Compositionality in subshifts of finite type

Mocqua PhD Day 2026

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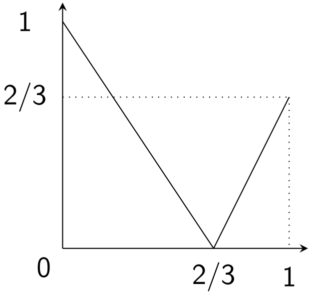
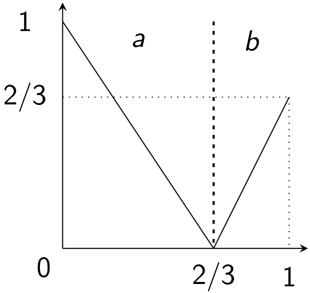
1. Introduction
2. Conjugacy
3. Compositionality
4. Conclusion

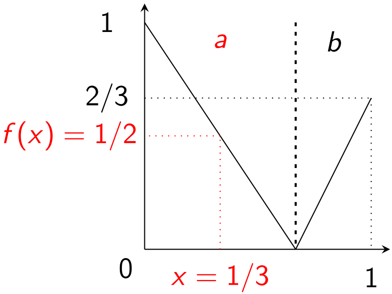
Introduction



Introduction

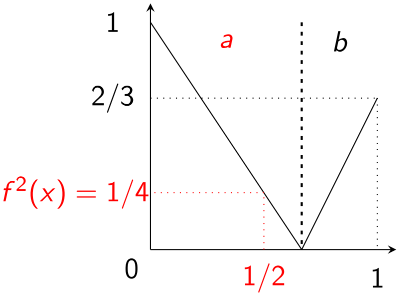


Introduction



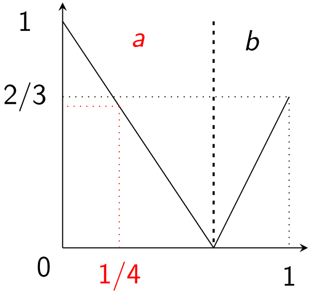
$x = 1/3$ a

Introduction



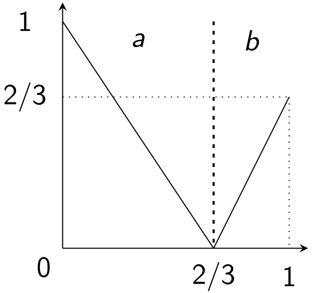
$$x = 1/3 \; a$$
$$f(x) = 0.5 \; a$$

Introduction



$$x = 1/3 \ \textcolor{red}{a}$$
$$f(x) = 0.5 \ \textcolor{red}{a}$$
$$f^2(x) = 0.25 \ \textcolor{red}{a}$$
$$\vdots$$
$$\textcolor{red}{aaaaa\ldots}$$

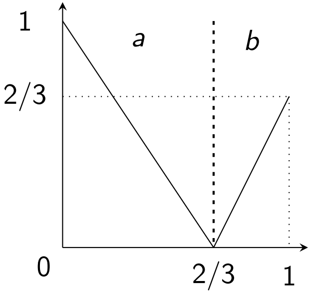
Introduction



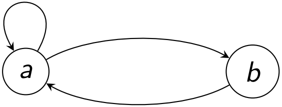
$$x = 1/3 \quad a$$
$$f(x) = 0.5 \quad a$$
$$f^2(x) = 0.25 \quad a$$
$$\vdots$$
$$aaaaa\ldots$$

$$x = 0.7 \quad b$$
$$f(x) \approx 0.066 \quad a$$
$$f^2(x) = \ldots \quad b$$
$$\vdots$$
$$babaaba\ldots$$

