Towards practical key exchange from ordinary isogeny graphs

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Isogeny-based protocols

Post-quantum candidates for key echange/encapsulation: e.g. SIDH/SIKE.

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Both: small keys.

Goals

CRS is worth improving.

- Key validation
- Security analysis
- Pre- and post-quantum parameter proposals
- Algorithmic improvements.

Cryptography with a group action

Hard Homogeneous Space (Couveignes): (G, X) where

- G finite commutative group
- $G \circlearrowright X$
- $g \mapsto g \cdot x_0$ is a 1-to-1 correspondence between G and X.

Hardness hypotheses:

- Given g and x, computing $g \cdot x$ is easy
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$x_a \leftarrow a \cdot x_0$		$x_b \leftarrow b \cdot x_0$

$$\begin{tikzcd} & x_0 \arrow[dl, dotted, "a"'] \arrow[dr, dotted, "b"] & \\ x_a & & x_b \end{tikzcd}$$

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$$s \leftarrow a \cdot x_b$$

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$$b \xleftarrow{R} G$$
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The same DH key exchange works:

- Sample $a \leftarrow G$ directly as a product $\prod s_i^{k_i}$, $s_i \in S$
- Compute $a \cdot x$ as the *sequence* of actions of s_i.

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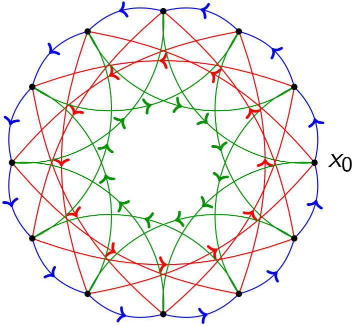

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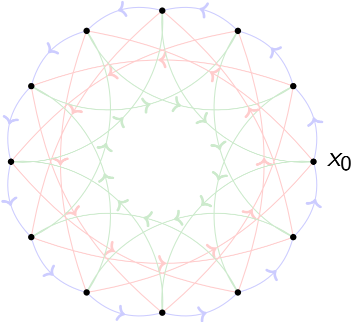

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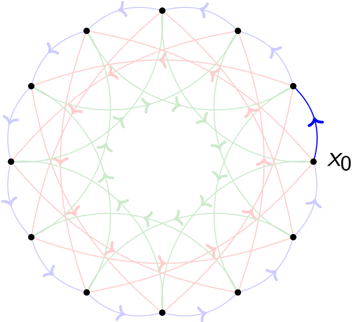

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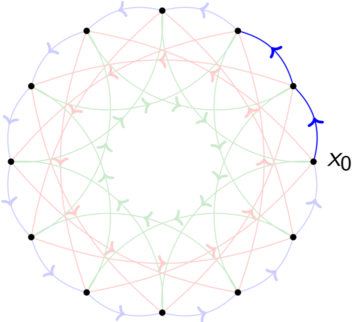

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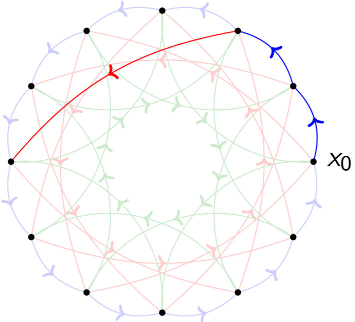

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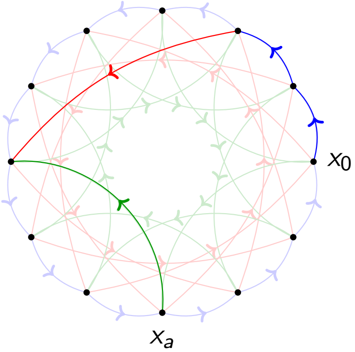

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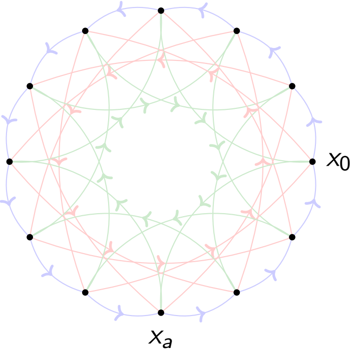

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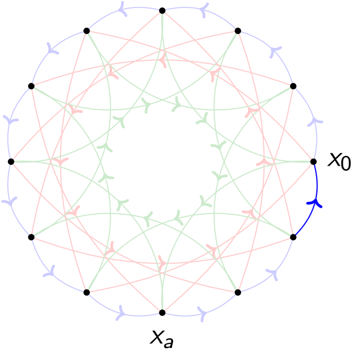

