Heights of rational fractions and interpolation

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Heights over the rationals

• For $\alpha \in \mathbb{Q}$, write $\alpha = \frac{p}{q}$ with $p$, $q$ coprime.

\[
h(\alpha) = \log \max\{|p|, |q|\}.
\]

• For $F \in \mathbb{Q}(X)$, write $F = \frac{P}{Q}$ with $P, Q \in \mathbb{Z}[X]$ coprime.

\[
h(F) = \max\{\log |c| : c \text{ nonzero coefficient of } P \text{ or } Q\}.
\]

• $\deg F = \max\{\deg P, \deg Q\}$.

Example

\[
h\left(\frac{103X^3 + 653X^2 + 383X + 175}{268X^3 + 197X^2 + 237X + 21}\right) = \log(653) = 6.48 \cdots
\]
Posing the problem

If $h(F)$ and $h(x)$ are known, it is easy to bound $h(F(x))$.

**Question**
Let $F \in \mathbb{Q}(X)$ of degree $d$. Let $x_1, \ldots, x_N \in [0, D]$ distinct that are not poles of $F$. Assume that

$$h(F(x_i)) \leq H \quad \text{for every } i.$$ 

What can we say about $h(F)$?

- When $N$ is minimal ($d + 1$ or $2d + 1$): analyze the interpolation algorithm (well known).
- Better bounds when $N$ is larger?

**Remarks**
- Easier for polynomials.
- Generalized to $[A, B]$, number fields.
Plan

1. The case of polynomials

2. Ideas of proof

3. The case of rational fractions
The case of polynomials
Proposition
Let $P \in \mathbb{Q}[X]$ of degree $d \geq 1$, and $x_1, \ldots, x_{d+1} \in [0, D]$ distinct. Assume that $h(P(x_i)) \leq H$ for every $i$. Then

$$h(P) \leq (d + 1)H + D \log(D) + d \log(2D) + \log(d + 1).$$

Height bound is multiplied by the degree.
Remarks on Lagrange interpolation

Height bound is multiplied by the degree.

• We cannot hope to do better in general:

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(x_i)$</td>
<td>261</td>
<td>496</td>
<td>16</td>
<td>683</td>
<td>278</td>
</tr>
<tr>
<td></td>
<td>758</td>
<td>543</td>
<td>143</td>
<td>258</td>
<td>495</td>
</tr>
</tbody>
</table>

gives

$$P(X) = -\frac{480830812799}{911120942160}X^4 + \frac{552317768903}{91112094216}X^3 + \cdots$$

• However, we actually see the huge coefficients when we add an evaluation point:

$$P(0) = -\frac{360058657852}{18981686295}.$$
Proposition
Let $P \in \mathbb{Q}[X]$ of degree $d \geq 1$, and $x_1, \ldots, x_{2d} \in [0, D]$ distinct. Assume that $h(P(x_i)) \leq H$ for every $i$. Then

$$h(P) \leq 2H + D \log(D) + d \log(2D) + \log(d + 1)$$

Using twice the number of points gives a linear bound on $h(P)$. 

Ideas of proof
Alternative definition of heights

- Let $\alpha_1, \ldots, \alpha_n \in \mathbb{Q}$ not all zero. The **projective height** of $(\alpha_1 : \cdots : \alpha_n)$ is

$$h(\alpha_1 : \cdots : \alpha_n) = \sum_{v \in \mathbb{P} \cup \{\infty\}} \log \max\{|\alpha_i|_v : 1 \leq i \leq n\}.$$ 

- For $F \in \mathbb{Q}(X)$, the **height** $h(F)$ is the projective height of the tuple formed by all the coefficients of $F$.

**Examples**

- For $\alpha = p/q \in \mathbb{Q}$,

$$h(\alpha) = h(\alpha : 1) = h(p : q) = \log \max\{|p|, |q|\}.$$ 

- For $P = \sum_{i=0}^{d} c_i X^i \in \mathbb{Q}[X]$,

$$h(P) = \sum_{v} \log \max\{1, |P|_v\}, \quad |P|_v = \max\{|c_i|_v : 0 \leq i \leq d\}.$$
**Proposition**

Let \( P \in \mathbb{Q}[X] \) of degree \( d \geq 1 \), and \( x_1, \ldots, x_{d+1} \in [0, D] \) distinct. Assume that \( h(P(x_i)) \leq H \) for every \( i \). Then

\[
h(P) \leq (d + 1)H + D \log(D) + d \log(2D) + \log(d + 1).
\]

**Proof.**

Lagrange: \( P = \sum_{i=1}^{d+1} \frac{\prod_{k \neq i}(X - x_k)}{\prod_{k \neq i}(x_i - x_k)} P(x_i). \)

- \( v \) archimedean: \( |P|_v \leq (d + 1)2^d D^d \prod_{i=1}^{d+1} \max\{1, |P(x_i)|_v\} \)
- \( v \) nonarchimedean: \( |P|_v \leq \left| \frac{1}{D!} \right| \prod_{i=1}^{d+1} \max\{1, |P(x_i)|_v\}. \)

Sum all contributions to conclude.
Proposition

Let $P \in \mathbb{Q}[X]$ of degree $d \geq 1$, and $x_1, \ldots, x_{2d} \in [0, D]$ distinct. Assume that $h(P(x_i)) \leq H$ for every $i$. Then

$$h(P) \leq 2H + D \log(D) + d \log(2D) + \log(d + 1)$$

Sketch of proof.

