CSE291-14: The Number Field Sieve

https://cseweb.ucsd.edu/classes/wi22/cse291-14

Emmanuel Thomé

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Part 4a

Polynomial selection in NFS

Plan

Introduction

Size of the coefficients: possible and impossible things

Searching for good polynomials with base-m

Skewness

Non-monic linear polynomials: Kleinjung's 2005 algorithm

Goal

Our goal: review the different methods for polynomial selection.

- Why is it important?
- What kind of game is it?
- What are the different methods?
- How do we measure the quality of the output?

What is polynomial selection

The polynomial selection phase is when we choose the pair of polynomials that define both sides of the NFS diagram.

- The algebraic polynomial f defines the number field.
- The rational polynomial (thus far, $x - m$) completes the picture.

In practice, this is more general

In fact, not even $x - m$ is monic. Several methods do relax this condition (but not the first ones).
In some cases, both polynomials are of degree > 1.

Importance of polynomial selection

Polynomial selection is important because it determines the size of the "norms" (actually, of the integers being checked for smoothness).

- Asymptotic analysis crudely reduced polynomial selection to the choice of the pair (D, δ).
- We eventually found out that δ was controlling the compromise between the size of $\mathrm{Res}(\phi, x - m)$ and of $\mathrm{Res}(\phi, f)$.

This general role is also true in practice

A good polynomial selection makes these "norms" small as $\phi(x) = a - bx$ ranges over the values we explore.

- Certainly, some things can be achieved, and some can't.
- Can we force these values to be smooth more often than on average?

A general workplan

Starting point: a method that can yield good polynomial pairs.

- Arrange so that the method has many degrees of freedom.
- Explore a huge search space to find exceptional situations.
- Find reasonable assessment criteria that make it possible to identify which are the “exceptionally good” polynomial pairs.

An extension: non-linear rational polynomial

What if we replace $x - m$ by $m_1 x - m_0$?

- $\mathrm{Res}(a - bx, x - m) = a - bm$ becomes
  $\mathrm{Res}(a - bx, m_1 x - m_0) = am_1 - bm_0$, which looks nicer.
- If we write $m = m_0/m_1 \in \mathbb{Q}$ and that $f(m) \equiv 0 \mod N$, everything works as before.
- The condition to meet is the existence of a common root:

$$\mathrm{Res}(f(x), m_1 x - m_0) \equiv 0 \mod N.$$

This extra degree of freedom has been part of all polynomial selection algorithms since the early 2000s.

An extension: higher degree polynomials

If a polynomial selection can find a pair of nonlinear polynomials:

- whose resultant is divisible by N with multiplicity 1
- and with a known common root in $\mathbb{Z}/N\mathbb{Z}$

Then we can work exactly along the lines of NFS.

Caveat: no such thing is known in general, EXCEPT for DLP.

- NFS for DLP (discrete logs in $\mathbb{Z}/p\mathbb{Z}^\times$): p replaces N.
- The existence of root finding $\bmod p$ is the key.
- In some cases (but not always), this wins.

Notations

Traditionally, notations are as follows:

- f is the algebraic polynomial.
  The coefficients are named $f_0, \ldots, f_d$, or $a_0, \ldots, a_d$.
- g is the linear polynomial.

Often, to highlight the symmetric roles played by the two sides:

- f_0 is "the polynomial on side 0" (typically $\deg f_0 = 1$).
- f_1 is "the polynomial on side 1".
- But this messes with the per-coefficient notations.
  Notations $a_0, \ldots, a_d$ are preferred for coefficients of the nonlinear polynomial in this case.

Implementations such as Cado-NFS are mostly agnostic w.r.t side numbering.

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Size matters

Notations: $(f_0, f_1), \deg f_1 = d, f_1 = \sum a_i x^i$.

Our first approach consists in searching for "small" polynomial pairs.