Introduction

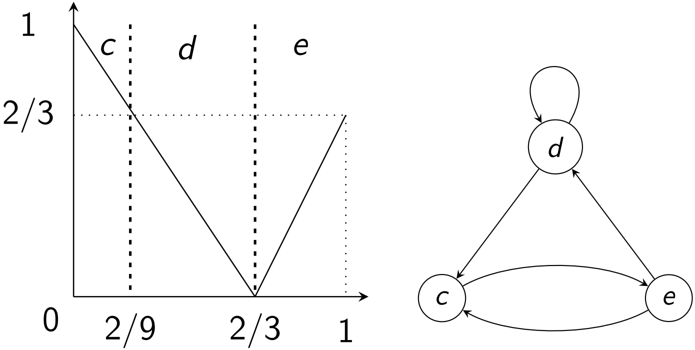


- $\mathcal{S}$ set of all trajectories of the system.
- Forbidden word: **bb**

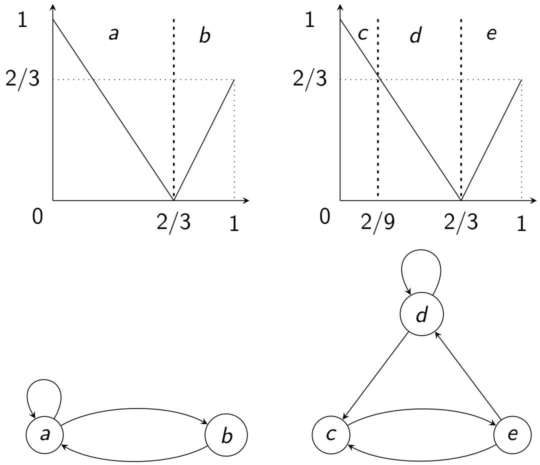


- **Subshift of finite type (SFT)**

- Another choice of discretization:

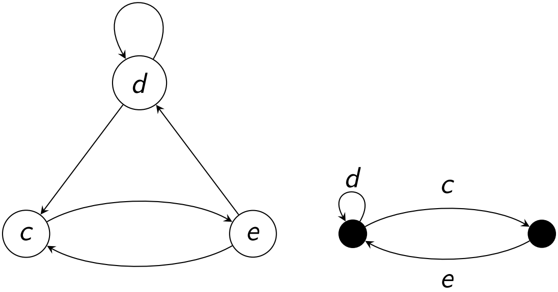


Conjugacy



- Open problem: decide if two SFTs are conjugate (for bi-infinite sequences).

Edge shifts



- We put labels on edges instead of vertices.

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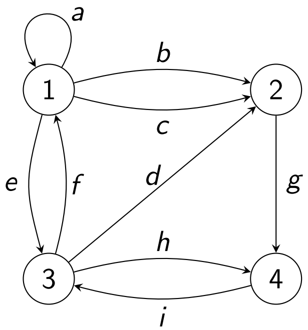
SFTs

- **Full-shift**: set of all bi-infinite sequences over $\mathcal{A}$:

$$\mathcal{A}^{\mathbb{Z}} = \{(x_i)_{i\in\mathbb{Z}} : x_i \in \mathcal{A}, \forall i \in \mathbb{Z}\}$$

- A **word**: finite sequence of symbols.
- $\mathcal{F}$: finite set of **forbidden words**.
- **Subshift of finite type (SFT)**: $X \subseteq \mathcal{A}^{\mathbb{Z}}$ contains all the sequences $x \in X$ that don't contain any words from $\mathcal{F}$.

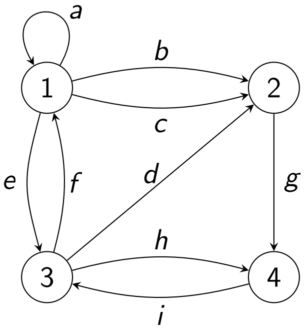
- Given a **graph** G, the **subshift of finite type** X_G associated to G is the set of **all bi-infinite paths** on G.



$$X_G = \{(x_k)_{k\in\mathbb{Z}} \in \mathcal{E}^{\mathbb{Z}} : t(x_k) = i(x_{k+1}), \forall k \in \mathbb{Z}\}$$

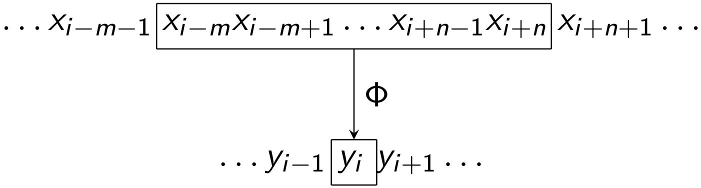
with $i(e)$ the vertex where e starts and $t(e)$ the vertex where e ends.

- We may also think of G as a matrix with nonnegative coefficients.



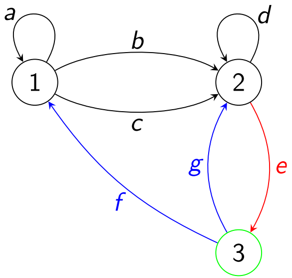
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

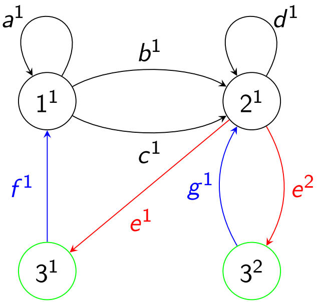
Sliding block codes



- **Sliding block code** of memory m and anticipation n.
- Similar behavior as a cellular automaton.
- ϕ is a **conjugacy** if it has an inverse.

