partial\Omega))\cap H^2(0,T;L^2(\partial\Omega))} + \|g\|^p_{H^1(0,T;H^{3/2}(\partial\Omega))\cap H^2(0,T;L^2(\partial\Omega))} + \|u_\circ\|_{H^2(\Omega)} + \|u_\circ\|^p_{H^2(\Omega)}\right).$$

Thus, if look at (1.1) as an elliptic problem, we get that $u \in C^0(0,T;H^2(\Omega))$ and we have the estimate

$$\begin{aligned} &\|u\|_{C^0(0,T;H^2(\Omega))} + \|u\|_{C^1(0,T;L^2(\Omega))} + \|u\|_{H^1(0,T;H^1(\Omega))} \\ &\leq C\left(\|\sigma h\|_{H^1(0,T;L^2(\Omega))} + \|\sigma h\|^p_{H^1(0,T;L^2(\Omega))} + \|u_\circ\|_{H^2(\Omega)} + \|u_\circ\|^p_{H^2(\Omega)}\right. \\ &\quad \left.\|g\|_{H^1(0,T;H^{3/2}(\partial\Omega))\cap H^2(0,T;L^2(\partial\Omega))} + \|g\|^p_{H^1(0,T;H^{3/2}(\partial\Omega))\cap H^2(0,T;L^2(\partial\Omega))}\right). \end{aligned} \tag{B.11}$$

Let us note that, since $u_\circ \in H^3(\Omega)$, the initial condition

$$\overline{w}(0,\cdot) = \Delta u_\circ - (u_\circ)^3 - \partial_t \tilde{u}(0,\cdot)$$

belongs to $H^1(\Omega)$. Then, if we multiply the equation (B.10) by $\partial_t \overline{w}$ and integrate in $(0,T)\times\Omega$, we obtain that $\partial_t \overline{w} \in L^2(0,T;L^2(\Omega))$ with

$$\|\partial_t \overline{w}\|_{L^2(0,T;L^2(\Omega))} \leq C\left(\|G\|_{L^2((0,T)\times\Omega)} + \|\overline{w}(0)\|_{H^1(\Omega)}\right).$$

Hence, if we look at (B.10) as an elliptic problem, we deduce that $\overline{w}$ belongs to $L^2(0,T;H^2(\Omega))$ with the following estimate

$$\|\overline{w}\|_{L^2(0,T;H^2(\Omega))} \leq C\left(\|G\|_{L^2((0,T)\times\Omega)} + \|u\|^2_{C^0(0,T;H^2(\Omega))}\|\overline{w}\|_{L^2((0,T)\times\Omega)} + \|\overline{w}(0)\|_{H^1(\Omega)}\right).$$

Besides, since $\overline{w}$ belongs to $H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$, we deduce that $\overline{w}$ belongs to $C^0(0,T;H^1(\Omega))$.

Coming back to $\partial_t u = \overline{w} + \partial_t \tilde{u}$, we finally deduce that u belongs to $H^1(0,T;H^2(\Omega)) \cap C^1(0,T;H^1(\Omega)) \cap H^2(0,T;L^2(\Omega))$ along with the estimate (2.8).

C Proof of Lemma 3.7

On $\widetilde{E}$, we consider the norm $\|\cdot\|_s$ defined by

$$\|z\|_s^2 = \int_0^T \int_\Omega e^{2s\varphi} \left(\frac{1}{s\theta} |\partial_t z|^2 + \frac{1}{s\theta} |\Delta z|^2 + s\theta |\nabla z|^2 + s^3\theta^3 |z|^2 \right) dxdt.$$

For any fixed $C > 0$, the set E_C is convex and closed in $(\widetilde{E}, \|\cdot\|_s)$.
Let z_1 and z_2 be given in E_C. We set $z = z_1 - z_2$. Then, we have

