Inverse Problems **28** (2012) 015005 (22pp) doi:10.1088/0266-5611/28/1/015005

On the identifiability of a rigid body moving in a stationary viscous fluid

Carlos Conca[1], **Erica L Schwindt**[1,2,3,4] **and Takéo Takahashi**[2,3]

[1] Facultad de Ciencias Físicas y Matemáticas, Departamento de Ingeniería Matemática, Universidad de Chile, Casilla 170/ 3-Correo 3, Santiago, Chile
[2] Institut Élie Cartan, UMR 7502, INRIA, Nancy-Université, CNRS BP239, 54506 Vandœuvre-lès-Nancy Cedex, France
[3] Team-Project CORIDA, INRIA Nancy—Grand Est, 615 rue du Jardin Botanique, 54600 Villers-lès-Nancy, France

E-mail: cconca@dim.uchile.cl, leris98@gmail.com and takeo.takahashi@inria.fr

Received 25 August 2011, in final form 24 October 2011
Published 15 December 2011
Online at stacks.iop.org/IP/28/015005

Abstract

This paper is devoted to a geometrical inverse problem associated with a fluid–structure system. More precisely, we consider the interaction between a moving rigid body and a viscous and incompressible fluid. Assuming a low Reynolds regime, the inertial forces can be neglected and, therefore, the fluid motion is modelled by the Stokes system. We first prove the well posedness of the corresponding system. Then we show an identifiability result: with one measure of the Cauchy forces of the fluid on one given part of the boundary and at some positive time, the shape of a convex body and its initial position are identified.

(Some figures may appear in colour only in the online journal)

1. Introduction

The aim of this paper is to consider a geometrical inverse problem associated with a fluid–structure system. More precisely, we want to identify the shape of a moving rigid body and its initial position. Geometrical inverse problems are frequent models in several applied areas such as medical imaging and non-destructive evaluation of materials.

In this work, we are interested in identifying an inaccessible solid structure, denoted by $\mathcal{S}(t)$, which is moving in a viscous incompressible fluid occupying a region denoted by $\mathcal{F}(t)$. We assume that both the fluid and the structure are contained in a bounded fixed domain (i.e. connected and open set) Ω of $\mathbb{R}^3$ so that $\mathcal{F}(t) = \Omega \setminus \overline{\mathcal{S}(t)}$.

We assume that the structure is a rigid body so that it can be described by its centre of mass $\boldsymbol{a}(t) \in \mathbb{R}^3$ and by its orientation $\boldsymbol{Q}(t) \in SO_3(\mathbb{R})$ as follows:

$$\mathcal{S}(t) := \mathcal{S}(\boldsymbol{a}(t), \boldsymbol{Q}(t)),$$

[4] Author to whom any correspondence should be addressed.

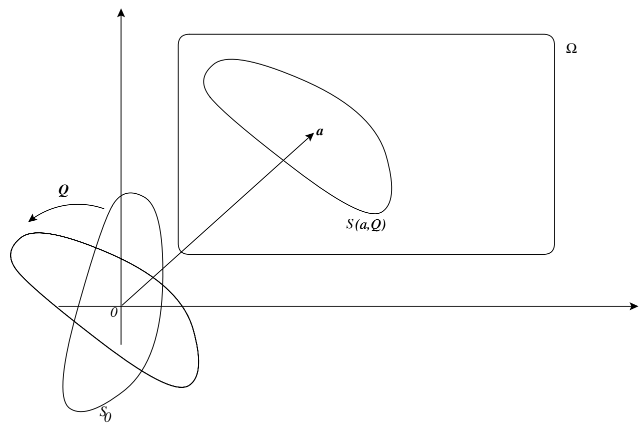

Figure 1. Domain of reference.

with

$$\mathcal{S}(\boldsymbol{a}, \boldsymbol{Q}) := \boldsymbol{Q}\mathcal{S}_0 + \boldsymbol{a}, \qquad (\boldsymbol{a}, \boldsymbol{Q}) \in \mathbb{R}^3 \times SO_3(\mathbb{R}). \tag{1}$$

Here, $\mathcal{S}_0$ is a smooth non-empty domain which is given and $SO_3(\mathbb{R})$ is the set of all rotation real square matrices of order 3. We can assume, without loss of generality, that the centre of mass of $\mathcal{S}_0$ is at the origin and in that case, $\boldsymbol{a}$ is the centre of mass of $\mathcal{S}(\boldsymbol{a}, \boldsymbol{Q})$. We also assume that there exists $(\boldsymbol{a}, \boldsymbol{Q}) \in \mathbb{R}^3 \times SO_3(\mathbb{R})$ such that $\overline{\mathcal{S}(\boldsymbol{a}, \boldsymbol{Q})} \subset \Omega$ and

$$\mathcal{F}(\boldsymbol{a}, \boldsymbol{Q}) := \Omega \setminus \overline{\mathcal{S}(\boldsymbol{a}, \boldsymbol{Q})},$$

is a smooth non-empty domain (see figure 1). In what follows, we will write

$$\mathcal{F}(t) := \mathcal{F}(\boldsymbol{a}(t), \boldsymbol{Q}(t)).$$

To write the equations governing the motion of this fluid–structure system, it is natural to model the fluid motion through classical Navier–Stokes equations and next apply Newton's laws to obtain the equations for the rigid body. The resulting system has been extensively studied in the last few years (see e.g. [9, 10, 13, 16, 20, 23, 26–28, 31] for a non-exhaustive list of articles on this subject). A large part of the literature associated with this system is devoted to its well posedness and in particular, up to our knowledge, there is no result for the geometrical inverse problems that we will consider in this paper. Indeed, this general case is quite difficult to handle directly and we will therefore limit ourselves to consider only a simplified version where it will be assumed that the Reynolds number is very small so that inertial forces can be neglected. In that case, the dynamics of the whole fluid–rigid solid body interaction writes as follows:

$$-\operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{u}, p)) = \boldsymbol{0} \quad \text{in } \mathcal{F}(t), \qquad t \in (0, T), \tag{2}$$

$$\operatorname{div}(\boldsymbol{u}) = 0 \quad \text{in } \mathcal{F}(t), \qquad t \in (0, T), \tag{3}$$

$$\boldsymbol{u} = \boldsymbol{\ell} + \boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{a}) \quad \text{on } \partial\mathcal{S}(t), \qquad t \in (0, T), \tag{4}$$

$$\boldsymbol{u} = \boldsymbol{u}_* \quad \text{on } \partial\Omega, \qquad t \in (0, T), \tag{5}$$

$$\int_{\partial\mathcal{S}(t)} \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n} \, \mathrm{d}\boldsymbol{\gamma}_{\boldsymbol{x}} = \boldsymbol{0} \qquad t \in (0, T), \tag{6}$$

$$\int_{\partial\mathcal{S}(t)} (\boldsymbol{x}-\boldsymbol{a}) \times \boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{n}\,\mathrm{d}\boldsymbol{\gamma}_{\boldsymbol{x}} = \boldsymbol{0} \qquad t \in (0,T), \tag{7}$$

$$\boldsymbol{a}' = \boldsymbol{\ell} \qquad t \in (0,T), \tag{8}$$

$$\boldsymbol{Q}' = \mathbb{S}(\boldsymbol{\omega})\boldsymbol{Q} \qquad t \in (0,T), \tag{9}$$

$$\boldsymbol{a}(0) = \boldsymbol{a}_0, \tag{10}$$

$$\boldsymbol{Q}(0) = \boldsymbol{Q}_0. \tag{11}$$

Here, $\mathbb{S}(\boldsymbol{\omega})$ is the skew symmetric matrix

$$\mathbb{S}(\boldsymbol{\omega}) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \qquad (\boldsymbol{\omega} \in \mathbb{R}^3)$$

which satisfies that $\mathbb{S}(\boldsymbol{\omega})\boldsymbol{z} = \boldsymbol{\omega} \times \boldsymbol{z}$, for every $\boldsymbol{z} \in \mathbb{R}^3$.

In the above system, $(\boldsymbol{u}, p)$ are the velocity and the pressure of the fluid, whereas $\boldsymbol{\ell}$ and $\boldsymbol{\omega}$ are respectively the linear and angular velocity of the solid. Moreover, we have denoted by $\boldsymbol{\sigma}(\boldsymbol{u}, p)$ the Cauchy stress tensor, which is defined by Stokes law

$$\boldsymbol{\sigma}(\boldsymbol{u},p) = -p\mathbf{Id} + 2\nu\,\boldsymbol{D}(\boldsymbol{u}),$$

where $\mathbf{Id}$ is the identity matrix of $\mathcal{M}_3(\mathbb{R})$, with $\mathcal{M}_3(\mathbb{R})$ being the set of all real square matrices of order 3, and $\boldsymbol{D}(\boldsymbol{u})$ is the strain tensor defined by

$$[\boldsymbol{D}(\boldsymbol{u})]_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right).$$

The positive constant ν is the kinematic viscosity of the fluid.

The velocity $\boldsymbol{u}_*$ is a given velocity satisfying the compatibility condition

$$\int_{\partial\Omega} \boldsymbol{u}_* \cdot \boldsymbol{n}\,\mathrm{d}\boldsymbol{\gamma}_{\boldsymbol{x}} = 0. \tag{12}$$

Despite the fact that equations (2)–(11) are simpler than the more general system which couples Navier–Stokes equations with Newton's laws, they do not represent a particular case of this one. Therefore, we first need to consider the well posedness of this system for a given shape $\mathcal{S}_0$ of the rigid body (see theorem 1).

Let us now give a more detailed description of the inverse problem we consider in this paper. Assume Γ is a non-empty open subset of $\partial\Omega$, where we can measure $\boldsymbol{\sigma}(\boldsymbol{u}, p)\ \boldsymbol{n}_{|\Gamma}$ at some time $t_0 > 0$. Is it possible to recover $\mathcal{S}_0$? In this paper, we show the identifiability of $\mathcal{S}_0$, proving that to each measurement of the Cauchy forces on Γ, there is a corresponding shape $\mathcal{S}_0$, which is unique up to a rotation matrix. More precisely, let us take two non-empty open sets $\mathcal{S}_0^{(1)}$ and $\mathcal{S}_0^{(2)}$. We prove that under certain assumptions on the rigid bodies and the function $\boldsymbol{u}_*$, we can identify the structure (see theorem 2).

Similar problems have been studied in the recent years. For example, in [1], the authors proved an identifiability result for a fixed smooth convex obstacle surrounded by a viscous fluid via the observation of the Cauchy forces on one given part of the boundary. Their method of proof is based on the unique continuation property for the Stokes equations due to Fabre–Lebeau [12]. They also obtained a weak stability result (directional continuity). In the case of a perfect fluid, the authors of [7] proved, in the two-dimensional case, an identifiability result when the obstacle is a ball which is moving around in an irrotational fluid by measuring the velocity of the fluid on a given subregion of the boundary. Precise stability results (linear stability estimates) are proved for this case using shape differentiation techniques due to Simon [29]. In [11] the authors considered the inverse problem of detecting the shape of a single rigid obstacle immersed in a fluid governed by the stationary Navier–Stokes equations by assuming that friction forces are known on one part of the outer boundary. They proved a uniqueness result when the obstacle is a simply connected open set. In this paper, we obtain, under some conditions, an identifiability result for moving obstacles in a viscous fluid. This is a first step towards detection results for the same problem. In the case of fixed obstacle in a viscous fluid, the authors of [17] gave a method to reconstruct obstacles. More precisely, using the geometrical optics solutions due to Silvester and Uhlmann [30], they can estimate the distance from a given point to an obstacle by means of boundary measurements. Another method was considered in [2] where the authors used the shape differentiation in a numerical method in order to recover a finite number of parameters of the obstacle.

In the papers cited above, both the obstacle and the fluid occupy a bounded domain, while in [8], an identifiability result for the case of a rigid solid moving around in a potential fluid filling $\mathbb{R}^2$ is proved. By assuming that the potential function is known at a given time, the authors showed that when the solid has some symmetry properties, it is possible to detect certain parameters of the solid: the angular velocity, the translational velocity, among others.

The paper is outlined as follows. In section 2, we state the main results of this paper given by theorems 1–2. In section 3, we consider an auxiliary system which is used to prove the well posedness of the system (2)–(11). More precisely, we can reduce the system (2)–(11) to a system of ODE for the position. Therefore, we show in section 4 that the solutions of the auxiliary system depend smoothly on the position, and by applying the Cauchy–Lipschitz–Picard theorem we deduce theorem 1. Section 5 is devoted to the proof of theorem 2. The method to prove the identifiability result is similar to the one developed in [1] for the case of a 'fixed' body. In the last section, section 6, we tackle the problem of stability by using the same approach as in [1].

2. Main results

The aim of this paper is to prove some results of identifiability for the system (2)–(11), but we first need to consider its well posedness for a given shape $\mathcal{S}_0$ of the rigid body.

Theorem 1. *Assume that $\boldsymbol{u}_* \in \boldsymbol{H}^{3/2}(\partial\Omega)$ satisfies (12), $\mathcal{S}_0$ is a smooth non-empty domain and $(\boldsymbol{a}_0, \boldsymbol{Q}_0) \in \mathbb{R}^3 \times SO_3(\mathbb{R})$ is such that $\overline{\mathcal{S}(\boldsymbol{a}_0, \boldsymbol{Q}_0)} \subset \Omega$ and $\mathcal{F}(\boldsymbol{a}_0, \boldsymbol{Q}_0)$ is a smooth non-empty domain. Then there exist a maximal time $T_* > 0$ and a unique solution*

$$(\boldsymbol{a}, \boldsymbol{Q}) \in \boldsymbol{C}^1([0, T_*); \mathbb{R}^3 \times SO(3)), \quad (\boldsymbol{\ell}, \boldsymbol{\omega}) \in \boldsymbol{C}([0, T_*); \mathbb{R}^3 \times \mathbb{R}^3),$$
$$(\boldsymbol{u}, p) \in \boldsymbol{C}\left([0, T_*); \boldsymbol{H}^2(\mathcal{F}(\boldsymbol{a}(t), \boldsymbol{Q}(t))) \times H^1(\mathcal{F}(\boldsymbol{a}(t), \boldsymbol{Q}(t)))/\mathbb{R}\right)$$

satisfying the system (2)–(11). Moreover, one of the following alternatives holds:

- $T_* = +\infty$;
- $\lim_{t\to T_*} \operatorname{dist}(\mathcal{S}(\boldsymbol{a}(t), \boldsymbol{Q}(t)), \partial\Omega) = 0$.

