The minimum number of spanning trees in regular multigraphs

Jakub Pekárek* Jean-Sébastien Sereni† Zelealem B. Yilma‡

Abstract

In a recent article, Bogdanowicz determines the minimum number of spanning trees a connected cubic multigraph on a fixed number of vertices can have and identifies the unique graph that attains this minimum value. He conjectures that a generalized form of this construction, which we here call a *padded paddle graph*, would be extremal for d-regular multigraphs where $d \geq 5$ is odd.

We prove that, indeed, the padded paddle minimises the number of spanning trees, but this is true only when the number of vertices, n, is greater than $\frac{9d+6}{8}$. We show that a different graph, which we here call the *padded cycle*, is optimal for $n < \frac{9d+6}{8}$. This fully determines the d-regular multi-graphs minimising the number of spanning trees for odd values of d.

We employ the approach we develop to also consider and completely solve the even degree case. Here, the parity of n plays a major role and we show that, apart from a handful of irregular cases when both d and n are small, the unique extremal graphs are padded cycles when n is even and a different family, which we call *fish graphs*, when n is odd.

1 Introduction

The celebrated Matrix-Tree Theorem establishes a link between the number of spanning trees of a (multi-)graph and linear algebra, thereby providing an efficient way to obtain the number of spanning trees of a given graph through the computation of the determinant of a certain matrix. It tells us little, however, about the extremal values taken by this number over graph classes. Following the extremal graph theory tradition, a number of works pursued this line of research.

In this context, a very natural class of graphs to consider is that of regular graphs. In particular, the question has been well studied for regular simple graphs and there are some results on asymptotic values for the minimum and maximum number of spanning trees a connected d-regular n-vertex simple graph can have.

Let us write $\tau(G)$ for the number of spanning trees of a graph G. Let

$$\delta_d = \liminf_{n\to\infty} \, (\tau(G))^{1/n} \quad \text{and} \quad \eta_d = \limsup_{n\to\infty} \, (\tau(G))^{1/n}$$

*Computer Science Institute, Charles University, Prague, Czech Republic. Email: pekarej@iuuk.mff.cuni.cz.

†Service public français de la recherche, Centre national de la recherche scientifique (ICube, CSTB), Strasbourg, France. Email: sereni@kam.mff.cuni.cz.

‡Carnegie Mellon University in Qatar. Email: zyilma@qatar.cmu.edu. Supported by the Qatar Foundation through Carnegie Mellon Qatar's Seed Research program.

where the infimum and supremum are taken over all connected d-regular n-vertex simple graphs. McKay [10] showed that

$$\eta_d = \frac{(d-1)^{d-1}}{(d(d-2))^{d/2-1}}$$

while Alon [1] proved that

$$\sqrt{2} \le \delta_d \le \Big((d+1)^{d-2}(d-1)\Big)^{1/(d+1)}. \tag{1}$$

For the case $d = 3$, Kostochka [8] showed, in a strong sense, that $\delta_d = 2^{3/4}$, by proving that $\tau(G) \ge 2^{3(n+2)/4}$ for all cubic simple graphs G on $n \ge 5$ vertices (this result was shown to hold for the class of 2-connected cubic graphs in earlier work by Valdes [12]). The value $\delta_3 = 2^{3/4}$ matches the upper bound given by (1). To our knowledge, the exact value of δ_d is not yet determined for $d \ge 4$ and, as underlined by Alon [1], it should indeed be a difficult question.

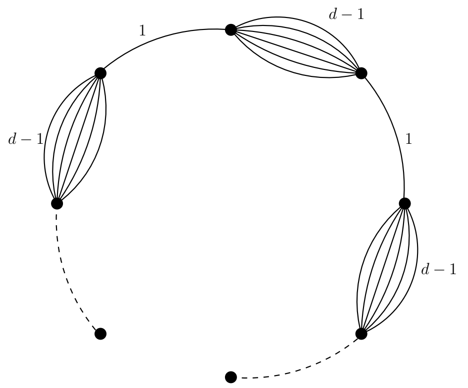
Alon [1] also proposed to study the question on multigraphs, noticing that loops should not be allowed — for otherwise, for odd d, there is always a d-regular n-vertex multigraph with a unique spanning tree. Let δ'_d be defined as δ_d except that the infimum is taken over all connected d-regular n-vertex multigraphs (without loops). Alon sketched a neat argument proving that δ'_d has order exactly $\sqrt{d}$, where the lower bound is obtained by a slight modification of his Theorem 1.1. Further, he explained that the conclusion of van der Waerden's conjecture, which had been already established by then [3, 4] (nowadays the reader can also consult Gurvits's proof [6] for an elementary and totally different argument) implies that any d-regular n-vertex loopless multigraph actually contains $(\Omega(\sqrt{d}))^n$ linear forests — that is, forests such that each connected component is a path.

We shall provide an exact formula for δ'_d for all values of d, and actually even the exact minimum value of τ over the class of connected d-regular n-vertex loopless multigraphs for all values of d and all possible values of n. In addition, we explicitly provide all graphs attaining this minimum value. Apart from a few exceptional cases, the extremal graphs belong to one of the three families of graphs which we introduce here.

Definition 1.1. Let d be an integer greater than 2 and n an integer greater than 3.

- If n is even, then $PC_{d,n}$ is the *padded cycle graph*, illustrated in Figure 1, left: it is the d-regular cycle graph on n vertices with edges of alternating multiplicities 1 and $d-1$.
- If n is even and d is odd, then $PP_{d,n}$ is the *padded paddle graph*, illustrated in Figure 2: it consists of a path of length $n-5$ with edges of alternating multiplicities 1 and $d-1$ and a pendant triangle at either end.
- If both n and d are even, then $F_{d,n}$ is the *d-regular fish graph* on n vertices, illustrated in Figure 1, right: it consists of a triangle and an odd cycle C_{n-2} that share one vertex, where the multiplicities of the edges in the triangle are $\frac{d}{2}-1$, $\frac{d}{2}-1$ and $\frac{d}{2}+1$ (and hence edge multiplicities in C_{n-2} are either 1 or $d-1$).

More precisely, our main results are the following.



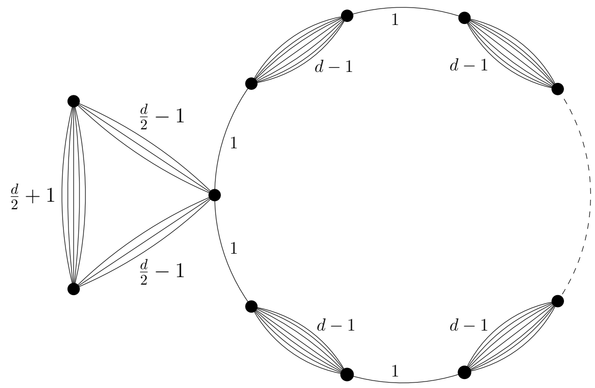


Figure 1: The padded cycle graph $PC_{d,n}$ (left) and the fish graph $F_{d,n}$ (right).

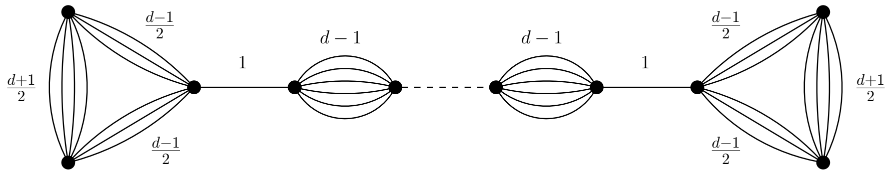


Figure 2: The padded paddle graph $PP_{d,n}$.

Theorem 1.2 (Odd degree case)**.** *Let G be a connected d-regular n-vertex multigraph where $d \geq 3$ is odd and $n \geq 4$. Then,*

$$\tau(G) \geq \min\{\tau(PC_{d,n}), \tau(PP_{d,n})\}$$

with equality if and only if $n < \frac{9d+6}{8}$ and G is isomorphic to the padded cycle graph or $n > \frac{9d+6}{8}$ and G is isomorphic to the padded paddle graph.

