Graphical Approach of Symbolic Dynamics

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MOCQUA

Outline

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Elements of combinatorics on words

$\mathcal{A}$ finite alphabet,

$\mathcal{A}^*$ set of all finite words on $\mathcal{A}$,

$\mathcal{A}^\omega$ set of all infinite words on $\mathcal{A}$ (ex: $x = x_1 \cdots x_i \cdots \in \mathcal{A}^\omega$),

$u \in \mathcal{A}^*, v \in \mathcal{A}^*$ (resp. $\mathcal{A}^\omega$), concatenation $uv \in \mathcal{A}^*$ (resp. $\mathcal{A}^\omega$).

Prefix

u is a *prefix* of w if it exists v such that:

$$w = uv$$

$u \in \mathcal{A}^*$, $w \in \mathcal{A}^*$ (resp. $\mathcal{A}^\omega$), $v \in \mathcal{A}^*$ (resp. $\mathcal{A}^\omega$).

Thompson group V

$W \subseteq \{0,1\}^*$ is a *prefix code* (for $X \subseteq \{0,1\}^\omega$) if:

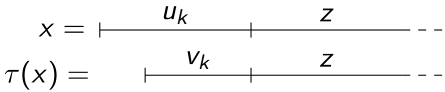
- every $w \in W$ is a prefix of a word in X;
- every $x \in X$ has a unique prefix w in W.

Example: $W = (00, 010, 1, 011)$ (with $X = \{0,1\}^\omega$).

Cylinder map on $X = \{0,1\}^\omega$

$U = (u_1, \ldots, u_n)$, $V = (v_1, \ldots, v_n)$ two prefix codes of same size n.
Define the *cylinder map* $\tau_{U,V} : X \longrightarrow X$ such that:

$$\forall z \in X, 1 \leqslant k \leqslant n, \qquad \tau_{U,V} : u_k z \mapsto v_k z$$



The Thompson group is $V := \{\tau \text{ cylinder map on } X\}$.

Example

Let $U = (00, 01, 10, 11)$ and $V = (00, 010, 1, 011)$. Then

$$\tau_{U,V} : \begin{cases} 00z & \mapsto & 00z \\ 01z & \mapsto & 010z \\ 10z & \mapsto & 1z \\ 11z & \mapsto & 011z \end{cases}$$

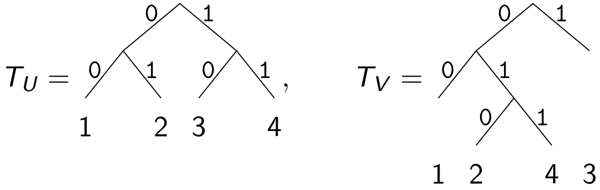
Ex:

$$\begin{array}{lcl} \color{red}{10}100\cdots & \mapsto & \color{red}{1}100\cdots \\ \color{red}{11}00\cdots & \mapsto & \color{red}{011}00\cdots \\ \color{red}{01}100\cdots & \mapsto & \color{red}{010}100\cdots \end{array}$$

Graphical representation of cylinder maps

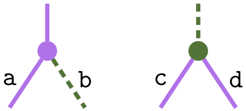
$$U = (00, 01, 10, 11)\,, V = (00, 010, 1, 011) \quad \tau_{U,V} : \begin{cases} 00z & \mapsto & 00z \\ 01z & \mapsto & 010z \\ 10z & \mapsto & 1z \\ 11z & \mapsto & 011z \end{cases}$$

Fact: the elements of a prefix code are the leaves of a binary tree.

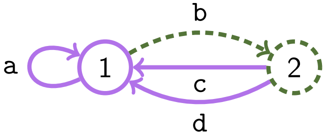


Other families of trees

Colored edges that divide differently, example:

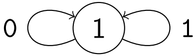


Encode the rules in a graph: $G =$



$X_G = ?$

For the Thompson group V: $H =$



$X_H = \{0, 1\}^\omega$

Outline

Edge shifts

Reminder: subshift

$X \subseteq \mathcal{A}^\omega$ is a *subshift* when:

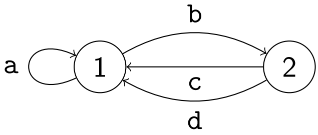
- X is shift-invariant: $\forall x \in X, \sigma(x) \in X$;
  (where σ is the shift: $\sigma(x_1 x_2 x_3 \cdots) = x_2 x_3 \cdots$)
- X is a closed set.

Edge shift

$G = (Q, \mathcal{A})$ a directed graph (Q is the vertex set and $\mathcal{A}$ is the edge set).

$$X_G := \{x \in \mathcal{A}^\omega : x \text{ is an infinite path on } G\}$$

is a subshift, called the *edge shift induced by G*.

Example: $G =$ , $\quad \mathtt{caabdbca} \cdots \in X_G$

Cylinder (???) maps

Let $X = X_G \subseteq \mathcal{A}^\omega$ an edge shift.

