Simultaneous Optimal Controls for Unsteady Stokes Systems

Claudia M. Gariboldi[†] and Erica L. Schwindt[‡]

[†]*Universidad Nacional de Río Cuarto, Facultad de Ciencias Exactas, Físico-Químicas y Naturales, Departamento de Matemática, 5800 - Río Cuarto, Argentina, cgariboldi@exa.unrc.edu.ar*
[‡]*Université d'Orléans, Laboratoire MAPMO, CNRS, UMR 7349, Fédération Denis Poisson, FR 2964, Bâtiment de Mathématiques, BP 6759, 45067 Orléans Cedex 2, France, leris98@gmail.com*

Abstract: This paper deal with optimal control problems for an unsteady Stokes system. We consider a simultaneous distributed-boundary optimal control problem with distributed observation. We prove the existence and uniqueness of an optimal control and we give the first order optimality condition for this problem. We also consider a distributed optimal control problem and a boundary optimal control problem and we obtain estimations between the simultaneous optimal control and the optimal controls of these last problems.

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1 Introduction

Let Ω be a bounded domain (i.e. connected and open set) of $\mathbb{R}^3$ with $\partial\Omega$ of class $\mathcal{C}^2$. We consider the following unsteady Stokes system

$$\begin{cases} \frac{\partial \boldsymbol{y}}{\partial t} - \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{y}, p) = \boldsymbol{u} & \text{in} \quad \Omega \times (0,T) \\ \operatorname{div} \boldsymbol{y} = 0 & \text{in} \quad \Omega \times (0,T) \\ \boldsymbol{y} = \boldsymbol{g} & \text{on} \quad \partial\Omega \times (0,T) \\ \boldsymbol{y}(0) = \boldsymbol{a} & \text{in} \quad \Omega. \end{cases} \tag{1}$$

Here, $(\boldsymbol{y}, p)$ are the velocity and the pressure of the fluid and $\sigma(\boldsymbol{y}, p)$ denotes the Cauchy stress tensor, which is defined by Stokes law $\boldsymbol{\sigma}(\boldsymbol{y}, p) = -p\,\boldsymbol{Id} + 2\nu \boldsymbol{D}(\boldsymbol{y})$, where $\boldsymbol{Id}$ is the identity matrix of order 3, ν is the kinematic viscosity of the fluid and $\boldsymbol{D}(\boldsymbol{y})$ is the strain tensor defined by

$$[\boldsymbol{D}(\boldsymbol{y})]_{kl} = \frac{1}{2}\left(\frac{\partial y_k}{\partial x_l} + \frac{\partial y_l}{\partial x_k}\right).$$

It is known (see [5, 3]) that System (1) admit a unique (up to a constant for p) solution $(\boldsymbol{y}, p) \in L^2(0,T; H^2(\Omega)) \times L^2(0,T; H^1(\Omega))$ (see below for the notation of these spaces), provided that $\boldsymbol{u} \in L^2(0,T; L^2(\Omega))$, $\boldsymbol{g} \in C^0(0,T; H^{3/2}(\Omega))$ satisfies $\int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{n}\, d\gamma = 0$ and $\boldsymbol{a} \in V$, where $V = \{\boldsymbol{v} \in H^1(\Omega) : \operatorname{div} \boldsymbol{v} = 0 \text{ and } \boldsymbol{v} = \boldsymbol{g}(0) \text{ on } \partial\Omega\}$ and $\boldsymbol{n}$ denotes the unit outer normal to fluid. Moreover, there exists a constant $K = K(\Omega, \nu)$ such that

$$\|\boldsymbol{y}\|_{L^2(H^2(\Omega))} + \|p\|_{L^2(H^1(\Omega))} \le K\left(\|\boldsymbol{u}\|_{L^2(H^2(\Omega))} + \|\boldsymbol{g}\|_{L^2(H^{3/2}(\partial\Omega))} + \|\boldsymbol{a}\|_{H^1(\Omega)}\right). \tag{2}$$

