The quest of the Domino problem

Guilhem Gamard

11th February 2021

Proving theorems by pattern recognition II

"In connection with the $\forall x \, \exists y \, \forall z$ *case, an amusing combinatorial problem is suggested in Section 4.1."*

— Hao Wang. *Proving theorems by pattern recognition II.* Bell Systems Technical Journal, XL(1), 1961.

$$\forall x,\ \exists y,\ \forall z : \Phi(x, y, z)$$

Φ is quantifier-free and may contain symbols of constant and of relation

No equality, no function symbols

Problem:

Find a model (ambient set, interpretations for symbols)

Example:

$$\forall x,\ \exists y,\ \forall z : \neg(x < x) \wedge (x < y) \wedge (y < z \implies x < z)$$

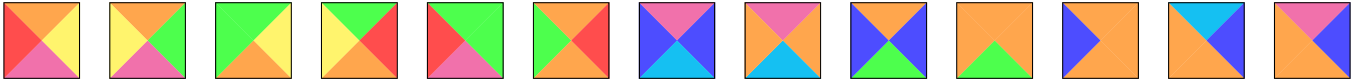
Satisfiability:

$x<x$	$x<y$	$x<z$	$y<x$	$y<y$	$y<z$	$z<x$	$z<y$	$z<z$
$\bot$	$\top$	$\top$			$\top$			
$\bot$	$\top$	$\top$			$\bot$			
$\bot$	$\top$	$\bot$			$\bot$			

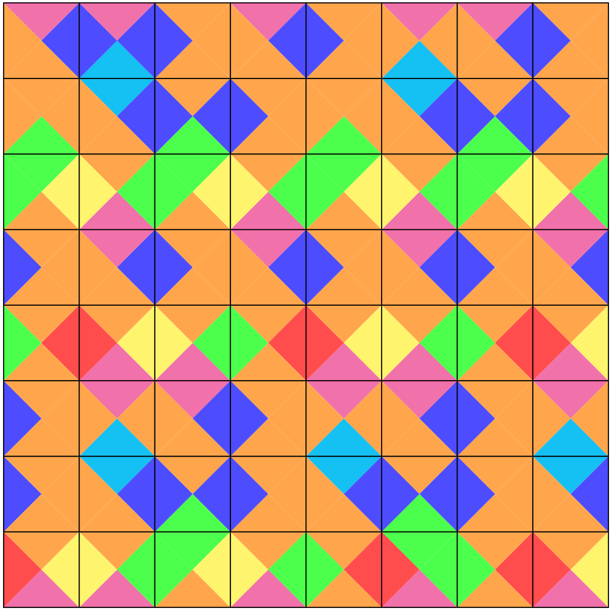
Wang tiles

Wang tile:

Tileset:



Tiling of a square:



- Borders in contact have the same color
- Tiles can be repeated
- Tiles cannot be turned nor flipped

The Domino problem

Theorem (Wang, 1964)

For all AEA formula F, there is a tileset T such that F has a model iff T has a tiling.

If F has a model, it is **satisfiable.** If T has a tiling, it is **valid.**

Problem definition

DOMINO

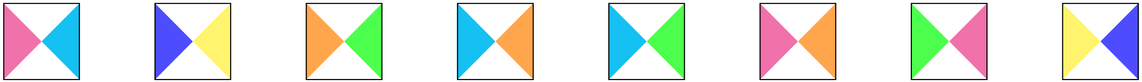
- **Input:** a tileset T
- **Output:** is T valid?

Let's find an algorithm for the Domino problem!

$\Longrightarrow$ **Automated theorem proving for AEA formulae!**

Warmup: the 1D case (1/2)

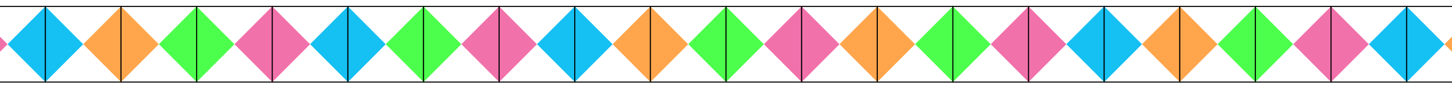
In 1D, tiles have 2 sides and we tile the line.



Lemma

If T is a valid 1D tileset, then it has a periodic tiling.

Infinitely many cells, finitely many colors: **a color is repeated.**



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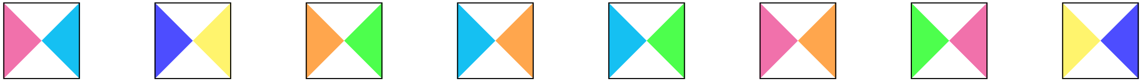

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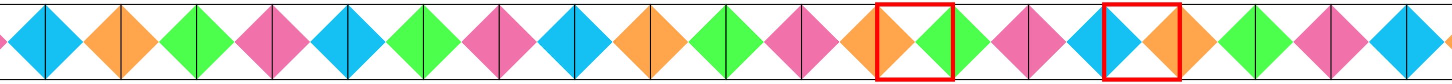

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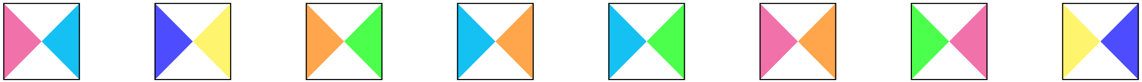

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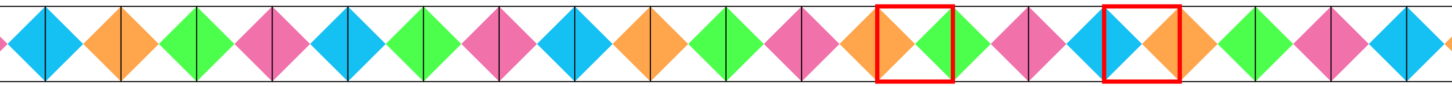
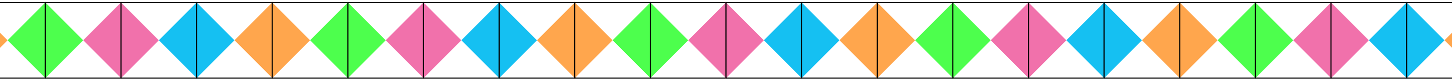


We can **repeat periodically.**



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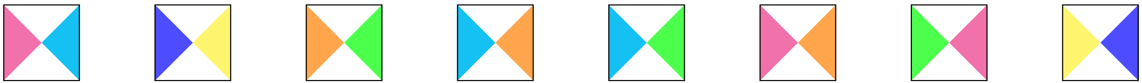

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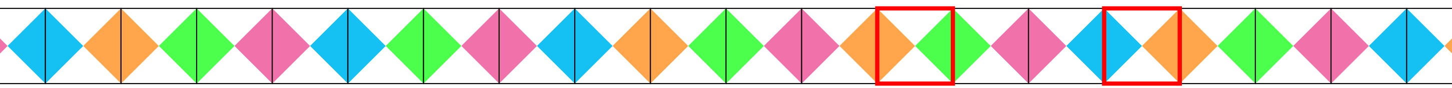

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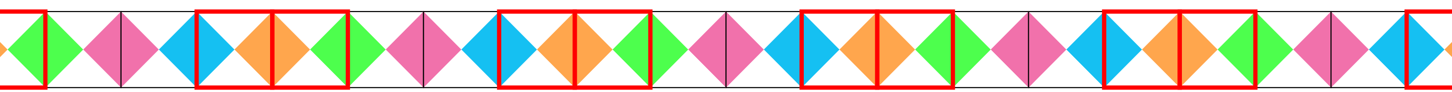
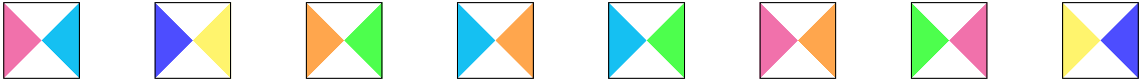
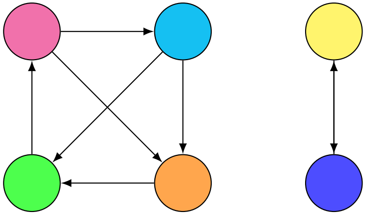


Warmup: the 1D case (2/2)



Given a 1D tileset T, we can build its **graph**.



- **Nodes:** colors
- **Edges:** tiles

Lemma

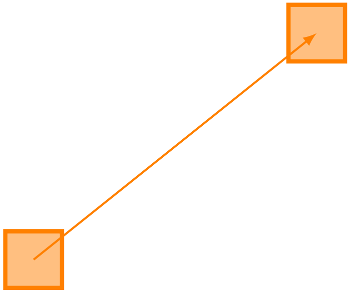
A 1D tileset is **valid** iff its graph has a **cycle.**

$\Longrightarrow$ **Polynomial time algorithm!**

Periodicity in 2D

Back to 2D!

Weakly periodic tiling: one direction of periodicity.