- Eventually, one of our guides will be the size of the integers we will try to factor.
- Given the power dependency in the degrees of the polynomials, we have only a few possible choices for the degree.
- Given a choice for $d = \deg f_1$, can we obtain polynomials f_0 and f_1 with small coefficients?

Two equations (Kleinjung, 2016)

A good question to ask

In order to reach all integers in a range $[M, 2M]$, how large do we have to choose the coefficients of f_0 and f_1?
Let M^{c_0} be a bound on the coefficients of f_0 (likewise M^{c_1} for f_1).

- First constraint: to reach M different values with the degrees of freedom that we have:

$$c_0 \cdot (d_0 + 1) + c_1 \cdot (d_1 + 1) \geq 1.$$

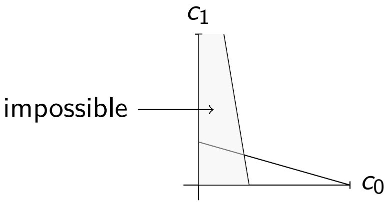
- Second constraint: $\mathrm{Res}(f_0, f_1)$ must be at least M.
Since $\mathrm{Res}(f_0, f_1) = M^{o(1)} \|f_0\|^{\deg f_1} \|f_1\|^{\deg f_0}$, we must have:

$$c_0 \cdot d_1 + c_1 \cdot d_0 \geq 1.$$

Two different constraints

Note that the constraints are of different nature.

- $c_0 \cdot (d_0 + 1) + c_1 \cdot (d_1 + 1) \geq 1$.
  Pairs not meeting this constraint may exist, but such a family cannot reach all integers.
- $c_0 \cdot d_1 + c_1 \cdot d_0 \geq 1$.
  It is outright impossible for pairs to not meet this constraint, and be useful for NFS.



Important example #1

Take what the naive polynomial selection method gives: $d_0 = 1$, $d_1 = d$, $c_0 = c_1 = \frac{1}{d+1}$.

- $c_0 \cdot (d_0 + 1) + c_1 \cdot (d_1 + 1) = \frac{2}{d+1} + \frac{d+1}{d+1} \geq 1$.
- $c_0 \cdot d_1 + c_1 \cdot d_0 = \frac{d}{d+1} + \frac{1}{d+1} = 1$.

Put otherwise, the resultant bound is tight, but there is immense legroom in the choice of f_0.

Important example #2

What can we obtain with $c_1 = 0$?
i.e., a family of algebraic polynomials with coefficients bounded by a constant.

The remaining constraint rewrites simply as

$$c_0 d_1 = 1,$$

which does not say much.

Does this do anything?

If we have access to a fictitious oracle that outputs such a polynomial f_1, what does it give?

SNFS: polynomial selection with an oracle

If we have access to a fictitious oracle that outputs such a polynomial f_1, what does it give?

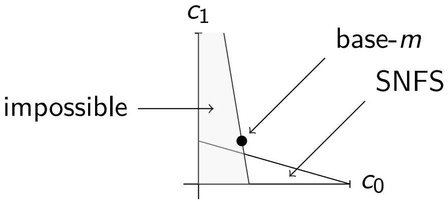
- We can do the entire NFS analysis based on that.
- The algebraic norm can be rewritten as $L_N[1/3, \cancel{\frac{1}{\delta}} + \alpha\delta]$.
- This changes the optimum δ from $\sqrt{2/\alpha}$ to $\sqrt{1/\alpha}$.
- Eventually, we end up with $L_N[1/3, (32/9)^{1/3} + o(1)]$.

This is called SNFS.

- The "special" integers are those that are precisely reached by this "ideal" choice.
- By extension, the SNFS term is also used for anything that is reached by a non-general polynomial selection.

The constraint space

Example for $d_0 = 1$ and $d_1 = 6$.



Note that $c_0 + c_1$ appears in the smoothness probability.