Let X be a Banach space, we will denote by $L^p(0,T;X)$ the space of the all measurable functions $\boldsymbol{y}$ such that $\boldsymbol{y} : [0,T] \to X$ defined by $\boldsymbol{y}(t)(x) = \boldsymbol{y}(t,x)$ satisfy

$$\|\boldsymbol{y}\|_{L^p(0,T;X)} = \left(\int_0^T \|\boldsymbol{y}(t)\|_X^p\, dt\right)^{1/p} < +\infty, \quad \text{if } p \in [1, +\infty)$$

$$\|\boldsymbol{y}\|_{L^\infty(0,T;X)} = ess \sup_{0 \le t \le T} \|\boldsymbol{y}(t)\|_X < +\infty, \quad \text{if } p = +\infty.$$

For the sake of simplicity of notation, we will often use $L^p(X)$ instead of $L^p(0,T,X)$. Lastly, we will denote $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_{\partial\Omega}$ the usual scalar products in $L^2(L^2(\Omega))$ and $L^2(L^2(\partial\Omega))$, respectively.

Now, we formulate the optimal control problems that we will study in this paper.

1. A *distributed* optimal control problem ($\mathcal{P}_u$):

$$\text{Find} \quad \overline{\boldsymbol{u}}^* \in L^2(0,T;L^2(\Omega)) \quad \text{such that} \quad \mathrm{J}_1(\overline{\boldsymbol{u}}^*) = \min_{\boldsymbol{u}\in L^2(L^2(\Omega))} \mathrm{J}_1(\boldsymbol{u}) \qquad (\mathcal{P}_u)$$

   where $\mathrm{J}_1 : L^2(0,T;L^2(\Omega)) \to \mathbb{R}_{\geq 0}$ is the cost function given by

$$\mathrm{J}_1(\boldsymbol{u}) := \frac{1}{2}\int_0^T \int_\Omega |\boldsymbol{y_u} - \boldsymbol{z}_d|^2 \; dx\, dt + \frac{\alpha}{2}\int_0^T \int_\Omega |\boldsymbol{u}|^2 \, dx\, dt.$$

   In this problem we consider $\boldsymbol{u}$ as the control variable; $\boldsymbol{z}_d$ is a given function, α is a positive constant and $\boldsymbol{y_u}$ is the unique solution of the problem (1) with fixed and known $\boldsymbol{g}$ and $\boldsymbol{a}$.

2. A *boundary* optimal control problem ($\mathcal{P}_g$):

$$\text{Find} \quad \overline{\boldsymbol{g}}^* \in L^2(0,T;H) \quad \text{such that} \quad \mathrm{J}_2(\overline{\boldsymbol{g}}^*) = \min_{\boldsymbol{g}\in L^2(H)} \mathrm{J}_2(\boldsymbol{g}) \qquad (\mathcal{P}_g)$$

   where $H = \{\boldsymbol{g} \in H^{1/2}(\partial\Omega) : \int_{\partial\Omega} \boldsymbol{g}\cdot\boldsymbol{n} = 0\}$ and the cost function $\mathrm{J}_2 : L^2(0,T;H) \to \mathbb{R}_{\geq 0}$ is given by

$$\mathrm{J}_2(\boldsymbol{g}) := \frac{1}{2}\int_0^T \int_\Omega \left|\boldsymbol{y_g} - \boldsymbol{z}_d\right|^2 \; dx\, dt + \frac{\beta}{2}\int_0^T \int_{\partial\Omega} |\boldsymbol{g}|^2 \, d\gamma\, dt.$$

   In this problem $\boldsymbol{g}$ is considered as the control variable; $\boldsymbol{z}_d$ is a given function, β is a positive constant and $\boldsymbol{y_g}$ is the unique solution of the problem (1) with fixed and known $\boldsymbol{u}$ and $\boldsymbol{a}$.