Now, let us take two smooth non-empty domains $\mathcal{S}_0^{(1)}$ and $\mathcal{S}_0^{(2)}$. Let us also consider $\left(\boldsymbol{a}_0^{(1)}, \boldsymbol{Q}_0^{(1)}\right), \left(\boldsymbol{a}_0^{(2)}, \boldsymbol{Q}_0^{(2)}\right) \in \mathbb{R}^3 \times SO_3(\mathbb{R})$ such that

$$\overline{\mathcal{S}^{(1)}\left(\boldsymbol{a}_0^{(1)}, \boldsymbol{Q}_0^{(1)}\right)} \subset \Omega \qquad \text{and} \qquad \overline{\mathcal{S}^{(2)}\left(\boldsymbol{a}_0^{(2)}, \boldsymbol{Q}_0^{(2)}\right)} \subset \Omega.$$

Applying theorem 1, we deduce that for any $\boldsymbol{u}_* \in \boldsymbol{H}^{3/2}(\partial\Omega)$ such that (12) holds, there exist $T_*^{(1)} > 0$ (respectively $T_*^{(2)} > 0$) and a unique solution $(\boldsymbol{a}^{(1)}, \boldsymbol{Q}^{(1)}, \boldsymbol{\ell}^{(1)}, \boldsymbol{\omega}^{(1)}, \boldsymbol{u}^{(1)}, p^{(1)})$

(respectively $(\boldsymbol{a}^{(2)}, \boldsymbol{Q}^{(2)}, \boldsymbol{\ell}^{(2)}, \boldsymbol{\omega}^{(2)}, \boldsymbol{u}^{(2)}, p^{(2)})$) of (2)–(11) in $[0, T_*^{(1)})$ (respectively in $[0, T_*^{(2)})$).

Then we have the following result.

Theorem 2. *With the above notation, assume that $\boldsymbol{u}_* \in \boldsymbol{H}^{3/2}(\partial\Omega)$ satisfies (12) and $\boldsymbol{u}_*$ is not the trace of a rigid velocity on Γ. Assume also that $\mathcal{S}_0^{(1)}$and $\mathcal{S}_0^{(2)}$ are convex. If there exists $0 < t_0 < \min(T_*^{(1)}, T_*^{(2)})$ such that*

$$\boldsymbol{\sigma}(\boldsymbol{u}^{(1)}(t_0), p^{(1)}(t_0))\, \boldsymbol{n}_{|\Gamma} = \boldsymbol{\sigma}(\boldsymbol{u}^{(2)}(t_0), p^{(2)}(t_0))\, \boldsymbol{n}_{|\Gamma},$$

then there exists $\boldsymbol{R} \in SO_3(\mathbb{R})$ such that

$$\boldsymbol{R}\mathcal{S}_0^{(1)} = \mathcal{S}_0^{(2)}$$

and

$$\boldsymbol{a}_0^{(1)} = \boldsymbol{a}_0^{(2)}, \qquad \boldsymbol{Q}_0^{(1)} = \boldsymbol{Q}_0^{(2)}\boldsymbol{R}.$$

In particular, $T_*^{(1)} = T_*^{(2)}$ and

$$\mathcal{S}^{(1)}(t) = \mathcal{S}^{(2)}(t) \qquad (t \in [0, T_*^{(1)})).$$

We recall that $\boldsymbol{v}$ is a rigid displacement in Ω if there exist two vectors $\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2 \in \mathbb{R}^3$ such that

$$\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2 \times \boldsymbol{x}, \quad \text{for all } \boldsymbol{x} \in \Omega.$$

In particular, if $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ and Ω is a bounded domain, we have that $\boldsymbol{v}$ is a rigid displacement if and only if

$$\boldsymbol{D}(\boldsymbol{v}) = \boldsymbol{0}.$$

Remark 1. In the above theorem, the hypothesis of convexity for the obstacle is technical and could probably be removed. The hypothesis $t_0 < \min(T_*^{(1)}, T_*^{(2)})$ means that we observe our data before any contact between the rigid body and the exterior boundary $\partial\Omega$.

To avoid this hypothesis, we should first model what happens when there is a contact. Unfortunately, this problem is quite complex: in particular, it can be proved (see [19, 21, 22]) that if Ω and S_0 are balls, then $T_* = \infty$, that is, there is no contact in finite time.

3. An auxiliary system

In this section, we consider and study an auxiliary system which is essential in the proof of theorem 1.

Let us fix $\boldsymbol{u}_* \in \boldsymbol{H}^{3/2}(\partial\Omega)$ satisfying (12) and consider $\mathcal{S}_0$ to be a non-empty smooth domain and assume $(\boldsymbol{a}_0, \boldsymbol{Q}_0) \in \mathbb{R}^3 \times SO_3(\mathbb{R})$ is such that $\overline{\mathcal{S}(\boldsymbol{a}_0, \boldsymbol{Q}_0)} \subset \Omega$ and $\mathcal{F}(\boldsymbol{a}_0, \boldsymbol{Q}_0)$ is a non-empty smooth domain. Let us consider the subset $\mathcal{A}$ of admissible positions of the rigid body in Ω,

$$\mathcal{A} := \{(\boldsymbol{a}, \boldsymbol{Q}) \in \mathbb{R}^3 \times SO_3(\mathbb{R}); \overline{\mathcal{S}(\boldsymbol{a}, \boldsymbol{Q})} \subset \Omega\}, \tag{13}$$

where $\mathcal{S}(\boldsymbol{a}, \boldsymbol{Q})$ is defined by (1).

For all $(\boldsymbol{a}, \boldsymbol{Q}) \in \mathcal{A}$, the following problem is well posed:

$$-\text{div}(\boldsymbol{\sigma}(\boldsymbol{u}, p)) = \boldsymbol{0} \quad \text{in } \mathcal{F}(\boldsymbol{a}, \boldsymbol{Q}), \tag{14}$$

$$\text{div}(\boldsymbol{u}) = 0 \quad \text{in } \mathcal{F}(\boldsymbol{a}, \boldsymbol{Q}), \tag{15}$$

$$\boldsymbol{u} = \boldsymbol{\ell} + \boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{a}) \quad \text{on } \partial\mathcal{S}(\boldsymbol{a}, \boldsymbol{Q}), \tag{16}$$

$$\boldsymbol{u} = \boldsymbol{u}_* \quad \text{on } \partial\Omega, \tag{17}$$

$$\int_{\partial\mathcal{S}(\boldsymbol{a},\boldsymbol{Q})} \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n}\, \mathrm{d}\boldsymbol{\gamma}_{\boldsymbol{x}} = \boldsymbol{0}, \tag{18}$$

$$\int_{\partial\mathcal{S}(\boldsymbol{a},\boldsymbol{Q})} (\boldsymbol{x} - \boldsymbol{a}) \times \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n}\, \mathrm{d}\boldsymbol{\gamma}_{\boldsymbol{x}} = \boldsymbol{0}. \tag{19}$$

More precisely, we have the following result.

Proposition 3. *Assume $\boldsymbol{u}_* \in \boldsymbol{H}^{3/2}(\partial\Omega)$ satisfying (12) and assume $(\boldsymbol{a}, \boldsymbol{Q}) \in \mathcal{A}$. Then there exists a unique solution $(\boldsymbol{u}, p, \boldsymbol{\ell}, \boldsymbol{\omega}) \in \boldsymbol{H}^2(\mathcal{F}(\boldsymbol{a}, \boldsymbol{Q})) \times H^1(\mathcal{F}(\boldsymbol{a}, \boldsymbol{Q}))/\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$ of (14)–(19).*

In order to prove the above proposition, we introduce for all $(\boldsymbol{a}, \boldsymbol{Q}) \in \mathcal{A}$, the following Stokes systems:

$$\begin{cases} -\mathrm{div}\left(\boldsymbol{\sigma}\left(\boldsymbol{u}^{(i)}, p^{(i)}\right)\right) = \boldsymbol{0} & \text{in } \mathcal{F}(\boldsymbol{a}, \boldsymbol{Q}) \\ \mathrm{div}\left(\boldsymbol{u}^{(i)}\right) = 0 & \text{in } \mathcal{F}(\boldsymbol{a}, \boldsymbol{Q}) \qquad (i = 1, 2, 3) \\ \boldsymbol{u}^{(i)} = \boldsymbol{e}^{(i)} & \text{on } \partial\mathcal{S}(\boldsymbol{a}, \boldsymbol{Q}) \\ \boldsymbol{u}^{(i)} = \boldsymbol{0} & \text{on } \partial\Omega, \end{cases} \tag{20}$$

$$\begin{cases} -\mathrm{div}\left(\boldsymbol{\sigma}\left(\boldsymbol{U}^{(i)}, P^{(i)}\right)\right) = \boldsymbol{0} & \text{in } \mathcal{F}(\boldsymbol{a}, \boldsymbol{Q}) \\ \mathrm{div}\left(\boldsymbol{U}^{(i)}\right) = 0 & \text{in } \mathcal{F}(\boldsymbol{a}, \boldsymbol{Q}) \qquad (i = 1, 2, 3) \\ \boldsymbol{U}^{(i)} = \boldsymbol{e}^{(i)} \times (\boldsymbol{x} - \boldsymbol{a}) & \text{on } \partial\mathcal{S}(\boldsymbol{a}, \boldsymbol{Q}) \\ \boldsymbol{U}^{(i)} = \boldsymbol{0} & \text{on } \partial\Omega, \end{cases} \tag{21}$$

where $\{\boldsymbol{e}^{(i)}\}_{i=1}^3$ is the canonical basis of $\mathbb{R}^3$, and

$$\begin{cases} -\mathrm{div}\left(\boldsymbol{\sigma}(\boldsymbol{V}^*, P^*)\right) = \boldsymbol{0} & \text{in } \mathcal{F}(\boldsymbol{a}, \boldsymbol{Q}) \\ \mathrm{div}\left(\boldsymbol{V}^*\right) = 0 & \text{in } \mathcal{F}(\boldsymbol{a}, \boldsymbol{Q}) \\ \boldsymbol{V}^* = \boldsymbol{0} & \text{on } \partial\mathcal{S}(\boldsymbol{a}, \boldsymbol{Q}) \\ \boldsymbol{V}^* = \boldsymbol{u}_* & \text{on } \partial\Omega. \end{cases} \tag{22}$$

Using that for all $i = 1, 2, 3$,

$$\int_{\partial\mathcal{S}(\boldsymbol{a},\boldsymbol{Q})} \boldsymbol{e}^{(i)} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\gamma}_{\boldsymbol{x}} = \int_{\partial\mathcal{S}(\boldsymbol{a},\boldsymbol{Q})} [\boldsymbol{e}^{(i)} \times (\boldsymbol{x} - \boldsymbol{a})] \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\gamma}_{\boldsymbol{x}} = \int_{\partial\mathcal{S}(\boldsymbol{a},\boldsymbol{Q})} \boldsymbol{u}_* \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\gamma}_{\boldsymbol{x}} = 0,$$

it is well known (see, for instance, [14, theorem 6.1 page 231]) that the systems (20), (21) and (22) admit unique solutions $(\boldsymbol{u}^{(i)}, p^{(i)})$, $(\boldsymbol{U}^{(i)}, P^{(i)})$, $(\boldsymbol{V}^*, P^*) \in \boldsymbol{H}^2(\mathcal{F}(\boldsymbol{a}, \boldsymbol{Q})) \times H^1(\mathcal{F}(\boldsymbol{a}, \boldsymbol{Q}))/\mathbb{R}$.

To solve (14)–(19), we search $(\boldsymbol{u}, p)$ as

$$\boldsymbol{u} := \sum_{i=1}^{3} \ell_i \boldsymbol{u}^{(i)} + \omega_i \boldsymbol{U}^{(i)} + \boldsymbol{V}^* \tag{23}$$

$$p := \sum_{i=1}^{3} \ell_i p^{(i)} + \omega_i P^{(i)} + P^*. \tag{24}$$

It is easy to check that $(\boldsymbol{u}, p) \in \boldsymbol{H}^2(\mathcal{F}(\boldsymbol{a}, \boldsymbol{Q})) \times H^1(\mathcal{F}(\boldsymbol{a}, \boldsymbol{Q}))/\mathbb{R}$ and satisfies

$$\begin{cases} -\mathrm{div}(\boldsymbol{\sigma}(\boldsymbol{u}, p)) = \boldsymbol{0} & \text{in } \mathcal{F}(\boldsymbol{a}, \boldsymbol{Q}), \\ \mathrm{div}(\boldsymbol{u}) = 0 & \text{in } \mathcal{F}(\boldsymbol{a}, \boldsymbol{Q}), \\ \boldsymbol{u} = \boldsymbol{\ell} + \boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{a}) & \text{on } \partial\mathcal{S}(\boldsymbol{a}, \boldsymbol{Q}), \\ \boldsymbol{u} = \boldsymbol{u}_* & \text{on } \partial\Omega. \end{cases}$$

Thus, $(\boldsymbol{u}, p)$ is a solution of (14)–(19) if and only if (18)–(19) hold, i.e. if and only if $(\boldsymbol{\ell}, \boldsymbol{\omega})$ satisfies

$$\sum_{i=1}^{3} \ell_i \int_{\partial\mathcal{S}(\boldsymbol{a},\boldsymbol{Q})} \boldsymbol{\sigma}(\boldsymbol{u}^{(i)}, p^{(i)})\boldsymbol{n}\, \mathrm{d}\boldsymbol{\gamma}_{\boldsymbol{x}} + \sum_{i=1}^{3} \omega_i \int_{\partial\mathcal{S}(\boldsymbol{a},\boldsymbol{Q})} \boldsymbol{\sigma}(\boldsymbol{U}^{(i)}, P^{(i)})\boldsymbol{n}\, \mathrm{d}\boldsymbol{\gamma}_{\boldsymbol{x}} + \int_{\partial\mathcal{S}(\boldsymbol{a},\boldsymbol{Q})} \boldsymbol{\sigma}(\boldsymbol{V}^{*}, P^{*})\boldsymbol{n}\, \mathrm{d}\boldsymbol{\gamma}_{\boldsymbol{x}} = \boldsymbol{0} \tag{25}$$

and

$$\sum_{i=1}^{3} \ell_i \int_{\partial\mathcal{S}(\boldsymbol{a},\boldsymbol{Q})} (\boldsymbol{x}-\boldsymbol{a}) \times \boldsymbol{\sigma}(\boldsymbol{u}^{(i)}, p^{(i)})\boldsymbol{n}\, \mathrm{d}\boldsymbol{\gamma}_{\boldsymbol{x}} + \sum_{i=1}^{3} \omega_i \int_{\partial\mathcal{S}(\boldsymbol{a},\boldsymbol{Q})} (\boldsymbol{x}-\boldsymbol{a}) \times \boldsymbol{\sigma}(\boldsymbol{U}^{(i)}, P^{(i)})\boldsymbol{n}\, \mathrm{d}\boldsymbol{\gamma}_{\boldsymbol{x}} + \int_{\partial\mathcal{S}(\boldsymbol{a},\boldsymbol{Q})} (\boldsymbol{x}-\boldsymbol{a}) \times \boldsymbol{\sigma}(\boldsymbol{V}^{*}, P^{*})\boldsymbol{n}\, \mathrm{d}\boldsymbol{\gamma}_{\boldsymbol{x}} = \boldsymbol{0}. \tag{26}$$

