Exact Comparison of Quadratic Irrationals

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Abstract

This paper discusses the relationship between quadratic irrational numbers and periodic continued fractions. By using some basic properties of this relationship, we will show how to compare exactly two quadratic irrationals. Instead of using their numerical values which are approximations of the real values, we use their corresponding continued fractions each of which contains only a finite set of integers.

Key words: *quadratic irrational, periodic continued fraction, exact comparison.*

1 Introduction

During our recent study about *discrete rotations* based on hinge angles for 2D and 3D digital images [1], we encounter the problem of comparing quadratic irrationals when we would like to observe the discretization of the rotation space induced by hinge angles. There exist various techniques for calculating quadratic irrationals; for instance Talor series transformation, Babyloian method, Exponential identity, Bakhshali approximation, etc. However these methods compare quadratic irrationals by using their approximative values, which means that the comparison is "*true*" with a given precision. For our purpose, we need to compare two quadratic irrationals with their "*exact values*". An exact approach, employing only integers, has been proposed in [1] by using the square function. The disadvantage is that square function generates large integers, which are out of range of types int, long in C/C++ and thus, that is a limitation to implement. Our approach is based on another representation altogether for quadratic irrationals by using *continued fraction expansion*, which are represented by sets of integers. Indeed, quadratic irrationals can be represented exactly using periodic continued fractions, and this representation is unique. This provides an exact method that represent such continuing fractions, as well as compare them, which allows us avoid numerical errors.

This paper is organized as follows: in section 2, we introduce some basic notions of *quadratic irrationals*. Section 3 is devoted to formulate the problem of comparing exactly two quadratic irrationals and to give a solution based on continued fraction representation. Then, in section 4 we compare our method with some others : Approximation method and square comparison method. Finally, we conclude the paper with discussion of future work and some applications in section 5.

2 Quadratic irrationals

In this section, we present several mathematical concepts that will be useful to understand the problem of comparison of quadratic irrationals. We first begin by the definition of *irrational numbers*.

Definition 2.1 *An **irrational number** is any real number which cannot be expressed as a fraction $\frac{a}{b}$, where a and b are integers with b non-zero, and is therefore not a rational number.*

Informally, this means that an irrational number cannot be represented as a simple fraction. In this paper, we are interested in a special kind of irrational numbers that are called *quadratic irrational numbers.* There are several ways in which quadratic irrationals can be represented. In this paper we refer the definition and results to mathematical analysis books given by K. Rosen [4].

Definition 2.2 *A real number $x \in \mathbb{R}$ is a* ***quadratic irrational*** [1] *if there exist $A, B, C \in \mathbb{Z}$ and $A \neq 0$ such that $Ax^2 + Bx + C = 0$ and $D = B^2 - 4AC > 0$ and D is not a perfect square* [2]*.*

Consequently, the solutions of this equation are quadratic irrational numbers, that can be expressed as:

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

According to Definition 2.2, quadratic irrationals have to satisfy two conditions: they must be the solution of a quadratic equation and be irrational.

We define now a special kind of quadratic irrationals, called *reduced surd*, because of their interesting properties associated with continued fraction that we will discuss in Section 3.

Definition 2.3 *A quadratic irrational Q is said to be* ***reduced*** *if $Q > 1$ is the root of a quadratic equation with integer coefficients whose conjugate root $\overline{Q}$ lies between -1 and 0.*

We will present as follow a lemma that will be used in the algorithms of Section 3 in other to solve the problem of exact comparison.

Lemma 2.1 *A quadratic irrational Q can be expressed in the form $Q = \frac{p+\sqrt{q}}{r}$, where q is not square free, $r \neq 0$, $p, q, r \in \mathbf{Z}$ that satisfies $r | q - p^2$.*

Proof : Suppose that $Q = \frac{p'+\sqrt{q'}}{r'}$ where $p', q', r' \in \mathbf{Z}$, $r' \neq 0$ and $r' \nmid q' - p'^2$. By multiplying both numerator and denominator of Q' with absolute value of denominator r', we obtain:

$$Q = \frac{|r'|p' + |r'|\sqrt{q'}}{|r'|r'} = \frac{|r'|p' + \sqrt{r'^2 q'}}{|r'|r'}.$$

Then, we have $(|r'|r')|(r'^2 q' - r'^2 p'^2)$.