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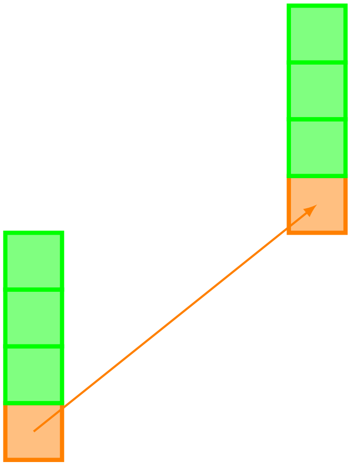

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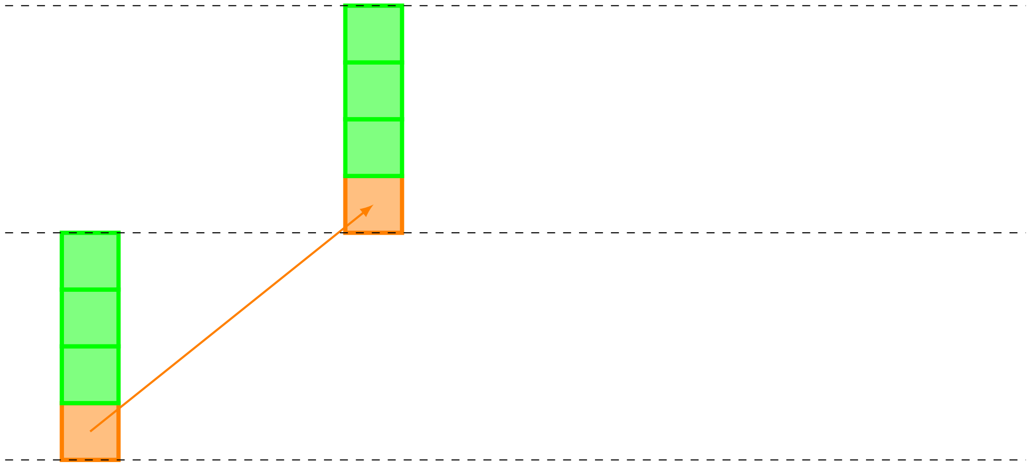

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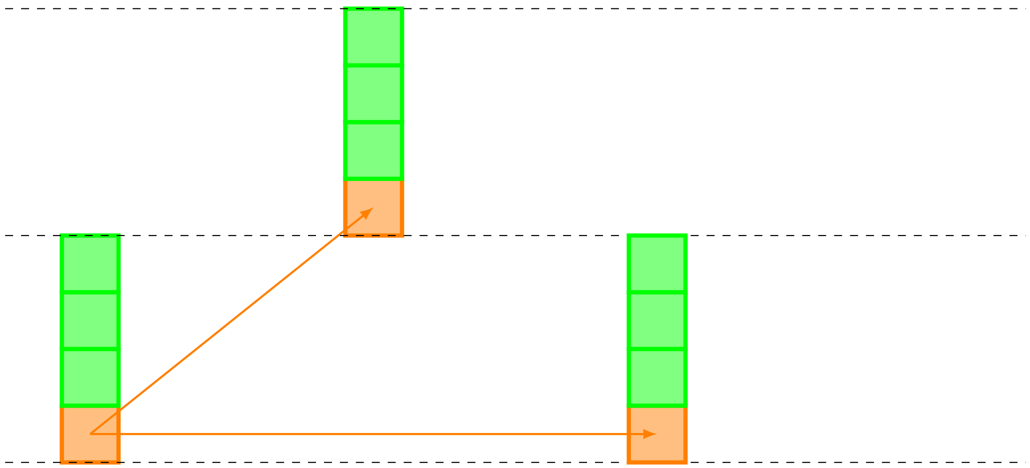

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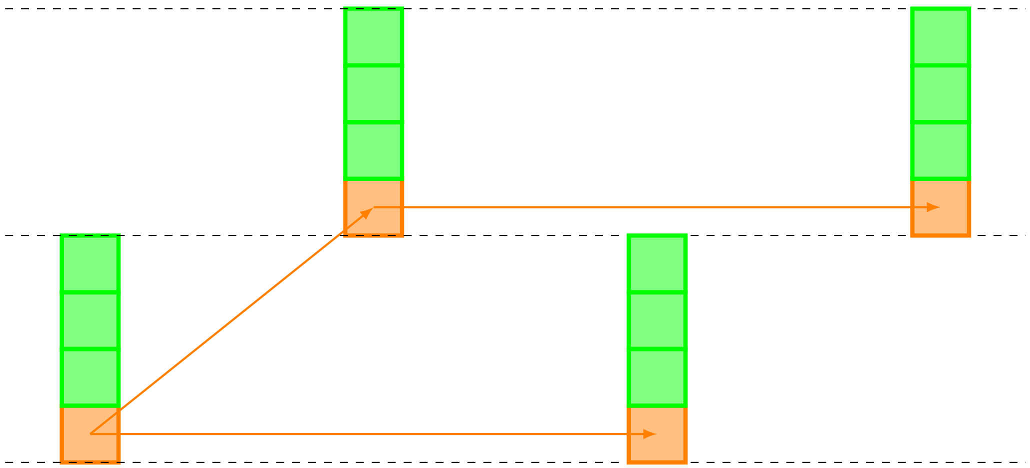

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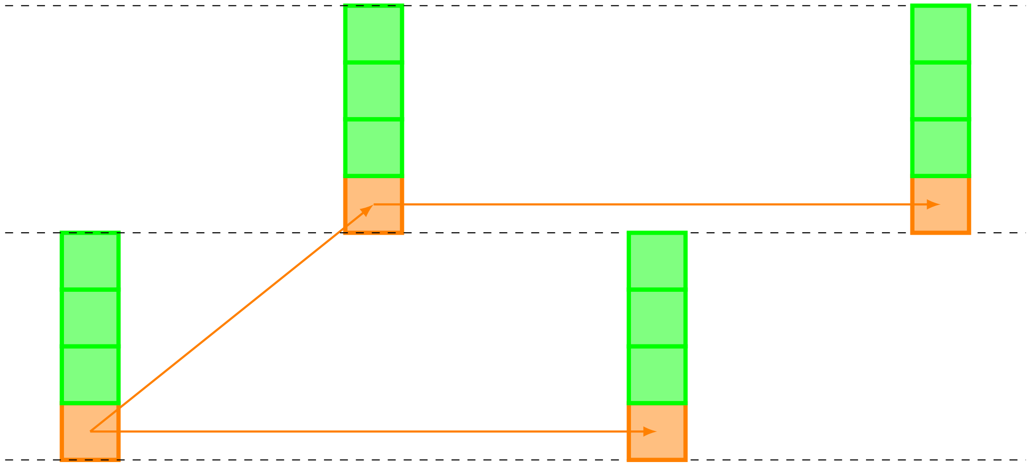

Strongly periodic tiling: two (noncolinear) directions of periodicity $\iff$ repeating blocks.

Periodicity in 2D

Back to 2D!

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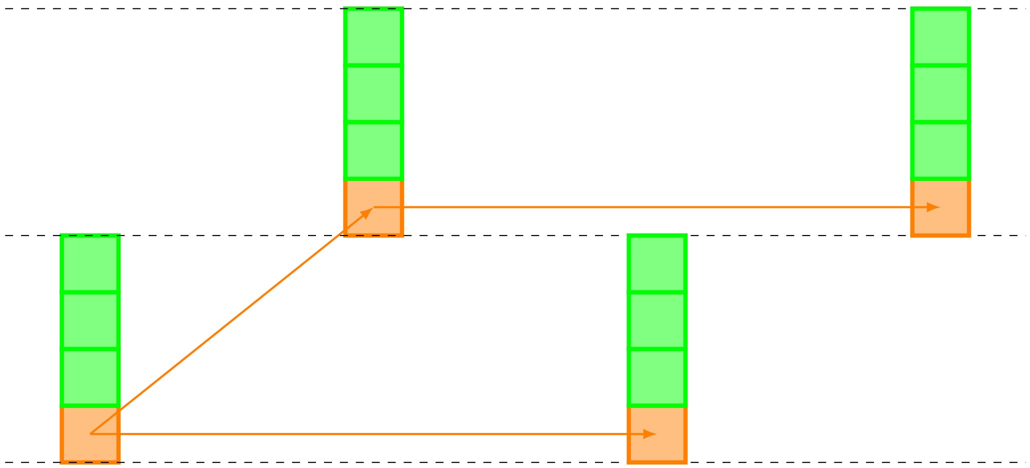


Strongly periodic tiling: two (noncolinear) directions of periodicity $\iff$ repeating blocks.

Lemma

If T has a **weakly** periodic tiling, then it has a **strongly** periodic tiling.

The compactness theorem

Theorem

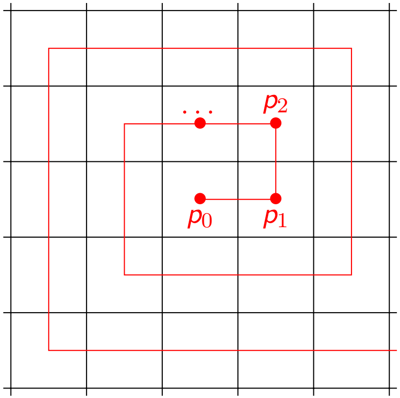
A tileset T is valid iff it tiles all finite squares.

The compactness theorem

Theorem

A tileset T is valid iff it tiles all finite squares.

- If T tiles the plane, it tiles all $n \times n$ squares (by restrictions)
- Let $p_0, p_1, \ldots$ be a spiral covering $\mathbb{Z}^2$ (say p_0 is the origin)
- Let s_n be a T-tiling of the $n \times n$ square
- Each s_n assigns a tile to p_0
- Infinitely many s_n assign a same tile to p_0



- Let $s_n^{(1)}$ be an infinite sequence of tiled squares that agree on p_0
- Let $s_n^{(2)}$ be an infinite sequence of tiled squares that agree on p_0 and p_1
- Et caetera

Wang's algorithm

Algorithm

1. $n \leftarrow 1$
2. Try all tilings of the $n \times n$ square
3. If there is a repeatable tiled square
   - Return `true`
4. If there is no tiled square
   - Return `false`
5. Else
   - $n \leftarrow n + 1$
   - Go to 2

- Finds **periodic** tilings

Plot twist

Theorem (Berger, 1966)

The Domino problem is undecidable.

Plot twist

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Corollary

Wang's algorithm doesn't stop on some inputs.

Plot twist

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There exist **aperiodic tilesets**.

(Valid tilesets with only aperiodic tilings.)

The long list of aperiodic tilesets

Tilings:

- Berger, 1964
- Knuth, 1968
- **Robinson, 1971**
- Penrose, 1974
- Ammann, 1977
- **Kari, 1996**
- Kari-Culik, 1996
- Jeandel-Rao, 2015
- Labbé, 2018

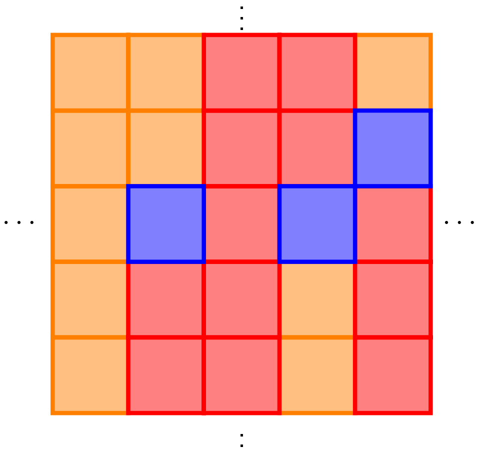
Methods:

- **Self-similarity**
- **Cut-and-project**
- **Computation on reals**

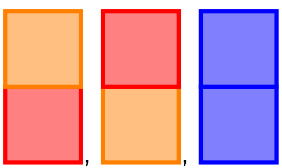
Intermission: Subshifts of Finite Type (SFTs)

Colors:

Configurations:



Patterns:



(possibly bigger)

Definition

A Subshift of Finite Type (**SFT**) is a set of configurations that avoids a finite set of **forbidden patterns.**

Proposition

Any tileset is an SFT.