Theorem 1.3 (Even d, even n)**.** *Let G be a connected d-regular n-vertex multigraph where $d, n \geq 2$ are both even. Then,*

$$\tau(G) \geq \tau(PC_{d,n})$$

with equality if and only if G is isomorphic to $PC_{d,n}$, unless $d = 4$ and $n \in \{6, 8, 10\}$.

Theorem 1.4 (Even d, odd n)**.** *Let G be a connected d-regular n-vertex multigraph where $d \geq 4$ is even and $n \geq 5$ is odd. Then*

$$\tau(G) \geq \tau(F_{d,n})$$

with equality if and only if G is isomorphic to $F_{d,n}$, unless the pair (d, n) belongs to $\{(4,7), (4,9), (4,11), (4,13), (6,7), (6,9), (6,11), (8,9)\}$.

We will mention here that the case $d = 3$ and $n \geq 6$ of Theorem 1.2 was recently solved by Bogdanowicz [2]. Indeed, our work was motivated by his conjecture that the padded paddle graph is the unique extremal structure that minimises the number of spanning trees in connected odd-regular graphs. The existence of two competing structures, contrary to the conjecture, differentiates the odd degree case and makes inductive approaches more difficult.

While our proofs of Theorems 1.2, 1.3 and 1.4 share the same general approach, some technicalities due to the parities of d and n and some irregular cases when d is even and n is small necessitate separating the exposition. We will start by collecting, in Section 2, the common notation and preliminary results (many of which we recall from the literature) which will be applicable in all cases. The proof of Theorem 1.2, which completely solves the odd degree case, is presented in Section 3. The even degree case is discussed in Section 4. This contains the proofs of Theorems 1.3 and 1.4 as well as all exceptional cases not covered in those theorems.

2 Preliminaries

As the majority of our work considers graphs that contain multiedges, we will use the terms "graph" and "multigraph" interchangeably. We use the term *simple graph* if we want to forbid multiedges (except where explicitly stated, all graphs in this paper are loopless).

Given a multigraph G and any two vertices $i, j \in V(G)$, we define $w_G(i,j)$ to be the number of edges of G between i and j. We write $i \sim_G j$ if $w_G(i,j) \geq 1$, that is, if i and j are adjacent in G. If $w_G(i,j) = 1$ then we may speak unambiguously of *the edge* ij *of* G. For convenience, if f is an edge of G then we define the *multiplicity of* f (in G) to be the number of edges of G with the same end-vertices as f.

2.1 Concavity of τ

Let $\mathcal{L}(G)$ be the Laplacian matrix of the graph G; that is,

$$\mathcal{L}(G)_{ij} = \begin{cases} \deg_G(i) & \text{if } i = j \\ -w_G(i,j) & \text{if } i \neq j \end{cases}$$

for all $i, j \in V(G)$. The Matrix-Tree Theorem, which applies to multigraphs also, gives the relationship between Laplacian matrices and the number of spanning trees.

Theorem 2.1 (Kirchhoff's Matrix-Tree Theorem)**.** *For every graph G, every cofactor of its Laplacian $\mathcal{L}(G)$ equals $\tau(G)$.*

The Matrix-Tree Theorem allows us to apply linear algebra tools and helps establish the next statement. For any graph G, let $2G$ be the graph obtained from G by doubling the multiplicity of each edge.

Proposition 2.2. *Let G, H_1, H_2 be connected graphs on the same vertex set. If $2G = H_1 + H_2$, then $\tau(G) \geq \tau(H_1)$ or $\tau(G) \geq \tau(H_2)$ with at least one strict inequality unless $G = H_1 = H_2$.*

Proof. The matrix form of the Brunn-Minkowski Inequality (see e.g. [7, p. 510]) states that if A and B are $m \times m$ positive definite matrices, then

$$\det(A+B)^{1/m} \geq \det(A)^{1/m} + \det(B)^{1/m},$$

with equality only if $A = cB$ for some $c > 0$. If we take A and B to be the $(n-1) \times (n-1)$ matrices obtained from $\mathcal{L}(H_1)$ and $\mathcal{L}(H_2)$ by deleting their first rows and first columns, thus creating positive definite matrices, we obtain (using also Theorem 2.1)

$$\left(2^{n-1}\tau(G)\right)^{1/(n-1)} = 2\tau(G)^{1/(n-1)} \geq \tau(H_1)^{1/(n-1)} + \tau(H_2)^{1/(n-1)},$$

and the result follows. $\square$

We will often apply the following consequence of Proposition 2.2.

Corollary 2.3. *Suppose G is a connected d-regular graph on n vertices that contains an even cycle with at least 4 vertices. Let M_1 and M_2 be the complementary perfect matchings of the even cycle. If G minimises the number of spanning trees over all connected d-regular graphs on n vertices, then at least one of the graphs $H_1 = G - M_1 + M_2$ and $H_2 = G + M_1 - M_2$ is not connected.*

2.2 Lifts

It is well known that if H is a graph and f is an edge in H, then

$$\tau(H) = \tau(H - f) + \tau(H/f),$$

where H/f is the graph obtained by contracting the edge f (that is, deleting all edges between the endpoints of f and then identifying the endpoints). As, in general, both deletion and contraction of an edge in a regular graph result in a graph which is not regular, we regain regularity by employing a 'lifting' operation similar to one used by Ok and Thomassen [11]. Let x, y_1 and y_2 be three distinct vertices in a graph H, and suppose that f_i is an edge in H between x and y_i, for $i \in \{1, 2\}$. *Lifting* f_1 and f_2 means deleting the two edges f_1 and f_2 and adding an edge between y_1 and y_2. If x is a vertex of degree $2m$ in H, a *complete lift* of x is the process of first performing a sequence of m lifts of pairs of edges incident with x and then deleting the vertex x (which is, by then, isolated), thereby producing a multigraph H_x. Observe that if there exists a vertex y such that $w_H(x, y) \geq m + 1$, then it is not possible to perform a complete lift of x since it will not be possible to pair up the edges incident with x such that edges in a same pair span three different vertices, as required in our definition of lift. Conversely, it is possible to perform a complete lift of x as soon as $w_H(x, y) \leq m$ for all vertices y. It is possible to produce a connected multigraph H_x via a complete lift of x if, in addition, H is connected and $H - x$ has at most $m + 1$ components.

Theorem 2.4 (Ok and Thomassen [11])**.** *Let H be a graph with a vertex x of degree $2m$. Let H_x be a graph obtained from H by a complete lift of x. Then*

$$\tau(H) \geq c_m \, \tau(H_x),$$

where

$$c_m = \min_{d_1, d_2, \ldots, d_k} \min_{X} \frac{\prod_{i=1}^{k} d_i}{\tau(X)},$$

where the minimum is taken over all sequences of positive integers $d_1, \ldots, d_k$ with varying length k such that $\sum_{i=1}^{k} d_i = 2m$, and over all connected k-vertex graphs X with degree sequence $d'_1, d'_2, \ldots, d'_k$ such that $d'_i \leq d_i$ for each $i \in \{1, \ldots, k\}$.

Ok and Thomassen [11] have determined the values $c_1 = 1$, $c_2 = 2$, $c_3 = 8/3$, and $c_4 = 18/5$. We observe that while in the definition of c_m the minimum is taken over all graphs X with degree sequence $d_i' \leq d_i$ (and, therefore, potentially fewer than m edges), it is enough to consider only graphs with exactly m edges, as we formalise and show next. This has the benefit of reducing the search space for an exhaustive computational search to determine more values of c_m and also allows us to combine Theorem 2.4 with a (direct extension to multigraphs of a) theorem of Grone and Merris [5] to find a general lower bound for c_m.

Proposition 2.5. *Using the notation of Theorem 2.4,*

$$c_m = \min_{d_1,d_2,\dots,d_k} \min_X \frac{\prod_{i=1}^k d_i}{\tau(X)},$$

where the minimum is taken over all sequences of positive integers $d_1,\dots,d_k$ with varying length k such that $\sum_{i=1}^k d_i = 2m$, and over all connected k-vertex graphs X with degree sequence $d_1, d_2, \dots, d_k$.