3 Continued fraction expansion

It is known that a quadratic irrational can be represented by a continued fraction. The research of this relationship began from 17th century. Some useful properties are known [2] [4]; For instance, Leonhard Euler proved that the infinite simple continued fraction of an irrational number is eventually periodic if this number is a quadratic irrational. Then in 1770, Joseph-Louis Lagrange found out the proof that the continued fraction expansion of any quadratic irrational is periodic.

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[1] It is also known as a quadratic irrationality or quadratic surd.

[2] A perfect square is a number that can be written as the product of some integer with itself, *i.e* : $\sqrt{x} = a$ with $a \in \mathbb{Z}$.

3.1 Continued fraction

Before describing our method for the exact comparison, we first provide definitions of *continued fractions* and some important properties that will be used in the method.

Definition 3.1 *A* ***simple continued fraction*** *has an expression of the form*

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + ...}}}$$

where $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{Z}_+$ for $i = 1, 2, 3....$ They are called the ***partial quotients*** *of the continued fraction.*

Hereafter, we will deal with the cases when partial quotients are all positive.

The number of terms a_i is limited for **finite continued fractions**, while it is unlimited for **infinite continued fractions**. In the case of finite continued fractions, we thus write

$$[a_0; a_1, a_2, a_3, ...a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + ... \; + \cfrac{1}{a_n}}}}.$$

Similarly, we denote infinite continued fractions by

$$[a_0; a_1, a_2, a_3, ...] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + ...}}}.$$

In this paper, when we say *continued fractions*, they mean *infinite continued fractions*; otherwise, we will specify the term for each case.

For the problem of comparison, there are two special forms of continued fractions that interest us: *periodic continued fraction* and *purely periodic*. The are defined as follow:

Definition 3.2 *An infinite continued fraction is called a* ***periodic continued fraction****, if its terms eventually repeat from some point until infinite. The minimal number of repeating terms is called the period of the continued fraction.*

A periodic continued fraction thus has the form:

$$[a_0; a_1, a_2, a_3, ..., a_{k-1}, a_k, a_{k+1}, ..., a_{k+m}, a_k, a_{k+1}, ..., a_{k+m}, ...],$$

for which we also write

$$[a_0; a_1, a_2, a_3, ..., a_{k-1}, \overline{a_k, a_{k+1}, ..., a_{k+m}}].$$

Definition 3.3 *A continued fraction which is periodic from the first partial quotient is called **purely periodic**.*

We denote as:

$$[a_0; a_1, a_2, a_3, ..., a_{k-1}, a_k, a_0, a_2, a_3, ..., a_{k-1}, a_k, ...] = [\overline{a_0, a_1, a_2, a_3, ..., a_{k-1}, a_k}].$$

If the period starts with the second partial quotient, the continued fraction is called **simply periodic** which is represented by:

$$[a_0; a_1, a_2, a_3, ..., a_{k-1}, a_k, a_1, a_2, a_3, ..., a_{k-1}, a_k, ...] = [a_0; \overline{a_1, a_2, a_3, ..., a_{k-1}, a_k}].$$

Then, a purely periodic is also simplify periodic but the reverse is not true.

3.2 Continued fraction expansions of quadratic irrationals

So far we have studied the basic notions of quadratic irrationals and continued fractions. We are now interested in their relationship to solve the exact comparison of two quadratic irrationals. For this, we will first present two fundamental theorems that indicate the bijection between a quadratic irrational and a continued fraction. We just formulate the result and refer to the literature [2] [3] for proofs.

The first one was proved by J-L Lagrange.

Theorem 3.1 ***Quadratic irrationals** are the real numbers that can be exactly represented by **periodic continued fractions**.*

The next one was shown by A. Ya. Khinchin [Theorem 14].

Theorem 3.2 *There is a **one-to-one correspondence** between a real number and a continued fraction, which is either finite or infinite.*

Galois then found out then an interesting property between reduce surds and purely periodic continued fractions.