Colors: Wang tiles
Forbidden patterns: Unmatched tiles

Any SFT is a tileset

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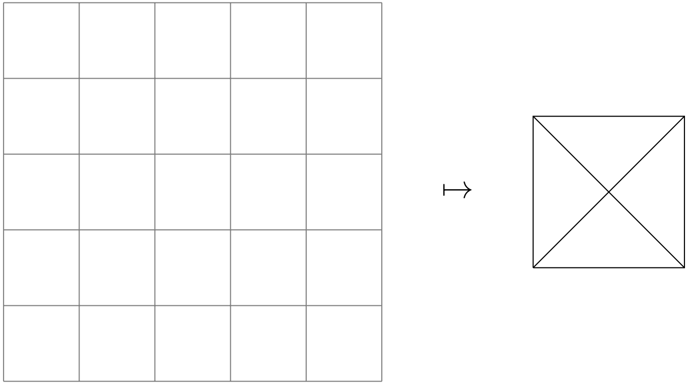
Any SFT is a tileset.

Suppose the largest forbidden pattern is $n \times n$.

Consider all $n \times n$ **allowed** patterns and make tiles as follows.



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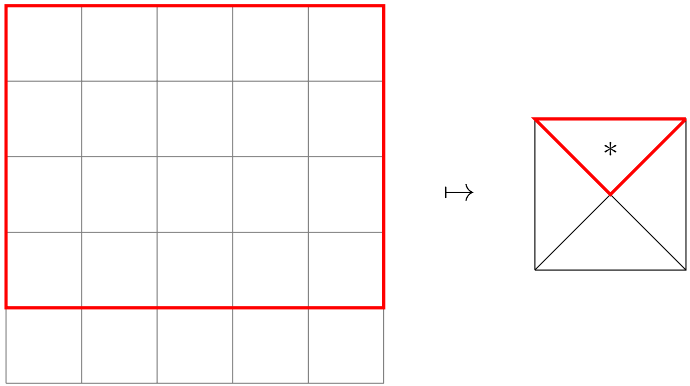

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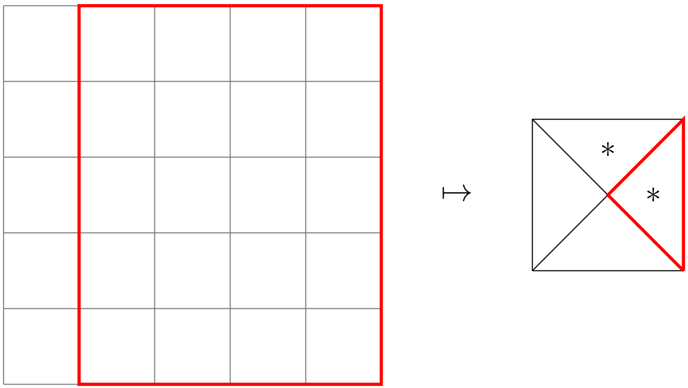

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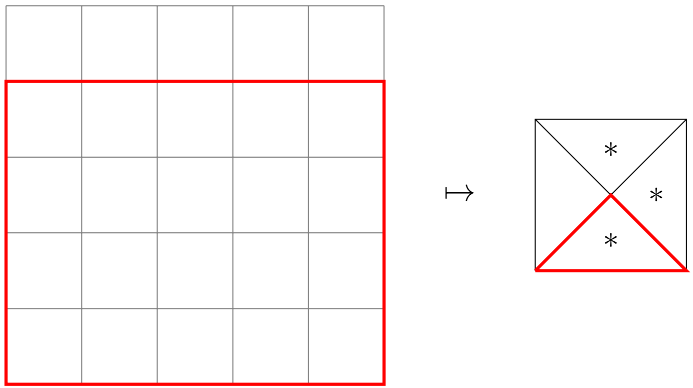

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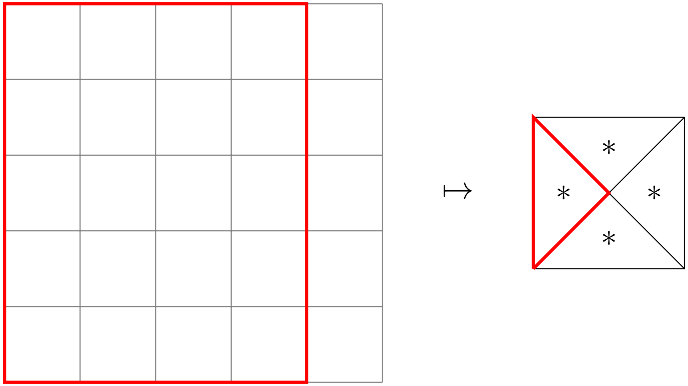

Domino with fixed origin

Definition

FIXED-ORIGIN DOMINO

- **Input:** a tileset T and a tile $t \in T$
- **Output:** is there a T-tiling with t at the origin?

Theorem

The FIXED-ORIGIN DOMINO problem is undecidable.

Fixed-origin Domino is undecidable

Fix a Turing machine $M = (\Sigma, Q, \delta, \dots)$.
Make a tileset such that every tiling is:

$\dots$

M after 3 steps

M after 2 steps

M after 1 step

Initial configuration of M

- Let $Q' = Q \sqcup \{\blacktriangleright, \blacktriangleleft\}$
- **Colors:** $Q' \times \Sigma$

- One head per line
  - Forbid: $q \blacktriangleright$; $\blacktriangleleft q$; $\blacktriangleleft\blacktriangleright$; $\blacktriangleright\blacktriangleleft$
- Transitions OK
  - Forbid all 3×2 patterns not allowed by δ

Fixed-origin Domino is undecidable

We have to ensure that the first line has state q_0 and only 0's on the tape

Suppose that M never reaches its initial state q_0 again.

New rules:

- Let $Q'' = Q' \sqcup \{\blacktriangleright_0, \blacktriangleleft_0\}$
- Mixing $\{\blacktriangleright_0, \blacktriangleleft_0\}$ with $\{\blacktriangleright, \blacktriangleleft\}$ on a same line is forbidden
- $\{\blacktriangleright_0, \blacktriangleleft_0\}$ ensure a single head like $\{\blacktriangleright, \blacktriangleleft\}$ did
- We're on a $\{\blacktriangleright_0, \blacktriangleleft_0\}$-line iff the head is q_0
- If we're on a $\{\blacktriangleright_0, \blacktriangleleft_0\}$-line, every tape letter is 0

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Now ask $(q_0, 0)$ at the origin.

Fixed-origin Domino is undecidable

But, wait! You're cheating! You only tile half of the plane!

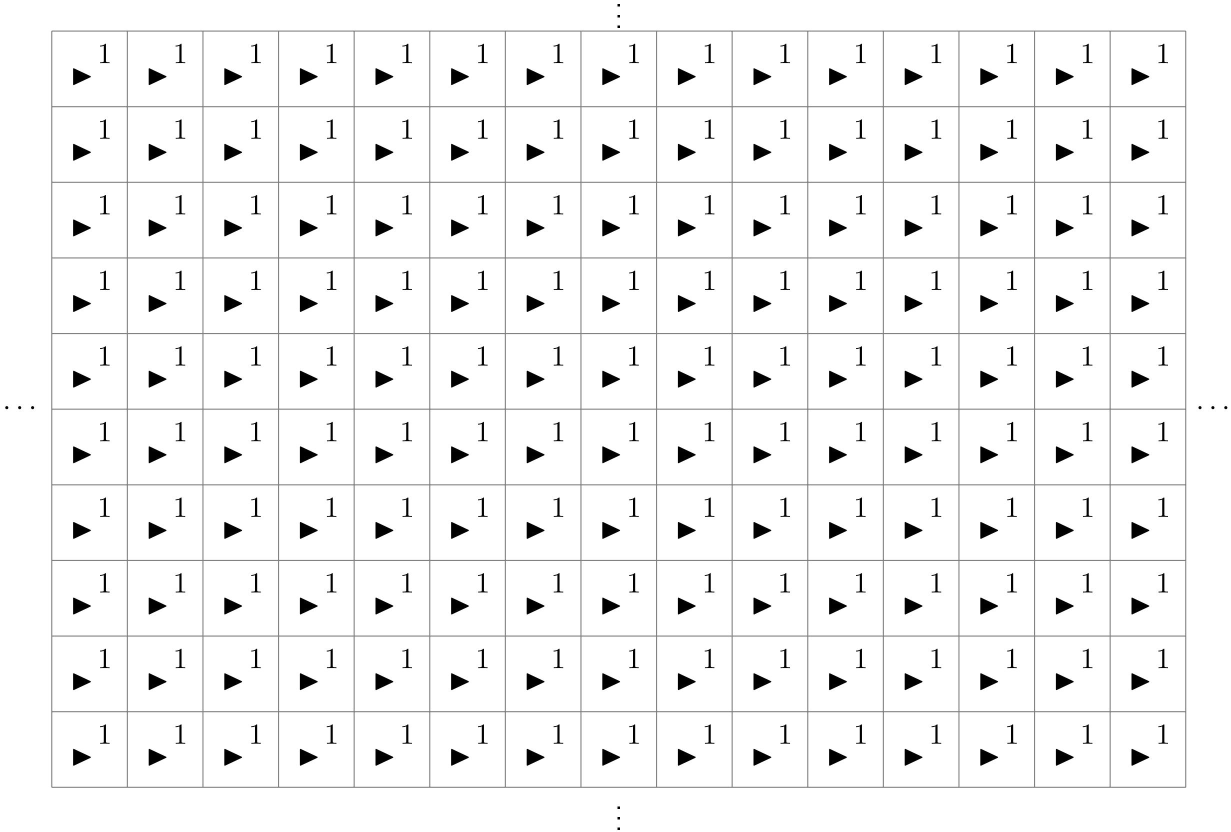
That's why compactness is for, baby!

- If we tile a half-plane,
- then we tile any $n \times n$ square,
- so by compactness we tile all the plane.

Lemma

- If M halts in n steps, we don't tile $(n+1) \times (n+1)$ squares.
- If M doesn't halt, we tile the plane.