We can rewrite the linear system (25)–(26) in $(\boldsymbol{\ell}, \boldsymbol{\omega})$ in a matricial way. In order to do this, we remark that from the boundary condition of (20),

$$\left(\int_{\partial\mathcal{S}(\boldsymbol{a},\boldsymbol{Q})} \boldsymbol{\sigma}(\boldsymbol{u}^{(i)}, p^{(i)})\boldsymbol{n}\, \mathrm{d}\boldsymbol{\gamma}_{\boldsymbol{x}}\right) \cdot \boldsymbol{e}^{(j)} = \int_{\partial\mathcal{S}(\boldsymbol{a},\boldsymbol{Q})} \boldsymbol{\sigma}(\boldsymbol{u}^{(i)}, p^{(i)})\boldsymbol{n} \cdot \boldsymbol{u}^{(j)}\, \mathrm{d}\boldsymbol{\gamma}_{\boldsymbol{x}} = 2\nu \int_{\mathcal{F}(\boldsymbol{a},\boldsymbol{Q})} \boldsymbol{D}(\boldsymbol{u}^{(i)}) : \boldsymbol{D}(\boldsymbol{u}^{(j)})\, \mathrm{d}\boldsymbol{x} \qquad (i, j = 1, 2, 3).$$

Similarly,

$$\left(\int_{\partial\mathcal{S}(\boldsymbol{a},\boldsymbol{Q})} (\boldsymbol{x}-\boldsymbol{a}) \times \boldsymbol{\sigma}(\boldsymbol{u}^{(i)}, p^{(i)})\boldsymbol{n}\, \mathrm{d}\boldsymbol{\gamma}_{\boldsymbol{x}}\right) \cdot \boldsymbol{e}^{(j)} = \int_{\partial\mathcal{S}(\boldsymbol{a},\boldsymbol{Q})} \boldsymbol{\sigma}(\boldsymbol{u}^{(i)}, p^{(i)})\boldsymbol{n} \cdot \boldsymbol{U}^{(j)}\, \mathrm{d}\boldsymbol{\gamma}_{\boldsymbol{x}} = 2\nu \int_{\mathcal{F}(\boldsymbol{a},\boldsymbol{Q})} \boldsymbol{D}(\boldsymbol{u}^{(i)}) : \boldsymbol{D}(\boldsymbol{U}^{(j)})\, \mathrm{d}\boldsymbol{x} \qquad (i, j = 1, 2, 3).$$

These relations and their counterparts for $(\boldsymbol{U}^{(i)}, P^{(i)})$ and $(\boldsymbol{V}^{*}, P^{*})$ allow us to write (25)–(26) as

$$\boldsymbol{A}\begin{pmatrix}\boldsymbol{\ell}\\ \boldsymbol{\omega}\end{pmatrix} = \boldsymbol{b},$$

where $\boldsymbol{A} \in \mathcal{M}_6(\mathbb{R})$ is defined by

$$A_{ij} = \int_{\mathcal{F}(\boldsymbol{a},\boldsymbol{Q})} \boldsymbol{D}(\boldsymbol{u}^{(i)}) : \boldsymbol{D}(\boldsymbol{u}^{(j)})\, \mathrm{d}\boldsymbol{x}, \qquad 1 \leqslant i, j \leqslant 3 \tag{27}$$

$$A_{ij} = \int_{\mathcal{F}(\boldsymbol{a},\boldsymbol{Q})} \boldsymbol{D}(\boldsymbol{u}^{(i-3)}) : \boldsymbol{D}(\boldsymbol{U}^{(j)})\, \mathrm{d}\boldsymbol{x}, \qquad 4 \leqslant i \leqslant 6, \quad 1 \leqslant j \leqslant 3 \tag{28}$$

$$A_{ij} = \int_{\mathcal{F}(\boldsymbol{a},\boldsymbol{Q})} \boldsymbol{D}(\boldsymbol{U}^{(i)}) : \boldsymbol{D}(\boldsymbol{u}^{(j-3)})\, \mathrm{d}\boldsymbol{x}, \qquad 1 \leqslant i \leqslant 3, \quad 4 \leqslant j \leqslant 6 \tag{29}$$

$$A_{ij} = \int_{\mathcal{F}(\boldsymbol{a},\boldsymbol{Q})} \boldsymbol{D}(\boldsymbol{U}^{(i-3)}) : \boldsymbol{D}(\boldsymbol{U}^{(j-3)})\, \mathrm{d}\boldsymbol{x}, \qquad 4 \leqslant i, j \leqslant 6 \tag{30}$$

and $\boldsymbol{b} \in \mathbb{R}^6$ is defined by

$$b_j = -\int_{\mathcal{F}(\boldsymbol{a},\boldsymbol{Q})} \boldsymbol{D}(\boldsymbol{V}^{*}) : \boldsymbol{D}(u^{(j)})\, \mathrm{d}\boldsymbol{x}, \qquad 1 \leqslant j \leqslant 3$$

$$b_j = -\int_{\mathcal{F}(\boldsymbol{a},\boldsymbol{Q})} \boldsymbol{D}(\boldsymbol{V}^{*}) : \boldsymbol{D}(\boldsymbol{U}^{(j-3)})\, \mathrm{d}\boldsymbol{x}, \qquad 4 \leqslant j \leqslant 6.$$

Lemma 4. *The matrix* $\boldsymbol{A}$ *defined by (27)–(30) is invertible.*

Proof. Since $\boldsymbol{A}$ is a symmetric matrix, it suffices to check that $\boldsymbol{A}$ is positive definite, i.e.

$$\left\langle \boldsymbol{A}\begin{pmatrix}\boldsymbol{\ell}\\ \boldsymbol{\omega}\end{pmatrix}, \begin{pmatrix}\boldsymbol{\ell}\\ \boldsymbol{\omega}\end{pmatrix}\right\rangle > 0, \qquad \forall \begin{pmatrix}\boldsymbol{\ell}\\ \boldsymbol{\omega}\end{pmatrix} \neq \begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{0}\end{pmatrix}.$$

Here, $\langle\cdot,\cdot\rangle$ denotes the canonical scalar product in $\mathbb{R}^6$. A short calculation shows that

$$\left\langle \boldsymbol{A}\begin{pmatrix}\boldsymbol{\ell}\\ \boldsymbol{\omega}\end{pmatrix}, \begin{pmatrix}\boldsymbol{\ell}\\ \boldsymbol{\omega}\end{pmatrix}\right\rangle = \int_{\mathcal{F}(\boldsymbol{a},\boldsymbol{Q})} \left|\boldsymbol{D}\left(\sum \ell_i \boldsymbol{u}^{(i)} + \omega_i \boldsymbol{U}^{(i)}\right)\right|^2 \,\mathrm{d}\boldsymbol{x} \geqslant 0 \quad \left(\begin{pmatrix}\boldsymbol{\ell}\\ \boldsymbol{\omega}\end{pmatrix} \in \mathbb{R}^6\right).$$

Suppose that $\binom{\boldsymbol{\ell}}{\boldsymbol{\omega}} \in \mathbb{R}^6$ satisfies

$$\int_{\mathcal{F}(\boldsymbol{a},\boldsymbol{Q})} \left|\boldsymbol{D}\left(\sum \ell_i \boldsymbol{u}^{(i)} + \omega_i \boldsymbol{U}^{(i)}\right)\right|^2 \,\mathrm{d}\boldsymbol{x} = 0.$$

Then, from Korn's inequality (see, for instance, [25, theorem 2.4–2, page 51]), we deduce

$$\sum_{i=1}^{3} \ell_i \boldsymbol{u}^{(i)} + \omega_i \boldsymbol{U}^{(i)} = \boldsymbol{0} \quad \text{in } \mathcal{F}(\boldsymbol{a},\boldsymbol{Q}).$$

From the boundary conditions in (20) and (21), it follows $\boldsymbol{\ell} + \boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{a}) = \boldsymbol{0}$ for all $\boldsymbol{x} \in \partial\Omega$. Lemma 5 (see below) yields $\boldsymbol{\ell} = \boldsymbol{\omega} = \boldsymbol{0}$. □

Lemma 5. *Let* $\mathcal{O}$ *be a non-empty bounded open smooth subset of* $\mathbb{R}^3$*. Then*

$$\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2 \times \boldsymbol{y} = \boldsymbol{0} \qquad (\boldsymbol{y} \in \partial\mathcal{O}) \;\Rightarrow\; \boldsymbol{\kappa}_1 = \boldsymbol{\kappa}_2 = \boldsymbol{0}.$$

Thanks to lemma 4, $(\boldsymbol{u}, p, \boldsymbol{\ell}, \boldsymbol{\omega})$ satisfies (14)–(19) if and only if $(\boldsymbol{u}, p)$ are defined by (23), (24) and $(\boldsymbol{\ell}, \boldsymbol{\omega})$ is given by

$$\begin{pmatrix}\boldsymbol{\ell}\\ \boldsymbol{\omega}\end{pmatrix} = \boldsymbol{A}^{-1}\boldsymbol{b}. \tag{31}$$

This ensures the existence and uniqueness of a solution $(\boldsymbol{u}, p, \boldsymbol{\ell}, \boldsymbol{\omega})$ satisfying (14)–(19) and this ends the proof of proposition 3.

4. Proof of theorem 1

Using the auxiliary system (14)–(19) introduced in the previous section, we easily check that

$$(\boldsymbol{a}, \boldsymbol{Q}) \in \boldsymbol{C}^1([0, T_*); \mathbb{R}^3 \times SO(3)), \qquad (\boldsymbol{\ell}, \boldsymbol{\omega}) \in \boldsymbol{C}([0, T_*); \mathbb{R}^3 \times \mathbb{R}^3),$$

$$(\boldsymbol{u}, p) \in \boldsymbol{C}\left([0, T_*); \boldsymbol{H}^2(\mathcal{F}(\boldsymbol{a}(t), \boldsymbol{Q}(t))) \times H^1(\mathcal{F}(\boldsymbol{a}(t), \boldsymbol{Q}(t)))/\mathbb{R}\right)$$

is the solution of the system (2)–(11) if and only if for all $t \in (0, T)$, $(\boldsymbol{a}(t), \boldsymbol{Q}(t)) \in \mathcal{A}$,

$$(\boldsymbol{\ell}(t), \boldsymbol{\omega}(t), \boldsymbol{u}(t), p(t)) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \boldsymbol{H}^2(\mathcal{F}(\boldsymbol{a}(t), \boldsymbol{Q}(t))) \times H^1(\mathcal{F}(\boldsymbol{a}(t), \boldsymbol{Q}(t)))/\mathbb{R}$$

satisfies (14)–(19) and

$$\boldsymbol{a}' = \boldsymbol{\ell}, \qquad \boldsymbol{Q}' = \mathbb{S}(\boldsymbol{\omega})\boldsymbol{Q} \qquad \text{in } (0, T),$$
$$\boldsymbol{a}(0) = \boldsymbol{a}_0, \qquad \boldsymbol{Q}(0) = \boldsymbol{Q}_0.$$

Therefore, to prove theorem 1, it suffices to prove that the solution $\left(\boldsymbol{u}_{[\boldsymbol{a},\boldsymbol{Q}]}, p_{[\boldsymbol{a},\boldsymbol{Q}]}\boldsymbol{\ell}_{[\boldsymbol{a},\boldsymbol{Q}]}, \boldsymbol{\omega}_{[\boldsymbol{a},\boldsymbol{Q}]}\right)$ of (14)–(19) depends smoothly on $\boldsymbol{a}$ and $\boldsymbol{Q}$. More precisely, the following proposition and the Cauchy–Lipschitz–Picard theorem allow to conclude the proof of theorem 1.

Proposition 6. *The mapping*

$$\mathcal{T}: \quad \mathcal{A} \to \mathbb{R}^6$$
$$(\boldsymbol{a}, \boldsymbol{Q}) \mapsto \left(\boldsymbol{\ell}_{[a,Q]}, \boldsymbol{\omega}_{[a,Q]}\right)$$

is of class $\boldsymbol{C}^1$.

In order to prove proposition 6, we use the following classical result introduced by Simon (see [29]).

Assume that W is a Banach space, B and C are reflexive Banach spaces, and $\mathcal{W}$ is a non-empty open subset of W. We also consider $\boldsymbol{g}_1 : \mathcal{W} \times B \to C$, $\boldsymbol{g}_2 : \mathcal{W} \to B$ and $\boldsymbol{g}_3 : \mathcal{W} \to C$ such that for all $\boldsymbol{w} \in \mathcal{W}$,

$$\boldsymbol{g}_1(\boldsymbol{w}, \cdot) \in \boldsymbol{\mathcal{L}}(B, C), \qquad \boldsymbol{g}_1(\boldsymbol{w}, \boldsymbol{g}_2(\boldsymbol{w})) = \boldsymbol{g}_3(\boldsymbol{w}).$$

Then we have the following result.

