Online estimation: application to localization and mapping

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14.10.2011

Bayesian filtering

Objective:

- estimate value,
- along time,
- according to an observation,
- taking control into account;

Hidden Markov Models:

- time series,
- discrete states,
- no control.

Bayesian filtering

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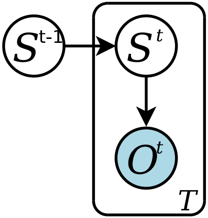
Baysian filtering

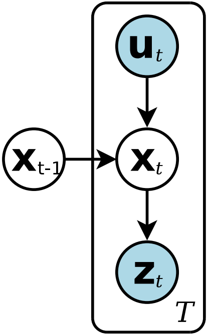
Adding control U_t:

HMM



$P(S^{t+1}|\mathbf{O}^{0:t+1})$
$\propto P(O^{t+1}|S^{t+1})$
$\times \sum_{S^t} P(S^{t+1}|S^t)P(S^t|\mathbf{O}^{0:t})$

Baysian filter



$P(\mathbf{x}_{t+1}|\mathbf{z}_{0:t+1}, \mathbf{u}_{0:t+1})$
$\propto P(\mathbf{z}_{t+1}|\mathbf{x}_{t+1})$
$\times \sum_{\mathbf{x}_t} P(\mathbf{x}_{t+1}|\mathbf{x}_t, \mathbf{u}_t)P(\mathbf{x}_t|\mathbf{z}_{0:t}, \mathbf{u}_{0:t})$

Bayesian filtering

Estimation:

$$\begin{aligned} & P(\mathbf{x}_t|\mathbf{z}_{0:t}, \mathbf{u}_{0:t}) \\ \propto\ & P(\mathbf{z}_t|\mathbf{x}_t) \\ \times\ & \sum_{\mathbf{x}_{t-1}} P(\mathbf{x}_t|\mathbf{x}_{t-1}, \mathbf{u}_{t-1}) P(\mathbf{x}_{t-1}|\mathbf{z}_{0:t-1}, \mathbf{u}_{0:t-1}) \end{aligned}$$

Distributions:

- $P(\mathbf{z}_t|\mathbf{x}_t)$: observation model,
- $P(\mathbf{x}_t|\mathbf{x}_{t-1}, \mathbf{u}_{t-1})$: transition model including command,
- $P(\mathbf{x}_{t-1}|\mathbf{z}_{0:t-1}, \mathbf{u}_{0:t-1})$: previous state estimate;

Characteristics:

- more general than HMM,
- can work with continuous state space,
- inference is integrating on the state space.

Bayesian filtering

Estimation:

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\begin{aligned}
& P(\mathbf{x}_t | \mathbf{z}_{0:t}, \mathbf{u}_{0:t}) \\
\propto \quad & P(\mathbf{z}_t | \mathbf{x}_t) \\
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Problem:

- complexity of the integration;

Solutions:

- make assumptions to find a closed-form expression,
- approximate (class 9).

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Bayesian filtering

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Kalman filter

Kalman filter:

- specific case of Bayesian filter,
- linear and Gaussian assumptions,
- closed-form expression for inference;

Uses (with variants):

- signal processing,
- attitude estimation,
- pose estimation,
- map estimation,
- computer vision,
- rockets,
- ...

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Kalman filter

Assumptions:

- states and observations: vectors of real numbers,
- all distributions are Gaussian,
- linear dynamic model,
- linear observation model,
- Markov assumption (Bayesian filter);

In other words:

- $P(\mathbf{x}_t|\mathbf{x}_{t-1},\mathbf{u}_t) = \mathcal{N}(\mathbf{F}\mathbf{x}_{t-1} + \mathbf{B}\mathbf{u}_t, \mathbf{Q})$,
- $P(\mathbf{z}_t|\mathbf{x}_t) = \mathcal{N}(\mathbf{H}\mathbf{x}_t, \mathbf{R})$

where:

- $\mathcal{N}(\mu, \mathbf{\Sigma})$ is the Gaussian distribution with mean μ and covariance matrix $\mathbf{\Sigma}$,
- $\mathbf{F}$, $\mathbf{B}$ and $\mathbf{H}$ are matrices,
- $\mathbf{Q}$ is the covariance matrix on the state transition model,
- $\mathbf{R}$ is the covariance matrix on the observation model.