What happens with free origin?

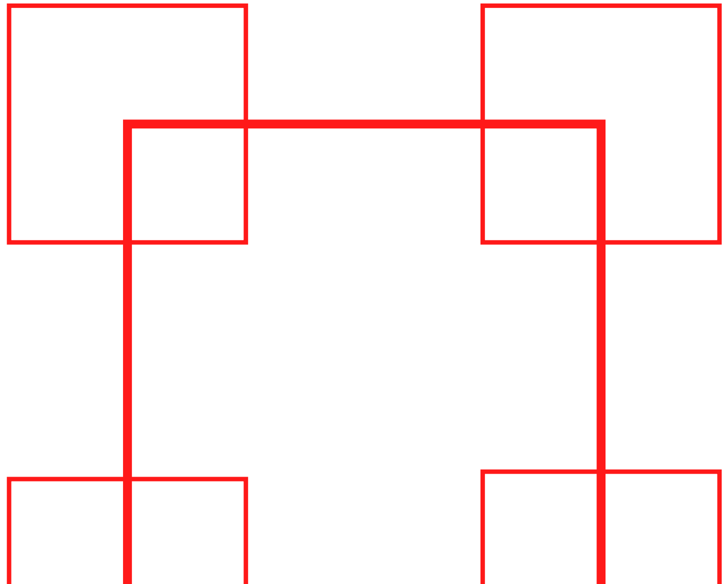


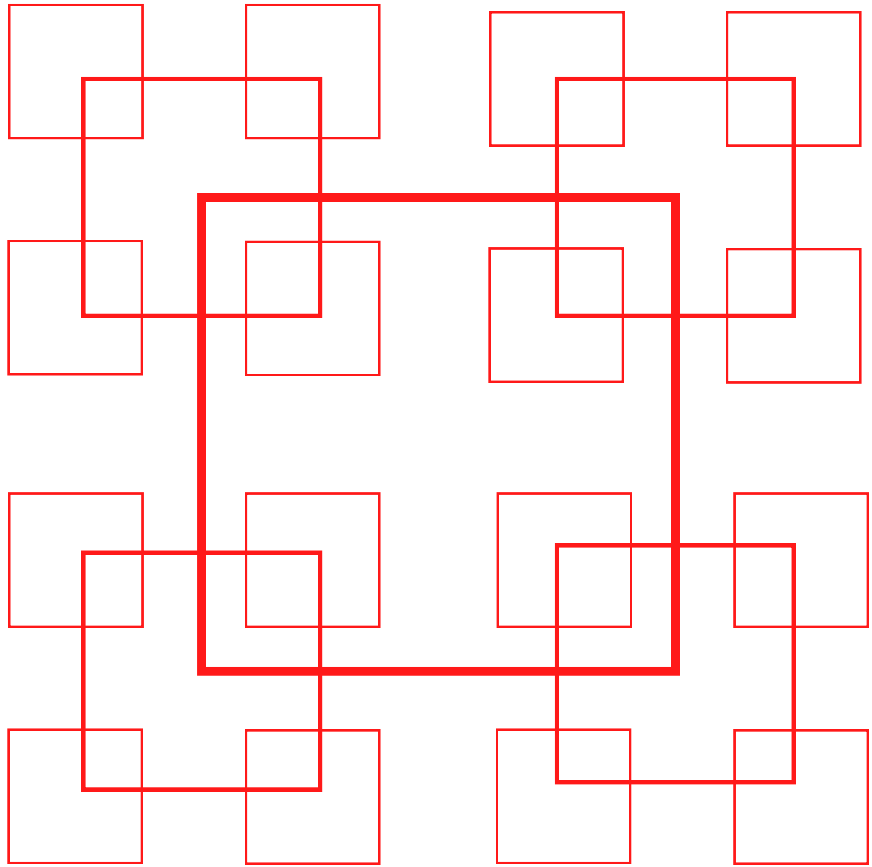
Aperiodic tilesets

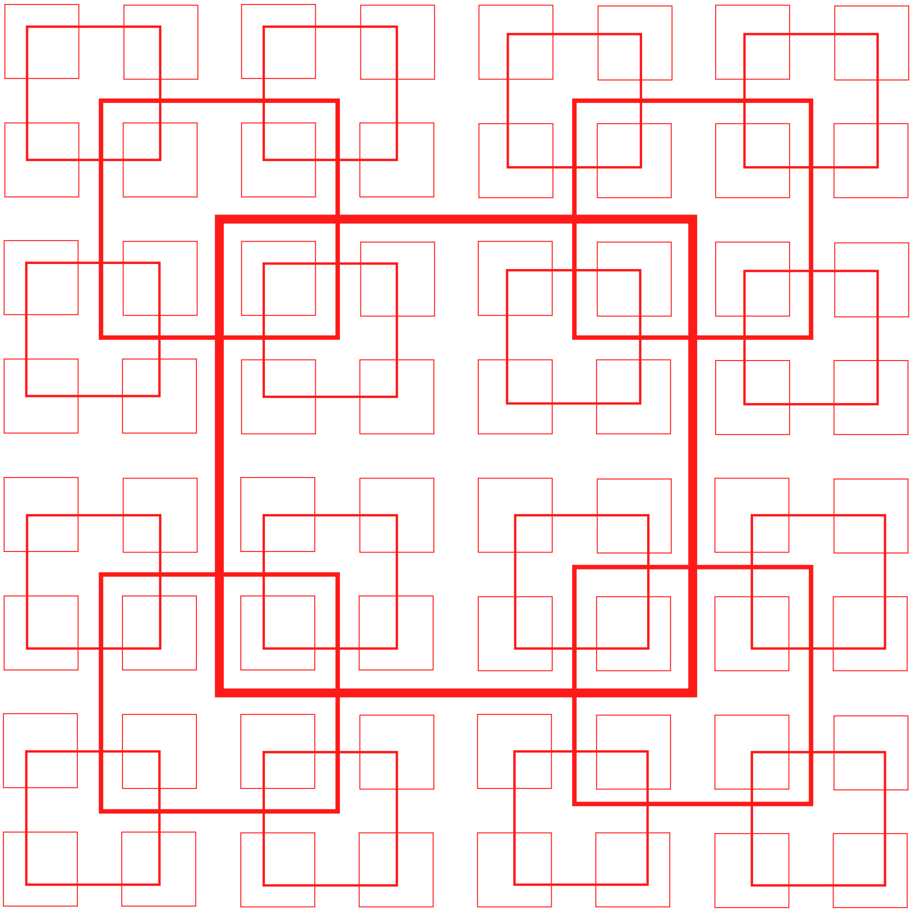
- We have “noncomputing” tilings
- They are all periodic
- Hence the link between aperiodic tilesets and undecidability

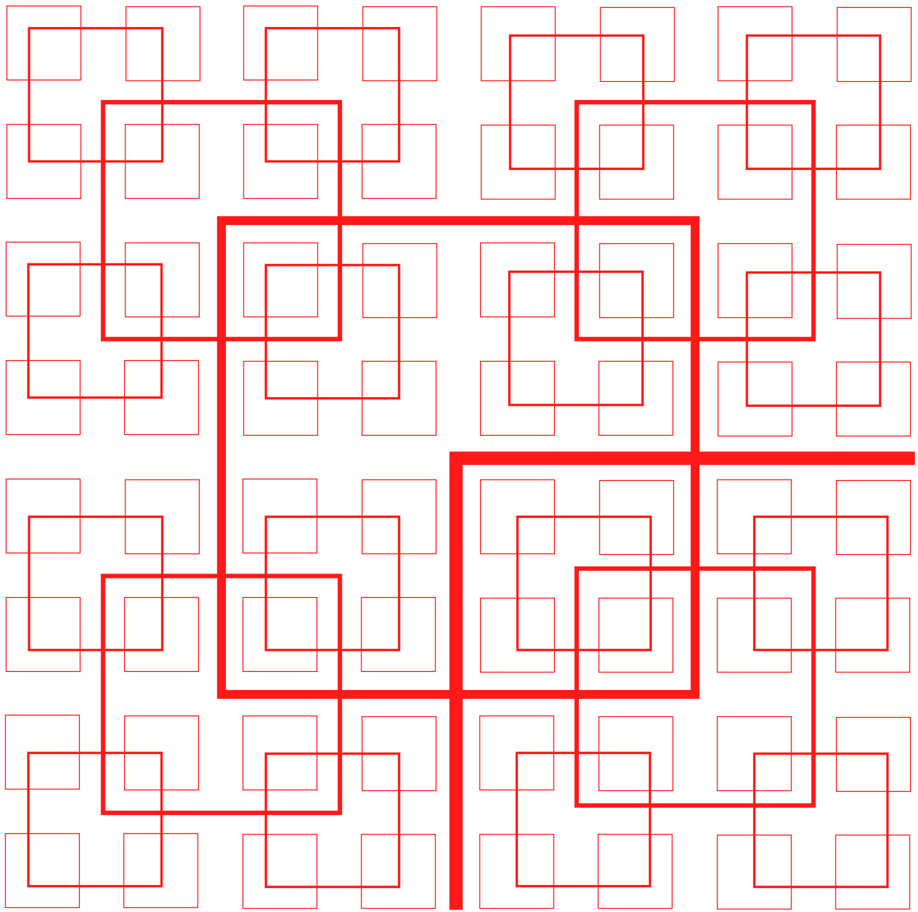
We need an aperiodic tileset
to prove that the Domino problem is undecidable

Pictures in the next slides are courtesy Daria S. Pchelina









Robinson tiling:

Fixed-point of that substitution

Proposition

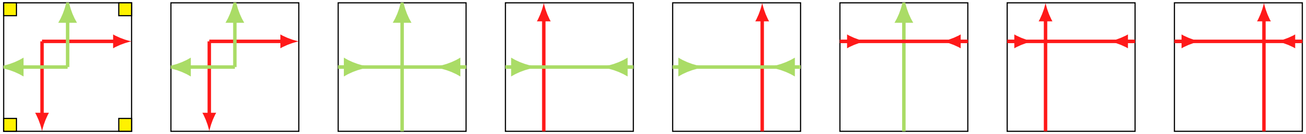
Any Robinson tiling is aperiodic.

- Any periodicity vector would send each red square to another
- There are arbitrarily large squares
- The periodicity vector would have to be infinite

How to implement this substitution with tiles?