- $c_0 + c_1 = \log(\|f_0\| \cdot \|f_1\|)/\log N$.
- $c_0 + c_1$ measures the polynomial-dependent part of the maximum size of the integers which are checked for smoothness.

Thus the intersection point P is "ideal". Alas, moving towards P is expensive.

Where are we?

- Base-m polynomial selection is a starting point.
- We have an argument that explains that it is not "optimal".
- SNFS numbers are really special.

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Non-monic linear polynomials: Kleinjung's 2005 algorithm

What can base-m do?

Notations: $(f_0, f_1), \deg f_1 = d, f_1 = \sum a_i x^i$.

Recall that the simplistic base-m method choses $m \approx N^{1/(d+1)}$.

- There is an immense degree of freedom in the choice of m.
- Can we do many trials and hope for something nice to happen?

Opportunities for improvement:

- It is not a very big deal if $\|f_0\|$ (max coefficient of the linear polynomial) increases by a tiny bit.
- Can this be compensated by a larger decrease of $\|f_1\|$?

Base-m, revisited

Instead of picking m first, and then the coefficients of f:

- Choose a_d first, slightly smaller than $N^{1/(d+1)}$.
- Then choose m, and deduce the rest of the coefficients.

Our game: correlation between effort and yield

Ultimately, we want to answer the question:

> *"If we generate C polynomial pairs, what is the best we can obtain, as a function of C?"*

Can also be phrased as: if $a_d \approx N^{1/(d+1)}/c$, how many trials does it take to have all coefficients of f close to a_d?

Base-m, revisited

Let c be an arbitrary number.

- Choose $a_d \approx N^{1/(d+1)}/c$ (many possible choices!)
- Let $m = \lfloor (N/a_d)^{1/d} \rceil = (N/a_d)^{1/d} + \mu$ with $|\mu| \leq 1$.

Lemma: $a_{d-1} \approx a_d$

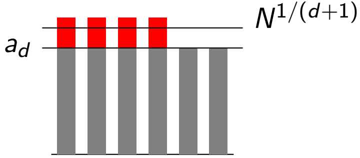
$$\begin{aligned} |a_{d-1}| &= \frac{\left|N - a_d m^d\right|}{m^{d-1}} = a_d \frac{\left|(m-\mu)^d - m^d\right|}{m^{d-1}} \\ &\leq a_d m \left|(1 - \mu/m)^d - 1\right| \\ &\leq d a_d \times \text{small constant bound.} \end{aligned}$$

And d is small as well, so we expect a_{d-1} to have roughly as many bits as a_d.

Other coefficients

The $d-1$ coefficients a_0 to a_{d-2} are a priori close to m, with:

$$m \approx (N/a_d)^{1/d} \approx (N^{1-1/(d+1)}c)^{1/d} \approx N^{1/(d+1)}c^{1/d}.$$



Heuristic: a_0 to a_{d-2} behave like random integers. With probability $(a_d/m)^{d-1} \approx \left(c^{-(1/d+1)}\right)^{d-1} = c^{-(d^2-1)/d}$, all are $\leq a_d$.

Conclusion

By trying $c^{(d^2-1)/d}$ values a_d, we expect to:

- change $\|f_1\|$ to $N^{1/(d+1)}/c$.
- change $\|f_0\|$ to $N^{1/(d+1)}\times c^{1/d}$.

Rewrite

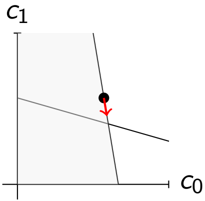
Conclusion (rewrite)

By trying $c^{(d^2-1)/d}$ values a_d, we expect to:

- change $\|f_1\|$ to $N^{1/(d+1)}/c$.
- change $\|f_0\|$ to $N^{1/(d+1)}\times c^{1/d}$.

We can also write: by trying C values a_d, we expect to:

- change $\|f_1\|$ to $N^{1/(d+1)}/C^{d/(d^2-1)}$.
- change $\|f_0\|$ to $N^{1/(d+1)}\times C^{1/(d^2-1)}$.
- change $\|f_0\|\|f_1\|$ to $N^{2/(d+1)}/C^{1/(d+1)}$.