If $\nu$ is nonarchimedean, then $|P(x_i)|_\nu \geq |D! \cdot P|_\nu$ for at least $d$ values of $i$ (Lagrange).

Therefore

$$\log \max\{1, |P|_\nu\} \leq \log \left| \frac{1}{D!} \right|_\nu + \frac{1}{d} \sum_{i=1}^{2d} \log \max\{1, |P(x_i)|_\nu\}. \square$$
The case of rational fractions
**Proposition**

Let $F \in \mathbb{Q}(X)$ of degree $d \geq 1$, and let $x_1, \ldots, x_{2d+1} \in [0, D]$ distinct that are not poles of $F$. Assume that $h(P(x_i)) \leq H$ for every $i$. Then

$$h(F) \leq (d + 1)(2d + 1)H + (d + 1)D \log(D) + (4d^2 + 3d) \log(2D) + (2d + 2) \log(2d + 1)$$

**Remark**

We cannot hope for better in general: when interpolating the same values as before by a fraction of degree 2, we obtain

$$F = \frac{22062125284572X^2 + \cdots}{57777551642321X^2 + \cdots}$$
Proof.

- Let $A$ be the polynomial interpolating the points $(x_i, F(x_i))$: 
  $$h(A) \leq (2d + 1)H + \cdots$$

- Let $Z = \prod_{i=1}^{2d+1}(X - x_i)$: 
  $$h(Z) \leq (2d + 1) \log(2D).$$

- Write a Bézout relation for the $d$-th subresultant of $A$ and $Z$: 
  $$QA + RZ = P.$$ 

  Height estimates on subresultants give 
  $$\max\{h(P), h(Q)\} \leq (d + 1)(2d + 1)H + \cdots$$

- $F = P / Q.$
Better bounds using more evaluation points

**Question**
Can we obtain better height bounds on $F$ when $h(F(x)) \leq H$ for more than $2d + 1$ evaluation points?

**Problems**

- No simple formula as was the case for Lagrange interpolation.
- Simplifications can occur: imagine the case where $F = P/Q$, with $P, Q \in \mathbb{Z}[X]$ coprime, has large height. Then $P(x_i)$ and $Q(x_i)$ are large (most of the time), but it could be that

$$F(x_i) = \frac{P(x_i)}{Q(x_i)}$$

is a very small rational number.
Main result

Proposition

Let $F \in \mathbb{Q}(X)$ of degree $d \geq 1$. Let $S \subset \mathbb{R}$ containing at least $2D/3$ elements and no poles of $F$.

Let $H \geq \max\{3, \log(2D)\}$, and assume that:

1. $h(F(x)) \leq H$ for every $x \in S$.
2. $D > \max\{d^4 H \log(dH), 6d\}$.

Then

$$h(F) \leq 3H + C \log(dH) + 3d \log(2D)$$

where $C$ is an absolute constant.
Sketch of proof

Write $F = P/Q$ with $P, Q \in \mathbb{Z}[X]$ coprime, and

$$F(x_i) = \frac{n_i}{d_i}, \quad P(x_i) = n_is_i, \quad Q(x_i) = d_is_i.$$ 

Idea: show that cancellations cannot happen too often, i.e. $s_i$ is (at least) sometimes small.

- The $s_i$ must divide $r = \text{Res}(P, Q)$. By fractional interpolation,

$$\max\{h(P), h(Q)\} \leq Cd^2H + \cdots,$$

so

$$h(r) \leq Cd^3H + \cdots$$

- For each prime $p|r$, estimate carefully the number of times that $P$ (or $Q$) can vanish mod $p$. We obtain

$$\log \left(\prod s_i\right) \leq C(d^4H \log(D) + D \log(dH)).$$

- Since we have $d^4H$ evaluation points, we can conclude.
• **Corollary:** if $F \in \mathbb{Q}(X)$ of degree $d \geq 1$ has the property that

\[ h(F(x)) = O(d \ h(x)) \]

for every $x$, then $h(F) = O(d \log(d))$.

• **Application:** height bounds for modular equations in a very general setting.

• The result about fractions seems suboptimal: would a smaller number of evaluation points, say $O(d)$, be sufficient?
Questions and remarks

Thank you!