Combinatorial structure for rigid transformations in 2D digital images

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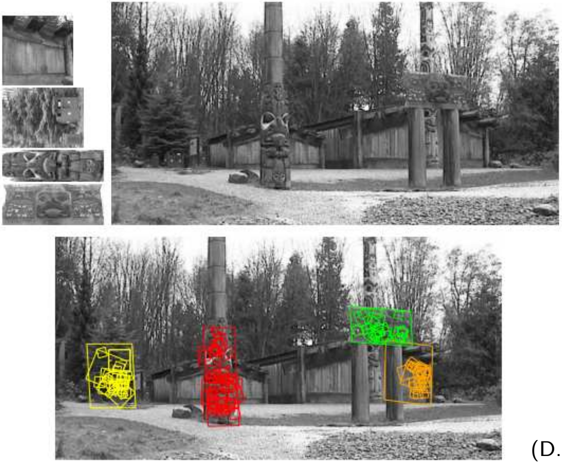
Rigid transformations

Rigid transformation is a function $\mathcal{T}_{ab\theta} : \mathbb{R}^2 \to \mathbb{R}^2$, such that

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} p\cos\theta - q\sin\theta + a \\ p\sin\theta + q\cos\theta + b \end{pmatrix}$$

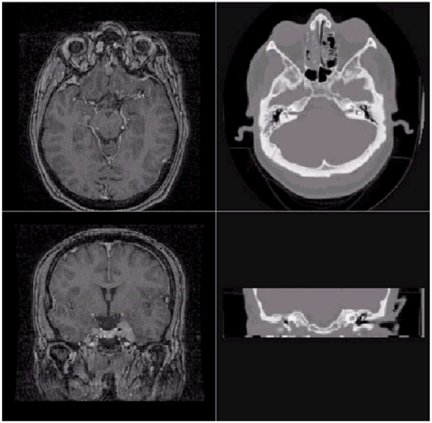
where $a, b, \theta \in \mathbb{R}$ and $(p, q), (p', q') \in \mathbb{R}^2$.

Applications of rigid transformations: Pattern matching

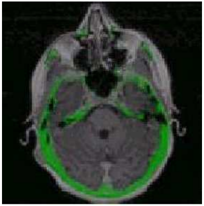


(D. G. Lowe '04)

Applications of rigid transformations: Image registration



MRI CT



Colored overlay

(J.B.Antoine Maintz and Max A. Viergever '98)

Motivation

Our questions

- Rigid transformations can be performed in a discrete space?

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- Rigid transformations can be performed in a discrete space?
- How many transformed images are there?
- How to generate all of them?

Previous works: Combinatorial image matching

For a 2D image of size of $N \times N$, the complexity of the generated images under different classes of transformations are:

Transformations	Complexity
Rotation (A. Amir '06)	$O(N^3)$
Scaling (A. Amir '03)	$O(N^3)$
Rotation and scaling (C. Hundt '09)	$O(N^6)$
Linear transformations (C. Hundt '08)	$O(N^{12})$
Affine transformations (C. Hundt '07)	$O(N^{18})$
Projective transformations (C. Hundt '08)	$O(N^{24})$

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- give an (exact computation) algorithm in linear time for construction this graph.

Rigid transformations for 2D digital images

Digital rigid transformation is the function $T_{ab\theta} : \mathbb{Z}^2 \to \mathbb{Z}^2$ such that

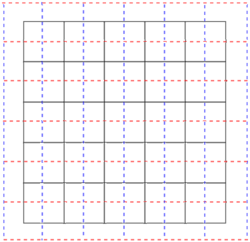
$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} \lfloor p\cos\theta - q\sin\theta + a + \frac{1}{2} \rfloor \\ \lfloor p\sin\theta + q\cos\theta + b + \frac{1}{2} \rfloor \end{pmatrix}$$

where $a, b, \theta \in \mathbb{R}$ and $(p, q), (p', q') \in \mathbb{Z}^2$.