This moves in the right direction!

- More work leads to smaller polynomials.
- This is woefully exponential, of course.

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Skewness

Notations: $(f_0, f_1), \deg f_1 = d, f_1 = \sum a_i x^i$, $\phi(x) = u - vx$.

Skewness is a way to add more flexibility to the polynomial selection process.

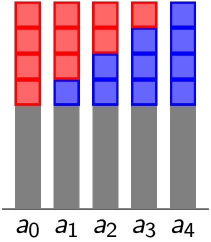
Observation: the polynomial $x - m$ is unbalanced. So is the expression $\mathrm{Res}(u - vx, x - m) = u - vm$.

- Can we work with larger u and smaller v?

Skewed polynomials

$\text{Res}(u - vx, f_1) = F(u, v) = (u^d a_d + \cdots + u^i v^{d-i} a_i + \cdots + v^d a_0)$.

- If coefficients of f_1 have roughly the same size and both coefficients of $\phi(x)$ are bounded by A, then all $a_i u^i v^{d-i}$ have the same size.



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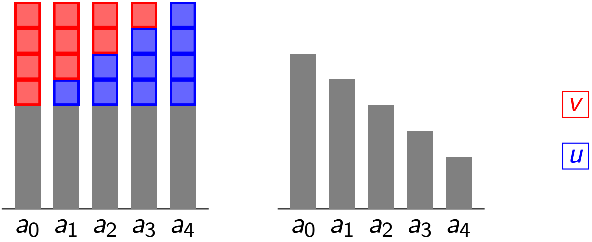
- If coefficients of f_1 have roughly the same size and both coefficients of $\phi(x)$ are bounded by A, then all $a_i u^i v^{d-i}$ have the same size.
- If the a_i are unbalanced, say $\frac{a_i}{a_{i+1}} \approx S > 1$, then with $|u| < A\sqrt{S}$ and $|v| < A/\sqrt{S}$, all $a_i u^i v^{d-i}$ have the same size.



The ratio S is called the skewness; the polynomials are skewed.

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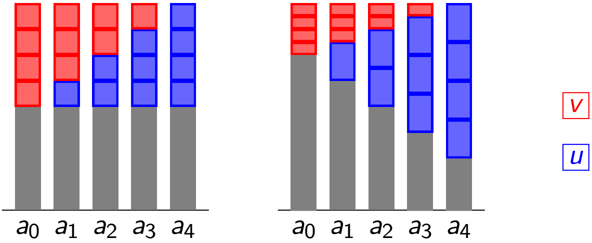

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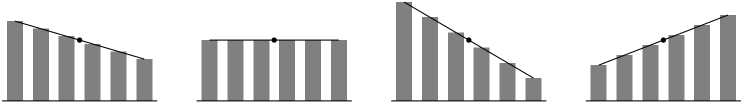
Skewed norm

Definition

Given $P = \sum p_i x^i \in \mathbb{R}[x]$, the S-skewed (infinity) norm of P is:

$$\|P\|_S = \|P\|_{\infty,S} = \max_{0 \leq i \leq \deg P} |p_i S^{i-(\deg P)/2}|.$$



$S = 100$ $S = 1$ $S = 10000$ $S = 0.01$

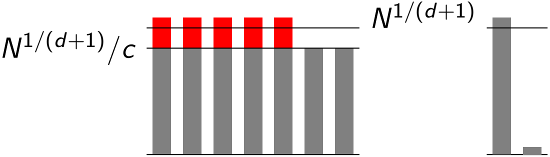
All polynomials above have the same S-skewed norms (with their respective S). If $\|P\| = \|Q\|_S$, then

$$\max\{\text{Res}(u - vx, P),\ (u, v) \in [0, A]^2\}$$
$$= \max\{\text{Res}(u - vx, Q), (u, v) \in [0, A\sqrt{S}] \times [0, A/\sqrt{S}]\}.$$

How do we find skewed polynomials?