Splittings and amalgamations





Strong Shift Equivalence

- Let A and B be two matrices with non-negative integer coefficients. An **elementary equivalence** between A and B is such that:

$$A = RS \quad \text{and} \quad B = SR$$

Strong Shift Equivalence

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$$A = RS \quad \text{and} \quad B = SR$$

- A **strong shift equivalence (SSE)** between A and B is a sequence of elementary equivalences $(R_1, S_1), (R_2, S_2), \ldots, (R_n, S_n)$ such that:

$$\begin{aligned} A = A_0 = R_1 S_1, &\qquad S_1 R_1 = A_1 \\ A_1 = R_2 S_2, &\qquad S_2 R_2 = A_2 \\ &\vdots \\ A_{n-1} = R_n S_n, &\qquad S_n R_n = A_n = B. \end{aligned}$$

Theorem (Williams, 1973)

The subshifts X_G and X_H are **conjugate**.

$\Leftrightarrow$ The graph H can be obtained from G by **a sequence of splittings and amalgamations**.

$\Leftrightarrow$ The matrices M_G and M_H are **SSE**.

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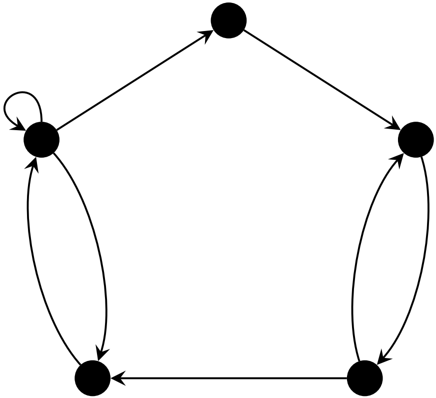
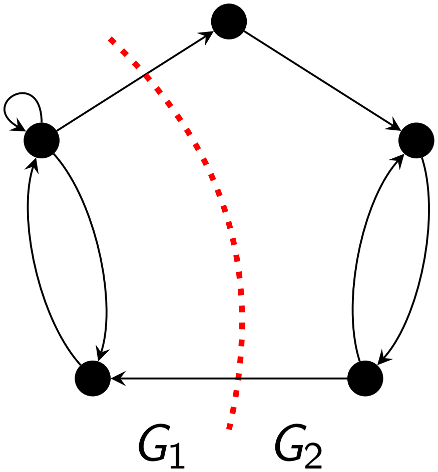
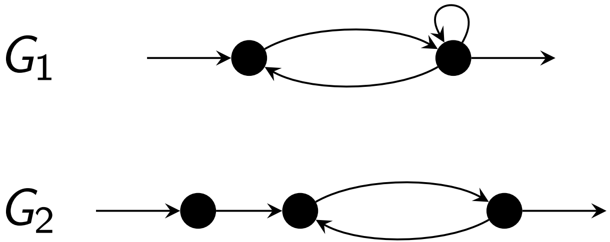
A compositional approach



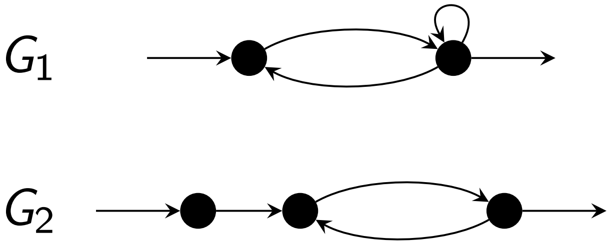
A compositional approach



A compositional approach



A compositional approach

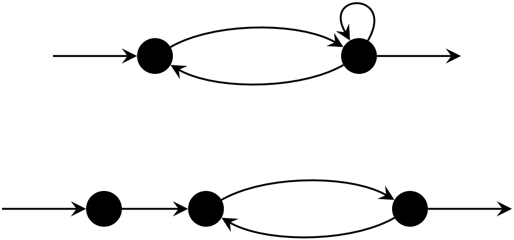


- We want to be able to compose these graphs: $G_1 \circ G_2$.