Theorem 7 (Simon)**.** *Assume that* $\boldsymbol{w} \mapsto \boldsymbol{g}_1(\boldsymbol{w}, \cdot)$ *is* C^1 *at* $\boldsymbol{w}_0$ *into* $\boldsymbol{\mathcal{L}}(B, C)$*,* $\boldsymbol{g}_3$ *is* C^1 *at* $\boldsymbol{w}_0$ *and there exists* $\alpha > 0$ *such that*

$$\|\boldsymbol{g}_1(\boldsymbol{w}_0, \boldsymbol{x})\|_C \geqslant \alpha \, \|\boldsymbol{x}\|_B \quad \forall \boldsymbol{x} \in B.$$

Then $\boldsymbol{g}_2$ *is* C^1 *at* $\boldsymbol{w}_0$*.*

Remark 2. Thanks to (31), proposition 6 is reduced to prove that the mappings

$$(\boldsymbol{a}, \boldsymbol{Q}) \mapsto \boldsymbol{A}^{-1}(\boldsymbol{a}, \boldsymbol{Q}) \quad \text{and} \quad (\boldsymbol{a}, \boldsymbol{Q}) \mapsto \boldsymbol{b}(\boldsymbol{a}, \boldsymbol{Q})$$

are of class $\boldsymbol{C}^1$.

4.1. Change of variables

In order to apply theorem 7, we first need to consider a local chart of $\mathcal{A}$ around an arbitrary $(\boldsymbol{a}_0, \boldsymbol{Q}_0) \in \mathcal{A}$.

Let us consider the skew-symmetric matrices

$$\boldsymbol{A}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \boldsymbol{A}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{A}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and for any given matrix $\boldsymbol{R} \in SO_3(\mathbb{R})$, let us define the mapping

$$\boldsymbol{\Psi}_{\boldsymbol{R}} : \quad \mathcal{U} \to SO_3(\mathbb{R})$$
$$\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3) \mapsto \exp(\theta_1 \boldsymbol{A}_1) \exp(\theta_2 \boldsymbol{A}_2) \exp(\theta_3 \boldsymbol{A}_3) \boldsymbol{R}$$

with $\mathcal{U} = (-\pi, \pi) \times (-\pi/2, \pi/2) \times (-\pi, \pi)$. It is easy to check that $\boldsymbol{\Psi}_{\boldsymbol{R}}$ is an infinitely differentiable diffeomorphism from $\mathcal{U}$ onto a neighbourhood of $\boldsymbol{R} \in SO_3(\mathbb{R})$ (see, for example, [15, pages 150 and 603] and [24], for more details).

Let us fix an arbitrary $(\boldsymbol{a}_0, \boldsymbol{Q}_0) \in \mathcal{A}$ and let us consider the following $\boldsymbol{C}^\infty$-diffeomorphism:

$$\boldsymbol{\Phi}_{(\boldsymbol{a}_0, \boldsymbol{Q}_0)} : \mathbb{R}^3 \times \mathcal{U} \to \mathbb{R}^3 \times SO_3(\mathbb{R})$$
$$(\boldsymbol{h}, \boldsymbol{\theta}) \mapsto \left(\boldsymbol{a}_0 + \boldsymbol{h}, \boldsymbol{\Psi}_{\boldsymbol{Q}_0}(\boldsymbol{\theta})\right).$$

It satisfies

$$\boldsymbol{\Phi}_{(\boldsymbol{a}_0, \boldsymbol{Q}_0)}(\boldsymbol{0}, \boldsymbol{0}) = (\boldsymbol{a}_0, \boldsymbol{Q}_0)$$

and there exists $r > 0$ such that $B_{\mathbb{R}^6}(\boldsymbol{0}, r) \subset \mathbb{R}^3 \times \mathcal{U}$ and

$$\boldsymbol{\Phi}_{(\boldsymbol{a}_0, \boldsymbol{Q}_0)} \left(B_{\mathbb{R}^6}(\boldsymbol{0}, r)\right) \subset \mathcal{A}. \tag{32}$$

Using the local chart introduced above, the proof of proposition 6 is reduced to prove that the mapping

$$\begin{aligned} \widetilde{\mathcal{T}}_{(a_0,Q_0)} &: B_{\mathbb{R}^6}(\mathbf{0}, r) \to \mathbb{R}^6 \\ (h, \theta) &\mapsto (\ell_{[\Phi_{(a_0,Q_0)}(h,\theta)]}, \omega_{[\Phi_{(a_0,Q_0)}(h,\theta)]}) \end{aligned} \tag{33}$$

is C^1 in $(\mathbf{0}, \mathbf{0})$.

To prove this, we combine theorem 7 with a change of variables. More precisely, let us construct a mapping $X : \Omega \to \Omega$ which transforms $\mathcal{S}(a_0, Q_0)$ onto $\mathcal{S}(\Phi_{(a_0,Q_0)}(h, \theta))$ and $\mathcal{F}(a_0, Q_0)$ onto $\mathcal{F}(\Phi_{(a_0,Q_0)}(h, \theta))$. We use this change of variables to transform systems (20), (21) and (22) into systems written in fixed domains. To construct our change of variables, we start with the mapping $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\begin{aligned} \varphi(h, \theta; y) &= \Psi_{Q_0}(\theta) Q_0^{-1}(y - a_0) + (a_0 + h) \\ &= \Psi_{\mathbf{Id}}(\theta)(y - a_0) + (a_0 + h) \\ &= \exp(\theta_1 A_1) \exp(\theta_2 A_2) \exp(\theta_3 A_3)(y - a_0) + (a_0 + h). \end{aligned}$$

It is easy to check that $\varphi(h, \theta; \cdot)$ maps $\mathcal{S}(a_0, Q_0)$ onto $\mathcal{S}(\Phi_{(a_0,Q_0)}(h, \theta))$ and, from the regularity of $\Phi_{(a_0,Q_0)}$, we deduce that $(h, \theta) \mapsto \varphi(h, \theta; \cdot)$ is a C^∞-mapping from $B_{\mathbb{R}^6}(\mathbf{0}, r)$ into $C^k(\mathbb{R}^3)$, for all $k \geqslant 0$.

We fix two open subsets $\mathcal{O}_1$ and $\mathcal{O}_2$ of Ω such that

$$\overline{\mathcal{S}(a_0, Q_0)} \subset \mathcal{O}_1, \qquad \overline{\mathcal{O}_1} \subset \mathcal{O}_2, \qquad \overline{\mathcal{O}_2} \subset \Omega$$

and we consider a function $\mathcal{Z} \in C_c^\infty(\Omega)$,

$$\mathcal{Z} \equiv 1 \text{ in } \overline{\mathcal{O}_1}, \qquad \mathcal{Z} \equiv 0 \text{ in } \Omega \setminus \mathcal{O}_2.$$

Then we set for $y \in \mathbb{R}^3$

$$X(h, \theta; y) := y + (\varphi(h, \theta; y) - y)\, \mathcal{Z}(y). \tag{34}$$

For all $y \in \mathbb{R}^3$, we have

$$\begin{aligned} |X(h, \theta; y) - y| &= |\, (\varphi(h, \theta; y) - y)\, \mathcal{Z}(y)| \\ &= |((\Psi_{\mathbf{Id}}(\theta) - \mathbf{Id})(y - a_0) + h)\, \mathcal{Z}(y)| \\ &\leqslant C(\Omega)|(h, \theta)|. \end{aligned}$$

Moreover, for all $y \in \mathbb{R}^3$

$$|\nabla_y(X(h, \theta; y)) - \mathbf{Id}| \leqslant C(\Omega)|(h, \theta)|,$$

where **Id** is the identity matrix of $\mathcal{M}_3(\mathbb{R})$ and where we have denoted by $|\cdot|$ the euclidean norm of $\mathbb{R}^k$ for $k = 3$ or 6.

In particular, for r small enough, for all $(h, \theta) \in B_{\mathbb{R}^6}(\mathbf{0}, r)$, $X(h, \theta; \cdot)$ is a C^∞-diffeomorphism from $\overline{\Omega}$ onto $\overline{\Omega}$ such that

$$X(h, \theta; \mathcal{S}(a_0, Q_0)) = \mathcal{S}(\Phi_{(a_0,Q_0)}(h, \theta)).$$

Furthermore, the mapping

$$\begin{aligned} B_{\mathbb{R}^6}(\mathbf{0}, r) &\to C^k(\overline{\Omega}) \\ (h, \theta) &\mapsto X(h, \theta; \cdot) \end{aligned}$$

is of class C^∞ for all $k \geqslant 0$.

It is well known that

$$X \mapsto X^{-1}$$

is a C^∞ mapping from the C^k-diffeomorphisms of $\overline{\Omega}$ onto itself.

Consequently, if we denote for all $(\boldsymbol{h}, \boldsymbol{\theta}) \in B_{\mathbb{R}^6}(\mathbf{0}, r)$, $\boldsymbol{Y}(\boldsymbol{h}, \boldsymbol{\theta}; \cdot)$ the inverse of $\boldsymbol{X}(\boldsymbol{h}, \boldsymbol{\theta}; \cdot)$, then

$$\begin{aligned} &B_{\mathbb{R}^6}(\mathbf{0}, r) \to \boldsymbol{C}^k(\overline{\Omega}) \\ &(\boldsymbol{h}, \boldsymbol{\theta}) \mapsto \boldsymbol{Y}(\boldsymbol{h}, \boldsymbol{\theta}; \cdot) \end{aligned}$$

is also of class $\boldsymbol{C}^\infty$ for all $k \geqslant 0$.

Now, we use the change of variables constructed above to write the systems of equations (20), (21) and (22) in fixed domains. In fact, we only detail this transformation for systems (20); the calculation is similar for systems (21) and (22).

Since $\boldsymbol{e}^{(i)} \in \boldsymbol{H}^{3/2}(\partial\Omega)$ satisfies

$$\int_{\partial\Omega} \boldsymbol{e}^{(i)} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\gamma} = 0,$$

there exists $\boldsymbol{\Lambda}^{(i)} \in \boldsymbol{H}^2(\Omega)$ (see, for instance [5]) such that

(1) $\boldsymbol{\Lambda}^{(i)} = -\boldsymbol{e}^{(i)}$ on $\partial\Omega$,
(2) $\mathrm{div}\boldsymbol{\Lambda}^{(i)} = 0$ in Ω,
(3) $\boldsymbol{\Lambda}^{(i)}(\boldsymbol{x}) = \mathbf{0}$ if $\boldsymbol{x} \in [\Omega]_\epsilon := \{z \in \Omega : \mathrm{dist}\,(z, \partial\Omega) \geqslant \epsilon\}$,

for some $\epsilon > 0$ such that $\mathcal{O}_2 \subset [\Omega]_\epsilon$.

Let us set

$$\widetilde{\boldsymbol{u}}^{(i)} := \boldsymbol{u}^{(i)} - \boldsymbol{e}^{(i)} - \boldsymbol{\Lambda}^{(i)}; \tag{35}$$

then $(\boldsymbol{u}^{(i)}, p^{(i)})$ satisfies (20) if and only if $(\widetilde{\boldsymbol{u}}^{(i)}, p^{(i)})$ satisfies

$$\begin{cases} -\mathrm{div}(\boldsymbol{\sigma}(\widetilde{\boldsymbol{u}}^{(i)}, p^{(i)})) = -\nu\Delta\boldsymbol{\Lambda}^{(i)} & \text{in } \mathcal{F}(\boldsymbol{\Phi}_{(\boldsymbol{a}_0, \boldsymbol{Q}_0)}(\boldsymbol{h}, \boldsymbol{\theta})) \\ \mathrm{div}(\widetilde{\boldsymbol{u}}^{(i)}) = 0 & \text{in } \mathcal{F}(\boldsymbol{\Phi}_{(\boldsymbol{a}_0, \boldsymbol{Q}_0)}(\boldsymbol{h}, \boldsymbol{\theta})) \\ \widetilde{\boldsymbol{u}}^{(i)} = \mathbf{0} & \text{on } \partial\mathcal{S}(\boldsymbol{\Phi}_{(\boldsymbol{a}_0, \boldsymbol{Q}_0)}(\boldsymbol{h}, \boldsymbol{\theta})) \\ \widetilde{\boldsymbol{u}}^{(i)} = \mathbf{0} & \text{on } \partial\Omega. \end{cases} \tag{36}$$

We set

$$\boldsymbol{v}^{(i)}(\boldsymbol{h}, \boldsymbol{\theta}; \boldsymbol{y}) = \det(\nabla\boldsymbol{X}(\boldsymbol{h}, \boldsymbol{\theta}; \boldsymbol{y}))\ (\nabla\boldsymbol{X}(\boldsymbol{h}, \boldsymbol{\theta}; \boldsymbol{y}))^{-1}\ \widetilde{\boldsymbol{u}}^{(i)}(\boldsymbol{h}, \boldsymbol{\theta}; \boldsymbol{X}(\boldsymbol{h}, \boldsymbol{\theta}; \boldsymbol{y})), \tag{37}$$

$$q^{(i)}(\boldsymbol{h}, \boldsymbol{\theta}; \boldsymbol{y}) = \det(\nabla\boldsymbol{X}(\boldsymbol{h}, \boldsymbol{\theta}; \boldsymbol{y}))p^{(i)}(\boldsymbol{h}, \boldsymbol{\theta}; \boldsymbol{X}(\boldsymbol{h}, \boldsymbol{\theta}; \boldsymbol{y})). \tag{38}$$

Let us remark that we do not use the change of variables

$$\boldsymbol{v}^{(i)}(\boldsymbol{h}, \boldsymbol{\theta}; \boldsymbol{y}) = \widetilde{\boldsymbol{u}}^{(i)}(\boldsymbol{h}, \boldsymbol{\theta}; \boldsymbol{X}(\boldsymbol{h}, \boldsymbol{\theta}; \boldsymbol{y}))$$

because of the divergence equation in (20). More precisely, we have the following result.

Lemma 8. *Assume $\boldsymbol{v}^{(i)}$ is defined by (37). Then*

$$(\mathrm{div}\,\boldsymbol{v}^{(i)})(\boldsymbol{h}, \boldsymbol{\theta}; \boldsymbol{y}) = \det(\nabla\boldsymbol{X}(\boldsymbol{h}, \boldsymbol{\theta}; \boldsymbol{y}))(\mathrm{div}\,\widetilde{\boldsymbol{u}}^{(i)})(\boldsymbol{h}, \boldsymbol{\theta}; \boldsymbol{X}(\boldsymbol{h}, \boldsymbol{\theta}; \boldsymbol{y})).$$

The proof can be found in [4] (see lemma 3.1).