3. A simultaneous *distributed-boundary* optimal control problem ($\mathcal{P}_{ug}$):

$$\text{Find} \quad (\boldsymbol{u}^*, \boldsymbol{g}^*) \in \mathcal{A}_{ad} \quad \text{such that} \quad \mathrm{J}(\boldsymbol{u}^*, \boldsymbol{g}^*) = \min_{(\boldsymbol{u},\boldsymbol{g})\in\mathcal{A}_{ad}} \mathrm{J}(\boldsymbol{u},\boldsymbol{g}) \qquad (\mathcal{P}_{ug})$$

   where $\mathcal{A}_{ad} = L^2(0,T;L^2(\Omega)) \times L^2(0,T;H)$ and the cost function $\mathrm{J} : \mathcal{A}_{ad} \to \mathbb{R}_{\geq 0}$ is given by

$$\mathrm{J}(\boldsymbol{u},\boldsymbol{g}) := \frac{1}{2}\int_0^T \int_\Omega \left|\boldsymbol{y_{ug}} - \boldsymbol{z}_d\right|^2 \; dx\, dt + \frac{\alpha}{2}\int_0^T \int_\Omega |\boldsymbol{u}|^2 \, dx\, dt + \frac{\beta}{2}\int_0^T \int_{\partial\Omega} |\boldsymbol{g}|^2 \, d\gamma\, dt.$$

   Here $(\boldsymbol{u},\boldsymbol{g})$ is considered as the control variable; $\boldsymbol{z}_d$ is a given function, α and β are the previous positive constants and $\boldsymbol{y_{ug}}$ is the unique solution of the problem (1) with fixed and known $\boldsymbol{a}$.

In [2], several optimal control problems of the type ($\mathcal{P}_u$) and ($\mathcal{P}_g$) have been studied. In [4], the authors studied a optimal control problem of the type ($\mathcal{P}_u$) for the heat equation with mixed boundary conditions.

The goal of our work is to study in detail the simultaneous optimal control problems for unsteady Stokes equations, a similar problem was studied in [1] for elliptic equations. In Section 2 we prove the existence and uniqueness of the solutions of the problem (P_{ug}). We obtain that the cost functions J_1, J_2 and J are Gâteaux-differentiable and we give the first order optimality conditions in terms of the adjoint states of the system. In Section 3 we get estimations between the unique solution of the problem (P_u) and the first component of the unique solution of the problem (P_{ug}). We also prove estimations between the unique solution of the problem (P_g) and the second component of the unique solution of the problem (P_{ug}).

2 Results of existence and uniqueness of optimal controls

2.1 Distributed optimal control and boundary optimal control

The proofs of existence and uniqueness of an optimal control $\overline{\boldsymbol{u}}^*$ for Problem ($\mathcal{P}_u$) and an optimal control $\overline{\boldsymbol{g}}^*$ for Problem ($\mathcal{P}_g$) follows similarly to what was done for example in [2] or [4], therefore we omit them. We only recall the optimality conditions that satisfy $\overline{\boldsymbol{u}}^*$ and $\overline{\boldsymbol{g}}^*$, which are expressed in terms of the Gâteaux derivative of their respective cost functions.

• For the distributed optimal control $\overline{\boldsymbol{u}}^*$:

$$\langle J_1'(\overline{\boldsymbol{u}}^*), \boldsymbol{w}\rangle = (\boldsymbol{y}_{\overline{\boldsymbol{u}}^*} - \boldsymbol{z}_d, \boldsymbol{y}_{\boldsymbol{w}} - \boldsymbol{y}_{\boldsymbol{0}})_\Omega + \alpha(\overline{\boldsymbol{u}}^*, \boldsymbol{w})_\Omega = 0, \quad \forall\, \boldsymbol{w} \in L^2(L^2(\Omega)) \tag{3}$$

we note that $\boldsymbol{y}_{\boldsymbol{w}} - \boldsymbol{y}_{\boldsymbol{0}}$ satisfies (1) with $\boldsymbol{u} = \boldsymbol{w}$, $\boldsymbol{g} = \boldsymbol{0}$ and $\boldsymbol{a} = \boldsymbol{0}$.

• For the distributed optimal control $\overline{\boldsymbol{g}}^*$:

$$\langle J_2'(\overline{\boldsymbol{g}}^*), \boldsymbol{f}\rangle = (\boldsymbol{y}_{\overline{\boldsymbol{g}}^*} - \boldsymbol{z}_d, \boldsymbol{y}_{\boldsymbol{f}} - \boldsymbol{y}_{\boldsymbol{0}})_\Omega + \beta(\overline{\boldsymbol{g}}^*, \boldsymbol{f})_{\partial\Omega} = 0, \quad \forall \boldsymbol{f} \in L^2(H) \tag{4}$$

we note that $\boldsymbol{y}_{\boldsymbol{f}} - \boldsymbol{y}_{\boldsymbol{0}}$ satisfies (1) with $\boldsymbol{u} = \boldsymbol{0}$, $\boldsymbol{g} = \boldsymbol{f}$ and $\boldsymbol{a} = \boldsymbol{0}$.