Theorem 3.3 *The continued fraction which represents a quadratic irrational Q is purely periodic if and only if Q is a reduced surd.*

In fact, Galois showed more than this result. He proved that if Q is a reduced quadratic surd and $\overline{Q}$ is its conjugate, then the continued fractions for Q and for $\frac{1}{\overline{Q}}$ are both purely periodic, and the repeating block in one of those continued fractions is the mirror image of the repeating block in the other.
From Theorem 3.3, we can state the following lemma that is useful for the exact comparison algorithm.

Lemma 3.1 *For any positive integer S that is not a perfect square, the continued fraction of $\sqrt{S}$ is simply periodic and more precisely it has the form*

$$\sqrt{S} = [a_0; \overline{a_1, a_2, ..., a_n, 2a_0}].$$

Proof. Let $a_0 = \lfloor\sqrt{S}\rfloor$. Since S is a positive integer, $\sqrt{S} + a_0 > 1$. Because S is not a perfect square, we have $0 < \sqrt{S} - a_0 < 1$ and its conjugate lies between -1 and 0, which means $-1 < -\sqrt{S} + a_0 < 0$. Therefore, from Definition 2.3, $\sqrt{S} + a_0$ is a *reduced quadratic irrational*. Applying Theorem 3.3 we then obtain:

$$\sqrt{S} + a_0 = [\overline{2a_0; a_1, a_2, ..., a_n}],$$

which is equivalent to

$$\sqrt{S} + a_0 = [2a_0; \overline{a_1, a_2, ..., a_n, 2a_0}].$$

Consequently, we have

$$\sqrt{S} = [a_0; \overline{a_1, a_2, ..., a_n, 2a_0}].$$

3.3 Algorithm for finding corresponding continued fraction of a quadratic irrational

Let us consider now two problems: finding the corresponding fraction of a square root, and of a quadratic irrational.

3.3.1 Periodic continued fraction of a square root

To find the continued fraction of a square root, one of the most known methods was proposed by K. Rosen, in [4]. This method base on the study of irrational numbers as continued fractions obtained by J.L. Lagrange. Let consider that we would link to calculate the continued fraction of a square root $\sqrt{S}$, where S is a positive integer and non perfect square. We obtain $\sqrt{S} = [a_0; a_1, a_2, a_3, ...]$ as result. The iterative algorithm, Algorithm 1, uses two sequential integer parameters : m_i and d_i, to obtain a_i.

Lemma 3.1 says that the corresponding continued fraction of square roots of S is simply periodic. It means the terms of expansion will repeat from some points until infinite. Because a program cannot be executed infinitely, we can stop it at the first repetition of the extension. Thus, the stop condition is given by verify the doublet m_i and d_i goes back as the same one that is encountered before.

Algorithm 1: Calculate the periodic continued fraction of a square root

Input: A positive integer S
Output: A periodic continued fraction of $\sqrt{S}$, $[a_0; \overline{a_1, a_2, a_3, ...}]$

$m_0 \longleftarrow 0$
$d_0 \longleftarrow 1$
$a_0 \longleftarrow \lfloor\sqrt{S}\rfloor$
$i \longleftarrow 0$
repeat
  $m_{i+1} \longleftarrow d_i a_i - m_i$
  $d_{i+1} \longleftarrow \frac{S - m_{i+1}^2}{d_i}$
  $a_{i+1} \longleftarrow \lfloor \frac{a_0 + m_{i+1}}{d_{i+1}} \rfloor$
  $i \longleftarrow i + 1$
until $(m_{i-1}, d_{i-1}) \neq (m_1, d_1)$;

According to Lemma 3.1, algorithm can terminate if we have $2a_0$ while calculate the a_i with $i > 0$. For this, in Algorithm 1 we replace the condition at line **10** by $a_i \neq 2a_0$.