$$\begin{aligned}
&DJ_{0,k}[\mu](z_1)(z) - DJ_{0,k}[\mu](z_2)(z) = \\
&\int_0^T \int_\Omega e^{2s\varphi} P_k z_1 DP_k(z_1)(z)\, dxdt + s \int_0^T \int_\Gamma e^{2s\varphi} \theta (\nabla z_1 \cdot n - \mu)(\nabla z \cdot n)\, d\gamma dt \\
&- \int_0^T \int_\Omega e^{2s\varphi} P_k z_2 DP_k(z_2)(z)\, dxdt - s \int_0^T \int_\Gamma e^{2s\varphi} \theta (\nabla z_2 \cdot n - \mu)(\nabla z \cdot n)\, d\gamma dt \\
= &\int_0^T \int_\Omega e^{2s\varphi} (P_k z_1 - P_k z_2) DP_k(z_2)(z)\, dxdt + \int_0^T \int_\Omega e^{2s\varphi} P_k z_1 (DP_k(z_1)(z) - DP_k(z_2)(z))\, dxdt \\
&+ s \int_0^T \int_\Gamma e^{2s\varphi} \theta |\nabla z \cdot n|^2\, d\gamma dt.
\end{aligned} \tag{C.1}$$

To estimate the two first terms, we will follow similar computations as in Section 3.4 since we can notice that these terms also appear in (3.37) if we replace z_1, z_2 and z respectively by w_k, Z_k and z_k. Since the assumptions on the functions are different (in Section 3.4, $w_k = \partial_t v_k$ where v_k satisfies (3.4) whereas here z_1 and z_2 belong to E_C), we detail the arguments below when they are different from the ones in Section 3.4.
For the first term, we follow the computations made in *Step 1* of Section 3.4 and the counterpart of (3.39) is:

$$\begin{aligned}
&\int_0^T \int_\Omega e^{2s\varphi} (P_k z_1 - P_k z_2) DP_k(z_2)(z) dxdt \geq \frac{3}{4} \int_0^T \int_\Omega e^{2s\varphi} |\partial_t z - \Delta z|^2 dxdt \\
&- M \int_0^T \int_\Omega e^{2s\varphi} |z|^2 dxdt - M \int_0^T \int_\Omega e^{2s\varphi} |\overline{y}|^2 (|\partial_t u_k|^2 + |z_1|^2 + |T'_{\overline{M}}(y_2)|^2 |\partial_t u_k|^2) dxdt \\
&- M \int_0^T \int_\Omega e^{2s\varphi} |\overline{y}|^2 |T'_{\overline{M}}(y_2)|^2 |z_2|^2 dxdt,
\end{aligned}$$

where

$$y_i(t,x) = v_k(T_0, x) + \int_{T_0}^t z_i(t', x) dt' \text{ for } 1 \leq i \leq 2 \text{ and } \overline{y}(t,x) = \int_{T_0}^t z(t', x) dt'.$$

Using (3.17), we deduce

$$\begin{aligned}
&\int_0^T \int_\Omega e^{2s\varphi} (P_k z_1 - P_k z_2) DP_k(z_2)(z) dxdt \geq \frac{3}{4} \int_0^T \int_\Omega e^{2s\varphi} |\partial_t z - \Delta z|^2 dxdt \\
&- M \int_0^T \int_\Omega e^{2s\varphi} |z|^2 dxdt - M \int_0^T \int_\Omega e^{2s\varphi} |\overline{y}|^2 (|\partial_t u_k|^2 + |z_1|^2 + |z_2|^2) dxdt.
\end{aligned}$$

For the last term of this inequality, we have

$$\begin{aligned}
&\int_0^T\int_\Omega e^{2s\varphi}|\overline{y}|^2(|\partial_t u_k|^2+|z_1|^2+|z_2|^2)dxdt\\
&\leq \|e^{s\varphi}\overline{y}\|^2_{L^\infty(0,T;L^2(\Omega))}(\|\partial_t u_k\|^2_{L^2(0,T;L^\infty(\Omega))}+\|z_1\|^2_{L^2(0,T;L^\infty(\Omega))}+\|z_2\|^2_{L^2(0,T;L^\infty(\Omega))})\\
&\leq M\|e^{s\varphi}\overline{y}\|^2_{L^\infty(0,T;L^2(\Omega))},
\end{aligned}$$

according to (3.21) and to the fact that z_1 and z_2 belong to E_C. Using the same arguments than in (3.42) with z_k and $\overline{y}_k$ replaced respectively by z and $\overline{y}$, we get

$$\int_0^T\int_\Omega e^{2s\varphi}|\overline{y}|^2(|\partial_t u_k|^2+|z_1|^2+|z_2|^2)dxdt\leq M\|e^{s\varphi}z\|^2_{L^2(0,T;L^2(\Omega))}.$$