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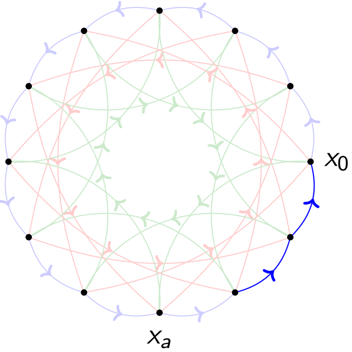

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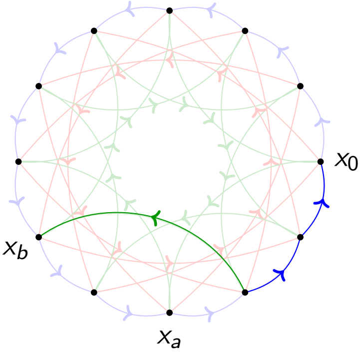

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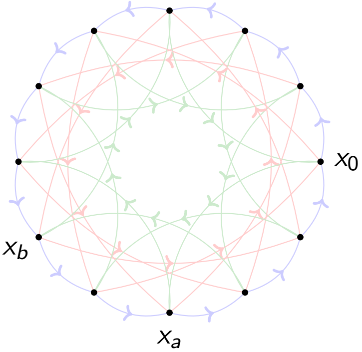

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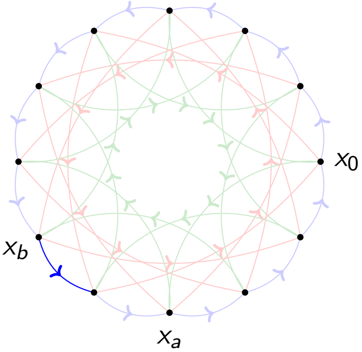

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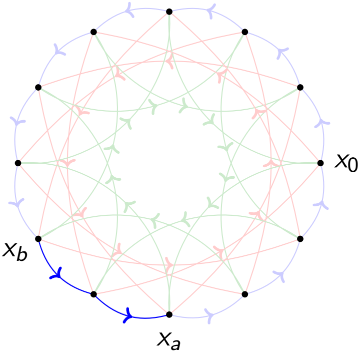

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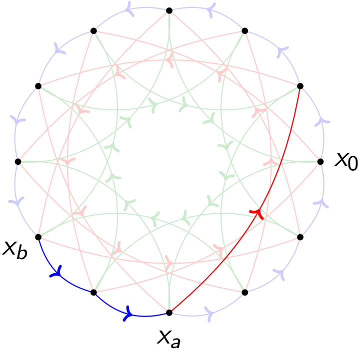

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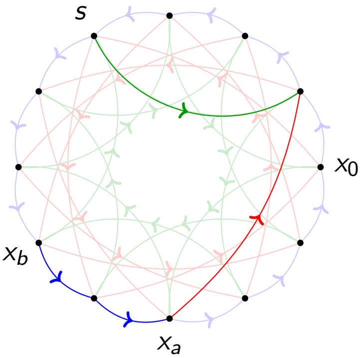

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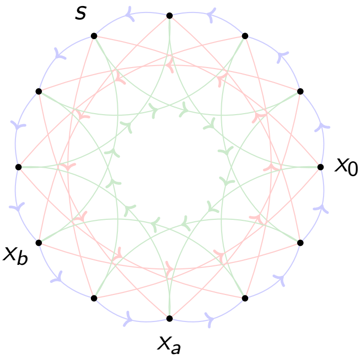

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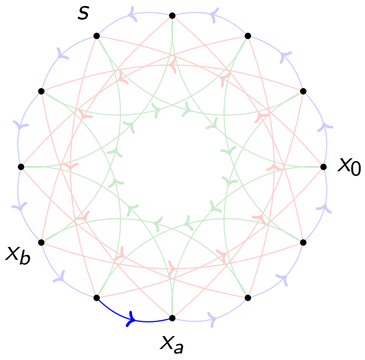

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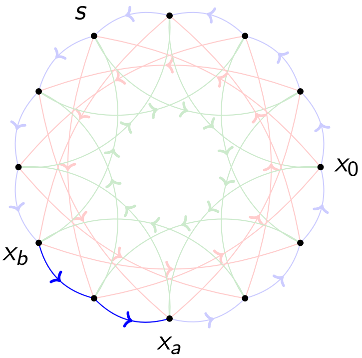

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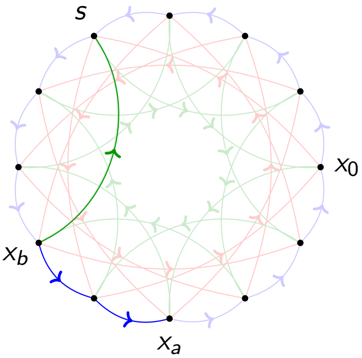

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Which HHS could we use?