Proof. Consider a positive integer m, a sequence $D = d_1,\dots,d_k$ for some positive integer k such that $\sum_{i=1}^k d_i = 2m$ and a connected graph X with degree sequence $D' = d_1',\dots,d_k'$ where $d_i' \leq d_i$ for each $i \in \{1,\dots,k\}$. Our goal is to show that if $D \neq D'$ then c_m is not attained by D and X.

As long as there are at least two indices $i \neq j$ such that $d_i' < d_i$ and $d_j' < d_j$, we can add a new edge between the two corresponding vertices of X to form a new connected graph X'. The number of spanning trees of X' is larger than that of X, and thus X' along with the sequence D show that c_m is not attained by D and X. We may thus assume that there exists a unique index $i \in \{1,\dots,k\}$ such that $d_i' < d_i$. It then follows that $d_i' \leq d_i - 2$ as both $\sum_{i=1}^k d_i$ and $\sum_{i=1}^k d_i'$ are even.

Consider the sequence $(s_j)_{1\leq j\leq k+1}$ defined by

$$s_j = \begin{cases} d_j & \text{if } j \neq i, \\ d_i - 1 & \text{if } j = i, \\ 1 & \text{if } j = k+1, \end{cases}$$

which satisfies that $\sum_{j=1}^{k+1} s_j = \sum_{j=1}^k d_j = 2m$. Note that

$$\prod_{j=1}^{k+1} s_j = \prod_{j=1}^{k} d_j \cdot \left(1 - \frac{1}{d_i}\right) < \prod_{j=1}^{k} d_j.$$

Let X' be the connected graph obtained from X by adding a new vertex of degree 1 joined to the vertex with degree d_i'. Then X' has degree sequence $s_1',\dots,s_{k+1}'$, where $s_j' = d_j'$ if $j \neq i$, while $s_i' = d_i' + 1$ and $s_{k+1}' = 1$. Consequently $s_j' \leq s_j$ for each $j \in \{1,\dots,k+1\}$. Moreover, $\tau(X') = \tau(X)$, and therefore c_m is not attained by D and X, which concludes the proof. □

Taking into consideration the result of Proposition 2.5, we have calculated a few more values of c_m using exhaustive computer search. The values, and the graphs attaining them, are given in Table 1. (For $m = 5$, the diamond graph, the graph obtained from K_4 by removing one edge, also attains the value $c_5 = 9/2$.)

Knowing that X has exactly m edges also allows us to use the following result of Grone and Merris [5], which was originally stated for simple graphs but the proof of which, using linear algebra, applies equally to multigraphs.

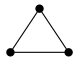
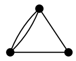
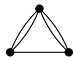
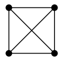
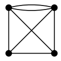
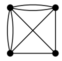
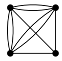
m	1	2	3	4	5	6	7	8	9
c_m	1	2	8/3	18/5	9/2	81/16	6	48/7	375/49
$\frac{2(m+1)}{3}$	4/3	2	8/3	10/3	4	14/3	16/3	6	20/3
extremal graphs									

Table 1: Values of the Ok-Thomassen c_m term, the lower bound from Proposition 2.7, and graphs attaining the c_m values.

Theorem 2.6 (Grone and Merris [5]). *If the degree sequence of a graph X is $d_1, d_2, \ldots, d_k$, then*

$$\tau(X) \le \left(\frac{k}{k-1}\right)^{k-1} \frac{\prod d_i}{\sum d_i}.$$

Theorem 2.6 and Table 1 give us the following bounds.

Proposition 2.7. *For all positive integers m we have $c_m > 2m/e$, where e is the base of the natural logarithm. In addition, $c_m \ge 2(m+1)/3$ for all $m \ge 2$ with strict inequality if $m \ge 4$.*

Proof. The first part of the statement directly follows from Proposition 2.5 and Theorem 2.6 by using the inequality $(1 + \frac{1}{k-1})^{k-1} < e$, valid for each positive integer $k \ge 2$.

The second inequality follows from the first by observing that $c_m > 2m/e \ge 2(m+1)/3$ for $m \ge \lceil e/(3-e) \rceil = 10$. For the remaining cases, namely $m \in \{2, \ldots, 9\}$, direct computations of c_m and $\frac{2(m+1)}{3}$ are provided in Table 1, which concludes the proof. □

2.3 Optimal Substructures

One may observe that all three of our extremal graph families contain long paths with edges of alternating multiplicity 1 and $d-1$. Additionally, two of these families contain what we call pendant triangles. To be precise, let G be a connected multigraph with at least 4 vertices, and suppose that $\{u, v, w\}$ is a set of three vertices inducing a triangle T in G. We say that T is a *pendant triangle* if at most, and hence exactly, one vertex in T has a neighbour not in T. This vertex is then an *articulation point* of G, and the other two vertices in T are the *terminal vertices* of T. The following two statements hint at the structural relevances of long alternating paths and of pendant triangles.

Proposition 2.8. *Let H be a connected multigraph on n vertices with $nd/2 - 1$ edges and maximum degree d. Then $\tau(H) \ge (d-1)^{n/2}$ with equality only if H is a path graph with edges of alternating multiplicities $d-1$ and 1.*

Proof. The degree condition and the number of edges imply that almost every vertex in H has degree exactly d. Let x be a vertex of degree at most $d-1$. Let $B_1, \ldots, B_r$ be the maximal 2-connected blocks of H with $x \in B_1$. Let n_i be the number of vertices in B_i. Observe that $\sum_{i=1}^{r} n_i = n + r - 1$.

Pick vertices $s_i, t_i \in B_i$ where $s_1 = x$, s_i is an articulation point for all $i \geq 2$, and $t_i \neq s_i$ is arbitrary. Let $s_i = y_{i,1}, y_{i,2}, \ldots, y_{i,n_i} = t_i$ be an *st-labeling* (also known as a *bipolar orientation*) of B_i. That is, for each $j \in \{2, \ldots, n_i - 1\}$, the vertex $y_{i,j}$ has at least one neighbour in B_i that comes before it and at least one neighbour in B_i that comes after it in the ordering. It is well known that there is such an ordering for any pair of vertices (s,t) in a 2-connected graph [9].

We can build a spanning tree of B_i by selecting, for each vertex $y_{i,j}$ with $j \leq n_i - 1$, one of its incident edges leading to a vertex that comes after $y_{i,j}$ in the ordering in B_i. For $j \in \{1, \ldots, n_i - 1\}$, let $d_{i,j}$ be the number of edges in B_i between $y_{i,j}$ and $\{y_{i,s} : s > j\}$. Observe that $\tau(B_i) \geq \prod_{j=1}^{n_i-1} d_{i,j}$ with equality only if B_i has exactly two vertices (otherwise, B_i contains a cycle and other spanning trees exist — where some vertex $y_{i,j}$ has more than one neighbour $y_{i,s}$ with $s > j$). Then,

$$\tau(H) = \prod_{i=1}^{r} \tau(B_i) \geq \prod_{i=1}^{r} \prod_{j=1}^{n_i-1} d_{i,j}. \tag{2}$$

The product of the rightmost side of (2) contains $\sum_{i=1}^{r}(n_i - 1) = n - 1$ terms. Note also that the number of edges in B_i is exactly $\sum_{j=1}^{n_i-1} d_{i,j}$, and hence the number of edges in H is $\sum_{i=1}^{r} \sum_{j=1}^{n_i-1} d_{i,j} = dn/2 - 1$. As $1 \leq d_{i,j} \leq d - 1$ for all i, j, the product is minimised when $n/2$ of the terms are $d - 1$ and the remaining $n/2 - 1$ terms are 1. Therefore, $\tau(H) \geq (d-1)^{n/2}$.

Note also that if the block tree of H is not a path, it will have at least three leaves. This would imply the existence of at least one terminal vertex t_{i,n_i} of degree d. This means that B_i has at least three vertices and the above inequality is again strict. Therefore, equality is attained only if H is the path graph as specified. $\square$

Given a subset $X \subseteq V(G)$, let ∂X be the number of edges with exactly one endpoint in X.

