Combinatorial Properties of 2D Discrete Rigid Transformations under Pixel-Invariance Constraints

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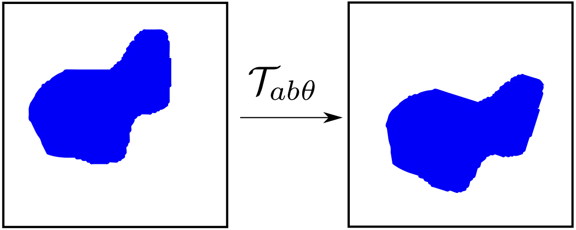
Rigid transformations

Rigid transformation is a function $\mathcal{T}_{ab\theta} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, such that

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} p\cos\theta - q\sin\theta + a \\ p\sin\theta + q\cos\theta + b \end{pmatrix}$$

where $a, b \in \mathbb{R}$, $\theta \in [0, 2\pi[$ and $(p, q), (p', q') \in \mathbb{R}^2$.

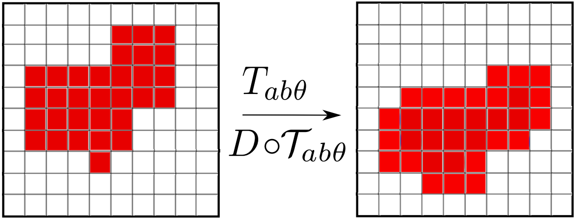


Digital rigid transformations

Digital rigid transformation is the function $T_{ab\theta} : \mathbb{Z}^2 \to \mathbb{Z}^2$ such that

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} \left[p\cos\theta - q\sin\theta + a + \frac{1}{2}\right] \\ \left[p\sin\theta + q\cos\theta + b + \frac{1}{2}\right] \end{pmatrix}$$

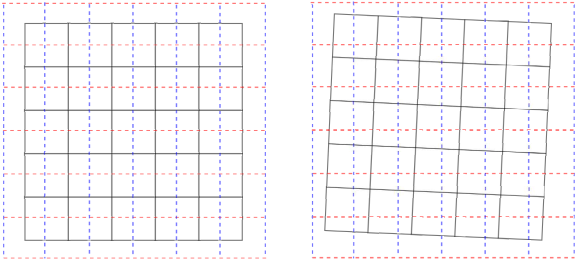
where $a, b \in \mathbb{R}$, $\theta \in [0, 2\pi[$ and $(p,q), (p',q') \in \mathbb{Z}^2$.



DRT

Definition

A **discrete rigid transformation** (DRT) is a set of all rigid transformations providing the same digitization of transformed grid of a given image.

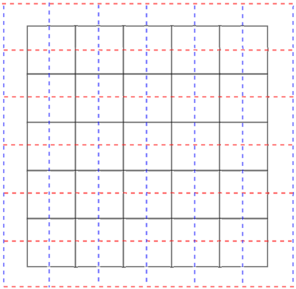


The parameter space is partitioned into the disjoint sets of DRTs.

Half-grid points

Definition

The **half-grid** $\mathcal{H}$ is the set of points (x, y) on the lines either $x = k + \frac{1}{2}$ or $y = l + \frac{1}{2}$ for any $k, l \in \mathbb{Z}$.

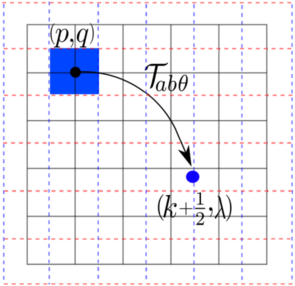


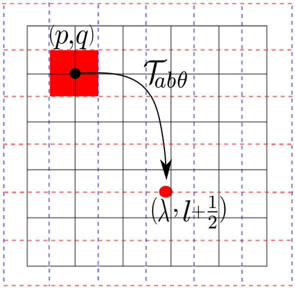
$\mathcal{H}$ divides the space $\mathbb{R}^2$ into unit squares, called *pixels*.

Critical transformations

Definition

A **critical rigid transformation** moves at least one integer point into either a vertical or a horizontal half-grid point.





