# Efficient Exact Computation for Incremental Discrete Rotations using Continued Fraction

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## Motivation

For **incremental discrete rotations** based on *hinge angles*. We need to sort all the hinge angles defined for a given digital image.

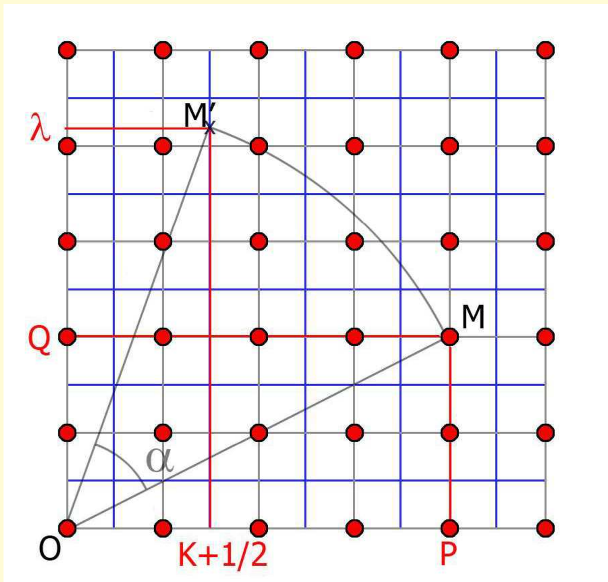
***Definition:*** *An angle $\alpha$ is a* **hinge angle** *for a discrete point $(p,q) \in \mathbb{Z}^2$ if the rotation of $(p,q)$ by $\alpha$ is a point in the half-grid.* [Nouvel,2006]

A hinge angle $\alpha$ for an image of size $N \times N$ can be presented by a triplet of integers $(p,q,k) \in N^3$, such that

$$\cos\alpha = \frac{p\lambda + q(k+\frac{1}{2})}{p^2+q^2}$$

$$\sin\alpha = \frac{p(k+\frac{1}{2}) - q\lambda}{p^2+q^2},$$

where $\lambda = \sqrt{p^2+q^2-(k+\frac{1}{2})^2}$ *and* $k < \sqrt{p^2+q^2}$.




In order to sort the hinge angles, we need to compare their cos and sin values, which are *quadratic irrational* numbers.

## Problem formulation

Given two quadratic irrationals $Q_1$ and $Q_2$ that have the forms:

$$Q_i(p_i,q_i,r_i) = \frac{p_i+\sqrt{q_i}}{r_i}$$

where $p_i, q_i, r_i \in \mathbb{Z}$, $q_i > 0$ and $r_i \neq 0$ for $i = 1,2$, we would like to **compare** $Q_1$ **and** $Q_2$, with an **exact calculation by using only integers**.

## Mathematical tools

### *Quadratic Irrational*

***Definition:*** *A real number $x \in \mathbb{R}$ is a* **quadratic irrational** *(QI) if there exist $A, B, C \in \mathbb{Z}$, $A \neq 0$ such that $Ax^2+Bx+C=0$ and $B^2-4AC>0$ is not a perfect square.*

The quadratic irrational can be expressed as:

$$x = \frac{-B \pm \sqrt{B^2-4AC}}{2A}.$$

### *Continued Fraction*

***Definition:*** *A* **continued fraction** *(CF) has an expression of the form*

$$[a_0;a_1,a_2,a_3,...] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2+\cfrac{1}{a_3+...}}}$$

*where $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{Z}_+$ for $i = 1,2,3....$*

Every rational number has a *finite continued fraction*, while every irrational number can be represented by an *infinite continued fraction*.

***Definition:*** *An infinite continued fraction is* **periodic**, *if its terms eventually repeat from some point.*

If the repeating terms are from $a_k$ to $a_{k+m}$, it is denoted by:

$$[a_0;a_1,a_2,...,a_{k-1},\overline{a_k,a_{k+1},...,a_{k+m}}].$$

### *Properties*

***Theorem:*** *There is a* ***unique correspondence*** *between a real number and a continued fraction, which is either finite or infinite.*

***Theorem:*** *A* ***quadratic irrational*** *is a real number that can be exactly represented by a* ***periodic continued fraction***.

## Our solution

In order to compare exactly two QIs, we use **periodic continued fractions**. Given two triplets of integers, representing two QIs, we calculate their corresponding periodic CFs. By examining the terms of these CFs, we obtain the comparison result. The algorithm is described as follows:

Two QIs are given with their corresponding CFs, such that:

$$Q_1(p_1,q_1,r_1) = [a_0;a_1,a_2,...],$$
$$Q_2(p_2,q_2,r_2) = [b_0;b_1,b_2,...].$$

Let $k \in \mathbb{Z}_+$ be the smallest index for which $a_k$ is unequal to $b_k$. Then,

1. $Q_1 = Q_2$ if and only if $p_1 = p_2$, $q_1 = q_2$ and $r_1 = r_2$,
2. $Q_1 < Q_2$ if $E < 0$,
3. $Q_1 > Q_2$ if $E > 0$,

where $E = (-1)^k(a_k - b_k)$.

**Advantage:** This method does not need to compute all the terms of the CF expansions. The comparison is made for each pair of terms $(a_i, b_i)$ in ascending order; it stops when $a_i \neq b_i$. There must be an $i$ such that $a_i \neq b_i$, otherwise $Q_1 = Q_2$.