Lemma 2.9. *If G is a connected d-regular multigraph minimising the number of spanning trees and $\partial\{u, v, w\} \leq d - 2$, then, without loss of generality, $N_G(v) \cup N_G(w) \subset \{u, v, w\}$.*

Proof. We are going to build a connected d-regular n-vertex multigraph G' that has fewer spanning trees than G unless the conclusion of the statement holds in G. To this end, let H be the subgraph of G induced by $\{u, v, w\}$, and note that H is necessarily a triangle. Let G_H be obtained from G by replacing u, v, w by a single vertex x joined to each vertex having a neighbour in $\{u, v, w\}$ in G (with multiplicities). Note that $\tau(G) \geq \tau(G_H)\tau(H)$.

Moreover, the degree d' of x in G_H is $\partial\{u, v, w\}$, which must have the same parity as d. Since $d' \leq d - 2$, we deduce that $d - d'$ is a positive and even integer. We can thus create a connected d-regular n-vertex multigraph G' from G_H by adding two vertices y and z such that $w_{G'}(x, y) = w_{G'}(x, z) = (d - d')/2$ and $w_{G'}(y, z) = (d + d')/2$. In particular, $T = \{x, y, z\}$ induces a pendant triangle in G', since $N_{G'}(y) \cup N_{G'}(z) \subseteq \{x, y, z\}$. Moreover, $G' - y - z$ is isomorphic to G_H. It follows that $\tau(G') = \tau(G_H)\tau(T)$, as every spanning tree of G' decomposes into a spanning tree of T and a spanning tree of $G_H = G' - y - z$.

We now observe that $\tau(T) \leq \tau(H)$ with equality if and only if H and T are isomorphic. Indeed, suppose that $w_G(u, v) = a$, $w_G(u, w) = b$ and $w_G(v, w) = c$, with $1 \leq a \leq b \leq c$. Then $\tau(H) = ab + ac + bc = (s^2 - a^2 - b^2 - c^2)/2$ where $s = a + b + c = \frac{1}{2}(3d - d') \geq d + 1$. With s

fixed and the degree conditions $a+b, a+c, b+c \le d$, the quantity $\tau(H)$ is minimised when, up to symmetry, $a' = b' = s-d$ and $c' = 2d-s$, that is, when H and T are isomorphic — and, in particular, all edges of G in $\partial\{u,v,w\}$ are incident to the same vertex of H. The conclusion follows. $\square$

3 The Odd Degree Case

We here prove Theorem 1.2, which completely solves the odd degree case. We will start by determining the number of spanning trees in padded paddles and padded cycles. It should be noted here that a misprint appears in the expression for $\tau(PP_{d,n})$ given by Bogdanowicz [2, Theorem 4], which should rather have been the expression given below. (Maybe this misprint is what caused the formulation of the incomplete conjecture, the incorrect expression being smaller than both $\tau(PP_{d,n})$ and $\tau(PC_{d,n})$.)

Lemma 3.1. *Let $d \ge 3$ be an odd integer.*

1. *For any integer $n \ge 6$,*

$$\tau(PP_{d,n}) = \frac{(3d+1)^2}{16}(d-1)^{n/2-1} = \frac{(3d+1)^2}{16(d-1)}(d-1)^{n/2}.$$

2. *For any integer $n \ge 4$,*

$$\tau(PC_{d,n}) = \frac{n}{2}(d-1)^{n/2} + \frac{n}{2}(d-1)^{n/2-1} = \frac{nd}{2(d-1)}(d-1)^{n/2}.$$

We find it convenient to define the terms

$$\alpha = \alpha_d = \frac{(3d+1)^2}{16(d-1)} \quad \text{and} \quad \beta(n) = \beta_d(n) = \frac{nd}{2(d-1)}$$

and at times use the following equivalent reformulation of Theorem 1.2.

Theorem 3.2. *Let G be a connected d-regular n-vertex multigraph where $d \ge 3$ is odd and $n \ge 4$. Then,*

$$\tau(G) \ge \min\{\alpha, \beta(n)\} \cdot (d-1)^{n/2}.$$

Furthermore, the only graphs attaining the minimum are the padded cycle graph if $n < \frac{9d+6}{8}$ and the padded paddle graph if $n > \frac{9d+6}{8}$.

The proof proceeds by induction on the number n of vertices. We show in our base case that the theorem holds for $n \in \{4,6\}$. The inductive step makes use of the observation that

$$\frac{\tau(PP_{d,n})}{\tau(PP_{d,n-2})} = d-1 \qquad \text{and} \qquad \frac{\tau(PC_{d,n})}{\tau(PC_{d,n-2})} = (d-1)\frac{n}{n-2} \le \frac{4}{3}(d-1) \quad \text{for } n \ge 8.$$

Lemma 3.3 (Base cases)**.** *Theorem 1.2 holds for all odd d and $n \in \{4,6\}$. More precisely,*

1. *For all odd $d \ge 3$, the padded cycle has the fewest spanning trees of all connected d-regular multigraphs on 4 vertices.*

2. For all odd $d \geq 5$, the padded cycle has the fewest spanning trees of all connected d-regular multigraphs on 6 vertices.

3. The padded paddle has the fewest spanning trees of all connected cubic multigraphs on 6 vertices.

Proof. By Proposition 2.2, any graph that minimises the number of spanning trees cannot be the convex combination of other graphs. In particular, Corollary 2.3 implies that if such a graph contains C_4 as a subgraph, then there cannot be a path, edge-disjoint from the cycle, that connects opposite vertices in the cycle. This implies, for one, that the underlying simple graph cannot contain the diamond graph. Note also that every vertex in the underlying simple graph must have degree at least 2.

For $n = 4$, it follows that the underlying simple graph must be C_4. Then, by Corollary 2.3, the multigraph minimising the number of spanning trees is the padded cycle.

For $n = 6$, we will show first that the only graphs that are not convex combinations of other graphs are the four graphs in Figure 3, which we call $PC_{d,6}, Gb_{d,6}, Gc_{d,6}, PP_{d,6}$. We reach this conclusion by looking at potential underlying simple graphs and considering some cases.

Case 1: The underlying simple graph does not contain C_4.

The connected C_4-free 6-vertex graphs with minimum degree at least two are C_6, two triangles connected by an edge, and the graph obtained from C_6 by adding a chord between two vertices at distance 2. If the underlying simple graph is C_6, then Corollary 2.3 implies the multigraph has to be $PC_{d,6}$. For two triangles connected by an edge, keeping in mind that d is odd, it is straightforward to see that $PP_{d,6}$ and $Gc_{d,6}$ are the extremal configurations. The third option, C_6 plus a chord forming a C_3 and a C_5, cannot be the underlying simple graph of a regular multigraph (the triple forming the triangle and the complementary triple should have induced graphs with an equal number of edges but that is not the case).

Case 2: The underlying simple graph contains C_4.

Suppose that $v_1v_2v_3v_4$ is a 4-cycle (necessarily chordless). We consider how to extend the graph creating neither a new path between v_1 and v_3 nor one between v_2 and v_4.

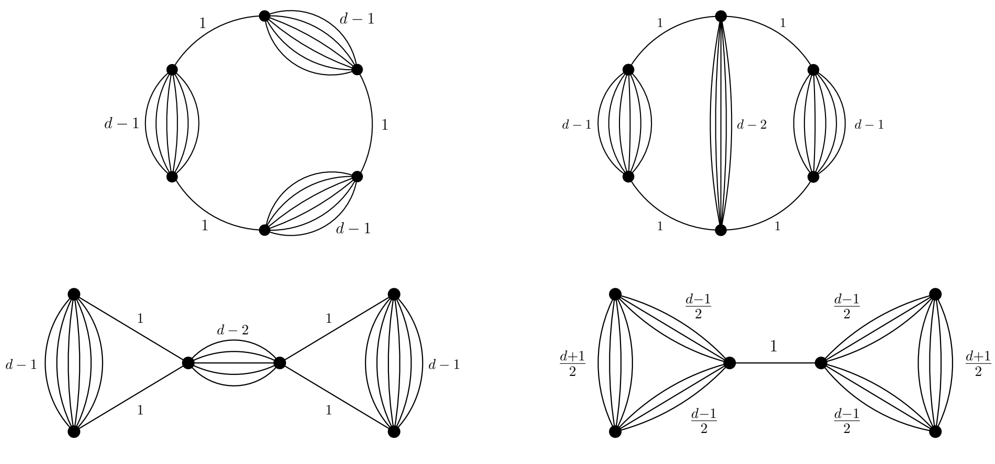


Figure 3: The graphs $PC_{d,6},\ Gb_{d,6},\ Gc_{d,6},\ PP_{d,6}$.