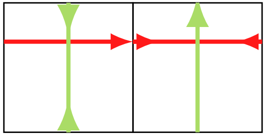
Robinson tiles

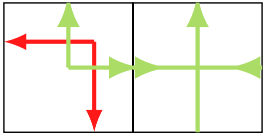
Tiles: (colors)

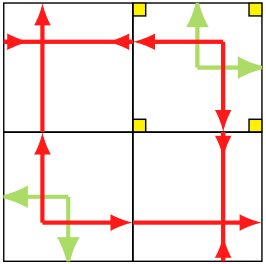


+ Rotations

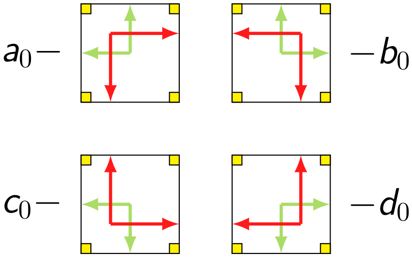
Rules: (forbidden patterns)



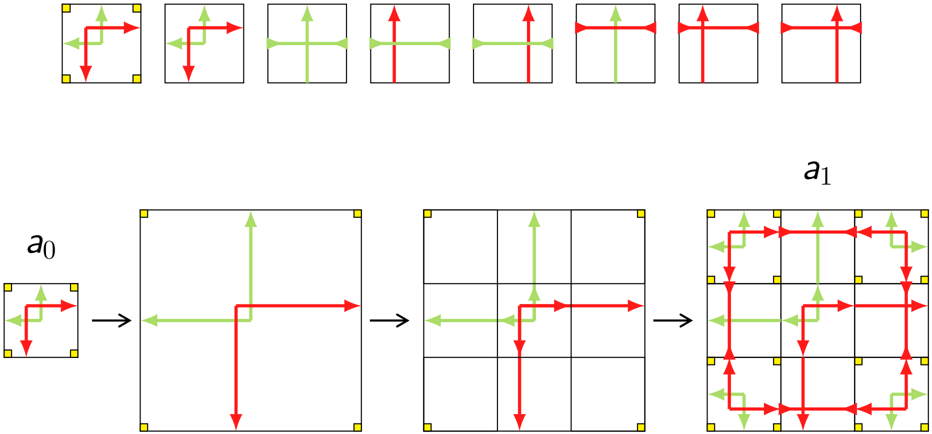




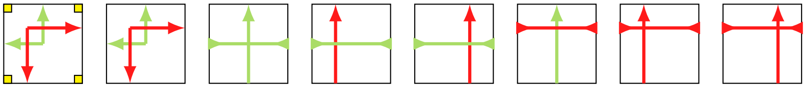
Orientations

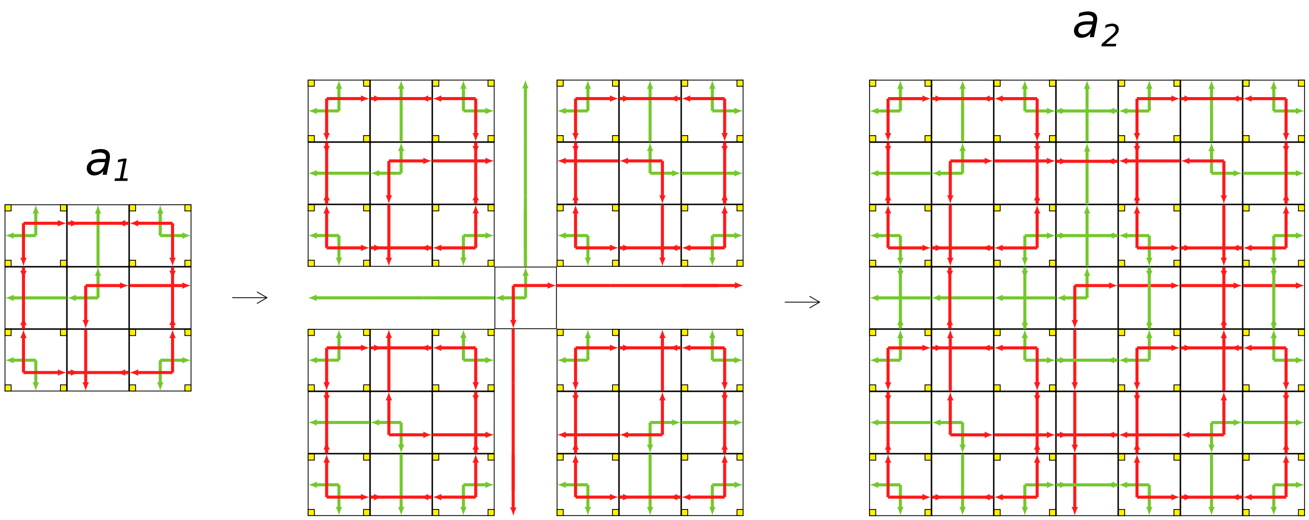


Macrotile of rank 2

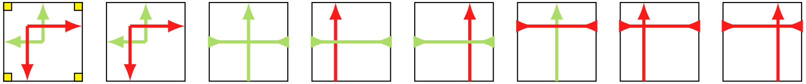


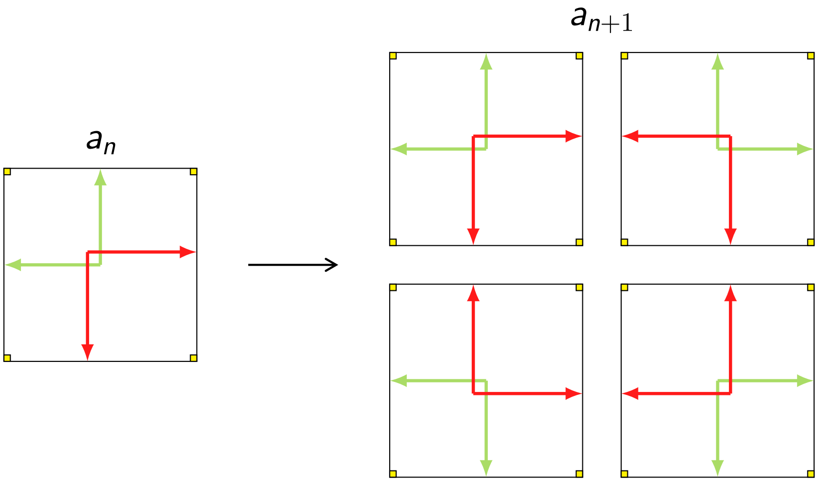
Macrotile of rank 3





Macrotile of rank n+1





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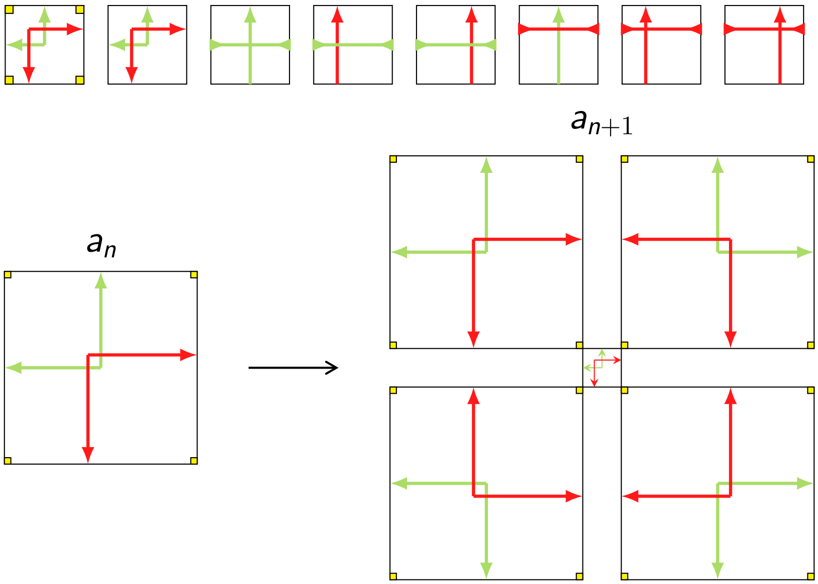

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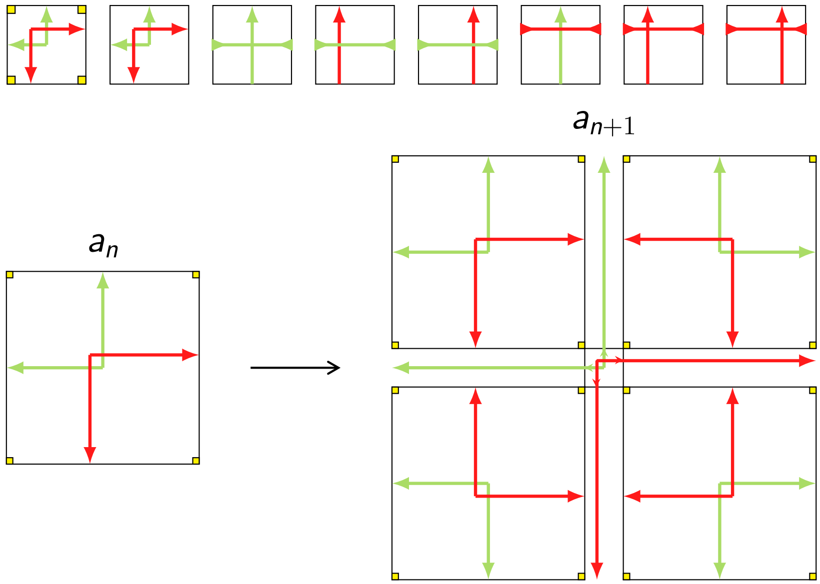