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Models

Transition model:

$$P(\mathbf{x}_t | \mathbf{x}_{t-1}, \mathbf{u}_t) = \mathcal{N}(\mathbf{F}\mathbf{x}_{t-1} + \mathbf{B}\mathbf{u}_t, \mathbf{Q})$$

$$\mathbf{x}_t = \mathbf{F}\mathbf{x}_{t-1} + \mathbf{B}\mathbf{u}_t + \mathbf{w}_t$$

where $\mathbf{w}_t \leftarrow \mathcal{N}(0, \mathbf{Q})$.

Observation model:

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Both models:

- linear,
- Gaussian noise.

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& P(\mathbf{x}_t|\mathbf{z}_{0:t}, \mathbf{u}_{0:t}) \\
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Note:

- convolution by a Gaussian,
- product with a Gaussian;

Consequence:

- if prior is Gaussian: posterior also,
- mean and covariance are sufficient,
- closed-form expression for mean and covariance.

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Inference

More notations:

- $\hat{\mathbf{x}}_{t|t}$: mean of $P(\mathbf{x}_t|\mathbf{z}_{0:t}, \mathbf{u}_{0:t})$,
- $\mathbf{P}_{t|t}$: covariance of $P(\mathbf{x}_t|\mathbf{z}_{0:t}, \mathbf{u}_{0:t})$,
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- $\tilde{\mathbf{y}}_t$: mean of innovation,
- $\mathbf{S}_t$: covariance of innovation,
- $\mathbf{K}_t$: Kalman gain.

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Kalman filter algorithm

Prediction:

$$\hat{\mathbf{x}}_{t|t-1} \leftarrow \mathbf{F}\hat{\mathbf{x}}_{t-1|t-1} + \mathbf{B}\mathbf{u}_t$$
$$\mathbf{P}_{t|t-1} \leftarrow \mathbf{F}\mathbf{P}_{t-1|t-1}\mathbf{F}^{\mathrm{T}} + \mathbf{Q}$$

Update:

$$\tilde{\mathbf{y}}_t \leftarrow \mathbf{z}_t - \mathbf{H}\hat{\mathbf{x}}_{t|t-1}$$
$$\mathbf{S}_t \leftarrow \mathbf{H}\mathbf{P}_{t|t-1}\mathbf{H}^{\mathrm{T}} + \mathbf{R}$$
$$\mathbf{K}_t \leftarrow \mathbf{P}_{t|t-1}\mathbf{H}^{\mathrm{T}}\mathbf{S}_t^{-1}$$
$$\hat{\mathbf{x}}_{t|t} \leftarrow \hat{\mathbf{x}}_{t|t-1} + \mathbf{K}_t\tilde{\mathbf{y}}_t$$
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Example

Train:

- estimate 1D position x_t,
- linear speed command u_t,
- noisy distance observation z_t;

Specifications:

- Transition model:
  - $\mathbf{F} = 1$,
  - $\mathbf{B} = \Delta t = 1$,
  - $\mathbf{Q} = 1$;
- Observation model:
  - $\mathbf{H} = 1$,
  - $\mathbf{R} = 1$.

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Example

Start:

- $\hat{x}_{0|0} = 0$,
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Prediction:

- $\hat{x}_{1|0} = \mathbf{F}\hat{x}_{0|0} + \mathbf{B}u_1 = 1$
- $\mathbf{P}_{1|0} = \mathbf{F}\mathbf{P}_{0|0}\mathbf{F}^{\mathrm{T}} + \mathbf{Q} = 2$

Example

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Update:

- $\tilde{y}_1 = z_1 - \mathbf{H}\hat{x}_{1|0} = 0.1$
- $\mathbf{S}_1 = \mathbf{H}\mathbf{P}_{1|0}\mathbf{H}^{\mathrm{T}} + \mathbf{R} = 3$
- $\mathbf{K}_1 = \mathbf{P}_{1|0}\mathbf{H}^{\mathrm{T}}\mathbf{S}_1^{-1} = 0.667$
- $\hat{x}_{1|1} = \hat{x}_{1|0} + \mathbf{K}_1\tilde{y}_1 = 1.067$
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Example

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- $u_1 = 1,$
- $z_1 = 1.1;$
- $\hat{x}_{1|1} = 1.067,$
- $\mathbf{P}_{1|1} = 0.667;$
- $u_2 = 1,$
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- $\tilde{y}_2 = 0.233$
- $\mathbf{S}_2 = 2.667$
- $\mathbf{K}_2 = 0.625$
- $\hat{x}_{2|2} = 2.213$
- $\mathbf{P}_{2|2} = 0.625$

Limitations

Limitations:

- planar pose update:

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \\ \theta_{t+1} \end{pmatrix} = \begin{pmatrix} x_t + \frac{v_t}{\omega_t}(\sin(\theta_t + \omega_t \Delta t) - \sin\theta_t) \\ y_t - \frac{v_t}{\omega_t}(\cos(\theta_t + \omega_t \Delta t) - \cos\theta_t) \\ \theta_t + \omega_t \Delta_t \end{pmatrix}$$

- uncertainty for distance measurements:
- throttle or acceleration command: $\mathbf{x} = x$ and $\mathbf{u} = a$

$$x_{t+1} = x_t + ???$$

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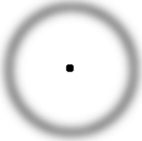

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$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \\ \theta_{t+1} \end{pmatrix} = \begin{pmatrix} x_t + \frac{v_t}{\omega_t}(\sin(\theta_t + \omega_t \Delta t) - \sin\theta_t) \\ y_t - \frac{v_t}{\omega_t}(\cos(\theta_t + \omega_t \Delta t) - \cos\theta_t) \\ \theta_t + \omega_t \Delta_t \end{pmatrix}$$

- uncertainty for distance measurements:
- throttle or acceleration command: $\mathbf{x} = x$ and $\mathbf{u} = a$

$$x_{t+1} = x_t + ???$$

Hypotheses and extensions

Hypotheses:

- linearity,
- Gaussian distributions,
- Order-1 Markov;

Extensions:

- Extended Kalman Filter (EKF) when slightly non-linear,
- Unscented Kalman Filter (UKF) when highly non-linear,
- enrich the state to transform order-k Markov into order-1 Markov,
- Kalman-Bucy filter for continuous time,
- (Extended) Information Filter for efficiency for several measurements,
- ...

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Extended Kalman Filter

Kalman Filter:

$$\mathbf{x}_t = \mathbf{F}\mathbf{x}_{t-1} + \mathbf{B}\mathbf{u}_t + \mathbf{w}_t$$

$$\mathbf{z}_t = \mathbf{H}\mathbf{x}_t + \mathbf{v}_t$$

Extended Kalman Filter:

$$\mathbf{x}_t = f(\mathbf{x}_{t-1}, \mathbf{u}_t) + \mathbf{w}_t$$

$$\mathbf{z}_t = h(\mathbf{x}_t) + \mathbf{v}_t$$

Define:

- $\mathbf{F}_t = \frac{\partial f}{\partial \mathbf{x}}\big|_{\hat{\mathbf{x}}_{t-1|t-1}, \mathbf{u}_t}$
- $\mathbf{H}_t = \frac{\partial h}{\partial \mathbf{x}}\big|_{\hat{\mathbf{x}}_{t|t-1}}$

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Extended Kalman Filter

Kalman filter algorithm

Prediction:

$$\hat{\mathbf{x}}_{t|t-1} \leftarrow \mathbf{F}\hat{\mathbf{x}}_{t-1|t-1} + \mathbf{B}\mathbf{u}_t$$
$$\mathbf{P}_{t|t-1} \leftarrow \mathbf{F}\mathbf{P}_{t-1|t-1}\mathbf{F}^{\mathrm{T}} + \mathbf{Q}$$

Update:

$$\tilde{\mathbf{y}}_t \leftarrow \mathbf{z}_t - \mathbf{H}\hat{\mathbf{x}}_{t|t-1}$$
$$\mathbf{S}_t \leftarrow \mathbf{H}\mathbf{P}_{t|t-1}\mathbf{H}^{\mathrm{T}} + \mathbf{R}$$
$$\mathbf{K}_t \leftarrow \mathbf{P}_{t|t-1}\mathbf{H}^{\mathrm{T}}\mathbf{S}_t^{-1}$$
$$\hat{\mathbf{x}}_{t|t} \leftarrow \hat{\mathbf{x}}_{t|t-1} + \mathbf{K}_t\tilde{\mathbf{y}}_t$$
$$\mathbf{P}_{t|t} \leftarrow (I - \mathbf{K}_t\mathbf{H})\mathbf{P}_{t|t-1}$$

Extended Kalman Filter

Extended Kalman filter algorithm

Prediction:

$$\hat{\mathbf{x}}_{t|t-1} \leftarrow f(\hat{\mathbf{x}}_{t-1|t-1}, \mathbf{u}_t)$$
$$\mathbf{P}_{t|t-1} \leftarrow \mathbf{F}_t \mathbf{P}_{t-1|t-1} \mathbf{F}_t^{\mathrm{T}} + \mathbf{Q}$$