2.2 Simultaneous distributed-boundary optimal control

We define the map $C : \mathcal{A}_{ad} \to L^2(H_0^1(\Omega))$ such that $(\boldsymbol{u}, \boldsymbol{g}) \mapsto \boldsymbol{y}_{\boldsymbol{ug}} - \boldsymbol{y}_{\boldsymbol{00}}$, where $\boldsymbol{y}_{\boldsymbol{00}}$ is the unique solution of (1) with $\boldsymbol{u} = \boldsymbol{g} = \boldsymbol{0}$ and fixed $\boldsymbol{a}$.

We also define the maps $\Pi : \mathcal{A}_{ad} \times \mathcal{A}_{ad} \to \mathbb{R}$ and $\Upsilon : \mathcal{A}_{ad} \to \mathbb{R}$ as

$$\Pi((\boldsymbol{u}, \boldsymbol{g}), (\boldsymbol{v}, \boldsymbol{h})) = (C(\boldsymbol{u}, \boldsymbol{g}), C(\boldsymbol{v}, \boldsymbol{h}))_\Omega + \alpha\,(\boldsymbol{u}, \boldsymbol{v})_\Omega + \beta\,(\boldsymbol{g}, \boldsymbol{h})_{\partial\Omega}\,, \quad \forall\,(\boldsymbol{u}, \boldsymbol{g}), (\boldsymbol{v}, \boldsymbol{h}) \in \mathcal{A}_{ad}$$

$$\Upsilon(\boldsymbol{u}, \boldsymbol{g}) = (C(\boldsymbol{u}, \boldsymbol{g}), \boldsymbol{z}_d)_\Omega\,, \quad \forall\,(\boldsymbol{u}, \boldsymbol{g}) \in \mathcal{A}_{ad}.$$

The following properties hold for the maps introduced above.

Proposition 1

a) C is a linear and continuous application.

b) Π is a bilinear, continuous, symmetric and coercive map.

c) Υ is a linear and continuous map.

After some calculations, it is possible to rewrite the cost function J as

$$J(\boldsymbol{u}, \boldsymbol{g}) = \frac{1}{2}\Pi((\boldsymbol{u}, \boldsymbol{g}), (\boldsymbol{u}, \boldsymbol{g})) - \Upsilon(\boldsymbol{u}, \boldsymbol{g}) + \frac{1}{2}\,\|\boldsymbol{z}_d\|^2_{L^2(L^2(\Omega))}\,. \tag{5}$$

Theorem 1 *There exists a unique solution $(\boldsymbol{u}^*, \boldsymbol{g}^*) \in \mathcal{A}_{ad}$ of the simultaneous distributed-boundary optimal control problem $(\mathcal{P}_{ug})$.*

Proof. Taking into account (5) and Proposition 1 we can deduce that J is a strictly convex function. Thus, by the classical theory of optimal control [2, Chapter 3], we have that there exists a unique solution $(\boldsymbol{u}^*, \boldsymbol{g}^*) \in \mathcal{A}_{ad}$ of the problem $(\mathcal{P}_{ug})$. □

Proposition 2 *The function J is Gâteaux-differentiable and its derivate is given by*

$$\langle J'(\boldsymbol{u}, \boldsymbol{g}), (\boldsymbol{v}, \boldsymbol{h})\rangle = (\boldsymbol{y}_{\boldsymbol{ug}} - \boldsymbol{z}_d, \boldsymbol{y}_{\boldsymbol{vh}} - \boldsymbol{y}_{\boldsymbol{00}})_\Omega + \alpha(\boldsymbol{u}, \boldsymbol{v})_\Omega + \beta(\boldsymbol{g}, \boldsymbol{h})_{\partial\Omega}, \quad \forall\,(\boldsymbol{v}, \boldsymbol{h}) \in \mathcal{A}_{ad}.$$