Now, we calculate the transformation of the gradient of $\widetilde{\boldsymbol{u}}^{(i)}$. Here, we do not write the dependence on the variables $\boldsymbol{h}$ and $\boldsymbol{\theta}$:

$$\frac{\partial \widetilde{u}_m^{(i)}}{\partial x_j} = (\det \nabla\boldsymbol{Y})\frac{\partial v_m^{(i)}}{\partial y_j}(\boldsymbol{Y}) + E_{mj}[\boldsymbol{v}^{(i)}] \tag{39}$$

with

$$E_{mj}\left[\boldsymbol{v}^{(i)}\right] = \sum_{k=1}^{3}\left[\frac{\partial}{\partial x_j}\left((\det\nabla\boldsymbol{Y})\frac{\partial X_m}{\partial y_k}(\boldsymbol{Y})\right)v_k^{(i)}(\boldsymbol{Y}) + \det\nabla\boldsymbol{Y}\left(\frac{\partial X_m}{\partial y_k}(\boldsymbol{Y}) - \delta_{mk}\right)\frac{\partial v_k^{(i)}}{\partial y_j}(\boldsymbol{Y})\right.$$
$$\left. + (\det\nabla\boldsymbol{Y})\frac{\partial X_m}{\partial y_k}(\boldsymbol{Y})\sum_{l=1}^{3}\frac{\partial v_k^{(i)}}{\partial y_l}(\boldsymbol{Y})([(\nabla\boldsymbol{X})^{-1}]_{lj}(\boldsymbol{Y}) - \delta_{lj})\right]. \tag{40}$$

Then,

$$\frac{\partial^2\widetilde{u}_m^{(i)}}{\partial x_j^2} = (\det\nabla\boldsymbol{Y})\frac{\partial^2 v_m^{(i)}}{\partial y_j^2}(\boldsymbol{Y}) + (\det\nabla\boldsymbol{Y})\sum_{l=1}^{3}\frac{\partial^2 v_m^{(i)}}{\partial y_j\partial y_l}(\boldsymbol{Y})([(\nabla\boldsymbol{X})^{-1}]_{lj}(\boldsymbol{Y}) - \delta_{lj})$$
$$+\frac{\partial}{\partial x_j}(\det\nabla\boldsymbol{Y})\frac{\partial v_m^{(i)}}{\partial y_j}(\boldsymbol{Y}) + \frac{\partial}{\partial x_j}(E_{mj}[\boldsymbol{v}^{(i)}]). \tag{41}$$

On the other hand, from (38) we have

$$\frac{\partial p^{(i)}}{\partial x_m} = \frac{\partial}{\partial x_m}(\det\nabla\boldsymbol{Y})\,q^{(i)}(\boldsymbol{Y}) + (\det\nabla\boldsymbol{Y})\frac{\partial q^{(i)}}{\partial y_m}(\boldsymbol{Y})$$
$$+ (\det\nabla\boldsymbol{Y})\sum_{l=1}^{3}\frac{\partial q^{(i)}}{\partial y_l}(\boldsymbol{Y})([(\nabla\boldsymbol{X})^{-1}]_{lm}(\boldsymbol{Y}) - \delta_{lm}). \tag{42}$$

Thereby, $(\boldsymbol{u}^{(i)}, p^{(i)})$ satisfies (20) if and only if $(\boldsymbol{v}^{(i)}, q^{(i)})$ satisfies

$$\begin{cases} -\nu[\boldsymbol{L}\boldsymbol{v}^{(i)}] + [\boldsymbol{G}q^{(i)}] = -\nu\Delta\boldsymbol{\Lambda}^{(i)} & \text{in } \mathcal{F}(\boldsymbol{a}_0, \boldsymbol{Q}_0)\\ \operatorname{div}(\boldsymbol{v}^{(i)}) = 0 & \text{in } \mathcal{F}(\boldsymbol{a}_0, \boldsymbol{Q}_0)\\ \boldsymbol{v}^{(i)} = \boldsymbol{0} & \text{on } \partial\mathcal{S}(\boldsymbol{a}_0, \boldsymbol{Q}_0)\\ \boldsymbol{v}^{(i)} = \boldsymbol{0} & \text{on } \partial\Omega \end{cases} \tag{43}$$

with

$$[\boldsymbol{L}\boldsymbol{v}^{(i)}]_m := \sum_{j=1}^{3}\left[\frac{\partial^2 v_m^{(i)}}{\partial y_j^2} + \sum_{l=1}^{3}\frac{\partial^2 v_m^{(i)}}{\partial y_j\partial y_l}([(\nabla\boldsymbol{X})^{-1}]_{lj} - \delta_{lj})\right.$$
$$\left. + (\det\nabla\boldsymbol{X})\frac{\partial}{\partial x_j}(\det\nabla\boldsymbol{Y})(\boldsymbol{X})\frac{\partial v_m^{(i)}}{\partial y_j} + (\det\nabla\boldsymbol{X})\frac{\partial}{\partial x_j}(E_{mj}[\boldsymbol{v}^{(i)}])(\boldsymbol{X})\right], \tag{44}$$

and

$$[\boldsymbol{G}q^{(i)}]_m := (\det\nabla\boldsymbol{X})\frac{\partial}{\partial x_m}(\det\nabla\boldsymbol{Y})(\boldsymbol{X})q^{(i)} + \frac{\partial q^{(i)}}{\partial y_m} + \sum_{l=1}^{3}\frac{\partial q^{(i)}}{\partial y_l}([(\nabla\boldsymbol{X})^{-1}]_{lm} - \delta_{lm}). \tag{45}$$

Let us remark that the right-hand side of the first equation of (43) is the same as in (36) since in $\Omega\setminus\mathcal{O}_2$, $\boldsymbol{X}(\boldsymbol{h}, \boldsymbol{\theta}; \cdot) = \mathbf{id}$ (see (34)) and by definition, the support of $\boldsymbol{\Lambda}^{(i)}$ is included in $\Omega\setminus\mathcal{O}_2$ (see property 3). In particular, this right-hand side is independent of $\boldsymbol{h}$ and $\boldsymbol{\theta}$.

4.2. Proof of proposition 6

We are now in position to prove proposition 6. We recall that proposition 6 yields theorem 1.

Proof of proposition 6. We apply theorem 7; let us take

$$W = \mathbb{R}^3\times\mathbb{R}^3,\quad \mathcal{W} = B_{\mathbb{R}^6}(\boldsymbol{0}, r),$$
$$\boldsymbol{B} = \left(\boldsymbol{H}^2(\mathcal{F}(\boldsymbol{a}_0, \boldsymbol{Q}_0))\cap\boldsymbol{H}^1_{\boldsymbol{\sigma}}(\mathcal{F}(\boldsymbol{a}_0, \boldsymbol{Q}_0))\right)\times H^1(\mathcal{F}(\boldsymbol{a}_0, \boldsymbol{Q}_0))/\mathbb{R},$$
$$\boldsymbol{C} = \boldsymbol{L}^2(\mathcal{F}(\boldsymbol{a}_0, \boldsymbol{Q}_0)),$$

where

$$\boldsymbol{H}^1_{\boldsymbol{\sigma}}(\mathcal{F}(\boldsymbol{a}_0,\boldsymbol{Q}_0)) = \left\{\boldsymbol{w} \in \boldsymbol{H}^1_0(\mathcal{F}(\boldsymbol{a}_0,\boldsymbol{Q}_0)) : \operatorname{div}(\boldsymbol{w}) = 0\right\}$$

and r is small enough (see (32) and the construction of $\boldsymbol{X}$).

We also set

$$\begin{aligned} &\boldsymbol{g}_1 : \mathcal{W} \times \boldsymbol{B} \to \boldsymbol{C} \\ &(\boldsymbol{h},\boldsymbol{\theta},\boldsymbol{v},q) \mapsto -\nu\,[\boldsymbol{L}(\boldsymbol{h},\boldsymbol{\theta})\boldsymbol{v}] + [\boldsymbol{G}(\boldsymbol{h},\boldsymbol{\theta})q]\,, \\ &\boldsymbol{g}_2 : \mathcal{W} \to \boldsymbol{B} \\ &(\boldsymbol{h},\boldsymbol{\theta}) \mapsto \left(\boldsymbol{v}^{(i)}(\boldsymbol{h},\boldsymbol{\theta}), q^{(i)}(\boldsymbol{h},\boldsymbol{\theta})\right), \end{aligned}$$

and

$$\begin{aligned} &\boldsymbol{g}_3 : \mathcal{W} \to \boldsymbol{C} \\ &(\boldsymbol{h},\boldsymbol{\theta}) \mapsto -\nu\Delta\boldsymbol{\Lambda}^{(i)}. \end{aligned}$$

Thus thanks to the regularity of the mappings $\boldsymbol{X}$ and $\boldsymbol{Y}$ we have that $(\boldsymbol{h},\boldsymbol{\theta}) \mapsto \boldsymbol{g}_1(\boldsymbol{h},\boldsymbol{\theta};\cdot)$ is $\boldsymbol{C}^1$ at $(\boldsymbol{0},\boldsymbol{0})$ into $\boldsymbol{\mathcal{L}}(\boldsymbol{B},\boldsymbol{C})$. Moreover, since $\boldsymbol{g}_3(\boldsymbol{h},\boldsymbol{\theta})$ does not depend on $(\boldsymbol{h},\boldsymbol{\theta})$, we deduce that $\boldsymbol{g}_3$ is $\boldsymbol{C}^1$ at $(\boldsymbol{0},\boldsymbol{0})$. Lastly, if

$$\boldsymbol{g}_1(\boldsymbol{0},\boldsymbol{0};(\boldsymbol{u},p)) = -\nu\Delta\boldsymbol{u} + \nabla p, \quad \forall(\boldsymbol{u},p) \in \boldsymbol{B},$$

then, thanks to the ellipticity regularity for Stokes systems (see, for instance, [5]), we have

$$\|\boldsymbol{g}_1(\boldsymbol{0},\boldsymbol{0};(\boldsymbol{u},p))\|_{\boldsymbol{C}} \geqslant K_0\|(\boldsymbol{u},p)\|_{\boldsymbol{B}},$$

where the constant K_0 depends on ν and the geometry of the domain.

Therefore, applying theorem 7 we conclude that the mapping

$$\begin{aligned} &\boldsymbol{g}_2 : B_{\mathbb{R}^6}(\boldsymbol{0},r) \to \left(\boldsymbol{H}^2(\mathcal{F}(\boldsymbol{a}_0,\boldsymbol{Q}_0)) \cap \boldsymbol{H}^1_{\boldsymbol{\sigma}}(\mathcal{F}(\boldsymbol{a}_0,\boldsymbol{Q}_0))\right) \times H^1(\mathcal{F}(\boldsymbol{a}_0,\boldsymbol{Q}_0))/\mathbb{R} \\ &\quad (\boldsymbol{h},\boldsymbol{\theta}) \mapsto (\boldsymbol{v}^{(i)}, q^{(i)}) \end{aligned}$$

is $\boldsymbol{C}^1$ in $(\boldsymbol{0},\boldsymbol{0})$.

From the definition of $\widetilde{\boldsymbol{u}}^{(i)}$ (see (35)), the theorem of change of variables and (39)–(40), we can rewrite (27) as follows:

$$\begin{aligned} A_{ij} &= \int_{\mathcal{F}(\boldsymbol{a}_0,\boldsymbol{Q}_0)} [\boldsymbol{D}(\widetilde{\boldsymbol{u}}^{(i)} + \boldsymbol{\Lambda}^{(i)})(\boldsymbol{X})] : [\boldsymbol{D}(\widetilde{\boldsymbol{u}}^{(j)} + \boldsymbol{\Lambda}^{(j)})(\boldsymbol{X})](\det\nabla\boldsymbol{X})\,\mathrm{d}\boldsymbol{y} \\ &= \int_{\mathcal{F}(\boldsymbol{a}_0,\boldsymbol{Q}_0)} \boldsymbol{T}^{(i)}(\boldsymbol{v}^{(i)},\boldsymbol{\Lambda}^{(i)},\boldsymbol{X},\boldsymbol{Y}) : \boldsymbol{T}^{(j)}(\boldsymbol{v}^{(j)},\boldsymbol{\Lambda}^{(j)},\boldsymbol{X},\boldsymbol{Y})(\det\nabla\boldsymbol{X})\,\mathrm{d}\boldsymbol{y}, \end{aligned}$$

where $\boldsymbol{T}^{(i)}$ is given by

$$\boldsymbol{T}^{(i)}(\boldsymbol{v}^{(i)},\boldsymbol{\Lambda}^{(i)},\boldsymbol{X},\boldsymbol{Y}) := \frac{1}{\det\nabla\boldsymbol{X}}\boldsymbol{D}(\boldsymbol{v}^{(i)}) + \frac{1}{2}(\boldsymbol{E}[\boldsymbol{v}^{(i)}] + \boldsymbol{E}[\boldsymbol{v}^{(i)}]^t)(\boldsymbol{X}) + \boldsymbol{D}(\boldsymbol{\Lambda}^{(i)}).$$

This proves that for $1 \leqslant i, j \leqslant 3$, the mappings

$$\begin{aligned} &B_{\mathbb{R}^6}(\boldsymbol{0},r) \to \mathbb{R} \\ &(\boldsymbol{h},\boldsymbol{\theta}) \mapsto A_{ij} = A_{ij}(\boldsymbol{\Phi}_{(\boldsymbol{a}_0,\boldsymbol{Q}_0)}(\boldsymbol{h},\boldsymbol{\theta})) \end{aligned}$$

are $\boldsymbol{C}^1$ in $(\boldsymbol{0},\boldsymbol{0})$. By similar calculations we obtain that for all $1 \leqslant i, j \leqslant 6$, the previous mappings are $\boldsymbol{C}^1$ in $(\boldsymbol{0},\boldsymbol{0})$. Likewise, for all $1 \leqslant j \leqslant 6$, the mappings

$$\begin{aligned} &B_{\mathbb{R}^6}(\boldsymbol{0},r) \to \mathbb{R} \\ &(\boldsymbol{h},\boldsymbol{\theta}) \mapsto b_j = b_j\left(\boldsymbol{\Phi}_{(\boldsymbol{a}_0,\boldsymbol{Q}_0)}(\boldsymbol{h},\boldsymbol{\theta})\right) \end{aligned}$$

are $\boldsymbol{C}^1$ in $(\boldsymbol{0},\boldsymbol{0})$.