Update:

$$\tilde{\mathbf{y}}_t \leftarrow \mathbf{z}_t - h\left(\hat{\mathbf{x}}_{t|t-1}\right)$$
$$\mathbf{S}_t \leftarrow \mathbf{H}_t \mathbf{P}_{t|t-1} \mathbf{H}_t^{\mathrm{T}} + \mathbf{R}$$
$$\mathbf{K}_t \leftarrow \mathbf{P}_{t|t-1} \mathbf{H}_t^{\mathrm{T}} \mathbf{S}_t^{-1}$$
$$\hat{\mathbf{x}}_{t|t} \leftarrow \hat{\mathbf{x}}_{t|t-1} + \mathbf{K}_t \tilde{\mathbf{y}}_t$$
$$\mathbf{P}_{t|t} \leftarrow (I - \mathbf{K}_t \mathbf{H}_t) \mathbf{P}_{t|t-1}$$

Summary

Kalman filter:

- Bayesian filter with Markov assumption,
- Linear models for update and observation,
- Gaussian distributions;

Extensions:

- Extended Kalman Filter for small non-linearities;

Uses:

- estimation,
- control,
- filtering,
- localization and mapping,
- GPS,
- ...

Localization and Mapping

Localization:

- definition:
  - find out where the robot is,
  - estimate the pose in a given reference frame given the sensor values;
- requirement:
  - map of the environment;

Mapping:

- definition:
  - build a map,
  - estimate a representation of the environment given the sensor values;
- requirement:
  - localization of the robot.

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Simultaneous Localization and Mapping

Simultaneous Localization and Mapping:

- localize in and map an unknown environment,
- jointly estimate pose and map given the sensor values.

Parallel Tracking and Mapping:

- localize in a partial map and update the map,
- computer vision and graphics.

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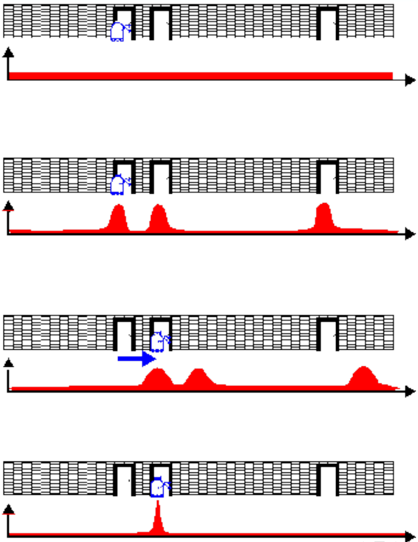
Markov Localization

Definition:

- localization with a Bayesian filter,
- state is the position of the robot,
- prediction based on actions,
- update based on observation;

Markov Localization



courtesy D. Fox

Markov Localization

Definition:

- localization with a Bayesian filter,
- state is the position of the robot,
- prediction based on actions,
- update based on observation;

Features:

- iterative computation,
- represent the state distribution:
  - Gaussian $\rightarrow$ Kalman filters,
  - histogram on fixed discretization,
  - samples: Monte-Carlo localization.

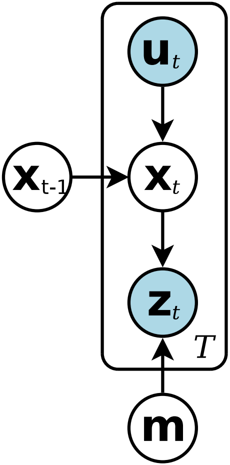
Simultaneous Localization and Mapping

Problem statement:

- joint estimation of pose and map;
- map parametrization:
  - dense: occupancy grid,
  - ...
  - sparse: landmarks;
- transition model;
- observation model;
- representation of the probablity distributions:
  - Gaussians: EKF-SLAM,
  - samples: fast-slam, Rao-Blackwellized Particle Filter...

Simultaneous Localization and Mapping

Common structure:



EKF-SLAM:

- landmarks: (x_i, y_i)
- map is the set of landmarks,
- include map into state vector,
- state: $\mathbf{x} = \begin{pmatrix} x \\ y \\ \theta \\ x_1 \\ y_1 \\ \vdots \\ x_N \\ y_N \end{pmatrix}$,
- apply EKF.

Summary on Localization and Mapping

Localization:

- estimate pose,

Mapping:

- estimate map,

SLAM:

- estimate both,
- several kinds of map:
  - occupancy grids,
  - landmarks,
- several algorithms:
  - EKF-SLAM,
  - Fast-SLAM,
  - Rao-Blackwellized Particle Filter (class 9),
  - PTAM,
  - ...

Summary

Bayesian filtering:

- generic estimation,
- Markov assumption,
- iterative inference;

Kalman filter:

- Bayesian filter,
- linear and Gaussian assumptions,
- extensions;

Localization and mapping:

- estimation problem,
- several variants.