3.3.2 Periodic continued fraction of a quadratic irrational

K. Rosen proposed in [4] a method which allows us to get the corresponding continued fraction of a quadratic irrational. Let $Q = \frac{p_0+\sqrt{q_0}}{r_0}$, where q_0 is not a perfect square, p_0 and r_0 are integers. Algorithm can be applied if p_0, q_0 and r_0 are satisfying $r_0 | q_0 - p_0^2$. For this condition, Lemma 2.1 says that any quadratic irrationals of non square free can be expressed in the form $\frac{p+\sqrt{q}}{r}$ that $r|q-p^2$. We have Algorithm 2 as follows:

Algorithm 2: Calculate periodic continued fraction of a quadratic irrational

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Input: p_0, q_0 and r_0 represent the quadratic irrational $Q = \frac{p_0+\sqrt{q_0}}{r_0}$
Output: Corresponding periodic continued fraction : $Q = [a_0; \overline{a_1, a_2, a_3, ...}]$

```
a_0 ⟵ ⌊ (p_0+√q_0)/r_0 ⌋
i ⟵ 0
repeat
    p_{i+1} ⟵ a_i r_i − p_i
    r_{i+1} ⟵ (q_0 − p_{i+1}^2)/r_i
    a_{i+1} ⟵ ⌊ (p_{n+1}+√q_0)/r_{n+1} ⌋
    i ⟵ i + 1
until a_{i−1} ≠ 2a_0;
```

Notice that p_n, r_n and a_n are integers.

4 Comparison of quadratic irrationals

In this section, we first formalize the problem of comparing quadratic irrationals, then explain how to solve this problem with an exact approach using the continued fraction expansion.

4.1 Problem statement

Using Definition 2.2 in previous section, we now formulate the problem as follows. Suppose that we have two quadratic irrationals Q_1 and Q_2 that have forms:

$$Q_i = \frac{p_i + \sqrt{q_i}}{r_i}$$

where $p_i, q_i, r_i \in \mathbb{Z}$, $q_i > 0$ and $r_i \neq 0$ for $i = 1, 2$. Then, we would like to compare Q_1 and Q_2, with an exact calculation by using only integers.

According to Theorems 3.1 and 3.2, for each quadratic irrational we can find out exactly one corresponding periodic continued fraction. Let us consider

$$Q_1 = \frac{p_1 + \sqrt{q_1}}{r_1} = [a_0; a_1, a_2, a_3, ..., a_{k-1}, \overline{a_k, a_{k+1}, ..., a_{k+m}}],$$

$$Q_2 = \frac{p_2 + \sqrt{q_2}}{r_2} = [b_0; b_1, b_2, b_3, ..., b_{l-1}, \overline{b_l, b_{l+1}, ..., b_{l+n}}]$$

where $a_0, b_0 \in \mathbb{Z}$ and $a_i, b_j \in \mathbb{Z}_+$ for $i = 1, 2, ..., k + m$ and $j = 1, 2, ..., l + n$. This representation allows us to describe the quadratic irrationals not by their numerical values, *i.e.* real values, but rather by their corresponding continued fractions which requires only integers. Thus, the comparison between quadratic irrationals becomes the problem of comparison their corresponding periodic continued fractions.

4.2 Algorithm for comparing two quadratic irrationals

We would like to compare two quadratic irrationals Q_1 and Q_2, given by:

$$Q_i = \frac{p_i + \sqrt{q_i}}{r_i}$$

where $p_i, q_i, r_i \in \mathbb{Z}$, $q_i > 0$ and $r_i \neq 0$ for $i = 1, 2$. As discussing, our exact comparison method is based on calculating the corresponding continued fraction. However, for comparing we dont need to compute all terms of continued fraction expansions but the comparison happen for each coup of terms having the same level of continued fractions. Precisely, let us consider

$$Q_1 = \frac{p_1 + \sqrt{q_1}}{r_1} = [a_0; a_1, a_2, ...] \quad and \quad Q_2 = \frac{p_2 + \sqrt{q_2}}{r_2} = [b_0; b_1, b_2, ...].$$

Without loss of generality, we can assume $k \in \mathbb{Z}_+$ is the smallest index for which a_k is unequal to b_k. In order to compare Q_1 and Q_2, we should calculate expression $E = (-1)^k (a_k - b_k)$. There three cases:

1. $Q_1 = Q_2$ if and only if $p_1 = p_2$, $q_1 = q_2$ and $r_1 = r_2$ (Theorem 3.2 about unique correspondence).
2. $Q_1 < Q_2$ if $E < 0$.
3. $Q_1 > Q_2$ otherwise.

