Parametrized complexity of relations between multidimensional subshifts

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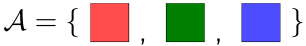
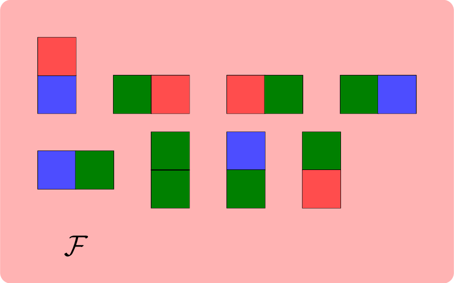
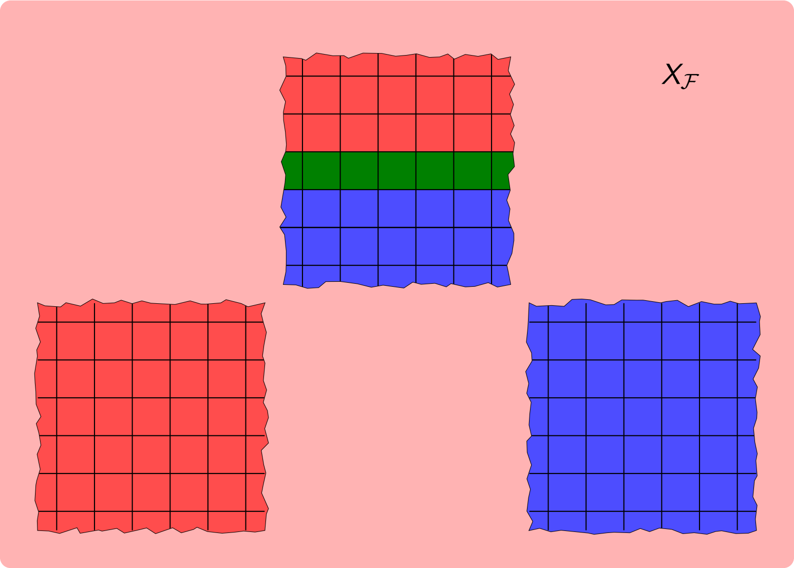
2026

Subshifts





Subshifts (2)



Basics about subshifts

Subshifts of Finite Type (SFT)

Subshifts $X_{\mathcal{F}}$ for which $\mathcal{F}$ is finite.

Effective subshifts

Subshifts $X_{\mathcal{F}}$ for which $\mathcal{F}$ is computably enumerable.

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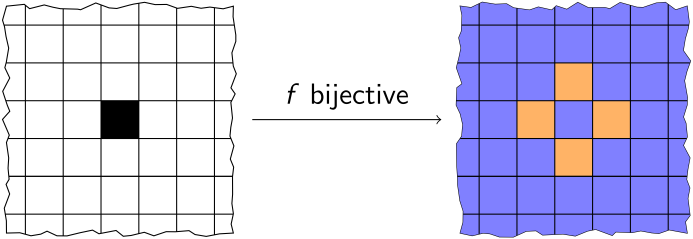
Local map

For X and Y subshifts, a map $F : X \to Y$ is local if there exists $r \in \mathbb{N}$ and a map $f : \mathcal{A}_X^{(2r+1)^2} \to \mathcal{A}_Y$ such that $\forall x \in X\ \forall \vec{v} \in \mathbb{Z}^2$
$F(x)_{\vec{v}} = f(x_{|[\![v_x - r, v_x + r]\!] \times [\![v_y - r, v_y + r]\!]})$.

Relations between subshifts

Conjugacy

Two subshifts X and Y are conjugate (denoted $X \simeq Y$) if there exists a bijective local map $f : X \to Y$.



Relations between subshifts (2)

Embedding

For two subshifts X and Y, we say that X embeds in Y (denoted $X \hookrightarrow Y$) if X is conjugate to a subset of Y (equivalently, if there exists an injective local map $f : X \to Y$).

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Basic relation	Dynamic version
Equality $=$	Conjugacy $\simeq$
Subset $\subseteq$	Embedding $\hookrightarrow$

Undecidable problems

Domino Problem:

Input: a SFT X

Question: $X \neq \emptyset$?

Extensivity problem:

Input: a SFT X and a pattern p

Question: $p \in \mathcal{L}(X)$?

Undecidability of the domino problem (Berger 1966)

The domino problem is undecidable, i.e. there exists no algorithm which solves the domino problem.

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Extensivity $\rightarrow$ Undecidable!

Parametrized extensivity... It depends.

Computability

The arithmetical hierarchy for decision problems is defined inductively as follows:

- Σ^0_0 and Π^0_0 are the set of decidable problems.
- R is in Π^0_{n+1} if there exists $S \in \Sigma^0_n$ such that $R \Leftrightarrow \forall n\ S(n)$.
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Computability

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where quantifiers range over natural numbers.
A problem P is harder than a problem Q if there exists a computable function f such that $Q(n) \Leftrightarrow P(f(n))$ for all inputs n.

C-hard

A problem P is C-hard if it is harder than every problem in C.

C-complete

A problem P is C-complete if it is C-hard and in C.

Complexity of our problems

We already know thanks to literature (Jeandel&Vanier 2015) the complexity of these problems:

Given X, Y	SFT	Effective
$X = Y$	Σ_1^0	Π_2^0
$X \simeq Y$	Σ_1^0	Σ_3^0
$X \subseteq Y$	Σ_1^0	Π_2^0
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They make stronger statement: For all SFT X, the problem $X \simeq Y$ for SFT inputs Y where X is a parameter is Σ_1^0-complete.

A bunch of new problems

In the following, Y is always a parameter and X the input.