When we revisited base-m, we chose a_d first, and then m.
This gave rise to:

- $a_d \approx a_{d-1} \approx N^{1/(d+1)}/c$.
- $m = \sqrt[d]{N/a_d} \geq N^{1/(d+1)} =$ the textbook base-m.



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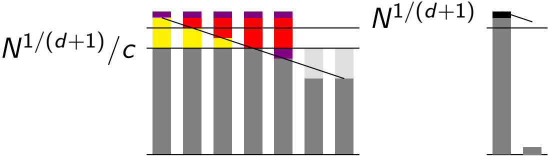

This polynomial pair is already somewhat skewed, we may turn it to our advantage.

- Aim at the same skew-norm, starting from a smaller a_d (■):
  - bits we still have to cancel (■),
  - bits we no longer care about (■),
  - new bits to cancel (■),
  - a moderately larger rational norm because m got larger (■),

Analysis of skewed base-m

Analysis is a bit painful, but the outcome is quite clear:

With the same number of trials, we can expect to find smaller skewed-norms that in the non-skewed base-m.
$C^{1/(d+1)}$ is replaced by a mildly larger number

Refinements:

- Do not optimize a_0.
- Rationale: this makes it possible to form many linear combinations like $f_0 + tf_1$ and choose the best one.
  We'll get to that with root properties and root sieving.

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Better polynomials: non-monic f_0

In fact, f_0 can be non-monic:

$$f_0 = m_1 x - m_0.$$

Then, the common root modulo N must be $m = m_0/m_1$ and

$$\mathrm{Res}(f_1, m_1 x - m_0) = a_d m_0^d + a_{d-1} m_1 m_0^{d-1} + \cdots + a_0 m_1^d.$$

Remark: if the latter is equal to N, it implies

$$a_d m_0^d \equiv N \mod m_1.$$

First ingredient of Kleinjung's algorithms (2006 and 2008) is called Kleinjung "Lemma 2.1". It computes a reasonably good f_1 from a fixed choice of N, d, m_1, m_0 and a_d.

Kleinjung "Lemma 2.1"

Input: N, d, m_1, m_0, and some fixed coefficients $[a_j, \ldots, a_d]$

Output: A polynomial f_1 such that $\text{Res}(f_1, m_1 x - m_0) = N$

First, compute

$$r_j = \frac{N - \sum_{i=j+1}^{d} a_i m_0^i m_1^{d-i}}{m_1^{d-j}}$$

Then, for $i = j-1, j-2, \ldots, 0$, compute:

- $r_i = \dfrac{r_{i+1} - a_{i+1} m_0^{i+1}}{m_1}$
- $a_i = \dfrac{r_i + t_i m_1}{m_0^i}$, where t_i is an integer such that

$$-\frac{m_0^i}{2} \leq t_i < \frac{m_0^i}{2} \text{ and } t_i \equiv -\frac{r_i}{m_1} \bmod m_0^i$$

The output is $f_1 = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0$.

Kleinjung "Lemma 2.1" – example

Kleinjung's "Lemma 2.1" algorithm applied to RSA-155 with $d = 5$ and

$$
\begin{aligned}
a_5 &= 358870426380 \\
m_0 &= 3139277687091176945951530919 8 \\
m_1 &= 823916492006383
\end{aligned}
$$

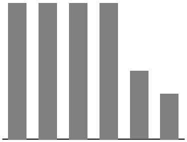
gives:

$$
\begin{aligned}
f_1 = {} & 358870426380x^5 \\
& + 428308592054328x^4 \\
& - 16336877672072510723154851996x^3 \\
& - 12601611387318107328006122118x^2 \\
& - 19621855499511523845845304751x \\
& - 83697637854955959853045028 99
\end{aligned}
$$

Output of Kleinjung "Lemma 2.1"

If we apply Kleinjung "Lemma 2.1" with only the leading coefficient a_d fixed and with m_0 close to $\tilde{m}_0 = \sqrt[d]{\frac{N}{a_d}}$, the algorithm yields:

- a_{d-1} rather small: $|a_{d-1}| < m_1 + da_d \frac{m_0 - \tilde{m}_0}{m_1}$.
- Other a_i's satisfy $|a_i| < m_1 + m_0$.