Half-grid

Definition

The **half grid** $\mathcal{H}$ is the set of points (x, y) on either of the lines $x = k + \frac{1}{2}$ or $y = l + \frac{1}{2}$ for any $k, l \in \mathbb{Z}$.

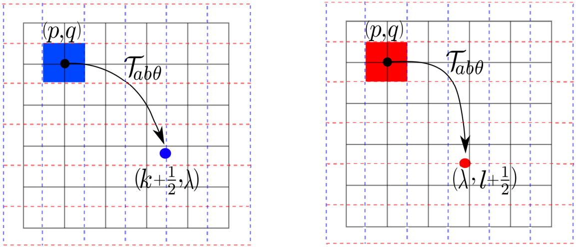


$\mathcal{H}$ divides the space $\mathbb{R}^2$ into unit squares, called *pixels*.

Critical transformations

Definition

A **critical rigid transformation** moves at least one integer point into a half-grid point.

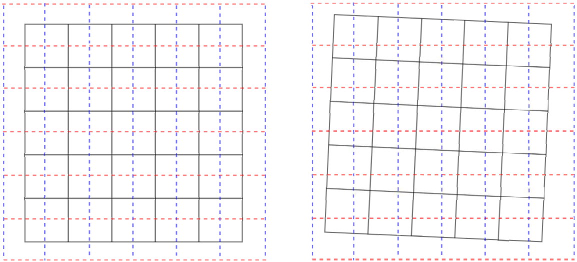


The set of the critical transformations corresponds to the *discontinuities* of digital rigid transformations.

DRT

Definition

A **discrete rigid transformation** (DRT) is a set of all rigid transformations given the same digital transformed image.



The parameter space is partitioned into the disjoint sets of DRT.

Tipping surfaces

Definition

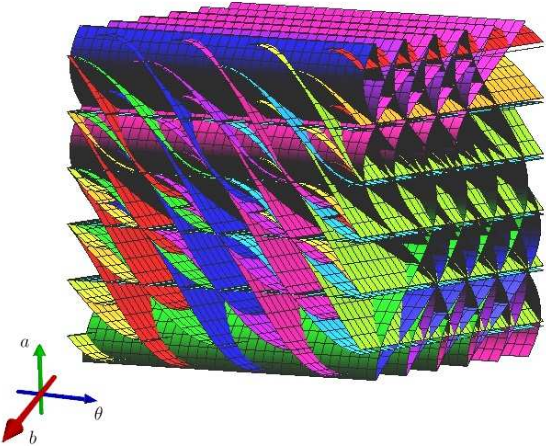
Tipping surfaces are the surfaces associated to the critical transformations in the parameter space (a, b, θ).

$$\left| \begin{array}{lccc} \Phi_{pqk} : & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (b, \theta) & \longmapsto & a = k + \frac{1}{2} + q \sin\theta - p \cos\theta, \end{array} \right.$$

$$\left| \begin{array}{lccc} \Psi_{pql} : & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (a, \theta) & \longmapsto & b = l + \frac{1}{2} - p \sin\theta - q \cos\theta, \end{array} \right.$$

for $p, q, k, l \in \mathbb{Z}$.

Example of tipping surfaces



Tipping curves

Definition

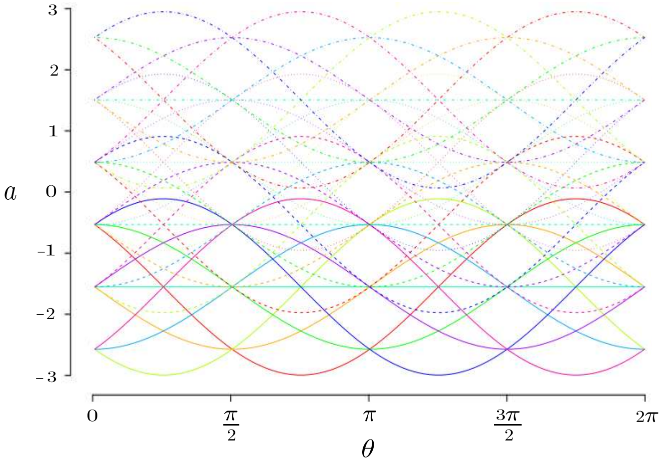
Tipping curves are the orthogonal a-axis (resp. b-axis) cross-sections of Φ_{pqk} (resp. ψ_{pql}) on the plane.