Tipping surfaces

Definition

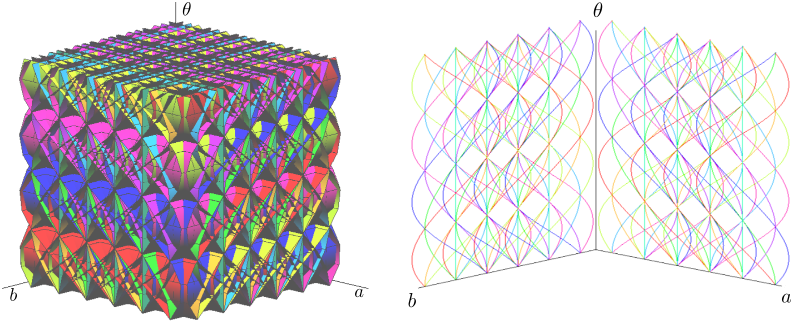
Tipping surfaces are the surfaces associated to the critical transformations in the parameter space (a, b, θ).

$$\Phi_{pqk} : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (b, \theta) & \longmapsto & a = k + \frac{1}{2} + q \sin\theta - p\cos\theta, \end{array} \qquad (\textit{vertical})$$

$$\Psi_{pql} : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (a, \theta) & \longmapsto & b = l + \frac{1}{2} - p \sin\theta - q\cos\theta, \end{array} \qquad (\textit{horizontal})$$

for $p, q, k, l \in \mathbb{Z}$.

Example of tipping surfaces



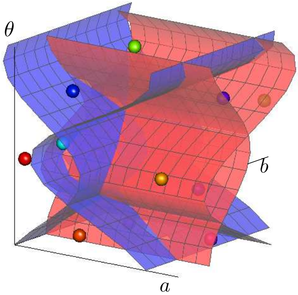
(a) Tipping surfaces (b) Orthogonal sections

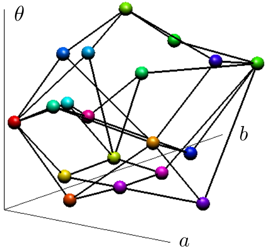
DRT graph

Definition

A **discrete rigid transformation graph** (DRT graph) is a graph $G = (V, E)$ such that

- each vertex in V corresponds to a DRT,
- each edge in E connects two vertices sharing a tipping surface.





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Given a digital image I of size $N \times N$:

Complexity of DRT graph (vertices)

The DRT graph associated to I has a space complexity of $O(N^9)$.

$$\underbrace{O(N^3) \times O(N^3)}_{\textit{Initial graph}} + \underbrace{O(N^6) \times O(N^3)}_{\#\ \textit{intersections}} = O(N^9)$$

Properties of DRT graphs

Advantages

- Rigid transformation has been studied as a **fully discrete process**.
- A **graph** G is used to model all the rigid transformations of a given digital image I.
- Using G, we can generate exhaustively and incrementally **all the transformed images** of I.

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- A **graph** G is used to model all the rigid transformations of a given digital image I.
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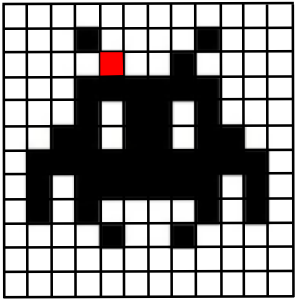
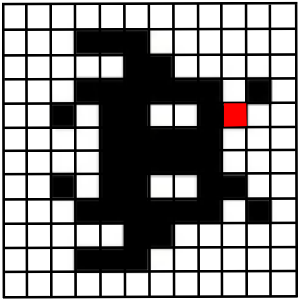
Disadvantages

The proposed structure has a **high complexity**.