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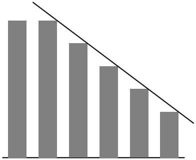

Our goal, and how we reach it

Have coefficient sizes which are a good match to some skewness.

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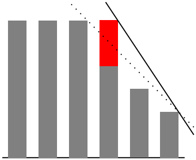

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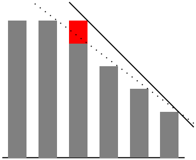

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- Find smart way to make a_{d-2} small.

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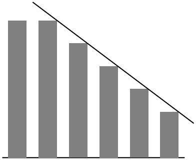

Our goal, and how we reach it

Have coefficient sizes which are a good match to some skewness.

- Find smart way to make a_{d-2} small.
- Rely on (mild) luck to make a_{d-3} small.

Making a_{d-2} small

Use the equation $a_d m_0^d \equiv N \mod m_1$.

Key idea

Build m_1 as a product of small primes. Use the combination of modular information to fabricate a small a_{d-2}.

- Let $\mathcal{P}$ be a set of small primes $\equiv 1 \mod d$ (m_1 will be a product of a subset of $\mathcal{P}$).
- Pick some a_d (e.g. smooth).
- Some primes $r \in \mathcal{Q} \subset \mathcal{P}$ give d solutions to $a_d x^d \equiv N \mod r$.
- Any choice of exactly one d-th root modulo each of those r's gives a value m_0 defined modulo $m_1 = \prod r$ by CRT.
  We may choose one which is close to $\tilde{m}_0 = \sqrt[d]{N/a_d}$.

Making a_{d-2} small

Many choices

Pick ℓ primes for which $a_d x^d \equiv N \mod r$ has d solutions.

- In total, d^ℓ possible choices for m_0.
- m_0 is a linear combination of ℓ values among $d \times \ell$.
  This follows from explicit Chinese Remainder Theorem.

Expand the value a_{d-2}/m_0 obtained by "Lemma 2.1" from the d-th roots of $m_0 \mod r$ that we have chosen.

- By restricting to 1st order terms, we get a linear combination.
- If a_{d-2}/m_0 ends up being close to an integer λ for some chosen m_0, then for $f_1' = f_1 - \lambda(m_1 x - m_0)x^{d-2}$, we have:
  - a'_{d-2}/m_0 close to 0,
  - a'_{d-1} does not change much.

Finding combinations that are close to $\mathbb{Z}$

The problem can be reduced to the following:

- ℓ sets $S_1, \ldots, S_\ell$, each containing d real numbers in $[0, 1)$.
- d^ℓ choices: $(s_1, \ldots, s_\ell)$ with each $s_i \in S_i$, and:

$$a_{d-2}/m_0 \bmod \mathbb{Z} \equiv \sum s_i.$$

- Naive complexity: $O(d^\ell)$.
- Better:

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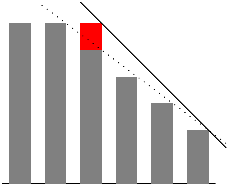
- Naive complexity: $O(d^\ell)$.
- Better: $O(d^{\ell/2})$.

What do small combinations give ?

Algorithm has:

- a_d chosen small.
- a_{d-1} small by construction, $\approx m_1$.
- a_{d-2} small thanks to small combinations.



With some extra luck,
a_{d-3} may be somewhat smaller than expected.