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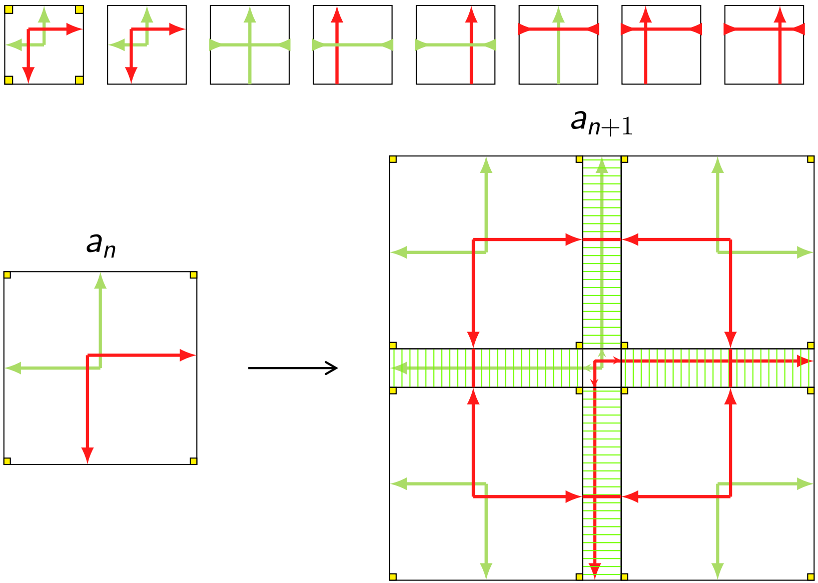
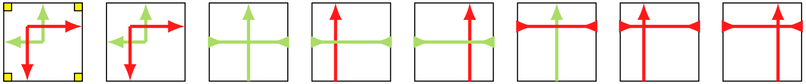
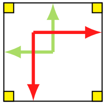


Local rules force macrotiles of rank 2





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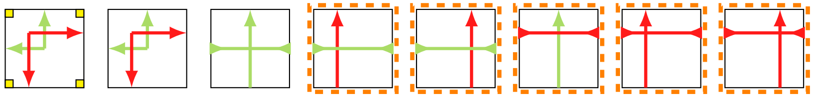

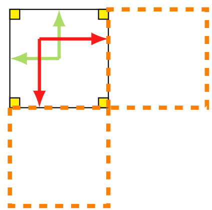

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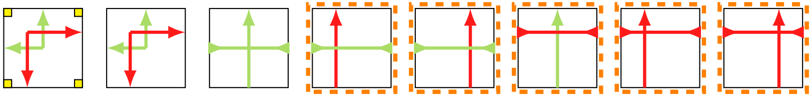

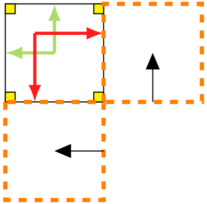

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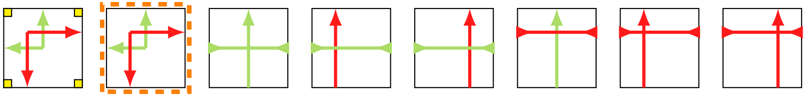

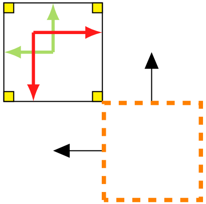

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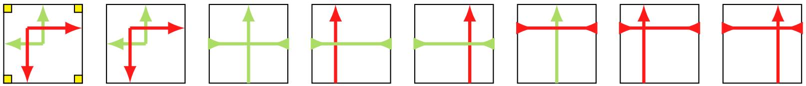

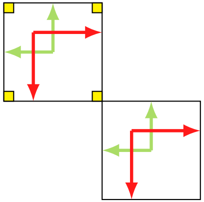

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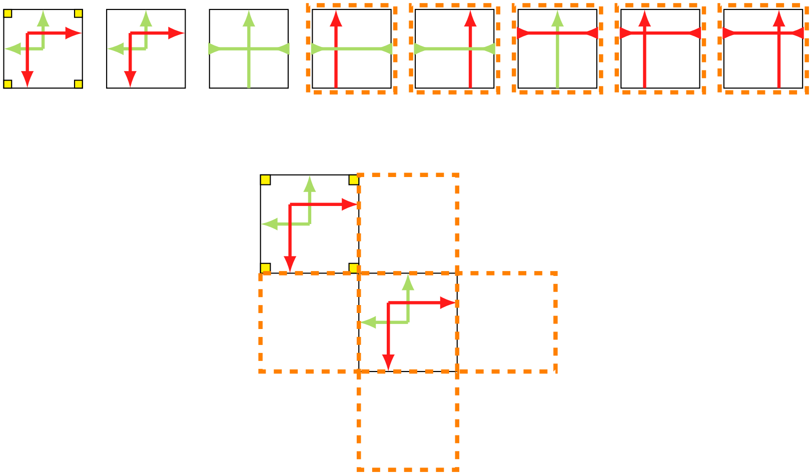

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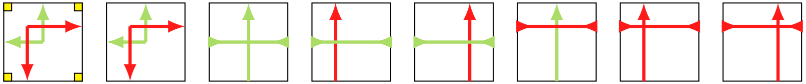

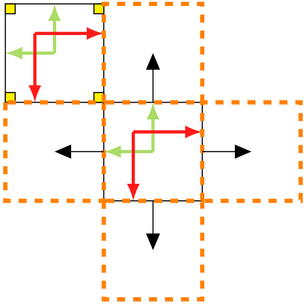

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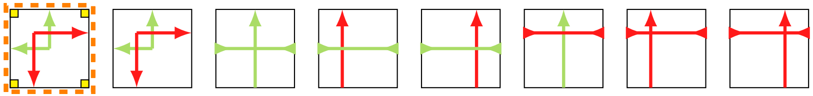

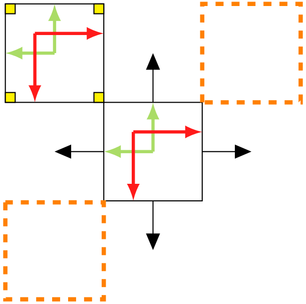

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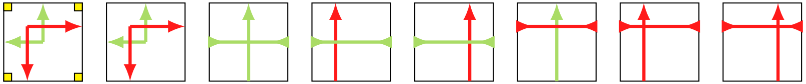

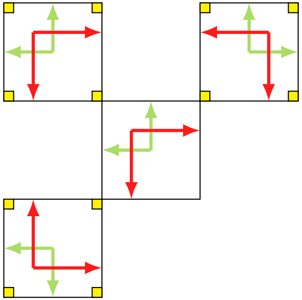

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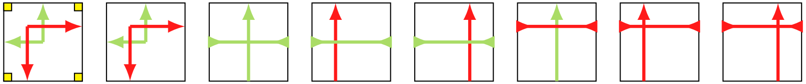

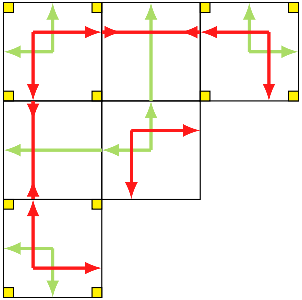

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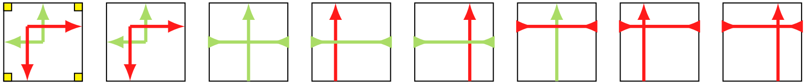

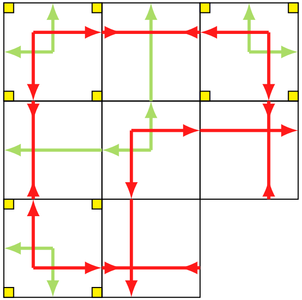

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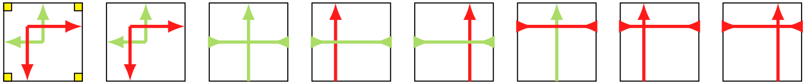
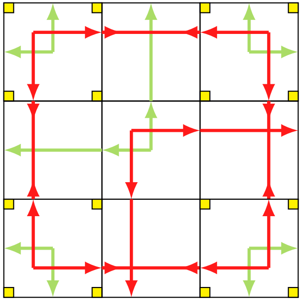
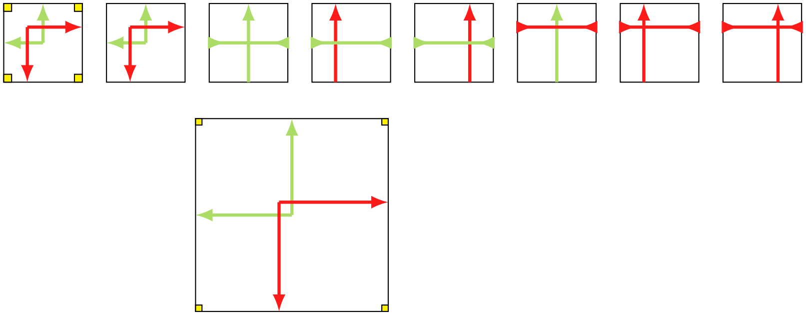
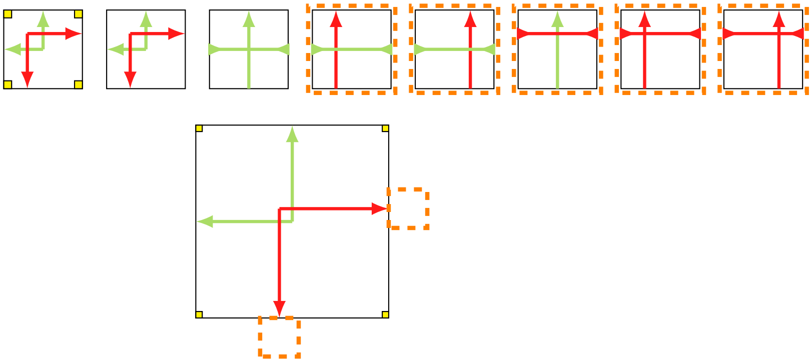




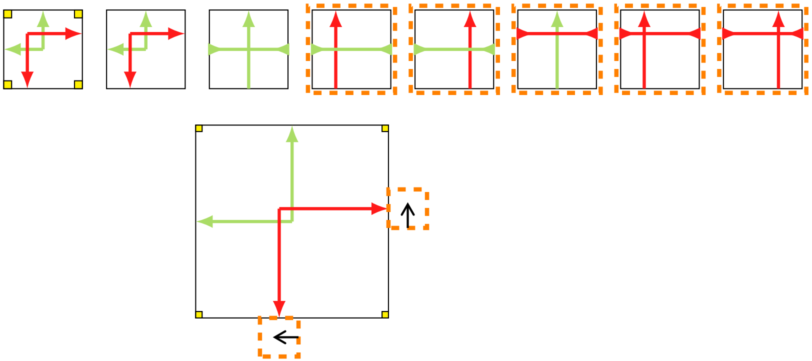
Local rules force macrotiles of rank n+1



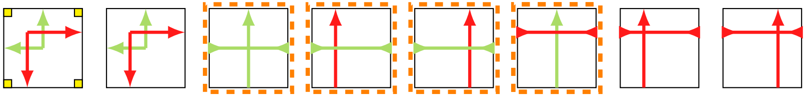
Local rules force macrotiles of rank n+1