Since the mapping

$$\begin{aligned} Gl_3(\mathbb{R}) &\to Gl_3(\mathbb{R}) \\ \boldsymbol{A} &\mapsto \boldsymbol{A}^{-1} \end{aligned}$$

is of class $\boldsymbol{C}^1$ and thanks to remark 2, we deduce that the mapping $\widetilde{\boldsymbol{\mathcal{T}}}$ defined by (33) is $\boldsymbol{C}^1$ in $(\mathbf{0}, \mathbf{0})$. Here, $Gl_3(\mathbb{R})$ denotes the set of all real invertible matrices of order 3. This concludes the proof of proposition 6. □

5. Proof of theorem 2

First, we recall a key result. We prove it for the sake of completeness.

Proposition 9. *Assume $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ are two convex smooth non-empty open sets such that*

$$\overline{\mathcal{S}^{(1)}} \subset \Omega \qquad \textit{and} \qquad \overline{\mathcal{S}^{(2)}} \subset \Omega. \tag{46}$$

Suppose $\boldsymbol{u}_* \in \boldsymbol{H}^{3/2}(\partial\Omega)$ satisfies (12) and consider

$$(\boldsymbol{u}^{(1)}, p^{(1)}) \in \boldsymbol{H}^2(\Omega \setminus \overline{\mathcal{S}^{(1)}}) \times H^1(\Omega \setminus \overline{\mathcal{S}^{(1)}})/\mathbb{R},$$
$$(\boldsymbol{u}^{(2)}, p^{(2)}) \in \boldsymbol{H}^2(\Omega \setminus \overline{\mathcal{S}^{(2)}}) \times H^1(\Omega \setminus \overline{\mathcal{S}^{(2)}})/\mathbb{R}$$

satisfying

$$\begin{cases} -\mathrm{div}\, \boldsymbol{\sigma}\left(\boldsymbol{u}^{(i)}, p^{(i)}\right) = \mathbf{0} & \text{in } \Omega \setminus \overline{\mathcal{S}^{(i)}} \\ \mathrm{div}\left(\boldsymbol{u}^{(i)}\right) = 0 & \text{in } \Omega \setminus \overline{\mathcal{S}^{(i)}} \\ \boldsymbol{u}^{(i)} = \boldsymbol{u}_* & \text{on } \partial\Omega. \end{cases} \qquad (i = 1, 2) \tag{47}$$

If Γ is a non-empty open subset of $\partial\Omega$ and

$$\boldsymbol{\sigma}(\boldsymbol{u}^{(1)}, p^{(1)})\boldsymbol{n}_{|\Gamma} = \boldsymbol{\sigma}(\boldsymbol{u}^{(2)}, p^{(2)})\boldsymbol{n}_{|\Gamma}, \tag{48}$$

then

$$\boldsymbol{u}^{(1)} \equiv \boldsymbol{u}^{(2)} \quad \textit{in } \Omega \setminus (\overline{\mathcal{S}^{(1)}} \cup \overline{\mathcal{S}^{(2)}}). \tag{49}$$

Proof. We write

$$\begin{aligned} \boldsymbol{u} &:= \boldsymbol{u}^{(1)} - \boldsymbol{u}^{(2)} \\ p &:= p^{(1)} - p^{(2)}. \end{aligned}$$

Combining (47) and (48), we deduce that $(\boldsymbol{u}, p)$ satisfies

$$\begin{cases} -\mathrm{div}(\boldsymbol{\sigma}(\boldsymbol{u}, p)) = \mathbf{0} & \text{in } \Omega \setminus (\overline{\mathcal{S}^{(1)}} \cup \overline{\mathcal{S}^{(2)}}) \\ \mathrm{div}(\boldsymbol{u}) = 0 & \text{in } \Omega \setminus (\overline{\mathcal{S}^{(1)}} \cup \overline{\mathcal{S}^{(2)}}) \\ \boldsymbol{u} = \mathbf{0} & \text{on } \partial\Omega \\ \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n} = \mathbf{0} & \text{on } \Gamma. \end{cases}$$

Applying the unique continuation property for the Stokes equations due to Fabre and Lebeau [12], (see also [1]), we deduce $\boldsymbol{u} \equiv \mathbf{0}$ in the connected component of $\partial\Omega$. Since Ω is connected

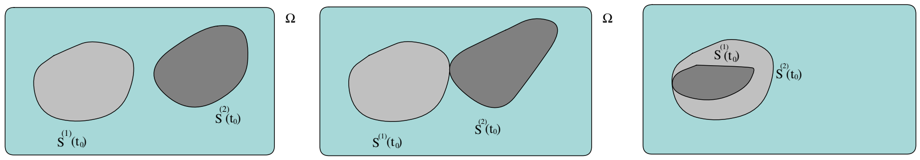

Figure 2. The intersection of the boundaries is contained in a straight line.

and $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ are convex and satisfy (46), we have that $\Omega \setminus (\overline{\mathcal{S}^{(1)}} \cup \overline{\mathcal{S}^{(2)}})$ is connected and thus

$$\boldsymbol{u}^{(1)} = \boldsymbol{u}^{(2)} \quad \text{in } \Omega \setminus (\overline{\mathcal{S}^{(1)}} \cup \overline{\mathcal{S}^{(2)}}). \qquad \square$$

Proof of theorem 2. In order to prove theorem 2, we reason by contradiction and we assume that there exists $0 < t_0 < \min(T_*^{(1)}, T_*^{(2)})$, such that

$$\boldsymbol{\sigma}(\boldsymbol{u}^{(1)}(t_0), p^{(1)}(t_0))\, \boldsymbol{n}_{|\Gamma} = \boldsymbol{\sigma}(\boldsymbol{u}^{(2)}(t_0), p^{(2)}(t_0))\, \boldsymbol{n}_{|\Gamma}$$

and $\mathcal{S}^{(1)}(t_0) \neq \mathcal{S}^{(2)}(t_0)$.

In that case, since $\mathcal{S}^{(1)}(t_0)$ and $\mathcal{S}^{(2)}(t_0)$ are convex sets, we have

- $\partial\mathcal{S}^{(1)}(t_0) \cap \partial\mathcal{S}^{(2)}(t_0)$ is included in a line,
- $\partial\mathcal{S}^{(1)}(t_0) \cap \partial\mathcal{S}^{(2)}(t_0)$ contains three noncollinear points.

The first case can be split into the three following subcases (see figure 2):

$$\mathcal{S}^{(1)}(t_0) \cap \mathcal{S}^{(2)}(t_0) = \emptyset \quad \text{or} \quad \mathcal{S}^{(1)}(t_0) \subsetneq \mathcal{S}^{(2)}(t_0) \quad \text{or} \quad \mathcal{S}^{(2)}(t_0) \subsetneq \mathcal{S}^{(1)}(t_0).$$

We will show that neither of these four cases are possible.

Case 1.1. $\mathcal{S}^{(1)}(t_0) \cap \mathcal{S}^{(2)}(t_0) = \emptyset$. Then, from (49), we deduce that in $\mathcal{S}^{(2)}(t_0) \subset \mathcal{F}^{(1)}(t_0)$, we have

$$\begin{cases} -\mathrm{div}(\boldsymbol{\sigma}(\boldsymbol{u}^{(1)}(t_0), p^{(1)}(t_0))) = \boldsymbol{0} & \text{in } \mathcal{S}^{(2)}(t_0) \\ \mathrm{div}\left(\boldsymbol{u}^{(1)}(t_0)\right) = 0 & \text{in } \mathcal{S}^{(2)}(t_0) \\ \boldsymbol{u}^{(1)}(t_0) = \boldsymbol{\ell}^{(2)}(t_0) + \boldsymbol{\omega}^{(2)}(t_0) \times \left(\boldsymbol{x} - \boldsymbol{a}^{(2)}(t_0)\right) & \text{on } \partial\mathcal{S}^{(2)}(t_0). \end{cases}$$

In particular,

$$\boldsymbol{v} = \boldsymbol{u}^{(1)}(t_0) - (\boldsymbol{\ell}^{(2)}(t_0) + \boldsymbol{\omega}^{(2)}(t_0) \times (\boldsymbol{x} - \boldsymbol{a}^{(2)}(t_0)))$$

satisfies the following Stokes system:

$$\begin{cases} -\mathrm{div}(\boldsymbol{\sigma}(\boldsymbol{v}, p^{(1)}(t_0))) = \boldsymbol{0} & \text{in } \mathcal{S}^{(2)}(t_0) \\ \mathrm{div}\,(\boldsymbol{v}) = 0 & \text{in } \mathcal{S}^{(2)}(t_0) \\ \boldsymbol{v} = \boldsymbol{0} & \text{on } \partial\mathcal{S}^{(2)}(t_0). \end{cases}$$

Multiplying by $\boldsymbol{v}$ the first equation of the above system, we deduce

$$\int_{\mathcal{S}^{(2)}(t_0)} |\boldsymbol{D}(\boldsymbol{v})|^2 \, \mathrm{d}\boldsymbol{x} = 0$$

and thus $\boldsymbol{v} = \boldsymbol{0}$ in $\mathcal{S}^{(2)}(t_0)$. Consequently, since $\boldsymbol{v}$ satisfies the Stokes system in $\mathcal{F}^{(1)}(t_0)$, we can apply again the result of Fabre and Lebeau [12] and we obtain

$$\boldsymbol{v} \equiv \boldsymbol{0} \quad \text{in } \mathcal{F}^{(1)}(t_0).$$

This yields that

$$\boldsymbol{u}_*(\boldsymbol{x}) = \boldsymbol{\ell}^{(2)}(t_0) + \boldsymbol{\omega}^{(2)}(t_0) \times (\boldsymbol{x} - \boldsymbol{a}^{(2)}(t_0)) \quad (\boldsymbol{x} \in \partial\Omega)$$

which contradicts that $\boldsymbol{u}_*$ is not the trace of a rigid velocity on Γ.

Case 1.2. $\mathcal{S}^{(1)}(t_0) \subsetneq \mathcal{S}^{(2)}(t_0)$ (*case 1.3.* $\mathcal{S}^{(2)}(t_0) \subsetneq \mathcal{S}^{(1)}(t_0)$ is identical).

Then, we have in $\mathcal{S}^{(2)}(t_0) \setminus \overline{\mathcal{S}^{(1)}}(t_0) \subset \mathcal{F}^1(t_0)$,

$$\begin{cases} -\operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{u}^{(1)}(t_0), p^{(1)}(t_0))) = \mathbf{0} & \text{in } \mathcal{S}^{(2)}(t_0) \setminus \overline{\mathcal{S}^{(1)}}(t_0) \\ \operatorname{div}(\boldsymbol{u}^{(1)}(t_0)) = 0 & \text{in } \mathcal{S}^{(2)}(t_0) \setminus \overline{\mathcal{S}^{(1)}}(t_0) \end{cases}$$

and using (49),

$$\begin{cases} \boldsymbol{u}^{(1)}(t_0) = \boldsymbol{\ell}^{(1)}(t_0) + \boldsymbol{\omega}^{(1)}(t_0) \times (\boldsymbol{x} - \boldsymbol{a}^{(1)}(t_0)) & \text{on } \partial\mathcal{S}^{(1)}(t_0) \\ \boldsymbol{u}^{(1)}(t_0) = \boldsymbol{\ell}^{(2)}(t_0) + \boldsymbol{\omega}^{(2)}(t_0) \times (\boldsymbol{x} - \boldsymbol{a}^{(2)}(t_0)) & \text{on } \partial\mathcal{S}^{(2)}(t_0). \end{cases}$$

Let us write $\boldsymbol{v} = \boldsymbol{u}^{(1)}(t_0) - (\boldsymbol{\ell}^{(2)}(t_0) + \boldsymbol{\omega}^{(2)}(t_0) \times (\boldsymbol{x} - \boldsymbol{a}^{(2)}(t_0)))$; then

$$-\operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{v}, p^{(1)}(t_0))) = \mathbf{0} \quad \text{in} \quad \mathcal{F}^{(1)}(t_0), \tag{50}$$

$$\operatorname{div}(\boldsymbol{v}) = 0 \quad \text{in} \quad \mathcal{F}^{(1)}(t_0), \tag{51}$$

$$\boldsymbol{v} = \boldsymbol{u}_* - (\boldsymbol{\ell}^{(2)}(t_0) + \boldsymbol{\omega}^{(2)}(t_0) \times (\boldsymbol{x} - \boldsymbol{a}^{(2)}(t_0))) \quad \text{on} \quad \partial\Omega, \tag{52}$$

$$\boldsymbol{v} = \widetilde{\boldsymbol{a}} + \widetilde{\boldsymbol{b}} \times (\boldsymbol{x} - \boldsymbol{a}^{(1)}(t_0)) \quad \text{on } \partial\mathcal{S}^{(1)}(t_0), \tag{53}$$

$$\int_{\partial\mathcal{S}^{(1)}(t_0)} \boldsymbol{\sigma}(\boldsymbol{v}, p^{(1)}(t_0))\boldsymbol{n} \, \mathrm{d}\boldsymbol{\gamma}_{\boldsymbol{x}} = \mathbf{0}, \tag{54}$$

$$\int_{\partial\mathcal{S}^{(1)}(t_0)} (\boldsymbol{x} - \boldsymbol{a}^{(1)}(t_0)) \times \boldsymbol{\sigma}(\boldsymbol{v}, p^{(1)}(t_0))\boldsymbol{n} \, \mathrm{d}\boldsymbol{\gamma}_{\boldsymbol{x}} = \mathbf{0}, \tag{55}$$

where

$$\widetilde{\boldsymbol{a}} = (\boldsymbol{\ell}^{(1)}(t_0) - \boldsymbol{\ell}^{(2)}(t_0)) + \boldsymbol{\omega}^{(2)}(t_0) \times (\boldsymbol{a}^{(1)}(t_0) - \boldsymbol{a}^{(2)}(t_0)) \tag{56}$$

and

$$\widetilde{\boldsymbol{b}} = (\boldsymbol{\omega}^{(1)}(t_0) - \boldsymbol{\omega}^{(2)}(t_0)). \tag{57}$$

Also, thanks to proposition 9, we have

$$\boldsymbol{v} = \mathbf{0} \quad \text{on } \partial\mathcal{S}^{(2)}(t_0). \tag{58}$$