Case 2a: v_5 and v_6 are adjacent.

Recall that the minimum degree in the underlying simple graph is at least 2. Now, if either v_5 or v_6 has two or more neighbours among v_1, v_2, v_3, v_4, it is not possible to avoid creating a diamond or a path joining opposite vertices of the 4-cycle. Let us, therefore, suppose that both v_5 and v_6 have exactly one neighbour each among v_1, v_2, v_3, v_4. If they have a common neighbour, say v_1, then the resulting graph cannot be extended to a regular multigraph (v_1 is forced to have degree d inside the C_4 because v_2, v_3 and v_4 do). Otherwise, the simple graph must be the domino graph (C_6 with a chord forming two copies of C_4). Applying Corollary 2.3 to the two copies of C_4, we see that $Gb_{d,6}$ is the only candidate with this underlying simple graph.

Case 2b: v_5 and v_6 are not adjacent.

Then both v_5 and v_6 have exactly two neighbours each among v_1, v_2, v_3, v_4. The only way to avoid creating a diamond or a path edge-disjoint from the 4-cycle and connecting two opposite vertices on it is, without loss of generality, to have edges v_1v_5, v_2v_5, v_3v_6, and v_4v_6 (thus forming the co-domino graph). But this graph contains a C_6 and the graph remains connected after the removal of either maximum matching of the C_6, contradicting Corollary 2.3.

Therefore, we need only compare the number of spanning trees of $PC_{d,6}, Gb_{d,6}, Gc_{d,6}$, and $PP_{d,6}$. We have

$$\begin{aligned}
\tau(PC_{d,6}) &= 3d^3 - 6d^2 + 3d, \\
\tau(Gb_{d,6}) &= 4d^3 - 8d^2 + 3d, \\
\tau(Gc_{d,6}) &= 4d^3 - 12d^2 + 9d - 2, \text{ and} \\
\tau(PP_{d,6}) &= (9d^4 - 12d^3 - 2d^2 + 4d + 1)/16,
\end{aligned}$$

with $PC_{d,6}$ attaining the minimum for $d \geq 5$. The graphs $Gc_{d,6}$ and $PP_{d,6}$, which are identical for $d = 3$, are optimal for that case. □

Now fix $n \geq 8$ and suppose that Theorem 1.2 holds for all d-regular multigraphs on at most $n-2$ vertices. Let G be a connected d-regular n-vertex multigraph minimising the number of spanning trees. We will prove statements on the structure of G which will eventually show that G must be isomorphic to $PC_{d,n}$ or $PP_{d,n}$ as required. We will start by showing that the existence of multiple bridges implies the presence of a structure common to both the padded paddle and the padded cycle, namely, a path with edges of alternating multiplicity 1 and $d-1$.

Lemma 3.4. *Suppose uv and xy are two distinct bridges (i.e., cutedges of multiplicity one) in G, such that $v \neq y$ and with u and x in the same component H of $G - uv - xy$ (possibly $u = x$). Then, H is a path graph between u and x with edges of alternating multiplicities $d-1$ and 1.*

Proof. Let C_1, C_2, H be the three components of $G - uv - xy$ with $v \in C_1$, $y \in C_2$, and $u, x \in H$. Let n' be the number of vertices in H. Note that n' is necessarily even, since d is odd and dn' equals twice the number of edges induced by H plus the two bridges. Let G' be the d-regular n-vertex multigraph obtained from the disjoint union of C_1 and C_2 by adding a path of length n' with edges of alternating multiplicities $d-1$ and 1 between v and y.

Since H is an n'-vertex graph with $n'd/2-1$ edges and maximum degree d, Proposition 2.8 implies that $\tau(H) \geq (d-1)^{n'/2}$ with equality only if H is isomorphic to the alternating path specified in the statement. It follows that

$$\tau(G) = \tau(H)\tau(C_1)\tau(C_2) \geq (d-1)^{n'/2}\tau(C_1)\tau(C_2) = \tau(G'),$$

with strict inequality unless H is the alternating path as described. □

One immediate consequence of Lemma 3.4 is the following.

Corollary 3.5. *If $u \sim_G v$ and $w_G(u,v) \leq d-2$ for two vertices $u, v \in V(G)$, then $G-u-v$ has at most $d - w_G(u,v)$ components. Moreover, for every vertex x, the graph $G-x$ has at most $(d+1)/2$ components.*

We proceed to show that every edge of G has one of the extreme multiplicities 1 or $d-1$ unless it is contained in a pendant triangle. For each of the three cases we then show necessary structural properties fully characterizing the only two possible structures for G.

Lemma 3.6. *If $u \sim_G v$, then one of the following holds:*

1. *$w_G(u,v) \in \{1, d-1\}$; or*

2. *u and v are part of a pendant triangle.*

Proof. The statement being trivial for $d=3$, we assume that $d \geq 5$. Recall that, by Lemma 2.9, if u and v have a common neighbour z such that $\{u,v,z\}$ induces at least $d+1$ edges in G, then the triple $\{u,v,z\}$ forms a pendant triangle.

Let us then set $m = w_G(u,v)$ and suppose, contrary to the statement, that $2 \leq m \leq d-2$ and yet there is no vertex z such that $\{u,v,z\}$ induces at least $d+1$ edges.

Let G' be obtained from G by first deleting all edges between u and v, and next identifying the vertices u and v into a new vertex x, which has thus degree $2(d-m)$ in G'. Observe that $\tau(G) \geq m \cdot \tau(G')$. Note that there is no vertex z in G' with $w_{G'}(x,z) \geq d-m+1$. In addition, Corollary 3.5 implies that $G-u-v = G'-x$ has at most $d-m$ components. As observed earlier, it follows that it is possible to produce a connected graph by performing a complete lift of x in G'.

Let G'_x be a connected graph obtained from G' by performing a complete lift of x. By Theorem 2.4 and Proposition 2.7,

$$\tau(G) \geq m \cdot \tau(G') \geq m \cdot c_{d-m} \cdot \tau(G'_x) \geq \frac{2m(d-m+1)}{3}\tau(G'_x) \geq \frac{4}{3}(d-1)\tau(G'_x).$$

By Proposition 2.7, and recalling that $m \in \{2, \dots, d-2\}$ with d being odd, there is equality if and only if $d=5$ and $m=2$.
Now, by our inductive hypothesis, $\tau(G'_x) \geq \min\{\alpha, \beta(n-2)\} \cdot (d-1)^{n/2-1}$. So, if $d \geq 7$ then

$$\tau(G) > \min\left\{\frac{4}{3}\alpha, \frac{4}{3}\beta(n-2)\right\} \cdot (d-1)^{n/2} \geq \min\{\alpha, \beta(n)\} \cdot (d-1)^{n/2}.$$

If $d = 5$, we notice that $\frac{4}{3}\beta_5(n-2) \geq \beta_5(n) > \alpha_5$ as $n \geq 8$. Therefore,

$$\tau(G) \geq \min\left\{\frac{4}{3}\alpha, \frac{4}{3}\beta(n-2)\right\} \cdot (d-1)^{n/2} > \alpha \cdot (d-1)^{n/2}.$$

□

We now consider edges of multiplicity 1 and $d-1$, starting with $d-1$.

Lemma 3.7. *If $w_G(u,v) = d-1$ for two vertices $u, v \in V(G)$, then one of the following holds:*

1. *$G - u - v$ is disconnected;*
2. *u and v are part of a pendant triangle; or*
3. *G is the padded cycle.*

Proof. Suppose that $G - u - v$ is connected. If u and v have a common neighbour w, then the triangle $\{u, v, w\}$ is a pendant triangle. So suppose that $a \sim_G u$ and $b \sim_G v$ with $v \neq a \neq b \neq u$. We need to show that G is the padded cycle graph. Let $G' = G - u - v + f$ where f is an edge joining a and b. Note that G' is a connected d-regular graph with $n-2$ vertices, to which the induction hypothesis thus applies, and $G' - f = G - u - v$ is a connected graph with $n-2$ vertices, maximum degree d and $nd/2 - (d+1) = (n-2)d/2 - 1$ edges, to which Proposition 2.8 thus applies. We make the following observations:

1. $\tau(G' - f) \geq (d-1)^{n/2-1}$ (by Proposition 2.8);
2. $\tau(G' - f) + \tau(G'/f) = \tau(G') \geq \min\{\alpha, \beta(n-2)\}(d-1)^{n/2-1}$; and
3. $\tau(G) = (d-1)\tau(G'/f) + (2d-1)\tau(G'-f) = (d-1)\tau(G') + d\,\tau(G'-f)$.