Constraint paradigm in discrete framework



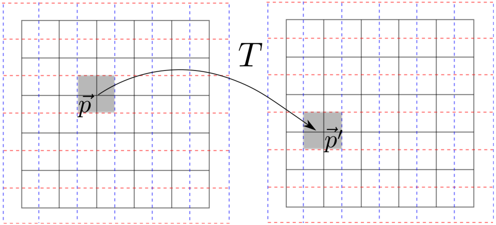


Pixel-invariance constraint

Definition

A **pixel-invariance constraint** between $\vec{p}$ and $\vec{p'}$ is defined by

$$p' - 1/2 < p\cos\theta - q\sin\theta + a < p' + 1/2,$$
$$q' - 1/2 < p\sin\theta + q\cos\theta + b < q' + 1/2.$$

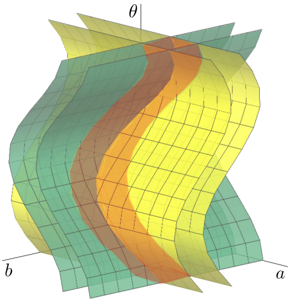


Pixel-invariance constraint

Definition

A **pixel-invariance constraint** between $\vec{p}$ and $\vec{p'}$ is the subspace in the parameter space (a, b, θ), defined as:

$$H^{+}_{pqp'} \cap H^{-}_{pqp'} \cap V^{+}_{pqq'} \cap V^{-}_{pqq'}.$$



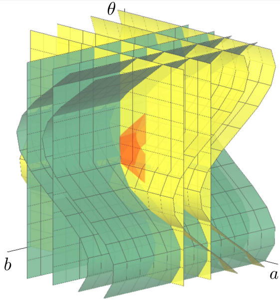
Feasible rigid transformation sets (FRTS)

Let $\mathcal{P} = \{(\vec{p}_i, \vec{p}_i')\}_{i=1}^{m}$ $(m \geq 1)$ be a set of pixel-invariance constraints.

Definition

The **feasible rigid transformation set** (FRTS) associated to $\mathcal{P}$ is the subspace $\mathcal{R} \subset \mathbb{R}^3$, defined as:

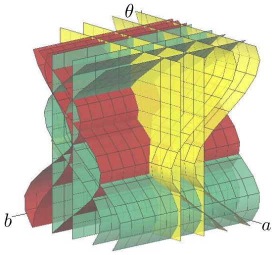
$$\mathcal{R} = \bigcap_{i \in [\![1,m]\!]} \left(H^{+}_{p_i q_i p_i'} \cap H^{-}_{p_i q_i p_i'+1} \cap V^{+}_{p_i q_i q_i'} \cap V^{-}_{p_i q_i q_i'+1} \right).$$

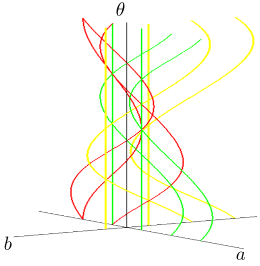


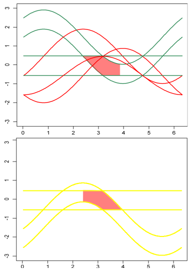
Construction of DRT graph in a FRTS

Problem

- **Input**: Two sets $\mathcal{P}$ and S of pixel-invariance constraints and of tipping surfaces.
- **Output**: The DRT graph G in the FRTS induced by $\mathcal{P}$.
- **Approach**: A modified sweeping method.



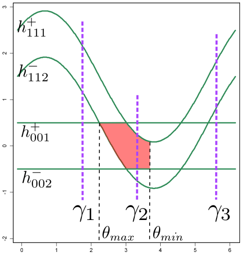




Construction of DRT graph in a FRTS

Algorithm

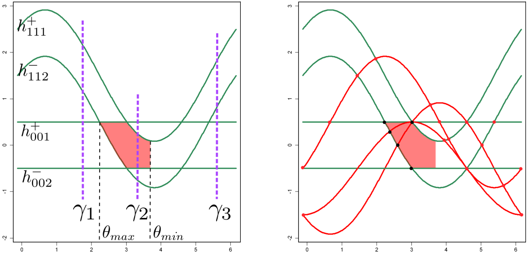
1. Finding the FRTS boundary induced by $\mathcal{P}$.
   - Sweeping a cut γ along θ-axis,
   - γ is in the FRTS when it is separated into two successive sequences of the upper and the lower half-planes.