Thanks to Proposition 2 we can give the first order optimality condition for $(\boldsymbol{u}^*, \boldsymbol{g}^*)$. Namely, the unique solution $(\boldsymbol{u}^*, \boldsymbol{g}^*)$ can be characterized as the unique pair of functions that satisfies

$$\langle J'(\boldsymbol{u}^*, \boldsymbol{g}^*), (\boldsymbol{v}, \boldsymbol{h})\rangle = (\boldsymbol{y}_{\boldsymbol{u}^*\boldsymbol{g}^*} - \boldsymbol{z}_d, \boldsymbol{y}_{\boldsymbol{vh}} - \boldsymbol{y}_{\boldsymbol{00}})_\Omega + \alpha(\boldsymbol{u}^*, \boldsymbol{v})_\Omega + \beta(\boldsymbol{g}^*, \boldsymbol{h})_{\partial\Omega} = 0, \tag{6}$$

for all $(\boldsymbol{v}, \boldsymbol{h}) \in \mathcal{A}_{ad}$.

Since the equation (6) does not permit to express the optimality condition easily, we will introduce the adjoint state $\boldsymbol{\phi}_{\boldsymbol{ug}}$, to rewrite this derivative into a more workable expression. For this, let us consider the following system

$$\begin{cases} -\frac{\partial \boldsymbol{\phi}}{\partial t} - \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{\phi}, q) = \boldsymbol{y}_{\boldsymbol{ug}} - \boldsymbol{z}_d & \text{in } \Omega \times (0,T), \\ \operatorname{div} \boldsymbol{\phi} = 0 & \text{in } \Omega \times (0,T), \\ \boldsymbol{\phi} = \boldsymbol{0} & \text{on } \partial\Omega \times (0,T) \\ \boldsymbol{\phi}(T) = \boldsymbol{a} & \text{in } \Omega. \end{cases} \tag{7}$$

Now, we are in position to establish the following result

Proposition 3 *The optimality condition* (6) *can be written in terms of the adjoint state as*

$$\langle J'(\boldsymbol{u}^*,\boldsymbol{g}^*),(\boldsymbol{v},\boldsymbol{h})\rangle = (\boldsymbol{\phi}_{\boldsymbol{u}^*\boldsymbol{g}^*}+\alpha\boldsymbol{u}^*,\boldsymbol{v})_\Omega + (\beta\boldsymbol{g}^* - \boldsymbol{\sigma}(\boldsymbol{\phi}_{\boldsymbol{u}^*\boldsymbol{g}^*},q)\boldsymbol{n},\boldsymbol{h})_{\partial\Omega} = 0, \quad \forall(\boldsymbol{v},\boldsymbol{h})\in\mathcal{A}_{ad}. \tag{8}$$

and the simultaneous optimal control $(\boldsymbol{u}^*,\boldsymbol{g}^*)$ *is given by*

$$\boldsymbol{u}^* = \frac{-1}{\alpha}\boldsymbol{\phi}_{\boldsymbol{u}^*\boldsymbol{g}^*} \ \text{ in } \ \Omega\times(0,T) \qquad \text{and} \qquad \boldsymbol{g}^* = \frac{1}{\beta}\boldsymbol{\sigma}(\boldsymbol{\phi}_{\boldsymbol{u}^*\boldsymbol{g}^*},q)\boldsymbol{n} \ \text{ on } \ \partial\Omega\times(0,T). \tag{9}$$

3 ESTIMATIONS

Proposition 4 *Let* $(\boldsymbol{u}^*,\boldsymbol{g}^*)$ *be the unique solution of the optimal control problem* $(\mathcal{P}_{ug})$ *with* $\boldsymbol{a}$ *given.*
(a) If $\overline{\boldsymbol{u}}^*$ *is the unique solution of the optimal control problem* $(\mathcal{P}_u)$ *for a function fixed* $\boldsymbol{g}$*, then*

$$\|\boldsymbol{u}^*-\overline{\boldsymbol{u}}^*\|_{L^2(L^2(\Omega))} \leq \frac{K}{\alpha}\left\|\boldsymbol{y}_{\boldsymbol{u}^*\boldsymbol{g}^*}-\boldsymbol{y}_{\overline{\boldsymbol{u}}^*}\right\|_{L^2(L^2(\Omega))}.$$