Plugging 1 and 2 into the last equality given by 3, we obtain

$$\begin{aligned}\tau(G) &\geq \min\{\alpha, \beta(n-2)\}(d-1)^{n/2} + \frac{d}{d-1}(d-1)^{n/2}\\ &= \min\left\{\alpha + \frac{d}{d-1},\ \beta(n-2) + \frac{d}{d-1}\right\}(d-1)^{n/2}\\ &\geq \min\left\{\alpha, \beta(n)\right\}(d-1)^{n/2}.\end{aligned}$$

We observe that equality holds only if all inequalities written are equalities, which cannot hold unless $\tau(G' - f) = (d-1)^{n/2-1}$. By Proposition 2.8, this happens only if $G' - f$ is a path with edges of alternating multiplicities $d-1$ and 1. This implies that G must be the padded cycle graph, as required.

As a side remark, notice also that $\beta(n) = \beta(n-2) + \frac{d}{d-1}$, and hence equality holds in the last inequality only if $\alpha + \frac{d}{d-1} \geq \beta(n)$, that is, $n \leq (9d^2 + 22d + 1)/(8d)$. □

Lemma 3.8. *If $w_G(u,v) = 1$ for two vertices $u, v \in G$, then one of the following must hold:*

1. *$G - uv$ is disconnected;*
2. *u and v are part of a pendant triangle; or*

3. G is the padded cycle.

Proof. Assume, on the contrary, that $w_G(u,v)=1$, the graph $G-uv$ is connected, u and v are not part of a pendant triangle and yet G is not the padded cycle. We will show that $\tau(G)$ is too large by writing

$$\tau(G)=\tau(G-uv)+\tau(G/uv)$$

and bounding from below each of the two terms in the right side. In both cases, we will perform a complete lifting operation so we first argue that it is possible to obtain connected graphs at the completion of the respective complete lifting operations. We start by establishing some facts on the multiplicities of edges incident with u or v.

(A). *Every edge of G incident to u or v has multiplicity less than $d-1$.*

Proof. Suppose, on the contrary, that $w_G(u,u')=d-1$, and hence u' and v are the only two neighbours of u. Let v' be the only neighbour of u' different from u. First notice that $v'\neq v$ because u and v are not part of a pendant triangle. Second, $G-u-u'$ must be connected because $G-uv$ is, and hence $G-uv$ contains a path from v to u' that avoids u, and hence contains v'. Consequently, Lemma 3.7 implies that G is the padded cycle, a contradiction. The same reasoning applies to v by symmetry. ⌟

It turns out that (A) actually leads to a stronger statement.

(B). *Neither u nor v is incident to an edge of multiplicity greater than $\frac{d-1}{2}$.*

Proof. Suppose, on the contrary, that $w_G(u,u')\geq\frac{d+1}{2}$. Since $\frac{d+1}{2}\geq 2$ as $d\geq 3$ and u is not incident to an edge of multiplicity $d-1$ by (A), we deduce from Lemma 3.6 the existence of a vertex w such that $T=\{u,u',w\}$ induces a pendant triangle. Since $\{u,v\}$ is not part of a pendant triangle, it follows that $w\neq v$ and $N_G(u')\cup N_G(w)\subset T$. In particular, $w_G(u,u')+w_G(u',w)=d=w_G(u',w)+w_G(u,w)$, which implies that $w_G(u,w)=w_G(u,u')\geq\frac{d+1}{2}$. Then the degree of u in G is at least $w_G(u,u')+w_G(u,w)\geq d+1$, a contradiction. ⌟

Let us now consider $\tau(G/uv)$. For simplicity of notation, let $H=G/uv$ be obtained by contracting u and v into a single vertex w. Note that w has degree $2(d-1)$ in H. As observed previously, it is possible to obtain a connected graph by completely lifting the vertex w if $H-w$ has at most d components and w is not incident to an edge of multiplicity at least d. The first condition holds by Corollary 3.5 as $H-w=G-u-v$ has at most $d-1$ components. Second, w is not incident to an edge of multiplicity greater than $d-1\geq 2$ in H, for otherwise u or v would be incident to an edge of multiplicity greater than $\frac{d-1}{2}$ in G, contradicting (B).

Let H_w be a connected d-regular $(n-2)$-vertex graph obtained by completely lifting the contracted vertex w in the graph $H=G/uv$. By Theorem 2.4 and the induction hypothesis, we have

$$\tau(G/uv)=\tau(H)\geq c_{d-1}\,\tau(H_w)\geq c_{d-1}\,\min\{\alpha,\beta(n-2)\}\cdot(d-1)^{n/2-1}. \tag{3}$$

We now turn our attention to $\tau(G-uv)$. In $G-uv$, both u and v have even degree $d-1$. We will first completely lift u in $G-uv$ and then proceed to completely lift the vertex v in the resulting graph. The assertion (B) guarantees that we can indeed sequentially completely lift the vertices u and v in $G - uv$. It directly follows from the definitions that a graph produced by this process will be d-regular and have $n-2$ vertices. What remains is to show that this process can lead to a connected graph.

Let G_{uv} be a graph obtained by first completely lifting u in $G - uv$ (yielding a connected graph G'_u) and next completely lifting v in G'_u. Suppose the lifts are performed so that G_{uv} has the smallest possible number of connected components among all graphs constructed in this way. We will show that G_{uv} is connected.

To this end, let E_u be the set of edges created by lifting u, that is, the edges in G'_u but not in G. Similarly, let E_v be the set of edges created by lifting v, that is, the edges in G_{uv} but not in G'_u. We build an auxiliary multigraph L (possibly containing loops) as follows. For each connected component of $G - u - v$ we create an associated vertex in L. For each edge e in $E_u \cup E_v$, we add an edge between the vertices associated to the end-vertices of e in L. (This may create loops if two edges leading to the same connected component of $G - u - v$ were lifted together.) It follows that G_{uv} and L have the same number of connected components. To lighten the writing, we shall canonically identify the edges of L with those in $E_u \cup E_v$.

Because G is connected, each connected component of $G - u - v$ contains a neighbour of u or a neighbour of v. Consequently, these connected components can be partitioned into $\mathcal{C}_u$, $\mathcal{C}_v$, and $\mathcal{C}_{uv}$, depending on whether they have an edge only to u, only to v, or to both u and v, respectively. Furthermore, $\mathcal{C}_{uv}$ is not empty because $G - uv$ is connected; let x be a vertex of L associated to a connected component in $\mathcal{C}_{uv}$.

By Lemma 3.4, at most one connected component in $\mathcal{C}_u \cup \mathcal{C}_v$ has exactly one edge to $\{u, v\}$. Consequently, at most one vertex of L has degree 1, all the others having degree at least 2. Therefore, every connected component of L contains a cycle (where a loop is considered to be a cycle).

Suppose now that G_{uv}, and hence L, is not connected. Then L contains an edge yz that is not in the same connected component as x and belongs to a cycle. Without loss of generality, assume that $yz \in E_u$. Let uu_1 and uu_2 be the two edges of $G - uv$ that were lifted to create yz. By the definition of x, there exists an edge xx' that belongs to E_u. Similarly, let uu_3 and uu_4 be the two edges of $G - uv$ that were lifted to create xx'.