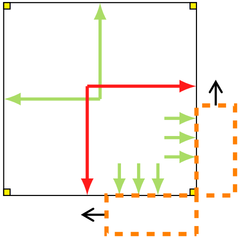


Local rules force macrotiles of rank n+1

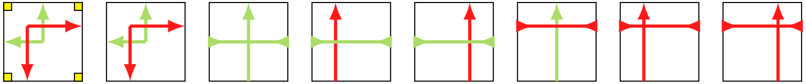


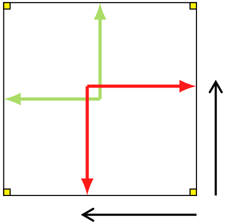
Local rules force macrotiles of rank n+1



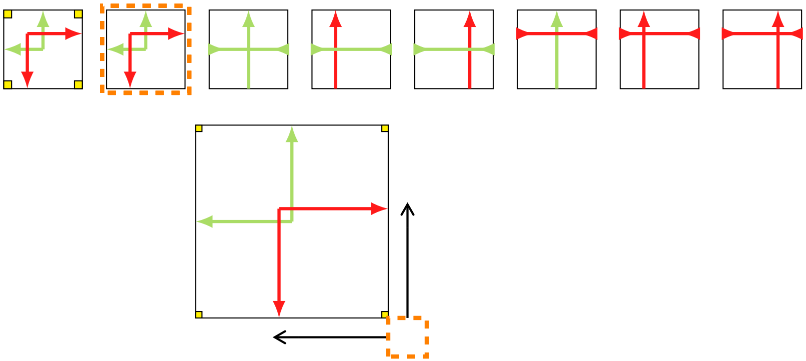


Local rules force macrotiles of rank n+1

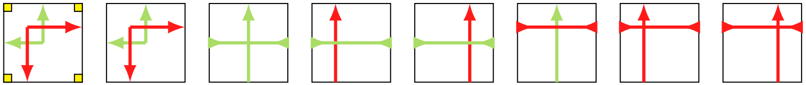


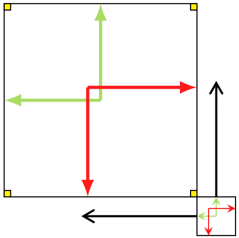


Local rules force macrotiles of rank n+1

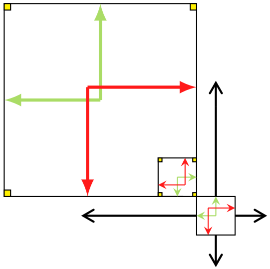


Local rules force macrotiles of rank n+1

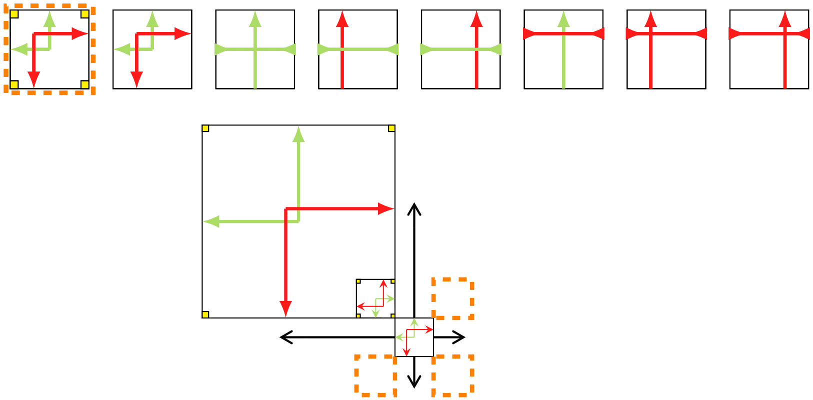




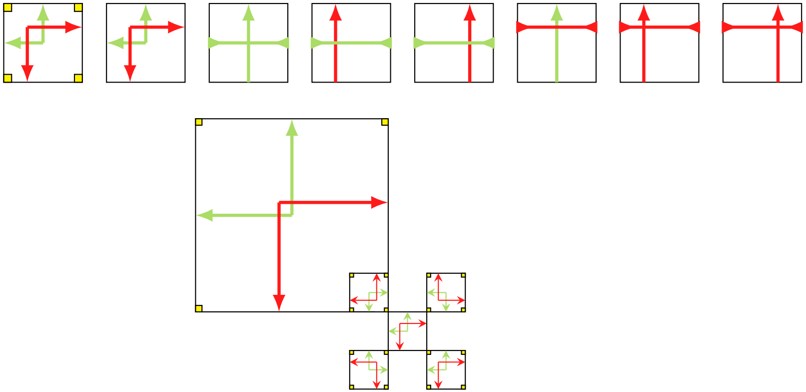
Local rules force macrotiles of rank n+1



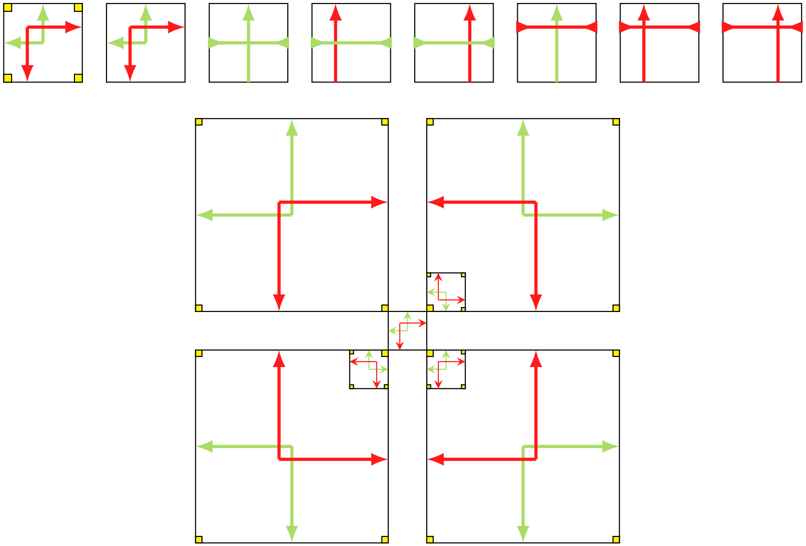
Local rules force macrotiles of rank n+1



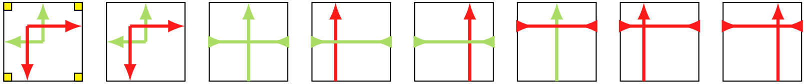
Local rules force macrotiles of rank n+1

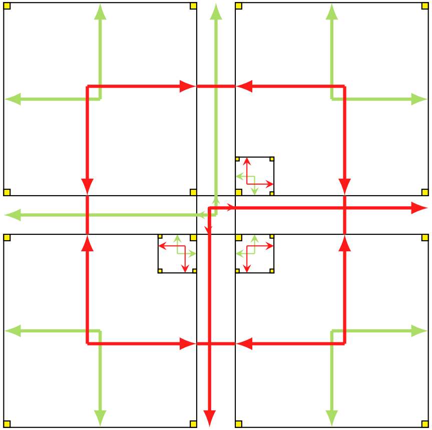


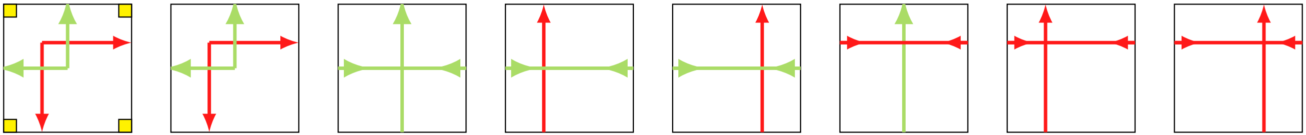
Local rules force macrotiles of rank n+1



Local rules force macrotiles of rank n+1







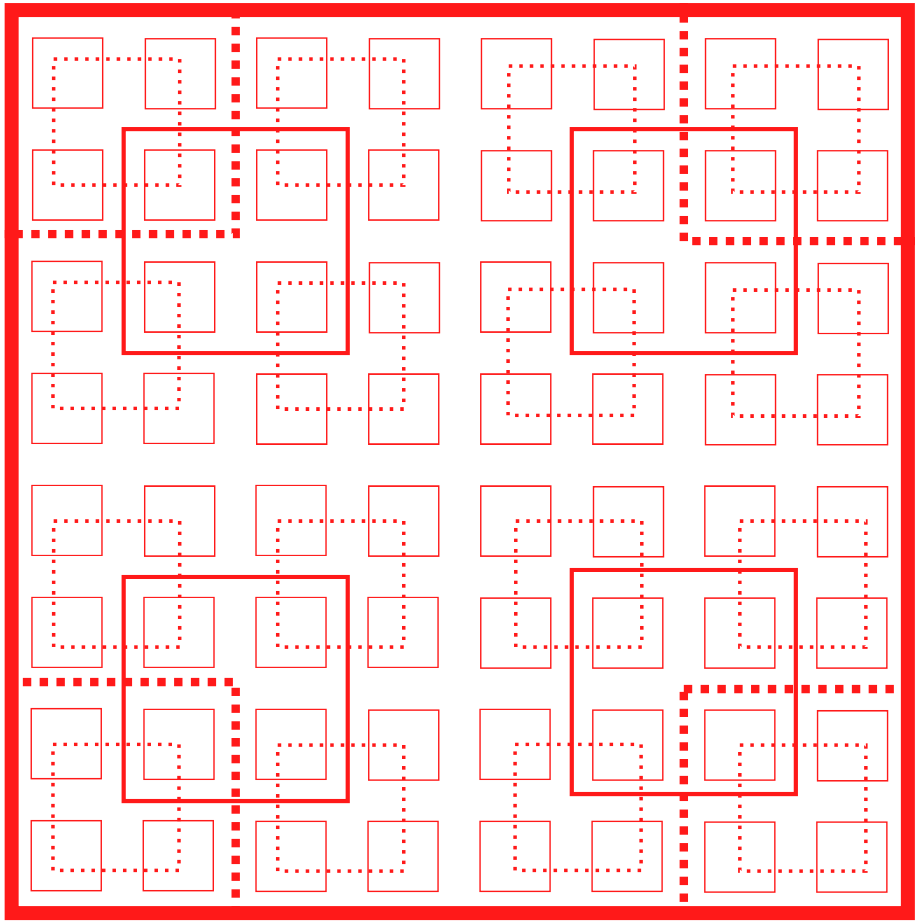
Theorem

The Robinson tileset is aperiodic.