## Recursive computing of continued fraction

Let us consider the CF of a QI: $Q(p_0,q_0,r_0) = [a_0;a_1,a_2,a_3,...]$. The following function is used to calculate the $i+1$-th term $a_{i+1}$ of $Q$.

**Function** Calculate $i+1$-th term of the CF of a QI

**Input**: A quadruplet of integer: $q_0, a_i, p_i$ and $r_i$
**Output**: A triplet of integer: $a_{i+1}, p_{i+1}$ and $r_{i+1}$

$$p_{i+1} \longleftarrow a_i r_i - p_i$$
$$r_{i+1} \longleftarrow \frac{q_0 - p_{i+1}^2}{r_i}$$
$$a_{i+1} \longleftarrow \lfloor \frac{p_{n+1}+\sqrt{q_0}}{r_{n+1}} \rfloor$$

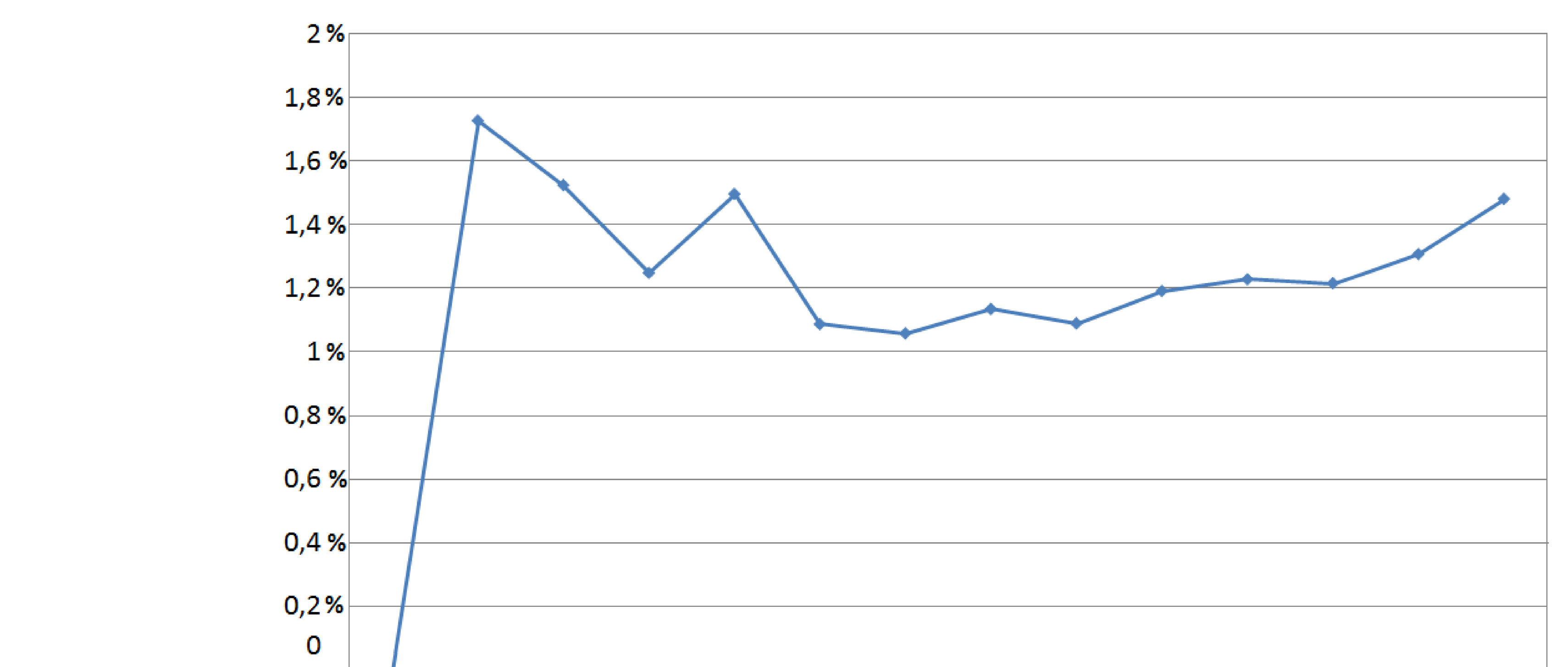
Note: It is assumed that $q_0$ is not a perfect square.

## Important remarks

1. The **algorithm complexity** depends on the number of terms of the periodic CF expansions to examine. In the worst case, it is the period of CF, which is $O(\sqrt{q}\ln q)$ for $Q(p,q,r)$.

2. The **largest partial denominator** $a_i$ of the CF of $Q(p,q,r)$ is smaller than $2\lfloor Q \rfloor$.

3. We carried out a comparison with **IEEE floating point numbers**, on image $N \times N$ for N=5,10,...,70. We observed that this approximative approach exhibit an average error rate of 1.12% for all sorted hinge angles.




| Image Size NxN | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 | 55 | 60 | 65 | 70 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Error rate (%) | 0 | 1,726 | 1,523 | 1,248 | 1,496 | 1,087 | 1,058 | 1,134 | 1,089 | 1,191 | 1,228 | 1,215 | 1,307 | 1,480 |
| Total generated HAs | 156 | 1390 | 4856 | 11692 | 23060 | 40098 | 63962 | 95784 | 13673 | 18794 | 25056 | 32574 | 41464 | 51837 |
| Total Error Comparison | 0 | 24 | 74 | 146 | 345 | 436 | 677 | 1087 | 1490 | 2239 | 3078 | 3960 | 5422 | 7676 |

Comparison Approximative Method and Exact Method