As explaining before, the comparison will be done for each coup of terms a_i and b_i. For calculating these terms we will use two methods that are described in section 3.

To apply Algorithm 1 for calculating continued factions, we need to modify quadratic irrationals Q_i into square root's form, as follows:

$$Q_1 = \frac{r_2 p_1 + r_2\sqrt{q_1}}{r_2 r_1} = \frac{r_2 p_1 + \sqrt{{r_2}^2 q_1}}{r_2 r_1},$$

$$Q_2 = \frac{r_1 p_2 + r_1\sqrt{q_2}}{r_2 r_1} = \frac{r_1 p_2 + \sqrt{{r_1}^2 q_2}}{r_2 r_1}$$

Thus, comparing Q_1 and Q_2 becomes comparing the numerators of Q_1 and Q_2. If we have

$$\sqrt{{r_2}^2 q_1} = [a'_0; \overline{a'_1, a'_2, ..., a'_n, 2a'_0}],$$

$$\sqrt{{r_1}^2 q_2} = [b'_0; \overline{b'_1, b'_2, ..., b'_n, 2b'_0}].$$

Then we obtain the two numerators as

$$r_2 p_1 + \sqrt{{r_2}^2 q_1} = [a'_0 + r_2 p_1; \overline{a'_1, a'_2, ..., a'_n, 2a'_0}],$$

$$r_1 p_2 + \sqrt{{r_1}^2 q_2} = [b'_0 + r_1 p_2; \overline{b'_1, b'_2, ..., b'_n, 2b'_0}].$$

Now we can use Algorithm 1 in order to calculate the periodic continued fractions of two square roots $\sqrt{{r_2}^2 q_1}$ and $\sqrt{{r_1}^2 q_2}$.

Function 1: Calculate i-th term of continued fraction using Algorithm 1

Input: A quartet of integer: a_0, a_i, m_i and d_i

Output: A triplet of integer: a_{i+1}, m_{i+1} and d_{i+1}

$m_{i+1} \longleftarrow d_i a_i - m_i$

$d_{i+1} \longleftarrow \frac{S - m_{i+1}^2}{d_i}$

$a_{i+1} \longleftarrow \lfloor \frac{a_0 + m_{i+1}}{d_{i+1}} \rfloor$

By using Algorithm 2, we can get directly the corresponding continued fraction of a quadratic irrational without modify quadratic form into square root's form.

Function 2: Calculate i-th term of continued fraction using Algorithm 2

Input: A quartet of integer: q_0, a_i, p_i and r_i

Output: A triplet of integer: a_{i+1}, p_{i+1} and r_{i+1}

$p_{i+1} \longleftarrow a_i r_i - p_i$

$r_{i+1} \longleftarrow \frac{q_0 - p_{i+1}^2}{r_i}$

$a_{i+1} \longleftarrow \lfloor \frac{p_{n+1} + \sqrt{q_0}}{r_{n+1}} \rfloor$

Finally, for comparing we have the algorithm using Algorithm 4 as follows:

Algorithm 3: Compare two quadratic irrationals

Input: Two triplets : (p_1, q_1, r_1) and (p_2, q_2, r_2) represent two quadratic irrational Q_1 and Q_2

Output: Value indicating result of the comparison: 0 if $Q_1 = Q_2$; 1 if $Q_1 > Q_2$; otherwise -1.

```
if p_1 = p_2 and q_1 = q_2 and r_1 = r_2 then
    return 0
else
    E ⟵ 0
    i ⟵ 0
    p_1i ⟵ p_1
    q_1i ⟵ q_1
    r_1i ⟵ r_1
    p_2i ⟵ p_2
    q_2i ⟵ q_2
    r_2i ⟵ r_2
    while E = 0 do
        (p_1(i+1), q_1(i+1), r_1(i+1)) ⟵ Function 2 (q_1, p_1i, q_1i, r_1i)
        (p_2(i+1), q_2(i+1), r_2(i+1)) ⟵ Function 2 (q_2, p_2i, q_2i, r_2i)
        E ⟵ (−1)^i (a_1i − a_2i)
        i ⟵ i + 1
    end
    if E > 0 then return 1
    else return -1
end
```