Let us multiply the first equation, (50), by $\boldsymbol{v}$

$$\begin{aligned} 0 = &-\int_{\mathcal{S}^{(2)}(t_0)\setminus\overline{\mathcal{S}^{(1)}(t_0)}} \operatorname{div}(\boldsymbol{\sigma}(\boldsymbol{v}, p^{(1)}(t_0))) \cdot \boldsymbol{v} \, \mathrm{d}\boldsymbol{x} \\ &= 2\nu \int_{\mathcal{S}^{(2)}(t_0)\setminus\overline{\mathcal{S}^{(1)}(t_0)}} |\boldsymbol{D}(\boldsymbol{v})|^2 \, \mathrm{d}\boldsymbol{x} - \int_{\partial(\mathcal{S}^{(2)}(t_0)\setminus\overline{\mathcal{S}^{(1)}(t_0)})} \boldsymbol{\sigma}(\boldsymbol{v}, p^{(1)}(t_0))\boldsymbol{n} \cdot \boldsymbol{v} \, d\boldsymbol{\gamma}_{\boldsymbol{x}} \\ &= 2\nu \int_{\mathcal{S}^{(2)}(t_0)\setminus\overline{\mathcal{S}^{(1)}(t_0)}} |\boldsymbol{D}(\boldsymbol{v})|^2 \, \mathrm{d}\boldsymbol{x} - \int_{\partial\mathcal{S}^{(2)}(t_0)} \boldsymbol{\sigma}(\boldsymbol{v}, p^{(1)}(t_0))\boldsymbol{n} \cdot \boldsymbol{v} \, d\boldsymbol{\gamma}_{\boldsymbol{x}} \\ &+ \int_{\partial\mathcal{S}^{(1)}(t_0)} \boldsymbol{\sigma}(\boldsymbol{v}, p^{(1)}(t_0))\boldsymbol{n} \cdot \boldsymbol{v} \, d\boldsymbol{\gamma}_{\boldsymbol{x}}. \end{aligned}$$

From (58) and (53)–(55), we have

$$\int_{\partial\mathcal{S}^{(2)}(t_0)} \boldsymbol{\sigma}(\boldsymbol{v}, p^{(1)}(t_0))\boldsymbol{n} \cdot \boldsymbol{v} \, d\boldsymbol{\gamma}_{\boldsymbol{x}} = 0 \quad \text{and} \quad \int_{\partial\mathcal{S}^{(1)}(t_0)} \boldsymbol{\sigma}(\boldsymbol{v}, p^{(1)}(t_0))\boldsymbol{n} \cdot \boldsymbol{v} \, d\boldsymbol{\gamma}_{\boldsymbol{x}} = 0,$$

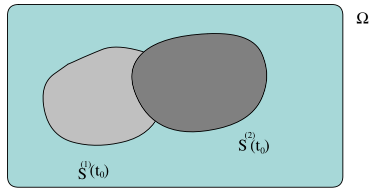

Figure 3. The intersection of the boundaries contains at least three noncollinear points.

respectively. Thereby, we deduce

$$2\nu \int_{\mathcal{S}^{(2)}(t_0)\setminus\overline{\mathcal{S}^{(1)}(t_0)}} |\boldsymbol{D}(\boldsymbol{v})|^2 \, \mathrm{d}\boldsymbol{x} = 0.$$

Consequently, $\boldsymbol{D}(\boldsymbol{v}) \equiv \boldsymbol{0}$ in $\mathcal{S}^{(2)}(t_0) \setminus \overline{\mathcal{S}^{(1)}(t_0)}$; this implies that there exist two vectors $\boldsymbol{\kappa}_1, \ \boldsymbol{\kappa}_2 \in \mathbb{R}^3$ such that

$$\boldsymbol{v} = \boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_1 \times \boldsymbol{y} \quad \text{in } \mathcal{S}^{(2)}(t_0) \setminus \overline{\mathcal{S}^{(1)}(t_0)}. \tag{59}$$

In particular, from (58) and lemma 5, we deduce

$$\boldsymbol{\kappa}_1 = \boldsymbol{\kappa}_2 = \boldsymbol{0}. \tag{60}$$

Therefore, we have $\boldsymbol{v} \equiv \boldsymbol{0}$ in $\mathcal{S}^{(2)}(t_0) \setminus \overline{\mathcal{S}^{(1)}(t_0)}$ and thus using again [12],

$$\boldsymbol{v} \equiv \boldsymbol{0} \quad \text{in } \mathcal{F}^{(1)}(t_0).$$

The above equation and (52) contradict that $\boldsymbol{u}_*$ is not the trace of a rigid velocity on Γ.

Case 2. $\partial\mathcal{S}^{(1)}(t_0) \cap \partial\mathcal{S}^{(2)}(t_0) \supset \{z_0, z_1, z_2\}$ where z_0, z_1, z_2 are three noncollinear points (see figure 3).

From proposition 9, we have

$$\boldsymbol{u}^{(1)} \equiv \boldsymbol{u}^{(2)} \quad \text{in } \Omega \setminus (\overline{\mathcal{S}^{(1)}(t_0) \cup \mathcal{S}^{(2)}(t_0)})$$

and

$$\boldsymbol{\ell}^{(1)}(t_0) + \boldsymbol{\omega}^{(1)}(t_0) \times (\boldsymbol{x} - \boldsymbol{a}^{(1)}(t_0)) \equiv \boldsymbol{\ell}^{(2)}(t_0) + \boldsymbol{\omega}^{(2)}(t_0) \times (\boldsymbol{x} - \boldsymbol{a}^{(2)}(t_0)) \quad \text{on } \partial\mathcal{S}^{(1)}(t_0) \cap \partial\mathcal{S}^{(2)}(t_0),$$

or equivalently

$$\widetilde{\boldsymbol{a}} + \widetilde{\boldsymbol{b}} \times (\boldsymbol{x} - \boldsymbol{a}^{(1)}(t_0)) = \boldsymbol{0} \quad \text{on } \partial\mathcal{S}^{(1)}(t_0) \cap \partial\mathcal{S}^{(2)}(t_0)$$

with $\widetilde{\boldsymbol{a}}$ and $\widetilde{\boldsymbol{b}}$ defined by (56)–(57). But, if $(\widetilde{\boldsymbol{a}}, \widetilde{\boldsymbol{b}}) \neq (\boldsymbol{0}, \boldsymbol{0})$, the set

$$\{\boldsymbol{y} : \widetilde{\boldsymbol{a}} + \widetilde{\boldsymbol{b}} \times \boldsymbol{y} = \boldsymbol{0}\}$$

is included in a straight line and since $\partial\mathcal{S}^{(1)}(t_0) \cap \partial\mathcal{S}^{(2)}(t_0)$ is not included in a straight line, we deduce

$$(\boldsymbol{\ell}^{(1)}(t_0) - \boldsymbol{\ell}^{(2)}(t_0)) + \boldsymbol{\omega}^{(2)}(t_0) \times (\boldsymbol{a}^{(1)}(t_0) - \boldsymbol{a}^{(2)}(t_0)) = (\boldsymbol{\omega}^{(1)}(t_0) - \boldsymbol{\omega}^{(2)}(t_0)) = \boldsymbol{0}. \tag{61}$$

Then, from (49) and (61), the function defined by

$$\boldsymbol{v} = \boldsymbol{u}^{(1)}(t_0) - (\boldsymbol{\ell}^{(2)}(t_0) + \boldsymbol{\omega}^{(2)}(t_0) \times (\boldsymbol{x} - \boldsymbol{a}^{(2)}(t_0)))$$

satisfies the following Stokes system:

$$\begin{cases} -\mathrm{div}\left(\boldsymbol{\sigma}\left(\boldsymbol{v}, p^{(1)}(t_0)\right)\right) = \mathbf{0} & \text{in } \mathcal{S}^{(2)}(t_0) \setminus \overline{\mathcal{S}^{(1)}(t_0)} \\ \mathrm{div}\,(\boldsymbol{v}) = 0 & \text{in } \mathcal{S}^{(2)}(t_0) \setminus \overline{\mathcal{S}^{(1)}(t_0)} \\ \boldsymbol{v} = \mathbf{0} & \text{on } \partial(\mathcal{S}^{(2)}(t_0) \setminus \overline{\mathcal{S}^{(1)}(t_0)}). \end{cases}$$

Arguing as in *case 1.1* we obtain a contradiction. Gathering *cases 1.1*, *1.2*, *1.3* and *2*, we deduce that

$$\mathcal{S}^{(1)}(t_0) = \mathcal{S}^{(2)}(t_0).$$

The above relation and (1) imply

$$\boldsymbol{a}^{(1)}(t_0) = \boldsymbol{a}^{(2)}(t_0)$$

and

$$\boldsymbol{Q}^{(1)}(t_0)\mathcal{S}_0^{(1)} = \boldsymbol{Q}^{(2)}(t_0)\mathcal{S}_0^{(2)}. \tag{62}$$

We set

$$\boldsymbol{R} = [\boldsymbol{Q}^{(2)}(t_0)]^{-1}\boldsymbol{Q}^{(1)}(t_0) \tag{63}$$

and

$$\boldsymbol{Q}(t) = \boldsymbol{Q}^{(2)}(t)\boldsymbol{R}, \qquad \boldsymbol{a}(t) = \boldsymbol{a}^{(2)}(t).$$

It is not difficult to see that

$$(\boldsymbol{a}(t_0), \boldsymbol{Q}(t_0)) = (\boldsymbol{a}^{(1)}(t_0), \boldsymbol{Q}^{(1)}(t_0)). \tag{64}$$

Moreover, (62) and (63) yield

$$\boldsymbol{R}\mathcal{S}_0^{(1)} = \mathcal{S}_0^{(2)}$$

and thus

$$\begin{aligned} \mathcal{S}^{(2)}(t) &= \boldsymbol{Q}^{(2)}(t)\mathcal{S}_0^{(2)} + \boldsymbol{a}^{(2)}(t) \\ &= \boldsymbol{Q}(t)\mathcal{S}_0^{(1)} + \boldsymbol{a}(t) \\ &= \mathcal{S}^{(1)}\left(\boldsymbol{a}(t), \boldsymbol{Q}(t)\right) \end{aligned}$$

and

$$\mathcal{F}^{(2)}(t) = \mathcal{F}^{(1)}\left(\boldsymbol{a}(t), \boldsymbol{Q}(t)\right).$$

In particular, since

$$\begin{aligned} &-\mathrm{div}\left(\boldsymbol{\sigma}\left(\boldsymbol{u}^{(2)}, p^{(2)}\right)\right) = \mathbf{0} \quad \text{in } \mathcal{F}^{(1)}(\boldsymbol{a}(t), \boldsymbol{Q}(t)), \\ &\mathrm{div}\left(\boldsymbol{u}^{(2)}\right) = 0 \quad \text{in } \mathcal{F}^{(1)}(\boldsymbol{a}(t), \boldsymbol{Q}(t)), \\ &\boldsymbol{u}^{(2)} = \boldsymbol{\ell}^{(2)} + \boldsymbol{\omega}^{(2)} \times (\boldsymbol{x} - \boldsymbol{a}) \quad \text{on} \quad \partial\mathcal{S}^{(1)}(\boldsymbol{a}(t), \boldsymbol{Q}(t)), \\ &\boldsymbol{u}^{(2)} = \boldsymbol{u}_* \quad \text{on } \partial\Omega, \\ &\int_{\partial\mathcal{S}^{(1)}(\boldsymbol{a}(t),\boldsymbol{Q}(t))} \boldsymbol{\sigma}(\boldsymbol{u}^{(2)}, p^{(2)})\boldsymbol{n}\,\mathrm{d}\boldsymbol{\gamma_x} = \mathbf{0}, \\ &\int_{\partial\mathcal{S}^{(1)}(\boldsymbol{a}(t),\boldsymbol{Q}(t))} (\boldsymbol{x} - \boldsymbol{a}) \times \boldsymbol{\sigma}(\boldsymbol{u}^{(2)}, p^{(2)})\boldsymbol{n}\,\mathrm{d}\boldsymbol{\gamma_x} = \mathbf{0}, \end{aligned}$$

we deduce that

$$\boldsymbol{\ell}^{(2)} = \boldsymbol{\ell}^{(1)}_{[\boldsymbol{a},\boldsymbol{Q}]} \qquad \text{and} \qquad \boldsymbol{\omega}^{(2)} = \boldsymbol{\omega}^{(1)}_{[\boldsymbol{a},\boldsymbol{Q}]}.$$

Consequently, $(\boldsymbol{a}, \boldsymbol{Q})$ is a solution of

$$\begin{cases} \boldsymbol{a}' = \boldsymbol{\ell}^{(1)}_{[\boldsymbol{a},\boldsymbol{Q}]} \\ \boldsymbol{Q}' = \mathbb{S}(\boldsymbol{\omega}^{(1)}_{[\boldsymbol{a},\boldsymbol{Q}]})\boldsymbol{Q} \end{cases}$$

and from (64) and the Cauchy–Lipschitz–Picard theorem, we deduce

$$\boldsymbol{a}(0) = \boldsymbol{a}^{(1)}(0), \qquad \boldsymbol{Q}(0) = \boldsymbol{Q}^{(1)}(0).$$

The above relation implies

$$\begin{cases} \boldsymbol{a}^{(1)}_0 = \boldsymbol{a}^{(2)}_0 \\ \boldsymbol{Q}^{(1)}_0 = \boldsymbol{Q}^{(2)}_0 \boldsymbol{R}. \end{cases}$$

This concludes the proof of theorem 2. □

6. Discussions and stability results

From the identifiability result we have obtained in theorem 2, it is possible to deduce several stability results. More precisely, with the notation of theorem 2, we would like to estimate from the difference

$$\|\boldsymbol{\sigma}(\boldsymbol{u}^{(1)}(t_0), p^{(1)}(t_0))\,\boldsymbol{n} - \boldsymbol{\sigma}(\boldsymbol{u}^{(2)}(t_0), p^{(2)}(t_0))\,\boldsymbol{n}\|_{\boldsymbol{H}^{1/2}(\Gamma)},$$

the differences between $\boldsymbol{R}\mathcal{S}^{(1)}_0, \boldsymbol{a}^{(1)}_0, \boldsymbol{Q}^{(1)}_0$ and $\mathcal{S}^{(2)}_0, \boldsymbol{a}^{(2)}_0, \boldsymbol{Q}^{(2)}_0\boldsymbol{R}$.