Cylinders

Let $u \in \mathcal{A}^*$. The *cylinder* of u in X is the set words having u as prefix.

$$[u] := \{uz \in X : z \in \mathcal{A}^\omega\}$$

Fact 1: the cylinders form a basis of *clopens* for the topology on $\mathcal{A}^\omega$.

Fact 2: if $U = (u_1, \ldots, u_n)$ is a prefix code for X, then we have a partition of X:

$$X = \bigsqcup_{k=1}^{n} [u_k]$$

So $\tau_{U,V}$ maps cylinders: $\tau_{U,V} : [u_k] \mapsto [v_k]$.

Cylinder maps on an edge shift

We generalize cylinder maps on edge shifts.

Let $G = (Q, \mathcal{A})$, $X = X_G \subseteq \mathcal{A}^\omega$.

Let $U = (u_1, \dots, u_n)$, $V = (v_1, \dots, v_n)$ two prefix codes for X.

For $\tau_{U,V} : u_k z \mapsto v_k z$ to be defined on X, we need $(\forall 1 \leqslant k \leqslant n)$:

$$\forall z \in \mathcal{A}^\omega,\ u_k z \in X \iff v_k z \in X$$

This is equivalent (in our case) to, $(\forall 1 \leqslant k \leqslant n)$:

The paths of the prefixes u_k and v_k in G have the same terminal vertex.

The condition on the prefix codes

Condition:

The paths of the prefixes u_k and v_k in G have the same terminal vertex.

We write:

- $t(u)$ the terminal vertex of u in G;
- X_q the set of infinite paths on G begining at the vertex q.

Then the condition is:

$$\forall 1 \leqslant k \leqslant n, t(u_k) = t(v_k)$$

Remarks:

- $z \in X_{t(u)} \iff uz \in X_G$
- with Q the set of vertices of G, $X_G = \bigsqcup_{q \in Q} X_q$.

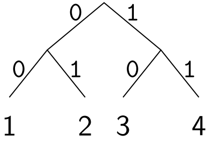
Outline

Diagrams for cylinder maps

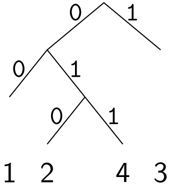
A way to represent an element of the Thompson group is to turn the trees and to join their leaves.

$$U = (00, 01, 10, 11) \qquad V = (00, 010, 1, 011)$$

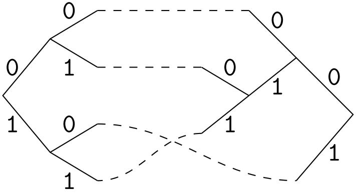
$T_U =$



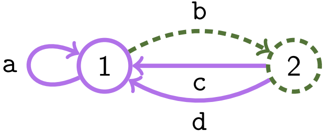
$T_V =$



$\tau_{U,V} =$

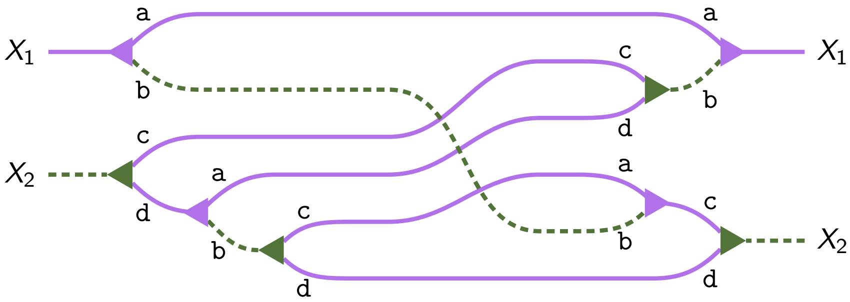


Diagrams for cylinder maps on edge shifts

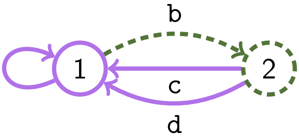
$G =$ 

, $X = X_G = X_1 \sqcup X_2$

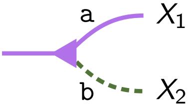
$U = (\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{da}, \mathtt{dbc}, \mathtt{dbd})$ $V = (\mathtt{a}, \mathtt{cb}, \mathtt{bc}, \mathtt{bd}, \mathtt{ca}, \mathtt{d})$



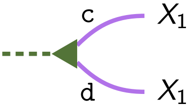
Generators *pop*...