Now, if we rather lift the pairs (uu_1, uu_3) and (uu_2, uu_4) instead of (uu_1, uu_2) and (uu_3, uu_4), and keep all other lifts the same, we obtain an auxiliary graph L' that has fewer connected components than L. Indeed, the edge yz belongs to a cycle in L, meaning that the two different connected components of x and y in L will become one in L'. This contradicts the definition of G_{uv}, and thus implies that L, and hence G_{uv}, is connected. Therefore, we can apply Theorem 2.4 to obtain

$$\tau(G - uv) \geq c_{\frac{d-1}{2}}\, \tau(G'_u) \geq \left(c_{\frac{d-1}{2}}\right)^2 \tau(G_{uv}) \geq \left(c_{\frac{d-1}{2}}\right)^2 \min\{\alpha, \beta(n-2)\} \cdot (d-1)^{n/2-1}. \tag{4}$$

Combining (3) and (4), we have

$$\tau(G) = \tau(G - uv) + \tau(G/uv) \geq \left[\left(c_{\frac{d-1}{2}}\right)^2 + c_{d-1}\right] \min\{\alpha, \beta(n-2)\} \cdot (d-1)^{n/2-1}$$

and, noting that $\beta(n) \le \frac{4}{3}\beta(n-2)$ for $n \ge 8$, we infer that G fails to be optimal if $\left(c_{\frac{d-1}{2}}\right)^2 + c_{d-1} > 4(d-1)/3$. For $d = 3$ and $d = 5$, we have

$$c_1^2 + c_2 = 1^2 + 2 = 3 > 8/3 \qquad \text{and} \qquad c_2^2 + c_4 = 2^2 + 18/5 = 38/5 > 16/3.$$

For $d \ge 7$, we apply Proposition 2.7 and obtain

$$\left(c_{\frac{d-1}{2}}\right)^2 + c_{d-1} > \left(\frac{2(\frac{d-1}{2})}{e}\right)^2 + \frac{2(d-1)}{e} = \left(\frac{d-1}{e^2} + \frac{2}{e}\right)(d-1) > \frac{4}{3}(d-1).$$

□

Let us now gather the implications of the lemmas above. One consistent theme is that the padded cycle is a candidate to be the graph with the fewest spanning trees. If G is not the padded cycle, then each edge must either disconnect the graph (and have multiplicity 1 or $d-1$) or belong to a pendant triangle. So the structure of G must be a tree with edges of multiplicity 1 or $d-1$ and some pendant triangles. However, by Lemma 3.4, the tree structure cannot have two adjacent edges of multiplicity 1. Therefore, G must be a long, alternating path with pendant triangles at either end (internal vertices cannot support a pendant triangle). It only remains to show that there is exactly one pendant triangle at each of the two ends of the long path.

Lemma 3.9. *A vertex in G belongs to at most one pendant triangle.*

Proof. Suppose that $T = \{x, u, v\}$ and $T' = \{x, y, z\}$ induce distinct pendant triangles. Let w be a vertex such that xw is a bridge. Such a vertex must exist because x cannot be incident to an edge of multiplicity $d-1$, and at least one edge incident to x does not belong to a pendant triangle, since d is odd. As G is not a convex combination of other graphs, we may assume that $w_G(x, u) = w_G(x, v) = 1$, up to swapping T and T'.

Let $w_G(x, y) = w_G(x, z) = a$. Then $\tau(G) = (2d-1)(a^2 + 2a(d-a))\tau(G - u - v - y - z)$.

Now consider the graph G' obtained from G by deleting u and v, altering the triangle induced by T' such that $w_{G'}(x, y) = w_{G'}(x, z) = a + 1$, deleting the edge xw, and introducing vertices u', v' such that $w_{G'}(x, u') = w_{G'}(x, v') = 1$ and $w_{G'} = (u', v') = d - 1$ (in other words, we remove one of the triangles at x and extend the path in $G - u - v - y - z$ by 2). Then $\tau(G') = (d-1)((a+1)^2 + 2(a+1)(d-a-1))\tau(G-u-v-y-z)$. So, we have $\tau(G) - \tau(G') = f(a)\tau(G-u-v-y-z)$ where $f(a) = (2d-1)(a^2+2a(d-a)) - (d-1)((a+1)^2+2(a+1)(d-a-1))$. Simplifying the above, we obtain $f(a) = -da^2 + 2a(d^2 + d - 1) + (-2d^2 + 3d - 1)$, and so $f(a)$ is a quadratic polynomial which achieves its maximum value at $a = (d^2 + d - 1)/d$. As $f(1) = 4d-3 > 0$, it follows that $f(a)$ is positive in the interval of interest, that is $a \in [1, (d-3)/2]$. Therefore, $\tau(G) > \tau(G')$, contradicting the optimality of G. □

It follows that if G is not the padded cycle graph, then it must be the padded paddle graph. Comparing the values α_d and $\beta_d(n)$, we see that $\tau(PC_{d,n}) < \tau(PP_{d,n})$ if and only if $n \le (3d+1)^2/8d$. The parity of d allows us to slightly simplify this expression to the one that appears in the statement of Theorem 1.2.

4 The Even Degree Case

The even degree case introduces some technical challenges not present in the odd degree case. For one, while the number of vertices, n, has to be even when d is odd, its parity is not restricted when d is even. As seen in the statements of Theorems 1.3 and 1.4, the extremal graphs are different depending on the parity of n.

In addition, the d even case contains a few irregular extremal graphs, specifically when the pair (d, n) belongs to $\{(4, 6), (4, 7), (4, 8), (4, 9), (4, 10), (4, 11), (4, 13), (6, 7), (6, 9), (6, 11), (8, 9)\}$, that are not covered in the statements of Theorems 1.3 and 1.4. To ease readability, the degree-4 case is given alone in Theorem 4.1, being peculiar in the sense that it is the only case where a version of the padded paddle graph can be optimal. Finally, the remaining cases excluded from Theorem 1.4 are presented in Proposition 4.3. Together these four parts, namely Theorem 1.3, Theorem 1.4, Theorem 4.1, and Proposition 4.3, give the unique minimiser graphs for each setting of n and d where d is even.

With a slight abuse of notation, for $n \geq 5$, let $PP_{4,n}$ be the *degree* 4 *padded paddle* graph illustrated in Figure 4: we start from an $(n-4)$-vertex path with end-vertices u_1 and u_2 and all edges of multiplicity 2, and then for each $i \in \{1, 2\}$ we add a triangle with edge multiplicities 1, 1 and 3 and identify its vertex of degree 2 with u_i (notice that $u_1 = u_2$ if $n = 5$, and then $PP_{4,5}$ is isomorphic to the 4-regular fish graph $F_{4,5}$).

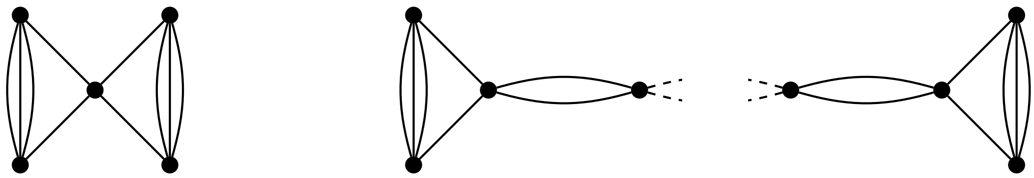


Figure 4: The 4-regular padded paddle graph $PP_{4,n}$ for $n = 5$ (left) and for larger values of n (right). The padded paddle graph $PP_{4,5}$ is isomorphic to the 4-regular fish graph $F_{4,5}$.

Theorem 4.1. *Let $n \geq 4$ be an integer and let G be a connected 4-regular n-vertex multigraph. Then G is the unique graph minimising $\tau(G)$ if and only if one of the following holds*

- $G = PP_{4,n}$ *and* $n \in \{5, \dots, 11\} \cup \{13\}$
- $G = PC_{4,n}$ *and n is even and either* 4 *or at least* 12
- $G = F_{4,n}$ *and n is odd and at least* 15.

Here, $\tau(PP_{4,n}) = 49 \cdot 2^{n-5}$; $\tau(PC_{4,n}) = 2n \cdot 3^{\frac{n}{2}-1}$ *and* $\tau(F_{4,n}) = 7 \cdot (2n-3) \cdot 3^{\frac{n-5}{2}}$.