(b) If $\overline{\boldsymbol{g}}^*$ *is the unique solution of the optimal control problem* $(\mathcal{P}_g)$ *for a function fixed* $\boldsymbol{u}$*, then*

$$\|\boldsymbol{g}^*-\overline{\boldsymbol{g}}^*\|_{L^2(L^2(\partial\Omega))} \leq \frac{K}{\beta}\left\|\boldsymbol{y}_{\boldsymbol{u}^*\boldsymbol{g}^*}-\boldsymbol{y}_{\overline{\boldsymbol{g}}^*}\right\|_{L^2(L^2(\Omega))}.$$

Here K *denote a positive constant depending on* Ω *and* ν*.*

Proof. (a) Taking $(\boldsymbol{v},\boldsymbol{h}) = (\overline{\boldsymbol{u}}^*-\boldsymbol{u}^*,\boldsymbol{0})$ in (6) and $\boldsymbol{w}=\boldsymbol{u}^*-\overline{\boldsymbol{u}}^*$ in (3), adding the two equations and taking into account that $\boldsymbol{y}_{\boldsymbol{w}}-\boldsymbol{y}_{\boldsymbol{0}} = -(\boldsymbol{y}_{\boldsymbol{vh}}-\boldsymbol{y}_{\boldsymbol{00}})$, we deduce

$$(\boldsymbol{y}_{\boldsymbol{u}^*\boldsymbol{g}^*}-\boldsymbol{y}_{\overline{\boldsymbol{u}}^*},\boldsymbol{y}_{\boldsymbol{vh}}-\boldsymbol{y}_{\boldsymbol{00}})_\Omega - \alpha(\boldsymbol{u}^*-\overline{\boldsymbol{u}}^*,\boldsymbol{u}^*-\overline{\boldsymbol{u}}^*)_\Omega = 0,$$

by using the Cauchy-Schwartz inequality and (2), we obtain

$$\begin{aligned}\alpha\|\boldsymbol{u}^*-\overline{\boldsymbol{u}}^*\|_{L^2(L^2(\Omega))} &\leq K\|\boldsymbol{y}_{\boldsymbol{u}^*\boldsymbol{g}^*}-\boldsymbol{y}_{\overline{\boldsymbol{u}}^*}\|_{L^2(L^2(\Omega))}\\ &\leq K\left(\|\boldsymbol{u}^*-\overline{\boldsymbol{u}}^*\|_{L^2(L^2(\Omega))}+\|\boldsymbol{g}^*-\boldsymbol{g}\|_{L^2(L^2(\partial\Omega))}\right).\end{aligned}\tag{10}$$

(b) By a similar way to (a) we obtain

$$\begin{aligned}\beta\|\boldsymbol{g}^*-\overline{\boldsymbol{g}}^*\|_{L^2(L^2(\partial\Omega))} &\leq K\|\boldsymbol{y}_{\boldsymbol{u}^*\boldsymbol{g}^*}-\boldsymbol{y}_{\overline{\boldsymbol{g}}^*}\|_{L^2(L^2(\Omega))}\\ &\leq K\left(\|\boldsymbol{g}^*-\overline{\boldsymbol{g}}^*\|_{L^2(L^2(\partial\Omega))}+\|\boldsymbol{u}^*-\boldsymbol{u}\|_{L^2(L^2(\Omega))}\right).\end{aligned}\tag{11}$$

□

Corollary 1 *If the coercitivity constants* α *and* β *satisfy* $\alpha,\beta > C$ *for some positive constant, then we have* $\boldsymbol{u}^*=\overline{\boldsymbol{u}}^*$ *and* $\boldsymbol{g}^*=\overline{\boldsymbol{g}}^*$*.*

Proof. It is sufficient to take $\boldsymbol{g}=\boldsymbol{g}^*$ in (10), $\boldsymbol{u}=\boldsymbol{u}^*$ in (11) and $C=K$ with $K=K(\Omega,\nu)$ the constant given in (2). □

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