Construction of DRT graph in a FRTS

Algorithm

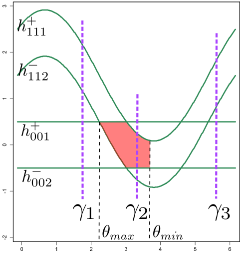
1. Finding the FRTS boundary induced by $\mathcal{P}$.
2. Finding tipping surfaces passing by the FRTS and the intersection points.
   - Calculating intersections of tipping surfaces in S and the boundary segments of the FRTS.

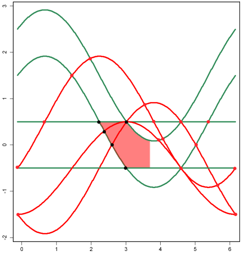


Construction of DRT graph in a FRTS

Algorithm

1. Finding the FRTS boundary induced by $\mathcal{P}$.
2. Finding tipping surfaces passing by the FRTS and the intersection points.
3. DRT-graph construction in the FRTS.
    - Using the sweeping method in the FRTS.





Complexity of DRT graph under one constraint

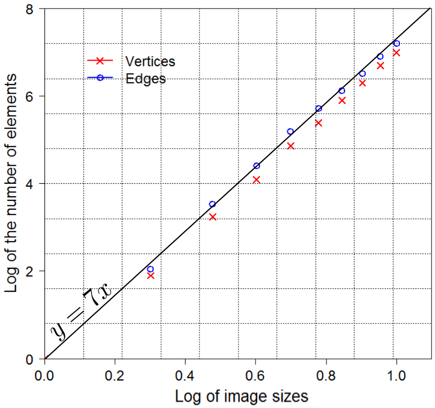
Given a digital image I of size $N \times N$ and one pixel-invariance constraint.

Complexity of DRT graph (vertices)

The DRT graph associated to I under one pixel-invariance constraint has a space complexity of $O(N^7)$.

$$\underbrace{O(N^2) \times O(N^2)}_{\textit{Initial graph}} \quad + \quad \underbrace{O(N^5) \times O(N^2)}_{\#\ \textit{intersections}} \quad = \quad O(N^7)$$

Experiments: One pixel-invariance constraint



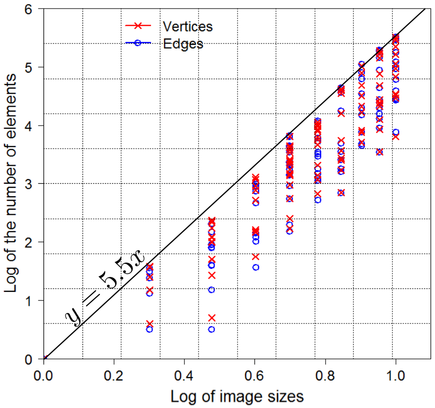
Complexity of DRT graph under more than one constraint

Given a digital image I of size $N \times N$ and more than one pixel-invariance constraint.

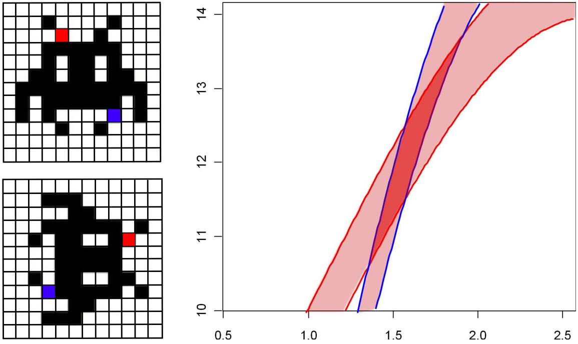
Complexity of DRT graph (vertices)

The theoretical space complexity of DRT graph associated to I under these pixel-invariance constraints ?

