A 2D extension of the Lyndon-Schützenberger theorem

Guilhem Gamard*, Gwenaël Richomme,
Jeffrey Shallit, Taylor J. Smith

*ENS Lyon

Workshop on bidimensional languages
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Warm up: a bit of 1D

Theorem (special case of Lyndon-Schützenberger)

Let x, y be finite words.
We have $xy = yx$ iff x and y are both powers of some word z.

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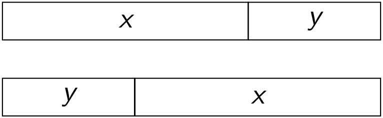

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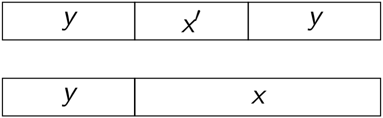

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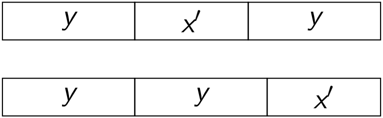

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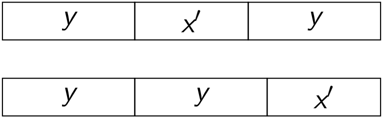


We have $x'y = yx'$ so $x' = z^m$ and $y = z^n$.
Therefore $x = yx' = z^{m+n}$.

Warm up: a bit of 1D

Theorem (Defect theorem)

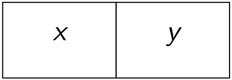
Let w, x, y be finite words.
If w may be written in two different ways over $\{x, y\}$,
then x, y are both powers of some z.

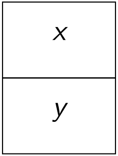
(Essentially a reduction to the previous result.)

Plan

A world of blocks

If x, y are **blocks**, then we have:

$x \obar y =$ 

$x \ominus y =$ 

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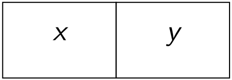
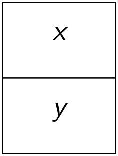
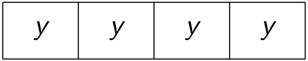
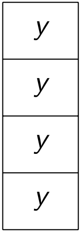
$x \varobar y =$ 

$x \ominus y =$ 

$$y^{m\times 1} = y \varobar y \varobar \dots \varobar y$$



$$y^{1\times n} = y \ominus y \ominus \dots \ominus y$$



The easy 2D theorem

Theorem

Let x, y be blocks with same height. We have

$$x \oplus y = y \oplus x \iff x = z^{m\times 1} \text{ and } y = z^{n\times 1}$$

for a block z and natural integers m, n.

Same for the vertical version.

Proof.

Use columns as letters and view it in 1D. □

Anything better?

Decompositions

Definition

A **pattern** is a finite 2D word with any shape.

a	c	b	a	
b	a	c	b	a
c	b	a	c	b
	c	b	a	c

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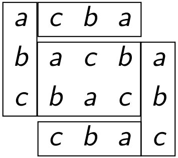

Definition

Let w be a pattern and $x_1, \ldots, x_k$ be blocks.
The x_i's **tile** w iff w can be partitionned into copies of the x_i's.

(No rotations, no reflections.)

A 2D defect theorem

Theorem

Let w be a pattern and x, y blocks.
Then x, y tile w in 2 different ways iff x, y are powers of some z.

Remark: it generalizes the easy 2D theorem.

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Proof.

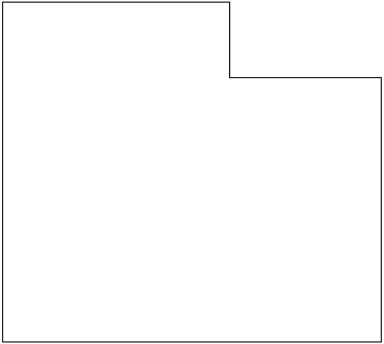
Outline:

- Assume not
- Take w counterexample with minimal $|w|$
- Also take x, y such that $|x| + |y|$ is minimal
- Find something even more minimal

□

Proof

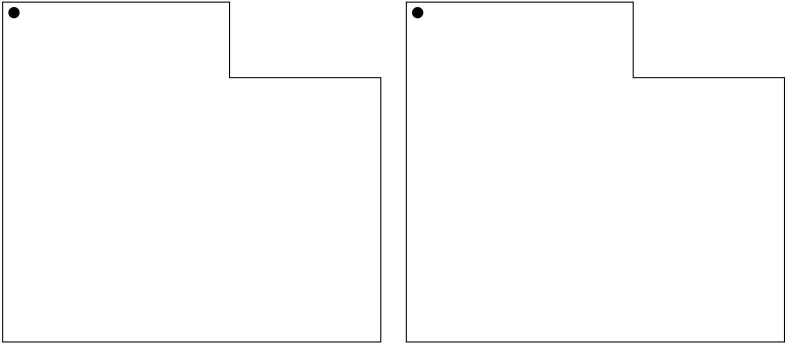
The shape of w...



...could be anything.

Proof

The shape of w...