A compositional approach

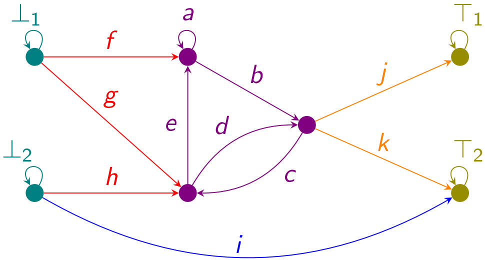
G_1

G_2



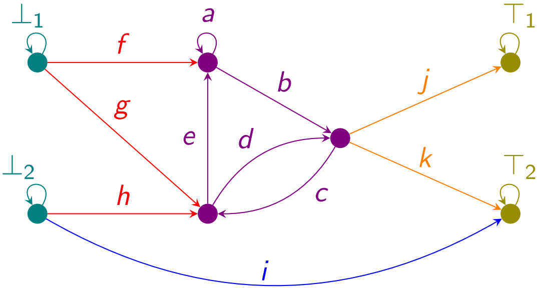
- We want to be able to compose these graphs: $G_1 \circ G_2$.
- But in a way that if we have $G_1 \cong G_1'$ then $G_1 \circ G_2 \cong G_1' \circ G_2$.

- **Open graph**: graph with **inputs** and **outputs**.



- **Boundary symbols**: $\bot$ and $\top$.
- Vertices subsets: $\mathcal{V}_\bot$, $\mathcal{V}_\top$ and $\mathcal{V}_x$.
- Edges subsets: $\mathcal{E}_{\bot\bot}$, $\mathcal{E}_{\bot x}$, $\mathcal{E}_{xx}$, $\mathcal{E}_{x\top}$, $\mathcal{E}_{\top\top}$, $\mathcal{E}_{\bot\top}$, $\mathcal{E}_{x\bot}$, $\mathcal{E}_{\top x}$ and $\mathcal{E}_{\top\bot}$, with $\mathcal{E}_{x\bot} = \mathcal{E}_{\top x} = \mathcal{E}_{\top\bot} = \emptyset$.

Matrices with boundaries



- For an open graph with n inputs and m outputs, we have an upper block triangular matrix:

$$M = \begin{pmatrix} I_n & M_{\bot x} & M_{\bot \top} \\ 0 & M_{xx} & M_{x\top} \\ 0 & 0 & I_m \end{pmatrix}$$

Conjugacy of SFT with boundaries

- **Edge shift with boundaries** from an open graph.
- Cellular automata: Sliding block codes with boundaries must **preserve the boundary symbols**.
- Open graphs: Splittings and amalgamations with boundaries must only affect the **internal vertices**.
- Matrices: Constraints on the form of the matrices:

$$R = \begin{pmatrix} I_n & 0 & 0 \\ 0 & R_1 & 0 \\ 0 & 0 & I_m \end{pmatrix} \text{ and } S = \begin{pmatrix} I_n & S_1 & S_2 \\ 0 & S_3 & S_4 \\ 0 & 0 & I_m \end{pmatrix}$$

Theorems with boundaries

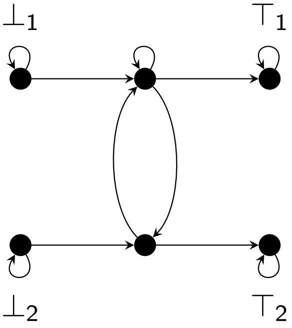
Theorem (proved during my internship)

The subshifts **with boundaries** X_G and X_H are conjugate.

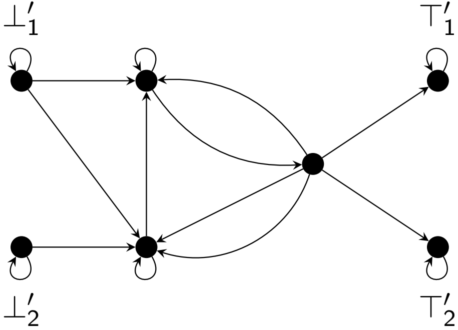
$\Leftrightarrow$ The **open** graph H can be obtained from G by a sequence of splittings and amalgamations **with boundaries**.

$\Leftrightarrow$ The matrices **with boundaries** M_G and M_H are SSE **with boundaries**.

