Combinatorial Properties of Rigid Transformations in 2D Digital Images

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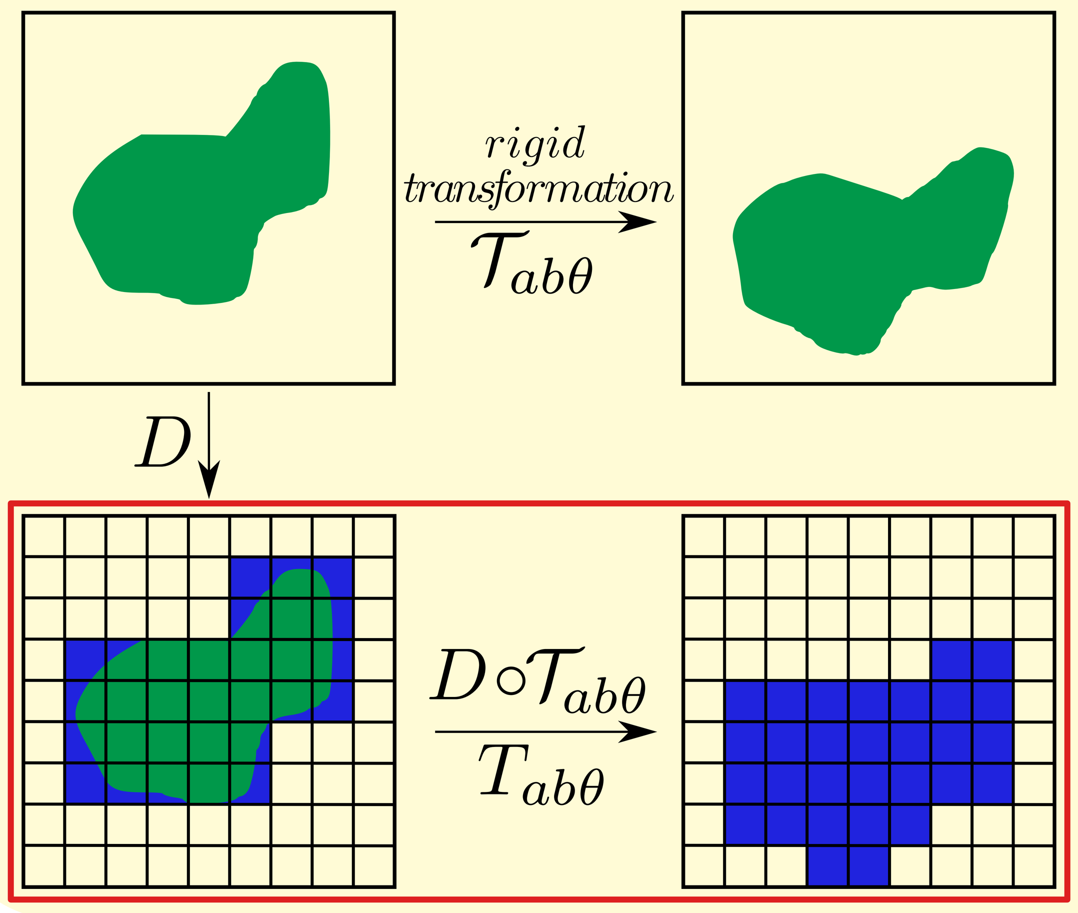
Motivations



Rigid transformations are frequently involved in image processing tasks. We study combinatorial aspects of rigid transformations in 2D digital images, in particular considering the following questions:

- Can rigid transformations be performed in a discrete space?
  - Yes! We propose a discrete version of rigid transformations for 2D digital images.
- What is the combinatorial structures of those transformations?
  - It is represented by a graph, which can be built by an (exact computation) algorithm.
- How many transformed images are there for a given image of size $N \times N$?
  - It is in the order of N^9 which is the complexity of the graph as well.
- Is it possible to generate all the transformed images?
  - Yes, by using the proposed graph.