Following the method of [1] and the classical theory on shape differentiation (see, for instance, [18]), in order to estimate the difference between the domains, one can consider particular deformations of $\mathcal{S}_0$ as follows: let us consider $\boldsymbol{\Psi} \in \boldsymbol{C}^2(\mathbb{R}^3)$ so that $\boldsymbol{\Psi} \equiv \boldsymbol{0}$ in a neighbourhood of $\partial\Omega$ and $\boldsymbol{\Psi} \not\equiv \boldsymbol{0}$ in $\mathcal{S}_0$. Then, for τ small, the mappings $\boldsymbol{\Psi}_\tau = \mathbf{id} + \tau\boldsymbol{\Psi}$ are $\boldsymbol{C}^2$-diffeomorphism and one can consider the domains $\mathcal{S}_{0,\tau} := \boldsymbol{\Psi}_\tau(\mathcal{S}_0)$. By fixing $\boldsymbol{a}_0$ and $\boldsymbol{Q}_0$, one can then consider the mapping

$$\boldsymbol{\Lambda}_\tau : \tau \mapsto \boldsymbol{\sigma}\,(\boldsymbol{u}_\tau(t_0), p_\tau(t_0))\;\boldsymbol{n}_{|\Gamma},$$

where $(\boldsymbol{u}_\tau, p_\tau)$ are the solutions of (2)–(11) associated with $\mathcal{S}_{0,\tau}$.

Using the change of variables introduced in subsection 4.1 (with $\boldsymbol{X} = \boldsymbol{\Psi}_\tau$), and the implicit function theorem for analytic functions (see, for instance, [3]), one can show that $\tau \in (0, \tau_1) \mapsto \boldsymbol{\Lambda}_\tau \in \boldsymbol{H}^{1/2}(\Gamma)$ is analytic. Then the idea is to use theorem 2 to get that it is a non-constant mapping. However, we need in that case that $\mathcal{S}_{0,\tau}$ is convex which implies imposing some conditions on $\boldsymbol{\Psi}$. Under these conditions and theorem 2, we can obtain as in [1] the existence of a positive constant c and of an integer $m \geqslant 1$ such that for all $\tau \in (0, \tau_1)$,

$$\|\boldsymbol{\Lambda}_\tau - \boldsymbol{\Lambda}_0\|_{\boldsymbol{H}^{1/2}(\Gamma)} \geqslant c|\tau|^m.$$

In what follows, we consider another alternative to the above one: we fix the shape of the rigid body $\mathcal{S}_0$ and use the difference

$$\|\boldsymbol{\sigma}(\boldsymbol{u}^{(1)}(t_0), p^{(1)}(t_0))\,\boldsymbol{n} - \boldsymbol{\sigma}(\boldsymbol{u}^{(2)}(t_0), p^{(2)}(t_0))\,\boldsymbol{n}\|_{\boldsymbol{H}^{1/2}(\Gamma)}$$

to estimate the difference of the centres of mass $\boldsymbol{a}^{(1)}_0 - \boldsymbol{a}^{(2)}_0$ and the difference of orientations $\boldsymbol{Q}^{(1)}_0 - \boldsymbol{Q}^{(2)}_0$. In order to achieve this, we first note that we can improve the result of proposition 6.

Proposition 10. *The mapping*

$$\mathcal{T} : \mathcal{A} \to \mathbb{R}^6$$
$$(\boldsymbol{a}, \boldsymbol{Q}) \mapsto (\boldsymbol{\ell}_{[\boldsymbol{a},\boldsymbol{Q}]}, \boldsymbol{\omega}_{[\boldsymbol{a},\boldsymbol{Q}]})$$

is analytic.

In order to prove the above proposition, we can follow the proof of proposition 6. More precisely, for a fix $(\boldsymbol{a}_0, \boldsymbol{Q}_0) \in \mathcal{A}$, we can consider the local chart $(\boldsymbol{h}, \boldsymbol{\theta}) \mapsto \boldsymbol{\Phi}_{(\boldsymbol{a}_0,\boldsymbol{Q}_0)}(\boldsymbol{h}, \boldsymbol{\theta})$ of $\mathcal{A}$. Using this chart, we can construct the change of variables $\boldsymbol{X}$ introduced in subsection 4.1 and note that

$$B_{\mathbb{R}^6}(\boldsymbol{0}, r) \to \boldsymbol{C}^k(\overline{\Omega})$$
$$(\boldsymbol{h}, \boldsymbol{\theta}) \mapsto \boldsymbol{X}(\boldsymbol{h}, \boldsymbol{\theta}; \cdot)$$

is analytic for all $k \geqslant 0$. Then, we can transform the solutions $(\boldsymbol{u}^{(i)}, p^{(i)})$ of (20) by using this change of variables and consider $(\boldsymbol{v}^{(i)}, q^{(i)})$ defined by (37)–(38). Then instead of applying theorem 7, we use the implicit function theorem for analytic functions (see, for instance, [3]) and deduce that

$$\boldsymbol{g}_2 : B_{\mathbb{R}^6}(\boldsymbol{0}, r) \to \big(\boldsymbol{H}^2(\mathcal{F}(\boldsymbol{a}_0, \boldsymbol{Q}_0)) \cap \boldsymbol{H}^1_{\sigma}(\mathcal{F}(\boldsymbol{a}_0, \boldsymbol{Q}_0))\big) \times H^1(\mathcal{F}(\boldsymbol{a}_0, \boldsymbol{Q}_0))/\mathbb{R}$$
$$(\boldsymbol{h}, \boldsymbol{\theta}) \mapsto (\boldsymbol{v}^{(i)}, q^{(i)}) \tag{65}$$

is analytic which implies proposition 10.

From proposition 10 and classical results on ordinary differential equations (see, for instance, [6]), we deduce that the trajectory $(\boldsymbol{a}, \boldsymbol{Q})$ of the rigid body is analytic in time. Moreover, using the analytic dependence on the initial conditions, we also obtain that

$$(\boldsymbol{a}_0, \boldsymbol{Q}_0) \mapsto (\boldsymbol{a}(t_0), \boldsymbol{Q}(t_0))$$

is analytic. Combining this with proposition 10 and with the analyticity of the mapping m defined in (65), we deduce that the mappings

$$(\boldsymbol{a}_0, \boldsymbol{Q}_0) \mapsto (\boldsymbol{\ell}(t_0), \boldsymbol{\omega}(t_0)) \in \mathbb{R}^6, \qquad (\boldsymbol{a}_0, \boldsymbol{Q}_0) \mapsto \boldsymbol{\sigma}(\boldsymbol{u}^{(i)}, p^{(i)})\boldsymbol{n}_{|\Gamma}(t_0) \in \boldsymbol{H}^{1/2}(\Gamma)$$

are also analytic. Using that the solution $(\boldsymbol{u}, p)$ of (2)–(11) can be decomposed as in (23) and (24), we deduce that

$$\boldsymbol{\Lambda} : (\boldsymbol{a}_0, \boldsymbol{Q}_0) \in \mathcal{A} \mapsto \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n}_{|\Gamma}(t_0) \in \boldsymbol{H}^{1/2}(\Gamma)$$

is analytic. In the above definition of $\boldsymbol{\Lambda}$, $(\boldsymbol{u}, p)$ is the solution of (2)–(11) associated with the initial conditions $(\boldsymbol{a}_0, \boldsymbol{Q}_0)$. Then, we can proceed as in the beginning of this section; let us fix $\boldsymbol{h} \in \mathbb{R}^3$. Then for τ small enough, $\tau \mapsto \boldsymbol{\Lambda}(\boldsymbol{a}_0 + \tau\boldsymbol{h}, \boldsymbol{Q}_0)$ is well defined, analytic and non-constant by using theorem 2. Consequently, there exist a positive constant c and an integer $m \geqslant 1$ such that for all $\tau \in (0, \tau_1)$,

$$\|\boldsymbol{\Lambda}(\boldsymbol{a}_0 + \tau\boldsymbol{h}, \boldsymbol{Q}_0) - \boldsymbol{\Lambda}(\boldsymbol{a}_0, \boldsymbol{Q}_0)\|_{\boldsymbol{H}^{1/2}(\Gamma)} \geqslant c|\tau|^m.$$

Similar calculations allow us to estimate the difference between the orientations $\boldsymbol{Q}_0^{(1)}$ and $\boldsymbol{Q}_0^{(2)}$.

Acknowledgments

The work of CC was partially supported by the MICDB through grant ICM-P05-001-F and by Basal-Conicyt. The work of ELS was partially supported by CONICyT and ANR grant BLAN07-2 202879 and by CMM Chile. The work of TT was partially supported by ANR grant BLAN07-2 202879 and by ANR grant 09-BLAN-0213-02.

References

[1] Alvarez C, Conca C, Friz L, Kavian O and Ortega J H 2005 Identification of immersed obstacles via boundary measurements *Inverse Probl.* **21** 1531–52

[2] Alvarez C, Conca C, Lecaros R and Ortega J H 2008 On the identification of a rigid body immersed in a fluid: a numerical approach *Eng. Anal. Bound. Elem.* **32** 919–25

[3] Berger M S 1977 *Nonlinearity and Functional Analysis (Lectures on Nonlinear Problems in Mathematical Analysis, Pure and Applied Mathematics)* (New York: Academic)

[4] Boulakia M, Schwindt E and Takahashi T 2011 Existence of strong solutions for the motion of an elastic structure in an incompressible viscous fluid *Interfaces and Free Boundaries* submitted

[5] Cattabriga L 1961 Su un problema al contorno relativo al sistema di equazioni di Stokes *Rend. Sem. Mat. Univ. Padova* **31** 308–40

[6] Coddington E A 1961 *An Introduction to Ordinary Differential Equations (Prentice-Hall Mathematics Series)* (Englewood Cliffs, NJ: Prentice-Hall)

[7] Conca C, Cumsille P, Ortega J and Rosier L 2008 On the detection of a moving obstacle in an ideal fluid by a boundary measurement *Inverse Probl.* **24** 045001

[8] Conca C, Malik M and Munnier A 2010 Detection of a moving rigid solid in a perfect fluid *Inverse Probl.* **26** 095010

[9] Conca C, San Martín H J and Tucsnak M 2000 Existence of solutions for the equations modelling the motion of a rigid body in a viscous fluid *Commun. Part. Differ. Equ.* **25** 1019–42

[10] Desjardins B and Esteban M J 1999 Existence of weak solutions for the motion of rigid bodies in a viscous fluid *Arch. Ration. Mech. Anal.* **146** 59–71

[11] Doubova A, Fernández-Cara E and Ortega J H 2007 On the identification of a single body immersed in a Navier–Stokes fluid *Eur. J. Appl. Math.* **18** 57–80

[12] Fabre C and Lebeau G 1996 Prolongement unique des solutions de l'équation de Stokes *Commun. Part. Differ. Equ.* **21** 573–96

[13] Feireisl E 2003 On the motion of rigid bodies in a viscous incompressible fluid *J. Evol. Equ.* **3** 419–41 (Dedicated to Philippe Bénilan)

[14] Galdi G P 1994 *An Introduction to the Mathematical Theory of the Navier–Stokes Equations: Vol I. Linearized Steady Problems (Springer Tracts in Natural Philosophy* vol 38*)* (New York: Springer)

[15] Goldstein H 1951 *Classical Mechanics* (Cambridge, MA: Addison-Wesley)

[16] Gunzburger M D, Lee H-C and Seregin G A 2000 Global existence of weak solutions for viscous incompressible flows around a moving rigid body in three dimensions *J. Math. Fluid Mech.* **2** 219–66

[17] Heck H, Uhlmann G and Wang J-N 2007 Reconstruction of obstacles immersed in an incompressible fluid *Inverse Probl. Imaging* **1** 63–76

[18] Henrot A and Pierre M 2005 *Variation et Optimisation de formes: Une analyse géométrique (Mathématiques & Applications,* vol 48*)* (Berlin: Springer)

[19] Hesla T I 2005 Collisions of smooth bodies in viscous fluids: a mathematical investigation *PhD Thesis* University of Minnesota

[20] Hillairet M 2007 Lack of collision between solid bodies in a 2D incompressible viscous flow *Commun. Part. Differ. Equ.* **32** 1345–71

[21] Hillairet M and Takahashi T 2009 Collisions in three-dimensional fluid structure interaction problems *SIAM J. Math. Anal.* **40** 2451–77

[22] Hillairet M and Takahashi T 2010 Blow up and grazing collision in viscous fluid solid interaction systems *Ann. Inst. Henri. Poincaré Anal. Non Linéaire* **27** 291–313

[23] Hoffmann K-H and Starovoitov V 1999 On a motion of a solid body in a viscous fluid. Two-dimensional case *Adv. Math. Sci. Appl.* **9** 633–48

[24] Houot J and Munnier A 2008 On the motion and collisions of rigid bodies in an ideal fluid *Asymptotic Anal.* **56** 125–58

[25] Raviart P-A and Thomas J-M 1983 *Introduction à l'analyse numérique des équations aux dérivées partielles (Collection Mathématiques Appliquées pour la Maîtrise)* (Paris: Masson)

[26] Martín J S, Starovoitov V and Tucsnak M 2002 Global weak solutions for the two-dimensional motion of several rigid bodies in an incompressible viscous fluid *Arch. Ration. Mech. Anal.* **161** 113–47

[27] Serre D 1987 Chute libre d'un solide dans un fluide visqueux incompressible. Existence *Japan. J. Appl. Math.* **4** 99–110

[28] Silvestre A L 2002 On the slow motion of a self-propelled rigid body in a viscous incompressible fluid *J. Math. Anal. Appl.* **274** 203–27

[29] Simon J 1991 Domain variation for drag in Stokes flow *Control theory of distributed parameter systems and applications (Shanghai, 1990) (Lecture Notes in Control and Information Science* vol 159*)* (Berlin: Springer) pp 28–42

[30] Sylvester J and Uhlmann G 1987 A global uniqueness theorem for an inverse boundary value problem *Ann. Math.* **125** 153–69

[31] Takahashi T and Tucsnak M 2004 Global strong solutions for the two-dimensional motion of an infinite cylinder in a viscous fluid *J. Math. Fluid Mech.* **6** 53–77