$$\phi_{pqk} : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ \theta & \longmapsto & a = k + \frac{1}{2} + q\sin\theta - p\cos\theta, \end{array}$$

$$\psi_{pql} : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ \theta & \longmapsto & b = l + \frac{1}{2} - p\sin\theta - q\cos\theta, \end{array}$$

for $p, q, k, l \in \mathbb{Z}$.

Example of tipping curves



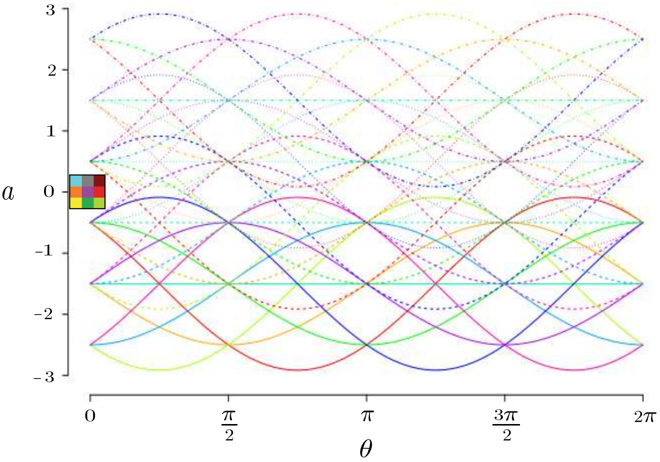
DRT graph

Definition

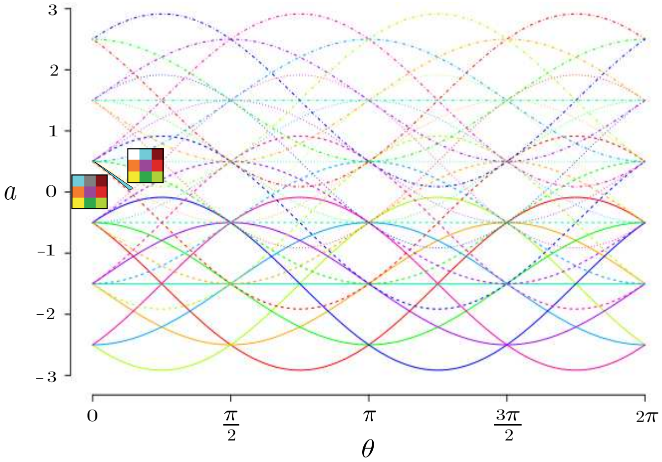
A **discrete rigid transformation graph** (DRT graph) is a graph $G = (V, E)$ such that

- each vertex in V corresponds to a DRT,
- each edge in E connects two vertices sharing a tipping surface.