The cell • is covered by x in a tiling and y in another.

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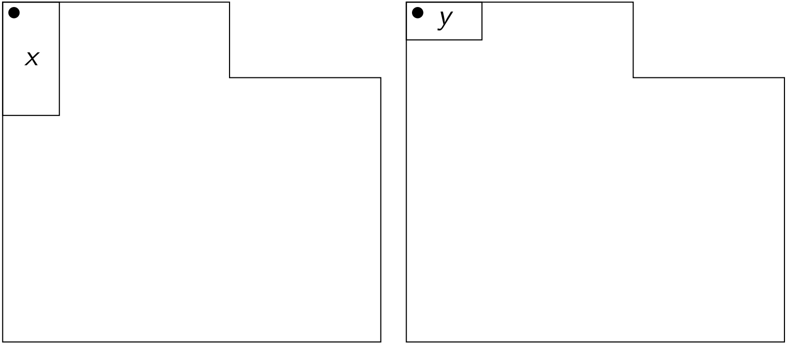

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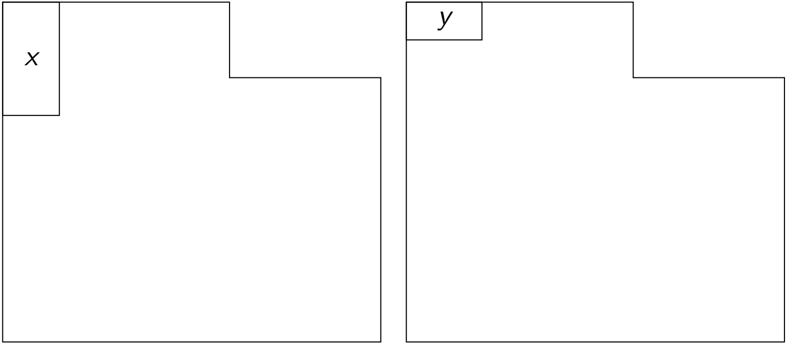
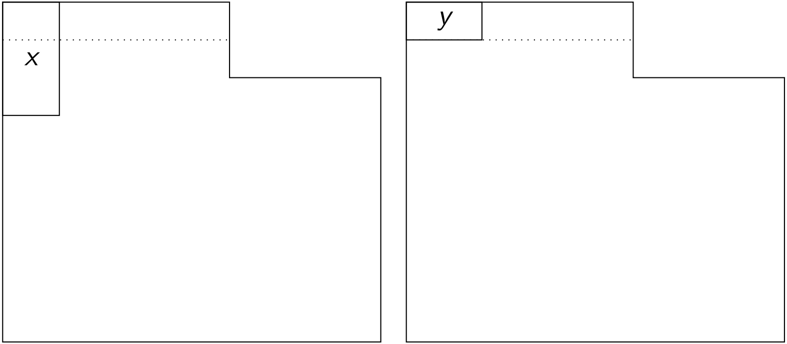


Assume x is taller than y, wlog.

Proof

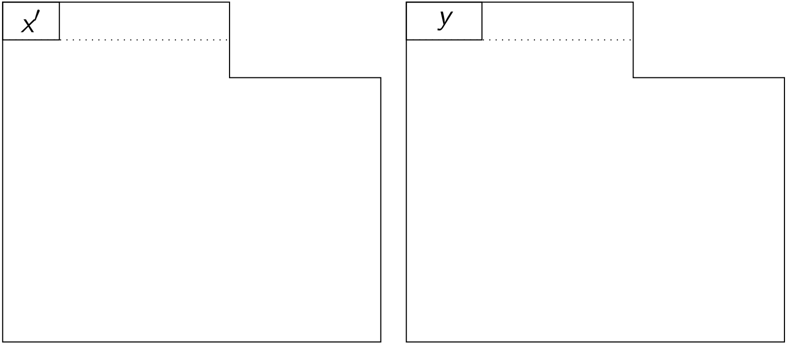
The shape of w...



Cut x at the height of y, let x' denote the result.

Proof

The shape of w...



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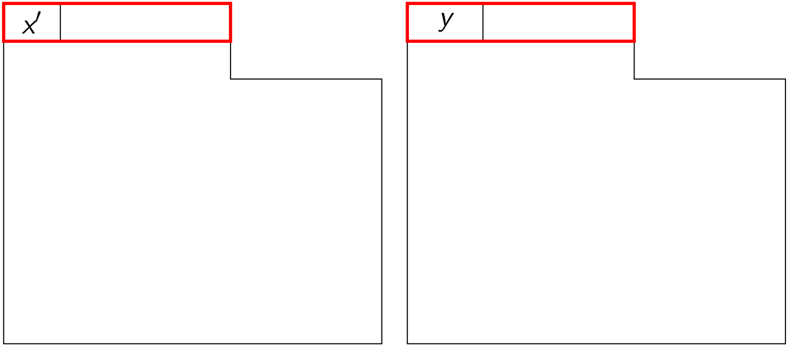
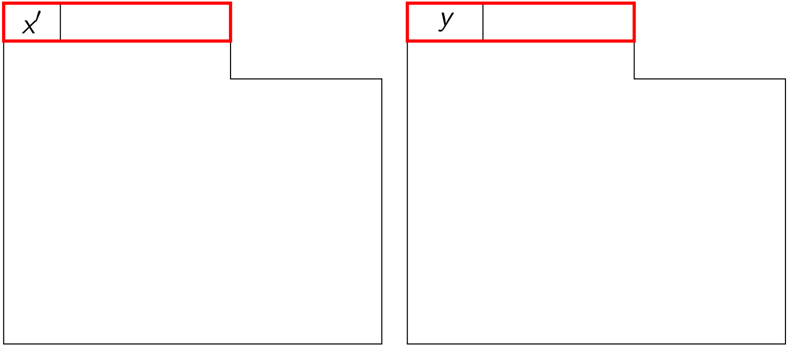
The shape of *w*...