Composition

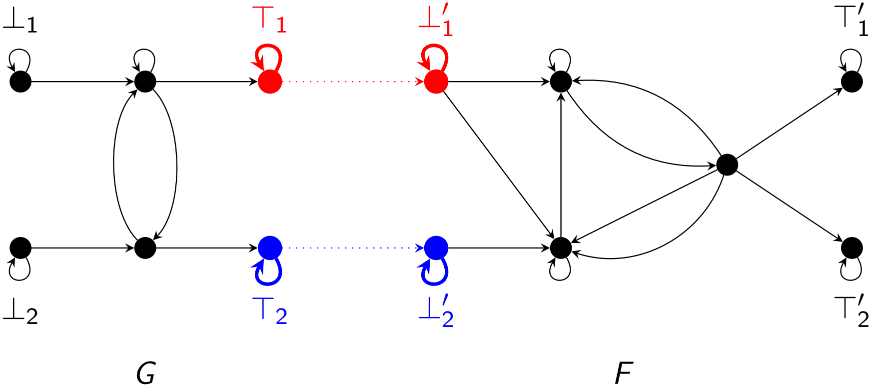


G

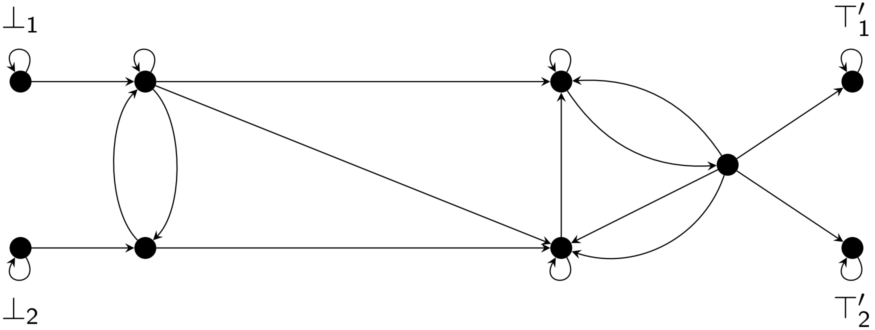


F

Composition



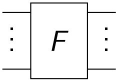
Composition



$F \circ G$

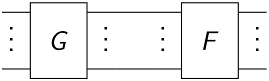
Composition

- We need a way to **compose** subshifts with boundaries while **preserving conjugacy**:
  - Sequentially,
  - In parallel.
- So we need a **symmetric monoidal category**.
- We can think of subshifts with boundaries as **boxes** with inputs and outputs:



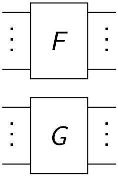
Composition

- We define these two operators:
  - A **composition** $F \circ G$:



$$\left(\begin{array}{c|cc|c} I_m & G_{\perp x} & G_{\perp\top}F_{\perp x} & G_{\perp\top}F_{\perp\top} \\ \hline 0 & G_{xx} & G_{x\top}F_{\perp x} & G_{x\top}F_{\perp\top} \\ 0 & 0 & F_{xx} & F_{x\top} \\ \hline 0 & 0 & 0 & I_p \end{array}\right)$$

  - A **tensor product** $F \otimes G$:

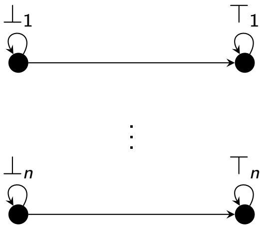


$$\left(\begin{array}{cc|cc|cc} I_k & 0 & F_{\perp x} & 0 & F_{\perp\top} & 0 \\ 0 & I_m & 0 & G_{\perp x} & 0 & G_{\perp\top} \\ \hline 0 & 0 & F_{xx} & 0 & F_{x\top} & 0 \\ 0 & 0 & 0 & G_{xx} & 0 & G_{x\top} \\ \hline 0 & 0 & 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & 0 & 0 & I_n \end{array}\right)$$

Composition

- We also need :
  - An **identity**:

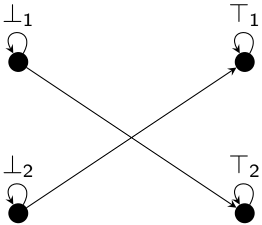
$id^{\otimes n}$



$$\left(\begin{array}{c|c} I_n & I_n \\ \hline 0 & I_n \end{array}\right)$$

  - A **swap**:

σ



$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Composition

- Both composition operators **preserve conjugacy**:

Theorem

Let F, F', G and G' four open graphs. We denote X_F, $X_{F'}$, X_G and $X_{G'}$ the edge shifts with boundaries associated with these graphs. If $X_F \cong X_{F'}$ and $X_G \cong X_{G'}$, then $X_{F \circ G} \cong X_{F' \circ G'}$ and $X_{F \otimes G} \cong X_{F' \otimes G'}$.

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Conclusion

- Trace operator
- Conjugacy invariants for SFTs with boundaries:
  - Quantity $\phi(M)$ such that $\phi(M) = \phi(N)$ whenever M and N are conjugate.