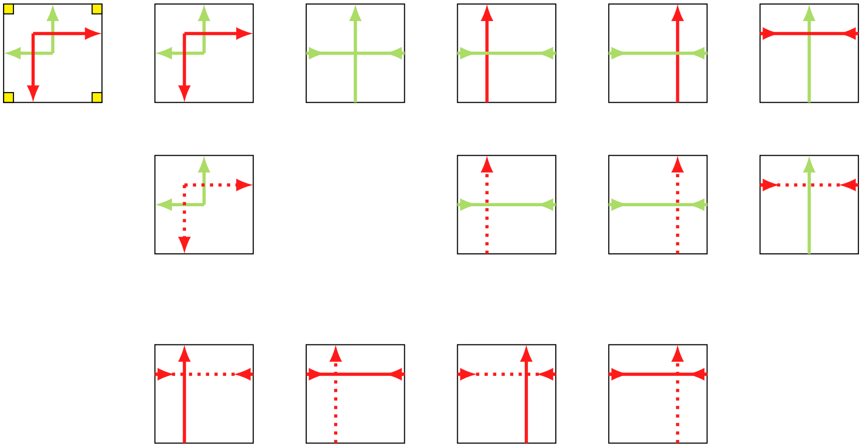
(It tiles the plane, but only aperiodically.)

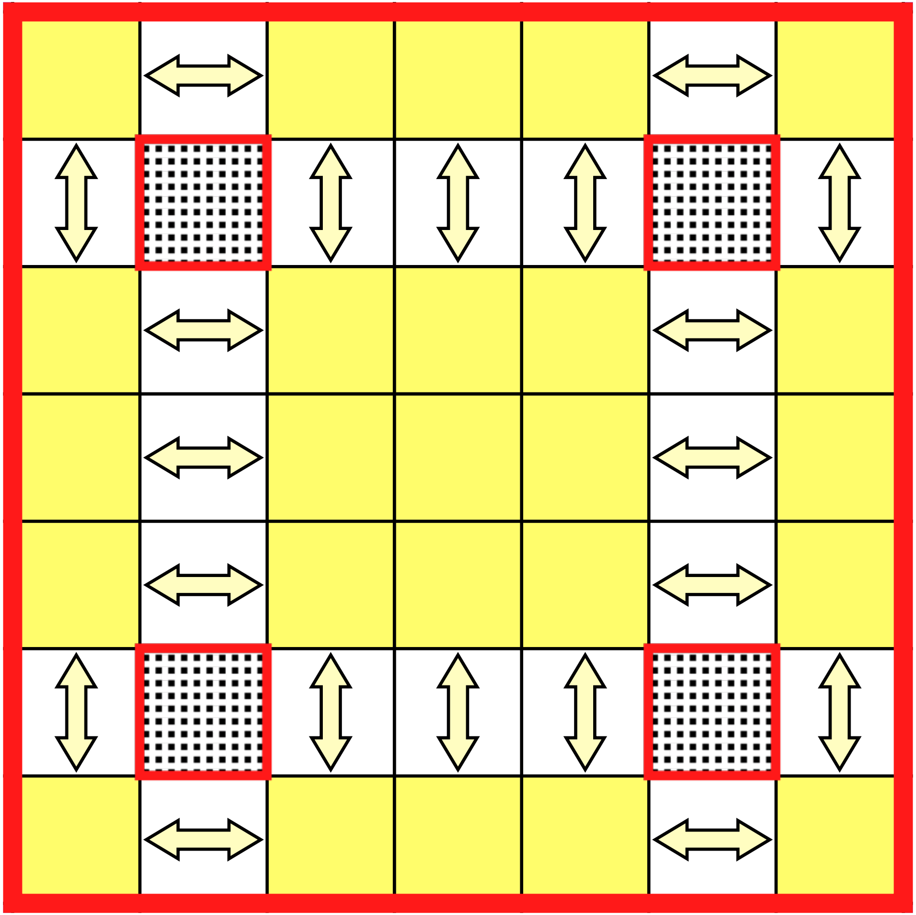
- Already explained why no periodicity
- One tile $\implies$ a 2-macrotile
- An n-macrotile $\implies$ an $(n+1)$-macrotile
- This continues for arbitrarily large n $\implies$ compactness $\implies$ tiles the plane

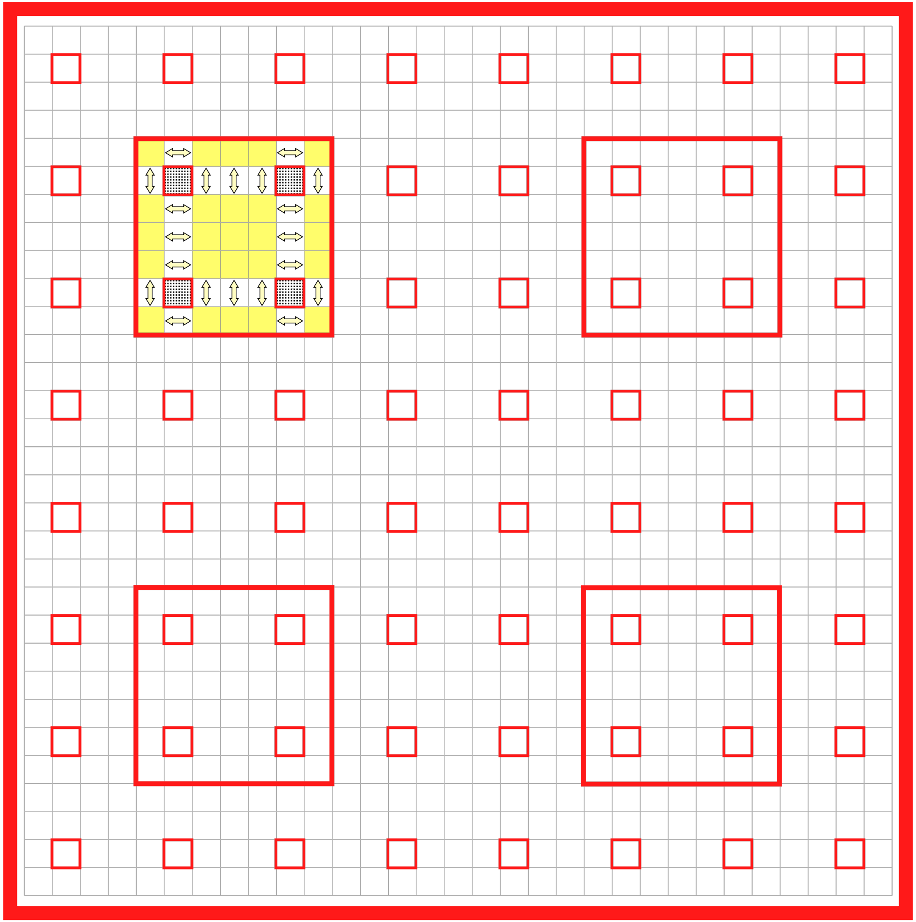
How to embark a Turing machine in there?

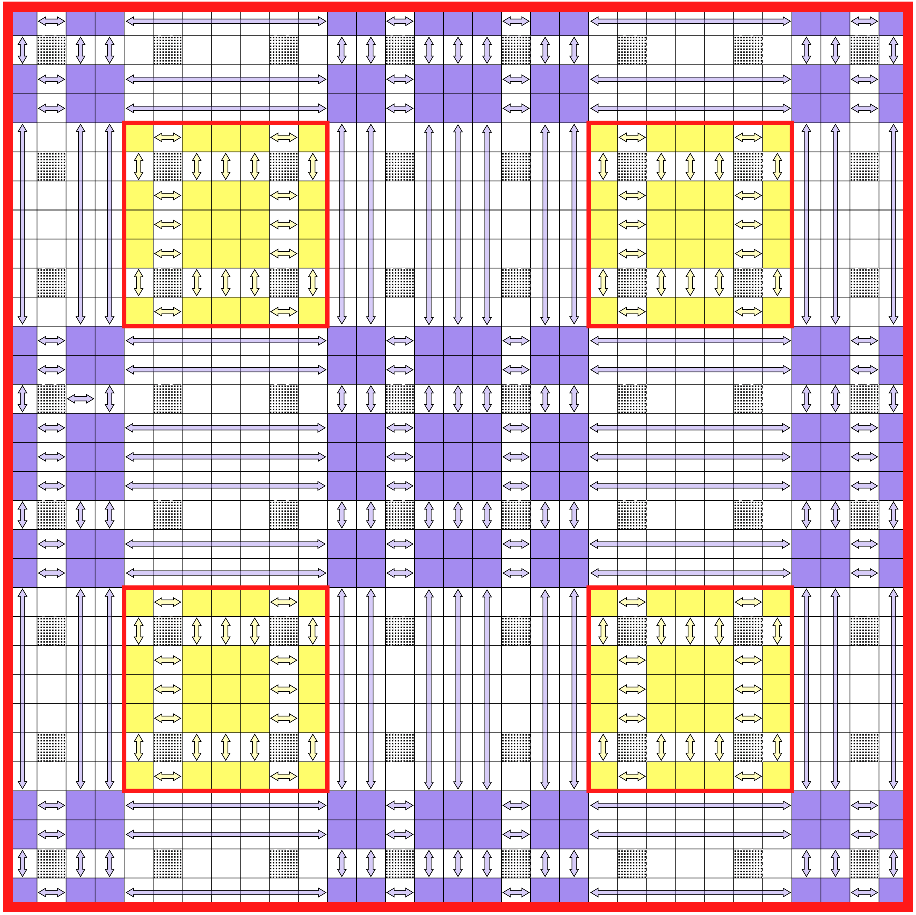


Extended Robinson tileset









Use the middle space to run the Turing machine

- Not a single spacetime diagram anymore
- Arbitrarily large spacetime diagrams
- Still: arbitrarily large diagrams iff the machine doesn't halt

Therefore:

Theorem

The Domino problem is undecidable

Conclusion

- AEA Formulae
- Domino problem decidable in 1D
- Weakly periodic $\implies$ strongly periodic
- Fixed-origin domino problem undecidable in 2D
- Substitutions
- Domino problem undecdiable in 2D

Thank you for your attention!