Experiments: Two pixel-invariance constraints



Experiments: Three pixel-invariance constraints



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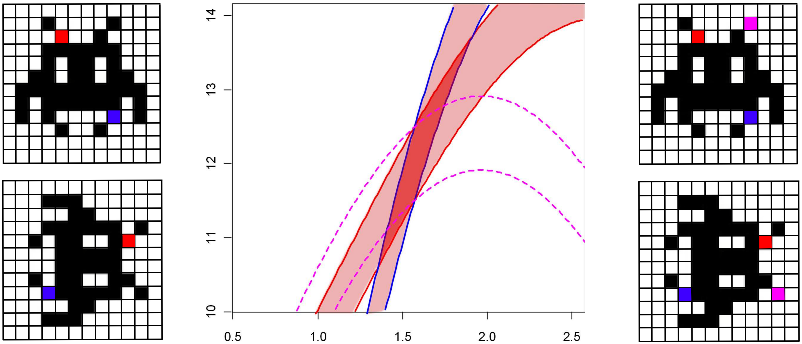
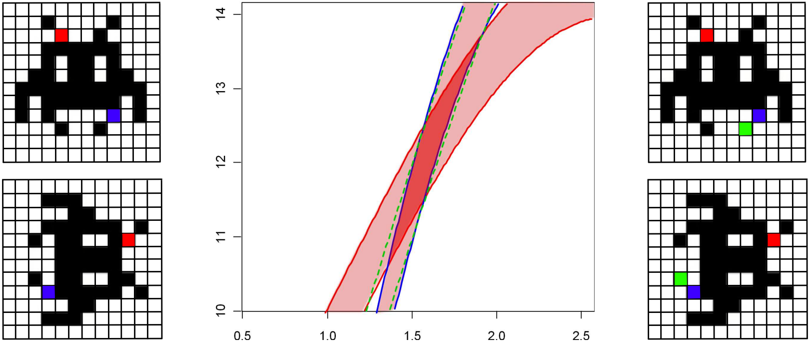
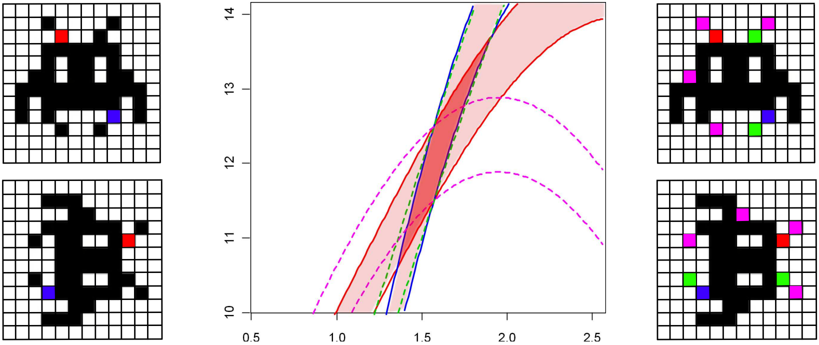


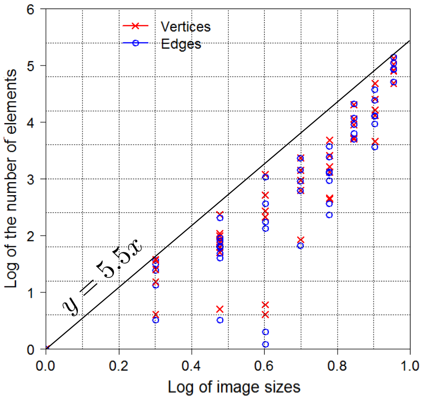
Experiments: Three pixel-invariance constraints



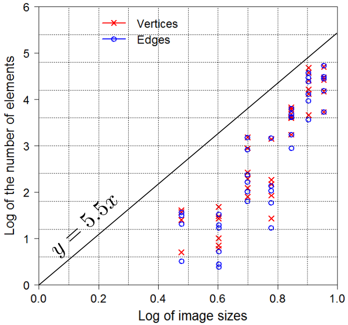
Experiments: Three pixel-invariance constraints