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Use isogenies between ordinary elliptic curves:

- X is a set of ordinary elliptic curves
- G is an arithmetic group: *class group*
- S is a set of "small" elements in G
- Computing $s \cdot E$ means computing an *isogeny*.

Why ordinary? Supersingular and ordinary isogeny graphs do not have the same structure.

Elliptic curves and isogenies

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- E *ordinary* elliptic curve ($\neq$ supersingular) over $\mathbb{F}_q$
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Fix $\mathcal{O}$ and take $X = \{E \text{ ordinary ell. curve} \mid \operatorname{End}(E) = \mathcal{O}\}$.

Isogenies/ideals correspondence

$E \in X$, i.e. $\mathrm{End}(E) = \mathcal{O}$.

Isogenies from E		Ideals in $\mathcal{O}$
ℓ-isogeny $\phi : E \to E'$	$\longleftrightarrow$	Ideal $\mathfrak{l}$ of norm ℓ in $\mathcal{O}$ $= \{\beta$ vanishing on $\mathrm{Ker}\,\phi\}$
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- G is the *class group* of $\mathcal{O}$: ideals modulo principal ideals.
- S is a set of ideals with small prime norms ℓ_i.
  When ℓ_i is nice (*split*), two ideals of norm ℓ_i: $\mathfrak{l}_i$ and $\mathfrak{l}_i^{-1}$.

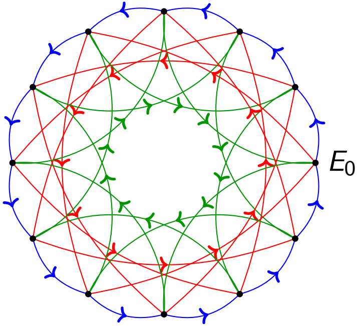
Group action of G on X, which we use as a HHS.

Isogeny walks

Computing the group action = walking in the *isogeny graph*:

- Vertices are elliptic curves,
- Edges are isogenies labelled per degree ℓ_i (arrows give the action of $\mathfrak{l}_i$).

$a = (2, 1, -1)$ represents the ideal $\mathfrak{a} = \mathfrak{l}_1{}^2 \mathfrak{l}_2{}^1 \mathfrak{l}_3{}^{-1}$:



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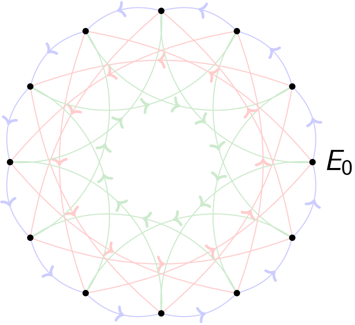

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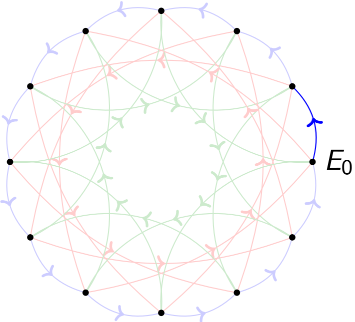
- Edges are isogenies labelled per degree ℓ_i (arrows give the action of $\mathfrak{l}_i$).

$a = (2, 1, -1)$ represents the ideal $\mathfrak{a} = \mathfrak{l}_1{}^2 \mathfrak{l}_2{}^1 \mathfrak{l}_3{}^{-1}$:



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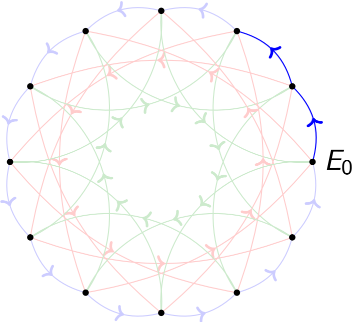