Definitions

A **rigid transformation** is a function $\mathcal{T}_{ab\theta} : \mathbb{R}^2 \to \mathbb{R}^2$, such that

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} p\cos\theta - q\sin\theta + a \\ p\sin\theta + q\cos\theta + b \end{pmatrix}$$

where $a, b \in \mathbb{R}$, $\theta \in [0, 2\pi[$ and $(p,q), (p',q') \in \mathbb{R}^2$.

A **digital rigid transformation** is the function $T_{ab\theta} : \mathbb{Z}^2 \to \mathbb{Z}^2$ such that

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} [p\cos\theta - q\sin\theta + a + \frac{1}{2}] \\ [p\sin\theta + q\cos\theta + b + \frac{1}{2}] \end{pmatrix}$$

where $a, b \in \mathbb{R}$, $\theta \in [0, 2\pi[$ and $(p,q), (p',q') \in \mathbb{Z}^2$.

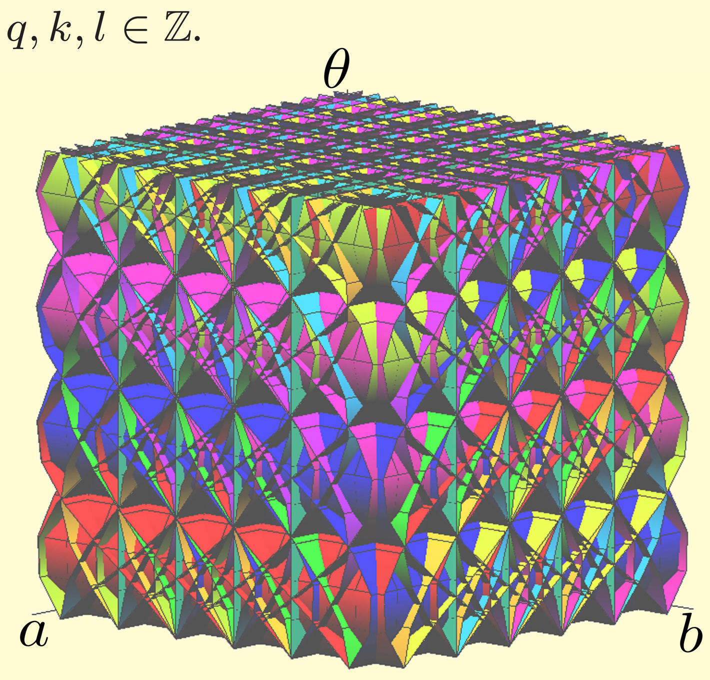
A **discrete rigid transformation** (DRT) is the set of all rigid transformations providing a same digitization of transformed grid of a given image.

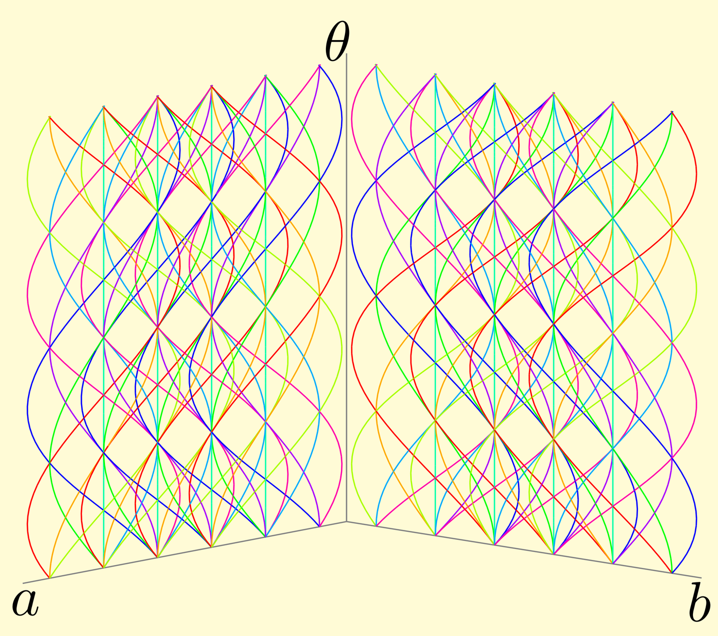
Tipping surfaces are the surfaces associated to the discontinuities of digital rigid transformations in the parameter space (a, b, θ).