$G =$ 

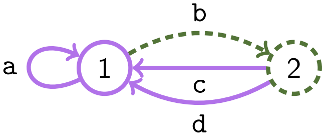
, $X = X_G = X_1 \sqcup X_2$

Generator *pop* of vertex 1: $x \in X_1$ 

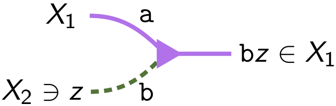
If $x \in X_1$, then $x = \begin{cases} \mathtt{a}z, z \in X_1 \longrightarrow \text{ return } z \in X_1 \text{ in the first wire.} \\ \mathtt{b}z, z \in X_2 \longrightarrow \text{ return } z \in X_2 \text{ in the second wire.} \end{cases}$

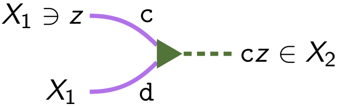
Same for vertex 2: $x \in X_2$ 

...and generators *push*

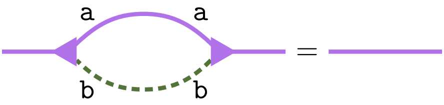
$G =$ 

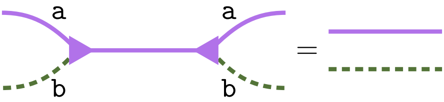
, $X = X_G = X_1 \sqcup X_2$





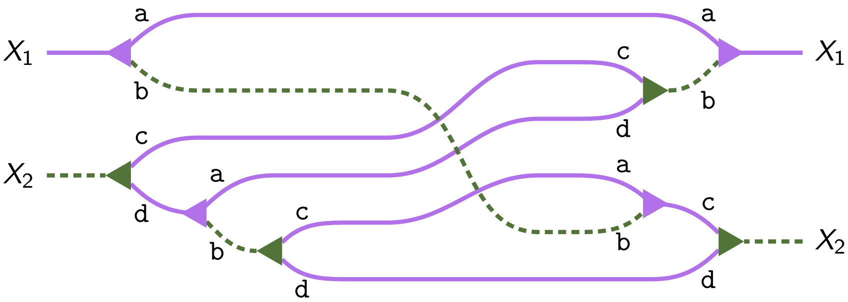
pop_q and $push_q$ are inverse for sequential composition:





Semantic: example

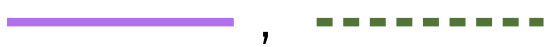
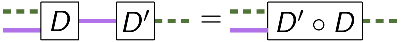
$$U = (\texttt{a}, \texttt{b}, \texttt{c}, \texttt{da}, \texttt{dbc}, \texttt{dbd}) \qquad V = (\texttt{a}, \texttt{cb}, \texttt{bc}, \texttt{bd}, \texttt{ca}, \texttt{d})$$

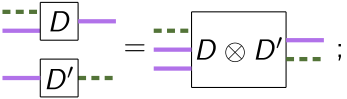


$\tau_{U,V}(\texttt{a}z) = ?, \quad \tau_{U,V}(\texttt{dbc}z) = ?$

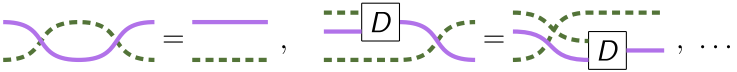
Colored PROP

Our diagrams are a *colored PROP* (= strict symetric monoidal category):

- Identities:  ;
- Sequential composition, 

- Parallel composition, 

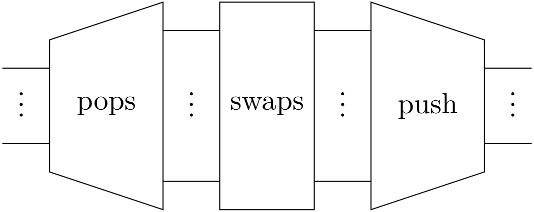
- Swaps (permutation of wires):  , ...

Such that: 

- Generators pop_q and $push_q$ ($\forall q \in Q$);
- Relations: $push_q \circ pop_q = \mathrm{id}$, $pop_q \circ push_q = \mathrm{id}$

Normal form

We can transform every diagram into a normal form:

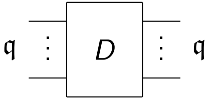


Outline

Isotropy groups

Fix $G = (Q, \mathcal{A})$ and a list of colors $\mathfrak{q} = \langle q_1, \ldots, q_n \rangle$ in Q.

$\mathcal{C}[\mathfrak{q}, \mathfrak{q}]$ is the set of morphisms (diagrams) $D : \mathfrak{q} \to \mathfrak{q}$.



Remark: if $\mathfrak{q} = Q$, then $\mathcal{C}[\mathfrak{q}, \mathfrak{q}]$ is the *topological full group*, the group of cylinder maps on X_G.

[Matui 2015]

$\mathcal{C}[\mathfrak{q}, \mathfrak{q}]$ is a finitely generated and presented group (and more: F_∞).

It is a group:

- we can compose the diagrams;
- inversion by vertical symmetry and *pop* $\leftrightarrow$ *push*.

Question: what does the generators/relations of the group look like?

Other interests

- Generalise to bigger classes of subshifts (SFT, sofic, ...).
- Generalise to bi-infinite words.
- Links with conjugacy problem (see Theo's presentation).
  Fact: the topological full group is a conjugacy invariant.
- Conjecture: the isomorphisms of topological full groups are conjugacy by specific transductors.

Thank you for listening

Please ask your questions!

Some references...

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