Thus,

$$\int_0^T\int_\Omega e^{2s\varphi}(P_kz_1-P_kz_2)DP_k(z_2)(z)dxdt\geq\frac{3}{4}\int_0^T\int_\Omega e^{2s\varphi}|\partial_t z-\Delta z|^2dxdt-M\int_0^T\int_\Omega e^{2s\varphi}|z|^2dxdt. \tag{C.2}$$

For the second term in the right hand side of (C.1), we follow the computations of *Step 3* in Section 3.4. We have

$$\begin{aligned}
&DP_k(z_1)(z)-DP_k(z_2)(z)=6zu_k\left(T_{\overline{M}}(y_1)-T_{\overline{M}}(y_2)\right)+3z\left(T_{\overline{M}}(y_1)^2-T_{\overline{M}}(y_2)^2\right)\\
&+6\partial_t u_k u_k\overline{y}\left(T'_{\overline{M}}(y_1)-T'_{\overline{M}}(y_2)\right)+6\partial_t u_k\overline{y}\left(T'_{\overline{M}}(y_1)-T'_{\overline{M}}(y_2)\right)T_{\overline{M}}(y_1)\\
&+6\partial_t u_k T'_{\overline{M}}(y_2)\overline{y}(T_{\overline{M}}(y_1)-T_{\overline{M}}(y_2))+6u_k\overline{y}\left(T'_{\overline{M}}(y_1)-T'_{\overline{M}}(y_2)\right)z_1+6u_kT'_{\overline{M}}(y_2)\overline{y}z\\
&+6\overline{y}\left(T'_{\overline{M}}(y_1)-T'_{\overline{M}}(y_2)\right)z_1T_{\overline{M}}(y_1)+6T'_{\overline{M}}(y_2)\overline{y}z_1(T_{\overline{M}}(y_1)-T_{\overline{M}}(y_2))+6T'_{\overline{M}}(y_2)\overline{y}zT_{\overline{M}}(y_2).
\end{aligned}$$

Using (3.21) and Proposition 3.5 which states the properties satisfied by $T_{\overline{M}}$, we get

$$|DP_k(z_1)(z)-DP_k(z_2)(z)|\leq M(|z||\overline{y}|+|\partial_t u_k||\overline{y}|^2+|z_1||\overline{y}|^2).$$

Thus, for the second term in the right hand side of (C.1), we have the bound

$$\begin{aligned}
&\int_0^T\int_\Omega e^{2s\varphi}|P_kz_1||DP_k(z_1)(z)-DP_k(z_2)(z)|\,dxdt\leq M\int_0^T\int_\Omega e^{2s\varphi}|P_kz_1||z||\overline{y}|\,dxdt\\
&+M\int_0^T\int_\Omega e^{2s\varphi}|P_kz_1|(|\partial_t u_k|+|z_1|)|\overline{y}|^2\,dxdt.
\end{aligned}\tag{C.3}$$

For the first term in the right hand side of (C.3), we have

$$\begin{aligned}
\int_0^T\int_\Omega e^{2s\varphi}|P_kz_1||z||\overline{y}|\,dxdt&\leq C\|P_kz_1\|_{L^2((0,T)\times\Omega)}\|e^{2s\varphi}z\overline{y}\|_{L^2((0,T)\times\Omega)}\\
&\leq C\|P_kz_1\|_{L^2((0,T)\times\Omega)}\|e^{s\varphi}z\|_{L^2(0,T;L^6(\Omega))}\|e^{s\varphi}\overline{y}\|_{L^\infty(0,T;L^3(\Omega))}\\
&\leq C\|P_kz_1\|_{L^2((0,T)\times\Omega)}(\|e^{s\varphi}z\|^2_{L^2(0,T;L^6(\Omega))}+\|e^{s\varphi}\overline{y}\|^2_{L^\infty(0,T;L^3(\Omega))})\\
&\leq M(\|e^{s\varphi}z\|^2_{L^2(0,T;L^6(\Omega))}+\|e^{s\varphi}\overline{y}\|^2_{L^\infty(0,T;L^3(\Omega))})
\end{aligned}$$

where we have used that z_1 belongs to E_C. For the second term in the right hand side of (C.3), we have