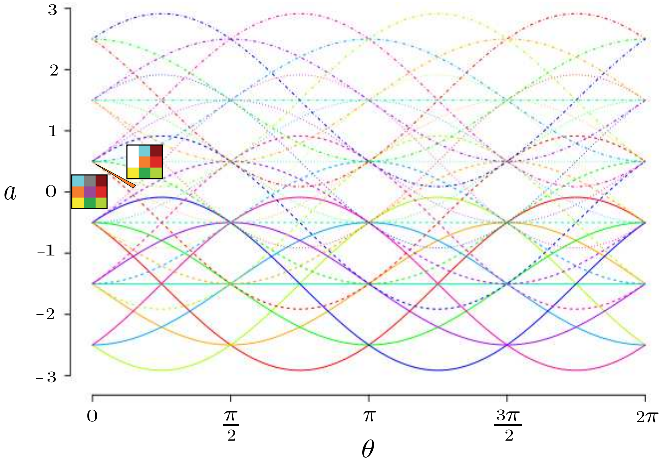
Example of DRT



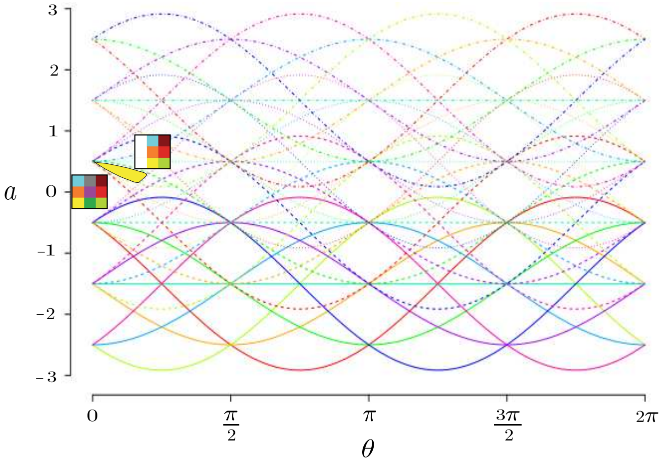
Example of DRT



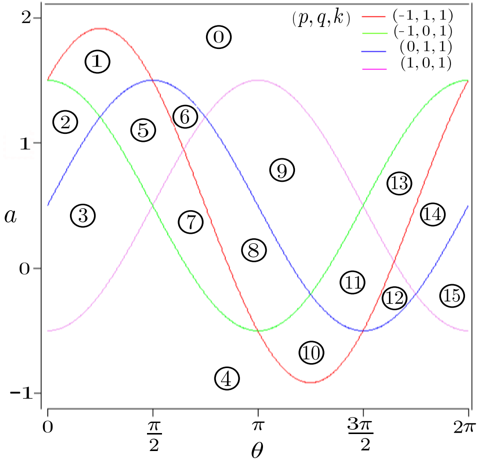
Example of DRT

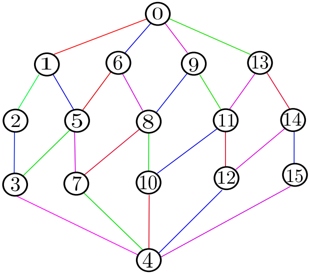


Example of DRT



Example of 2D DRT graph for the tipping curves





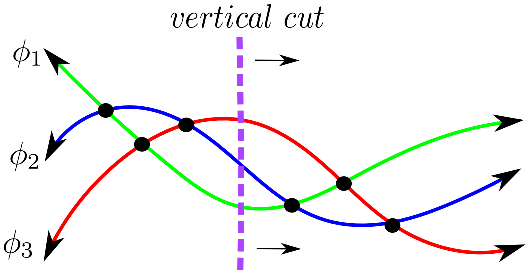
Construction of 2D DRT graph for the tipping curves

Problem

- **Input**: A collection of tipping curves C.
- **Output**: The 2D DRT graph G_C for C.
- **Approach**: Sweeping method.

Sweeping approach

The sweeping method uses a vertical cut to sweep throughout C for constructing G_C.



Incremental graph construction

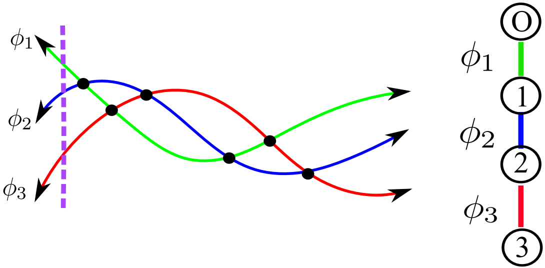
Proposition

Let G_C be the 2D DRT graph of C. Then we have

$$G_C = \cup \sum_{i=0}^{m} \delta G_{C_i}$$

where δG_{C_i} is the partial graph at the i-th step and m is the total number of intersections.