$$\Phi_{pqk} : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (b,\theta) & \longmapsto & a = k + \frac{1}{2} + q\sin\theta - p\cos\theta, \end{array} \quad (1)$$

$$\Psi_{pql} : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (a,\theta) & \longmapsto & b = l + \frac{1}{2} - p\sin\theta - q\cos\theta, \end{array} \quad (2)$$

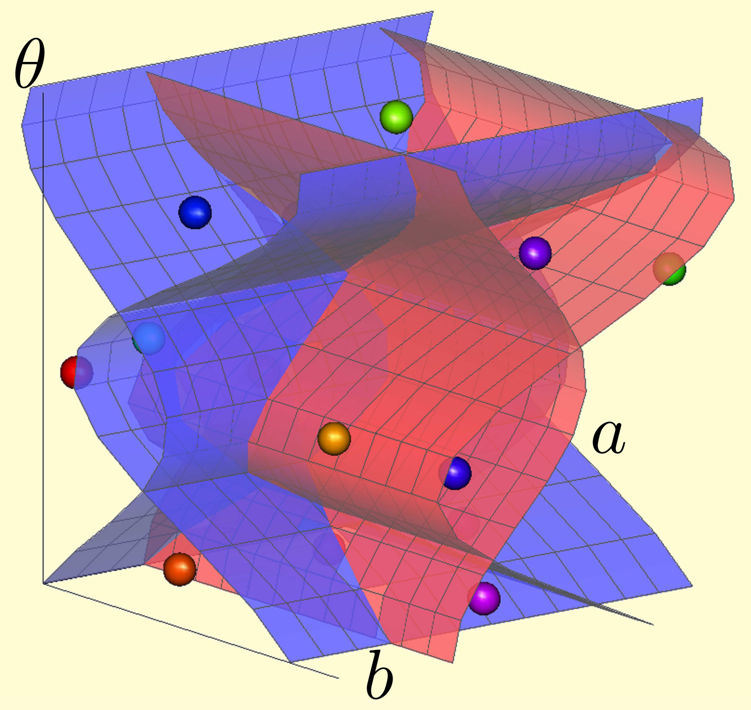
for $p, q, k, l \in \mathbb{Z}$.

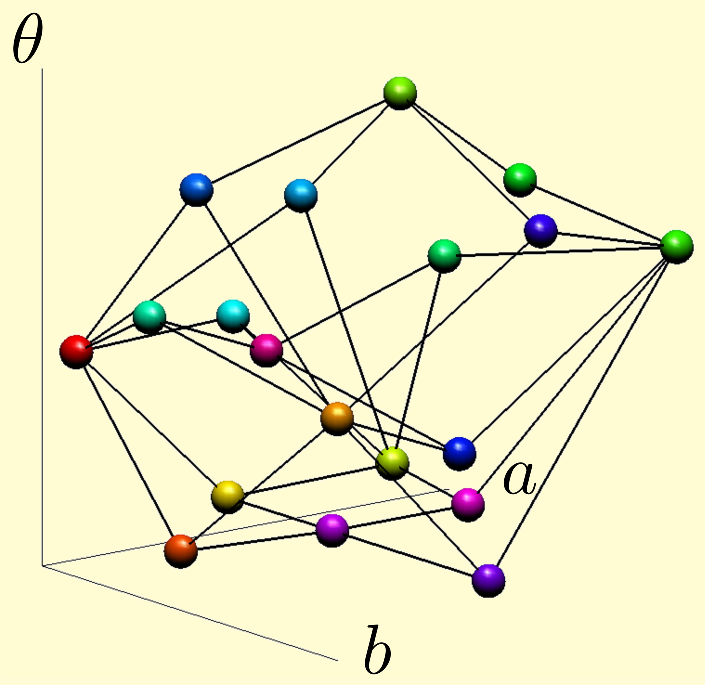




A **discrete rigid transformation graph** (DRT graph) is defined as a graph $G = (V, E)$, such that