Consider the zone above the dotted line as an 1D word (letters = columns).

Proof

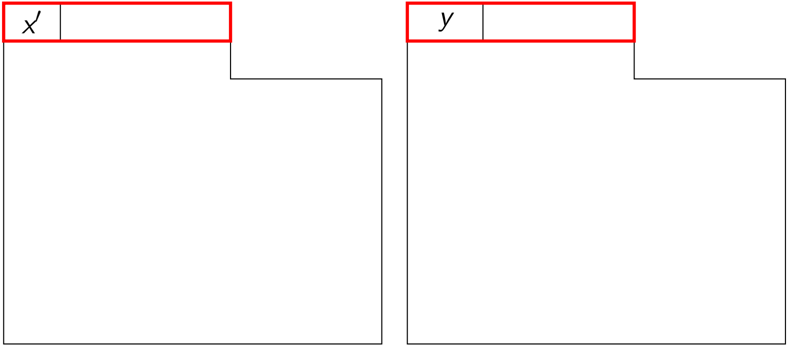
The shape of w...



It decomposes over x', y in 2 ways, so we have $x' = z^m$ and $y = z^n$.

Proof

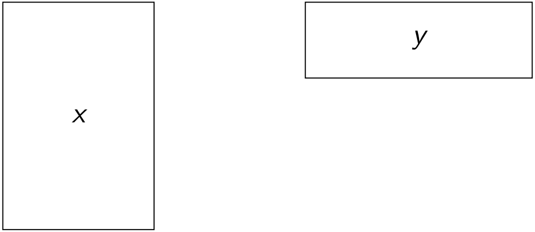
The shape of w...



In 2D terms, $x' = z^{m \times 1}$ and $y = z^{n \times 1}$.

Proof (recap)

- Assume a counterexample with $|w|$ and $|x| + |y|$ minimal



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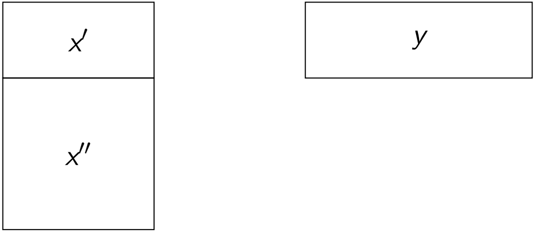

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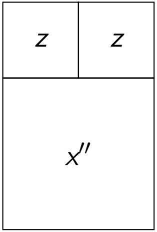

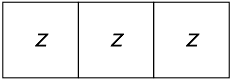

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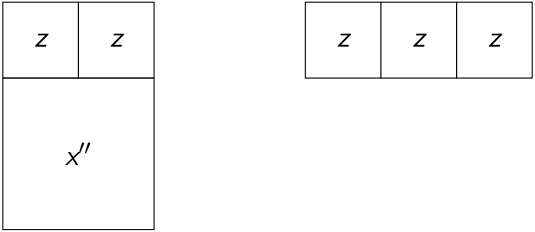
- We proved that $x' = z^{m\times 1}$ and $y = z^{n\times 1}$ for some z, m, n
- Decompose w over z and x'': we have $|z| + |x''| < |x| + |y|$



$\Longrightarrow$ Contradiction!

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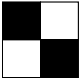
A block x is **primitive** if $x = y^{m \times n}$ implies $m = n = 1$.



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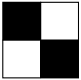

Corollary

For all block x, there is a unique primitive y such that $x = y^{m \times n}$.

(Suppose we had y and y'; apply theorem.)

Computing the primitive root of a block

Let x denote a block. What is its primitive root?

Algorithm

Let (m, n) be the size of x.

- $r_i \leftarrow$ primitive root of row i
- $c_i \leftarrow$ primitive root of column i
- $p \leftarrow \text{lcm}(|r_1|, \ldots, |r_m|)$
- $q \leftarrow \text{lcm}(|c_1|, \ldots, |c_m|)$
- return $x[1, \ldots, p\,;\ 1, \ldots, q]$.

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- Fails with 3 blocks instead of 2
- Allows to work with primitive roots

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