Illustration for graph construction



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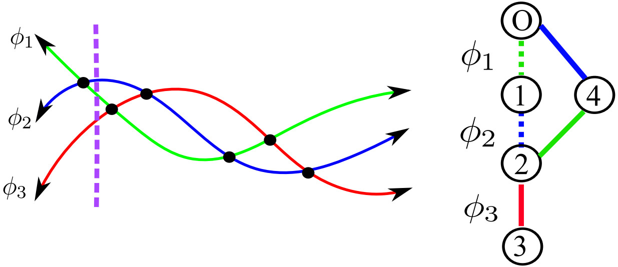

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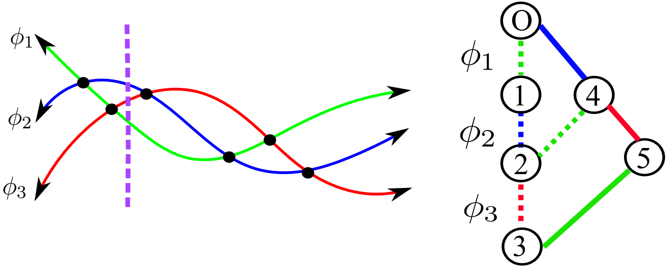

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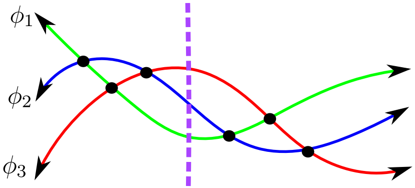

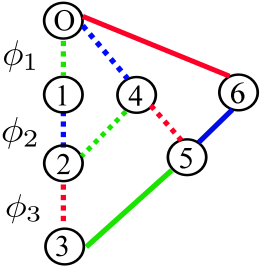

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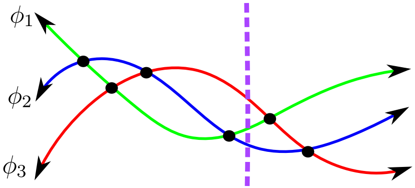

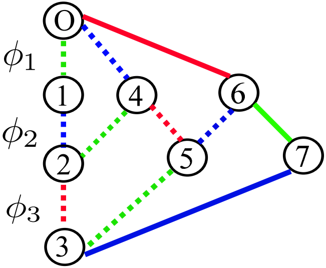

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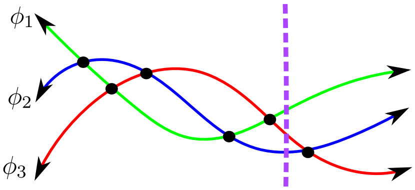

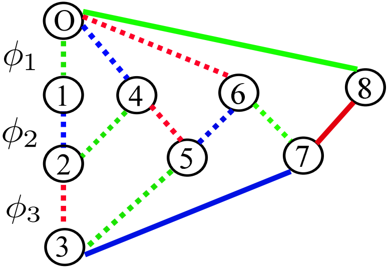

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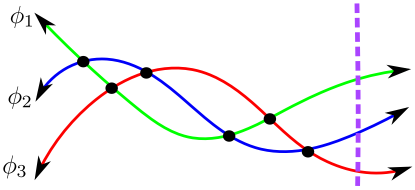