- each vertex in V corresponds to a DRT,
- each edge in E connects two vertices sharing a tipping surface.





Properties

Let (p, q, k) (resp. (p, q, l)) be the integer triplet modelling the tipping surface Φ_{pqk} (resp. Ψ_{pqk}) in Equation 1 (resp. 2).

Property 1. *There exists a unique integer quadruple $(p, q, k, i) \in \mathbb{Z}^4$ for each tipping surface, where $i = \{0, 1\}$ indicates whether it is either a vertical or a horizontal tipping surface.*

Let $F_\phi(\theta)$ and $F_\psi(\theta)$ denote two families of vertical and horizontal tipping surfaces respectively, such that for $\theta \in \mathbb{R}$

$$F_\phi(\theta) = \{\Phi_{pqk}(\theta) \mid p, q, k \in \mathbb{Z}\}, \text{ and } F_\psi(\theta) = \{\Psi_{pql}(\theta) \mid p, q, l \in \mathbb{Z}\}.$$

Property 2. *The families of tipping surfaces F_ϕ and $F_\psi(\theta)$ are symmetric:*

$$F_\phi(\theta) = -F_\phi(\theta), \text{ and } F_\phi(\theta) = F_\phi(-\theta + m_1\frac{\pi}{4}), \quad \forall m_1 \in \mathbb{Z},$$
$$F_\psi(\theta) = -F_\psi(\theta), \text{ and } F_\psi(\theta) = F_\psi(-\theta + m_2\frac{\pi}{4}), \quad \forall m_2 \in \mathbb{Z}.$$

Property 3. *The families of tipping surfaces F_ϕ and $F_\psi(\theta)$ are periodic:*

$$F_\phi(\theta) = F_\phi(\theta + m_1\frac{\pi}{2}), \text{ and } F_\phi(\theta) = F_\phi(\theta) + m_2, \quad \forall m_1, m_2 \in \mathbb{Z},$$
$$F_\psi(\theta) = F_\psi(\theta + m_3\frac{\pi}{2}), \text{ and } F_\psi(\theta) = F_\psi(\theta) + m_4, \quad \forall m_3, m_4 \in \mathbb{Z}.$$

Property 4. *Given a image of size $N \times N$, the family F_ϕ (resp. F_ψ) has:*

- *$N^2(N+1)$ tipping surfaces,*
- *N^2 sets of vertical offset tipping surfaces,*
- *$N+1$ vertically offset tipping surfaces in each set.*

Property 5. *The DRT graph G associated to an image of size $N \times N$ has a space complexity of $\mathcal{O}(N^9)$.*

Property 6. *Any two transformed images associated to two connected vertices of DRT graph differ by only one pixel.*

Property 7. *The smallest cycle of a DRT graph is of length 4.*

Property 8. *Let $P(v_1, v_2)$ be the set of all the paths between two vertices v_1 and v_2 of DRT graph. For any path $p \in P(v_1, v_2)$, let $|p|$ be the number of vertices in p, we have $|p| = min_{p' \in P(v_1,v_2)} \{|p'|\} + 2n$, for $n \in \mathbb{N}$.*

Property 9. *The minimum (resp. maximum) degree of vertices in a DRT graph is 6 (resp. $\mathcal{O}(N^2)$).*

Perspectives

- Extending the method for 3D digital images.
- Using DRT graph for image matching, registration problems.
- Topology preservation of images under rigid transformations.