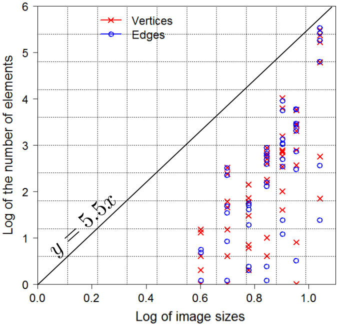
Experiments: Three pixel-invariance constraints



Experiments: Five and ten pixel-invariance constraints



(e) Five constraints



(f) Ten constraints

Conclusion and perspectives

Conclusion

- We studied the effects of geometric constraints on the structure of rigid transformations for digital images;
- we gave an (exact computation) algorithm in linear time for constructing this structure under some imposed constraints; and
- we analysed, theoretically and practically, the complexity of the introduced structure.

Conclusion and perspectives

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- We studied the effects of geometric constraints on the structure of rigid transformations for digital images;
- we gave an (exact computation) algorithm in linear time for constructing this structure under some imposed constraints; and
- we analysed, theoretically and practically, the complexity of the introduced structure.

Perspectives

- Proving theoretical complexity for two (and more) constraints.
- Evaluating the pixel-invariance constraints.
- Multi-resolution approach.

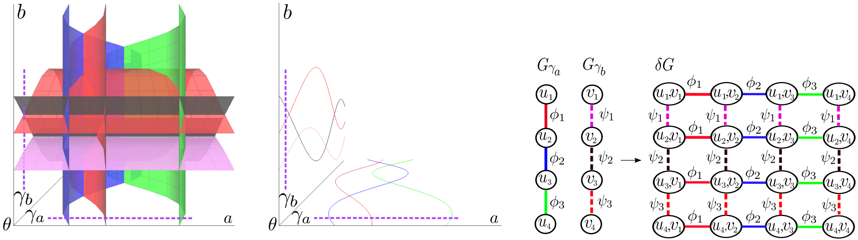
Thank you for your attention

We welcome your questions, suggestions and comments!

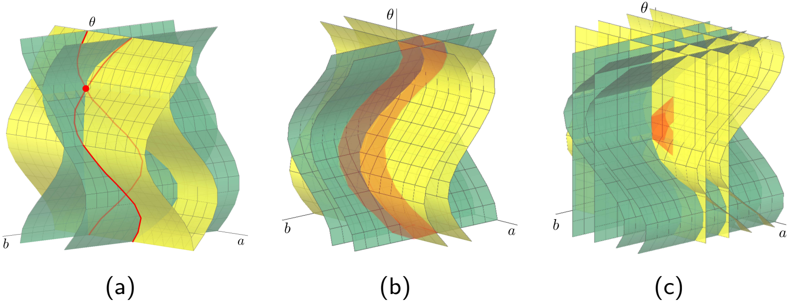
Construction of DRT graph

Problem

- **Input**: A collection of tipping surfaces S.
- **Output**: The DRT graph G for S.
- **Approach**: A sweeping method.

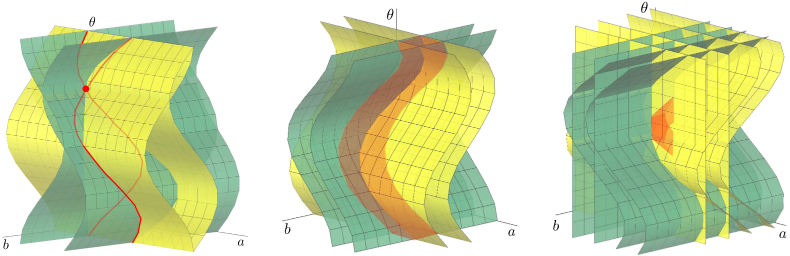


Evaluation of pixel-invariance constraints



Rigid transformation in continuous and discrete frameworks

Evaluation of pixel-invariance constraints



(a) (b) (c)

Rigid transformation in continuous and discrete frameworks

Evaluate the correspondence of interest points between images.

Evaluation of pixel-invariance constraints