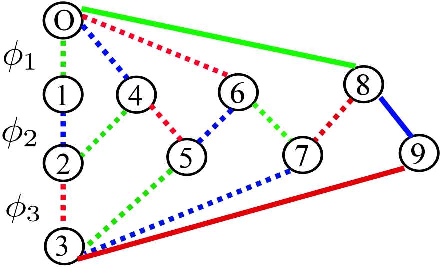

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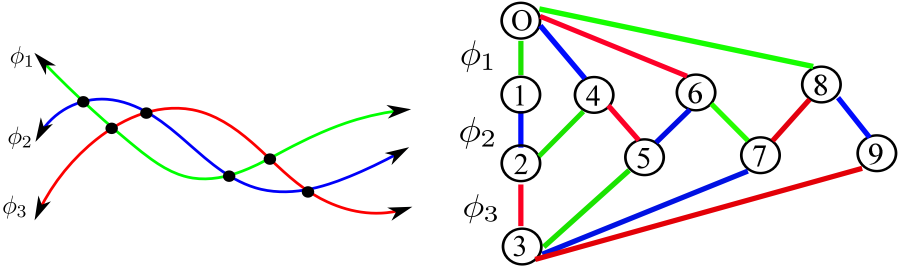
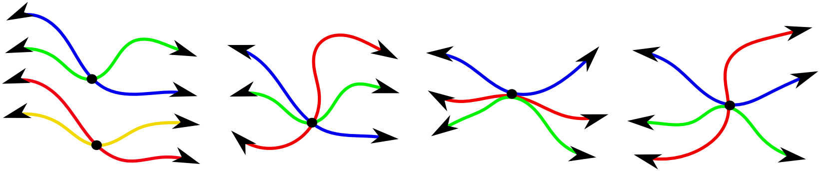


Degeneracies



(a) (b) (c) (d)

Remark: Intersection of two tipping curves can be expressed by a quadratic irrational.

Exact comparison

We can compare the intersections with an exact computation using the continued fractions.

Complexity

Given a digital image of size $N \times N$

Properties of tipping curves

- There are $N^2(N+1)$ tipping curves.
- Two tipping curves can intersect at two points maximum.

Complexity of 2D DRT graph (vertices)

$$\underbrace{O(N^3)}_{Initial\ graph} + \underbrace{O(N^6)}_{Number\ of\ intersections} = O(N^6)$$

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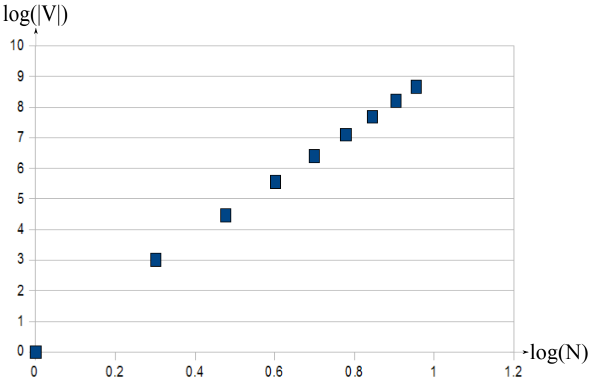
Complexity of DRT graph (vertices)

$$\underbrace{O(N^3) \times O(N^3)}_{\textit{Initial graph}} \;+\; \underbrace{O(N^6) \times O(N^3)}_{\textit{Number of intersections}} \;=\; O(N^9)$$

Experiments

Image size	# Vertices
1×1	1
2×2	1033
3×3	29631
4×4	357421
5×5	2487053
6×6	12550225
7×7	48604267
8×8	160554101
9×9	457270393



Conclusion

- We proposed a discrete version of rigid transformations for 2D digital image of size $N \times N$,
- defined a graph for representing the combinatorial structure of discrete rigid transformations,
- showed that the complexity of the graph is $O(N^9)$,
- gave an (exact computation) algorithm in linear time for construction this graph.

Perspectives

- Extending the method for 3D digital image.
- Integrating topology information.
- Application of DRT graph for denoising problem, patch approach.

Bibliography

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[2] Amihood Amir, Ayelet Butman, Moshe Lewenstein, and Ely Porat, 2003.

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[5] Hundt Christian, and Liśkiewicz Maciej, 2008.

[6] David G. Lowe, 2004.

[7] J.B.Antoine Maintz and Max A. Viergever, 1998.