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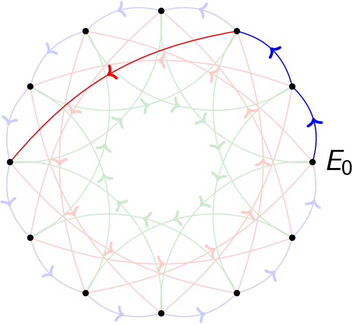

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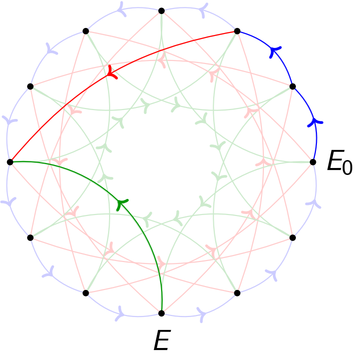
$a = (2, 1, -1)$ represents the ideal $\mathfrak{a} = \mathfrak{l}_1{}^2 \mathfrak{l}_2{}^1 \mathfrak{l}_3{}^{-1}$: $\quad E = \mathfrak{a} \cdot E_0$.



Key validation

E is valid protocol data iff $\mathrm{End}(E) = \mathcal{O}$.

This can be checked using

- a few scalar multiplications on E,
- a few small-degree isogenies.

Key validation is easy and efficient.

Hardness assumptions

Isogeny DH-analogues:

- Class Group Action-DDH (CGA-DDH)
- CGA-CDH

Sampling in G using products of small ideals is a probability distribution σ.

- Distinguish σ from the uniform distribution: Isogeny Walk Distinguishing (IWD).

Security analysis

Theorem (assuming GRH, IWD, CGA-DDH)

The key exchange protocol is session-key secure in the authenticated-links adversarial model of Canetti–Krawczyk.

Theorem (assuming IWD, CGA-CDH)

The derived hashed ElGamal protocol is IND-CPA secure in the random oracle model.

Key validation gives CCA-secure encryption. In contrast, CCA attack against SIKE.PKE (Galbraith et al., AsiaCrypt 2016).

Classical security

CGA-DDH

Compute an isogeny between two curves to recover the key.
Best classical algorithm: $O(\sqrt{N})$ where $N = \#G \simeq \sqrt{q}$.

- Choose $\log_2(q) \simeq 4n$.

IWD

Heuristic: it is enough to have keyspace size $\geq \sqrt{q}$.
We cannot prove this even under GRH.

- Keyspace size: isogeny degrees $\ell_i = O(\log q)$.

Quantum security

Key recovery is an instance of the Hidden Shift Problem.

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Key recovery is an instance of the Hidden Shift Problem.

- Kuperberg's algorithm solves HShP in subexponential time.
- This does not mean that CRS is broken.
- Estimates on query complexity alone:
  $\log_2(q) = 688,\ 1656,\ 3068$ for NIST levels 1, 3, 5.

Computing small-degree isogenies

The basic building block of CRS is computing ℓ-isogenies.

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The CRS approach

Use *modular equations* linking E and E'.

- Find the roots of a degree $\ell + 1$ polynomial over $\mathbb{F}_q$.

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Suppose there is some $P \in E(\mathbb{F}_q)$ of order ℓ.

- Find one such P using a scalar multiplication on E,
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Why? The Frobenius on $E[\ell_i]$ is $\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}$, so the Frobenius on $E^t[\ell_i]$ is $\begin{pmatrix} -1 & 0 \\ 0 & -q \end{pmatrix}$ and $-q = 1$.

Finding good initial curves

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Only exponential algorithms are known to find ordinary curves with smooth order (no CM method here).

We look for E_0 using

- early-abort point counting
- curve selection with modular curves

but we cannot use our improvements in full even after 2 years CPU time searching.

Best results

512-bit prime $q = 7\prod \ell_i - 1$, where the ℓ_i are all primes ≤ 380.

Best E_0:

$$\#E_0(\mathbb{F}_q) = 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 103 \cdot 523 \cdot 821 \cdot R$$
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Type	Isogeny degrees	#steps
Torsion ($\mathbb{F}_q$)	11: see above	409
Torsion ($\mathbb{F}_{q^r}$)	13: $19, 661$ ($r = 3$), ...	81 down to 10
General	25: $73, 89, \ldots$ up to 359	6 down to 1

Not enough primes in the first two lines: walk $\simeq 520$ s.

Take away messages

- Isogeny graphs can be used to construct post-quantum key exchange protocols, and post-quantum NIKE.
- Our improvements speed up CRS considerably, but we cannot use them in full with ordinary curves (not enough torsion points!)
  See next talk on CSIDH.