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Some remarks:

- These problems are always easier than their non parametrized versions,
- The complexity of a problem parametrized by Y is a property of Y,
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"Problems of this kind are always undecidable" -Le Figaro

Existence of decidable problems?

Take $Y = \{\square^{\mathbb{Z}^2}\}$, the problem $Y \subseteq X$ is decidable (for SFT inputs).

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Theorem 17 [Hellouin, Carrasco-Vargas, P.]

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Berger property

A property $\mathcal{P}$ of SFTs is <u>Berger</u> if there are two SFTs X_+ and X_- such that X_+ satisfies $\mathcal{P}$, every SFT that factor on X_- does not satisfy $\mathcal{P}$ and there is a local map $f : X_+ \to X_-$.

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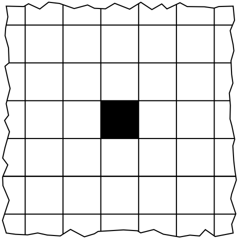
Theorem 23 [Hellouin, Carrasco-Vargas, P.]

If Y is a finite subshift, then $Y \hookrightarrow X$ is decidable.

$Y \hookrightarrow X$

Theorem 21 [Hellouin, Carrasco-Vargas, P.]

Assume that Y contains a configuration that is not strongly periodic, but is strongly periodic outside of a finite region. Then $Y \hookrightarrow X$ for SFT inputs is Σ_1^0-complete (maximal complexity).



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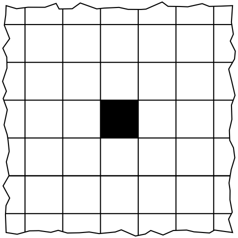

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Do we have $Y \hookrightarrow X$ harder than $Y \subseteq X$?

$Y \hookrightarrow X$ (2)

Do we have $Y \hookrightarrow X$ harder than $Y \subseteq X$?

Theorem 24 [Hellouin, Carrasco-Vargas, P.]

There exists an effective $\mathbb{Z}$-subshift Y such that $Y \hookrightarrow X$ is decidable for SFT inputs but $\mathcal{L}(Y)$ is not decidable (and therefore $Y \subseteq X$ for SFT inputs is not decidable).

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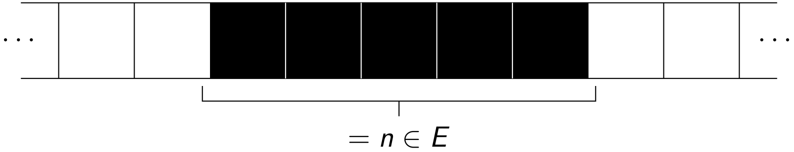

With $E \subseteq \mathbb{N}^*$, a Π^0_1 undecidable set.

Conclusion

	Problem	Complexity min	Complexity max	Non parametrized
$Y \neq \emptyset$ SFT	$X = Y$	$D(\Sigma_1^0)$	Π_2^0	Π_2^0
X effective	$X \simeq Y$	$D(\Sigma_1^0)$	Σ_3^0	Σ_3^0
	$Y \subseteq X$	Δ_1^0	Σ_1^0	Σ_1^0
Y effective	$Y \hookrightarrow X$	Δ_1^0	Σ_1^0	Σ_1^0
X SFT	$X \subseteq Y$	Σ_1^0	Π_2^0	Π_2^0
	$X \hookrightarrow Y$	Σ_1^0	Σ_3^0	Σ_3^0

Further work: Factor, Morphism.

Parenthesis on dimension 1

Conjugacy and embedding are closely linked.

Krieger's embedding theorem (1982)

Let Y be a nonempty mixing $\mathbb{Z}$-SFT. We have $Y \hookrightarrow X$ harder than $X \simeq Y$ for SFT inputs.

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Moreover, the following question stands for quite a long time:

Open question

Is the (non parametrized) conjugacy problem decidable on $\mathbb{Z}$-SFT?

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~~Interesting case? Y effective and X SFT.~~
Interesting case: $Y \in$ Class of the inputs.
i.e. Y SFT or effective and X effective.

$X \simeq Y$ (2)

If $Y = \emptyset$, then $Y \simeq X$ (i.e. $X \simeq \emptyset$) is exactly the coDomino problem known to be Σ_1^0-complete.

Otherwise, $Y \simeq X$ is always Σ_3^0 and $D(\Sigma_1^0)$-hard.

Definition

A set $E \subseteq \mathbb{N}$ is $D(\Sigma_1^0)$ if there exists $A, B \in \Sigma_1^0$ such that $E = A \backslash B$.

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Lower bound is reached:

Theorem 39 [Hellouin, Carrasco-Vargas, P.]

Let Y be a P-minimal SFT for inclusion for a Π_1^0 monotonous, conjugacy invariant SFT property P. Then $Y \simeq X$ is $D(\Sigma_1^0)$ (minimal complexity).

Works for the following P:

- Being minimal,
- Having computable topologial entropy and being entropy-minimal,
- Having a dense set of strongly periodic points.

$X \simeq Y$ (3)

Upper bound of $X \simeq Y$ is reached too.

Theorem 37 [Hellouin, Carrasco-Vargas, P.]

There exists an SFT Y such that $Y \simeq X$ for effective inputs is Σ^0_3-complete (maximal complexity).

$X \simeq Y$ (3)

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Theorem 37 [Hellouin, Carrasco-Vargas, P.]

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Take $Y = (\mathcal{A}, \mathcal{F})$ for $\mathcal{A} = \{0, 1, 2\}$ and $\mathcal{F} = \{20, 21\}$.

$\cdots$	0	1	0	1	1	0	0	2	2	2	2	2	2	2	2	$\cdots$