$$\int_0^T \int_\Omega e^{2s\varphi} |P_k z_1| (|\partial_t u_k| + |z_1|) |\overline{y}|^2 \, dx dt \leq C \|P_k z_1\|_{L^2((0,T)\times\Omega)} \|(|\partial_t u_k| + |z_1|) e^{2s\varphi} |\overline{y}|^2\|_{L^2((0,T)\times\Omega)}$$
$$\leq C \|P_k z_1\|_{L^2((0,T)\times\Omega)} (\|\partial_t u_k\|_{L^2(0,T;L^\infty(\Omega))} + \|z_1\|_{L^2(0,T;L^\infty(\Omega))}) \|e^{2s\varphi}|\overline{y}|^2\|_{L^\infty(0,T;L^2(\Omega))}$$
$$\leq M \|e^{s\varphi}\overline{y}\|^2_{L^\infty(0,T;L^4(\Omega))},$$

where we have used that z_1 belongs to E_C and u_k satisfies (3.21). Therefore, inequality (C.3) becomes

$$\int_0^T \int_\Omega e^{2s\varphi} |P_k z_1| |DP_k(z_1)(z) - DP_k(z_2)(z)| \, dx dt \leq M (\|e^{s\varphi} z\|^2_{L^2(0,T;L^6(\Omega))} + \|e^{s\varphi}\overline{y}\|^2_{L^\infty(0,T;L^4(\Omega))}). \tag{C.4}$$

For the terms in the right hand side, we first have

$$\|e^{s\varphi} z\|^2_{L^2(0,T;L^6(\Omega))} \leq \|e^{s\varphi} z\|^2_{L^2(0,T;H^1(\Omega))} \leq \int_0^T \int_\Omega e^{2s\varphi} \big((1 + s^2\theta^2)|z|^2 + |\nabla z|^2\big) \, dx dt$$

thanks to (A.2). Moreover,

$$\|e^{s\varphi}\overline{y}\|^2_{L^\infty(0,T;L^4(\Omega))} \leq C \|e^{s\varphi}\overline{y}\|^2_{L^\infty(0,T;H^1(\Omega))} \leq C \sup_{0\leq t\leq T} \int_\Omega e^{2s\varphi} \big((1 + s^2\theta^2)|\overline{y}|^2 + |\nabla \overline{y}|^2\big) \, dx.$$

We notice that, for all $t \in (0, T)$

$$\int_\Omega e^{2s\varphi(t,x)} \big((1 + s^2\theta^2(t, x))|\overline{y}(t, x)|^2 + |\nabla \overline{y}(t, x)|^2\big) \, dx$$
$$\leq C \int_\Omega e^{2s\varphi(t,x)} (1 + s^2\theta^2(t, x)) \left| \int_{T_0}^t |z(t', x)|^2 \, dt' \right| dx + C \int_\Omega e^{2s\varphi(t,x)} \left| \int_{T_0}^t |\nabla z(t', x)|^2 \, dt' \right| dx$$
$$\leq C \int_0^T \int_\Omega e^{2s\varphi(t,x)} (1 + s^2\theta^2(t, x)) |z(t, x)|^2 \, dx dt + C \int_0^T \int_\Omega e^{2s\varphi(t,x)} |\nabla z(t', x)|^2 \, dx dt$$

according to the fact that, in both cases (2.1) and (2.4), for s large enough, for all t' between T_0 and t, for all $x \in \Omega$

$$e^{2s\varphi(t,x)} (1 + s^2\theta^2(t, x)) \leq e^{2s\varphi(t',x)} (1 + s^2\theta^2(t', x)). \tag{C.5}$$

Thus, inequality (C.4) becomes

$$\int_0^T \int_\Omega e^{2s\varphi} |P_k z_1| |DP_k(z_1)(z) - DP_k(z_2)(z)| \, dx dt \leq M \int_0^T \int_\Omega e^{2s\varphi} \big((1 + s^2\theta^2)|z|^2 + |\nabla z|^2\big) \, dx dt.$$

Coming back to (C.1), using (C.2), this last inequality and the Carleman estimate (2.1), we conclude that, for s large enough, there exists $\delta > 0$ such that, for all z_1 and z_2 in E_C

$$DJ_{0,k}[\mu](z_1)(z_1 - z_2) - DJ_{0,k}[\mu](z_2)(z_1 - z_2) \geq \delta \|z_1 - z_2\|^2_s.$$

This proves that $J_{0,k}[\mu]$ is strongly convex in E_C.

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