4.3 Complexity

The complexity of comparing two periodic continued fractions depends on the number of terms to examine. In the worst case, we have to compare all terms in the periodic expansion. Thus, the complexity belongs to length of the repeating block of quadratic surd Q, given by:

$$Q = \frac{p + \sqrt{q}}{r} = [a_0; \overline{a_1, a_2, ..., a_n, 2a_0}].$$

If the length of the partial quotients of Q and q are denoted by $L(Q)$ and $L(q)$; we have $L(Q) = L(q) = n + 1$. Lagrange proved that the largest partial denominator a_i in the expansion is less than $2\sqrt{q}$, and that $L(q) < 2q$. More recently Hickerson [5] and Podsypanin [6], based on the divisor function[3], have shown that $L(q)$ is given by $L(q) = O(\sqrt{(q)} \ln q)$.

5 Discussion

In this section, we would like to compare our approach with several other methods: *Approximation method* and *Square comparison method* to show why this approach is more efficient than the others.

[3]divisor function is an arithmetical function related to the divisor of an integer. It counts the number of divisors of an integer.

5.1 Approximation method vs. Exact method

Square root of a positive integer number S which is not a perfect square is an irrational number. There exist many methods for evaluating this value. It can be classified into two categories : **approximation methods** and **exact methods**. We say *approximation method* if we represent $\sqrt{S}$ by an equivalence which is a infinitive real number[4]. For instance Newton's method, Exponential identity, Rough estimation, Babylonian method, etc. However, there are two problems with these approaches : (1) Computers normally use the IEEE[5] representation for real numbers; this standard represents a floating point number up to some precision as supported by the computing hardware. However, irrational numbers, such as $\sqrt{2}$, or even many rational numbers cannot be expressed exactly in this format. They must be approximated. (2) An algorithm must terminate at some point when a sufficient approximation is reached; as a result, only an approximation of quadratic irrational is provided. For these reasons, the comparison of two quadratic irrationals is true only within *some precision.* By using periodic continued fraction to represent irrational numbers, our approach is an *exact method* because it employs only integers to represent and to compare them.

5.2 Square comparison vs. Continued Faction comparison

Square approach is an exact comparison by using square function to avoid the irrational numbers. For instance, we have two quadratic irrational Q_1 and Q_2

$$Q_1 = \frac{p_1 + \sqrt{q_1}}{r_1} \quad and \quad Q_2 = \frac{p_2 + \sqrt{q_2}}{r_2}.$$

If $Q_1 > Q_2$ then $Q_1 - Q_2 > 0$

$$Q_1 - Q_2 = \frac{p_1 + \sqrt{q_1}}{r_1} - \frac{p_2 + \sqrt{q_2}}{r_2} = \frac{r_2 p_1 + r_2\sqrt{q_1}}{r_2 r_1} - \frac{r_1 p_2 + r_1\sqrt{q_2}}{r_2 r_1} > 0,$$

because $r_1 r_2 > 0$, thus

$$r_2 p_1 + r_2\sqrt{q_1} > r_1 p_2 + r_1\sqrt{q_2},$$

take the square two times

$$(r_1^2 q_2 + r_2^2 q_1 - (r_2 p_1 - r_1 p_2)^2)^2 > 4 r_1^2 r_2^2 q_1 q_2.$$

The below equation contains only integers. Therefore, we can verify $Q_1 - Q_2$ with integer calculation. However the fact of using the square function, computation perform on big values which are squares of integer values ! The comparison of quadratic irrationals with continued factions approach (Algorithm 1) permit to calculate with integers that is smaller than $r_2 p_1 + r_2\sqrt{q_1}$ and $r_1 p_2 + r_1\sqrt{q_2}$.

6 Conclusion

In this paper we presented a discussion on relation between quadratic irrationals and continued fractions, from that we introduced an exact comparison of two quadratic irrationals that

[4]Irrational number can't be represented as terminating or repeating decimals.

[5]Standard for Floating-Point Arithmetic is the most widely-used standard for floating-point computation.

is important in dividing parameter space of digital image transforms. We have determined the complexity of this comparison that permit to reduce the complexity of general discrete rigid transformation's algorithm.

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