Definition 4.2. For n odd and d even, let $F^*_{d,n}$ be the d-regular multigraph on n vertices obtained from the disjoint union of $\frac{d}{2} - 1$ triangles and one odd cycle by identifying one vertex from each into a single central vertex. Note that the central vertex has exactly d distinct neighbours (hence it is incident only to edges of multiplicity 1) and all the other edge multiplicities alternate between 1 and $d-1$ on each cycle.

Proposition 4.3. *Let G be a connected d-regular n-vertex multigraph. If the pair (d, n) belongs to $\{(6, 7), (6, 9), (6, 11), (8, 9)\}$, then*

$$\tau(G) \geq \frac{1}{2} \cdot (2d-1)^{d/2-1}(d(n-d+3)-2)(d-1)^{(n-d-1)/2},$$

with equality if and only if G is isomorphic to $F^*_{d,n}$ (see Figure 5 for illustration).

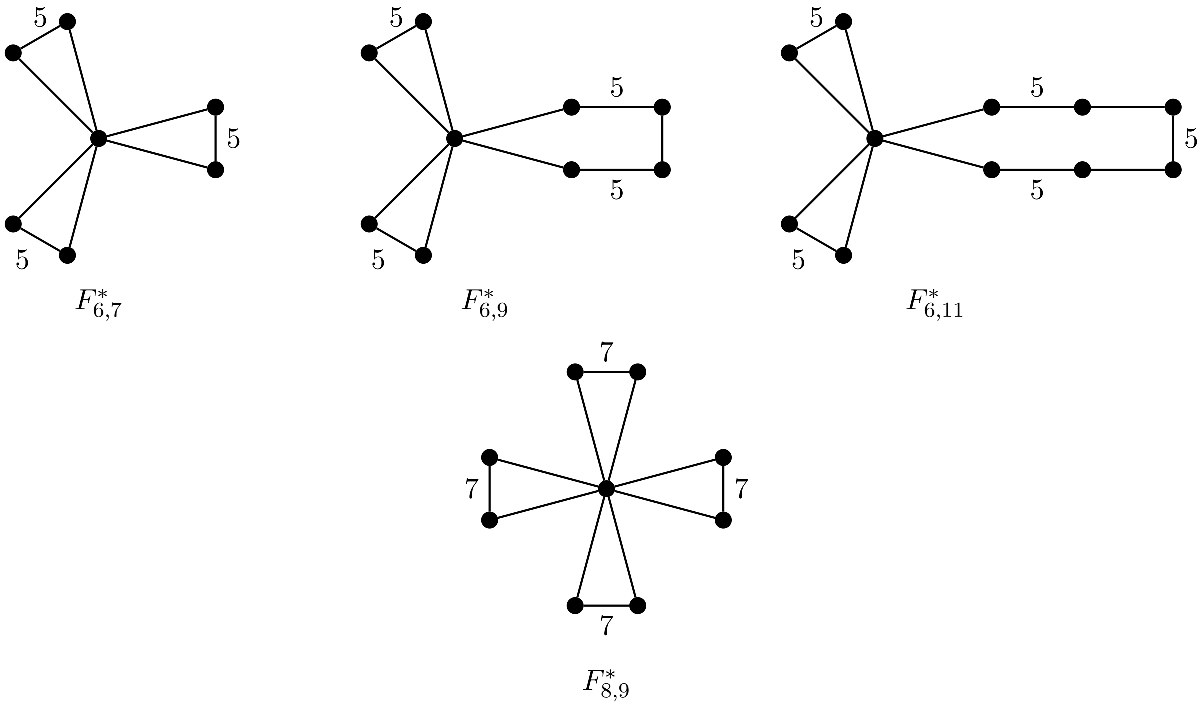


Figure 5: The unique minimiser $F^*_{d,n}$ in each of the four sporadic cases when $d \geq 6$. Parallel edges are represented by a single edge with the multiplicity written next to it.

The rest of this article is organised as follows. Section 4.1 contains results that are applicable to all subsequent cases, regardless of the parity of n. In particular, Section 4.1 quickly clears out the cases where the number of vertices is at most 5, which serve as the base for an induction on the number of vertices. Sections 4.2 and 4.3 then respectively establish, when $d \geq 6$, Theorem 1.3, and Theorem 1.4 along with Proposition 4.3. The case of connected 4-regular multigraphs is kept separated and dealt with in Section 4.4. Contrary to the general case, when $d = 4$ the analysis is cleaner if it is not split with respect to the parity of the number of vertices of the multigraphs considered.

4.1 Base Cases and Preliminary Results

Lemma 4.4. *For all even $d \geq 4$, the unique connected n-vertex d-regular multigraph minimising the number of spanning trees is*

1. *for $n = 2$: an edge of multiplicity d;*

2. *for $n = 3$: a triangle with all edges of multiplicity $d/2$;*

3. for $n = 4$: the padded cycle graph $PC_{d,4}$;

4. for $n = 5$: the fish graph $F_{d,5}$.

Proof. Let G be a connected n-vertex d-regular multigraph that minimises the number of spanning trees. For $n = 2$ and $n = 3$, the graphs given in the statement are, in fact, the unique connected d-regular n-vertex multigraphs. It is possible to look at these graphs as degenerate forms of the padded cycle and of $F_{d,n}$, respectively.

For $n = 4$ and $n = 5$, consider first the underlying simple graph H of G. As G is connected and regular, every vertex must have degree at least 2 in H. In addition, we know, by Corollary 2.3 of Proposition 2.2, that a graph containing a cycle of length 4 and a path (edge-disjoint from the cycle) connecting opposite vertices of the cycle cannot be optimal. When $n = 4$, this leaves C_4 as the only option for H. Applying Corollary 2.3 again, we see that G must be the padded cycle graph $PC_{d,4}$.

For $n = 5$, we may also disallow, by Lemma 2.9, those underlying simple graphs containing a triangle with only one vertex of degree 2. Consequently, we infer that H is isomorphic to either C_5 or the butterfly graph (obtained from the disjoint union of two triangles by identifying two vertices belonging to distinct triangles). If H is isomorphic to C_5, then in G all edges have equal multiplicity $d/2$, implying that $\tau(G) = 5 \cdot (d/2)^4 = \frac{5}{16} \cdot d^4$. If H is isomorphic to the butterfly graph, then Proposition 2.2 implies that the two triangles have to be as lopsided as possible. It follows that G is isomorphic to $F_{d,5}$, and hence $\tau(G) = \frac{1}{4} \cdot (3d^2 - 4d - 4)(2d - 1)$, which is smaller than $\frac{5}{16} \cdot d^4$. Therefore G must be $F_{d,5}$, as announced. □

Just as in the odd degree case, the lifting operation will remain an important component of our proofs by induction. In fact, as every vertex now has even degree, we can more directly perform complete lifts at vertices without any preprocessing (such as deletion or contraction of edges). Recall that, given a connected graph H and a vertex x of even degree $2m$, we can completely lift x and obtain a connected graph H_x as long as x is not incident to any edge of multiplicity greater than m and the graph $H - x$ has at most $m + 1$ components. As the next lemma will show, when all vertices in a graph have even degree, the second condition is automatically satisfied.

Lemma 4.5. *If d is an even integer and G is a connected d-regular graph, then, for every vertex x, the graph $G - x$ has at most $d/2$ components. Moreover, if xy is an edge of multiplicity $w_G(x, y)$, then $G - x - y$ has at most $d - w_G(x, y)$ components.*

Proof. If C is a component of $G - x$, or of $G - x - y$, then because d is even there must be an even (and positive, as G is connected) number of edges in G between C and $\{x\}$, or between C and $\{x, y\}$, respectively. The statements follow. □

We also find it useful at times to sequentially perform complete lifts of two vertices. While this might not be possible in general, we show in the lemma below that we can do so if we further assume that the given graph is optimal with respect to the number of spanning trees.

Proposition 4.6. *Suppose G is a connected d-regular multigraph minimising the number of spanning trees and let u and v be two vertices in G not incident to edges of multiplicity more than $d/2$. Then it is possible to sequentially completely lift u and v to produce a connected d-regular graph.*

Proof. First, suppose that there is no vertex x such that $\{u, v, x\}$ forms a triangle in G. Let G_u be any connected multigraph obtained by completely lifting u in G. Then for every neighbour w of v in G different from u, the multiplicities of the edge vw in G and in G_u are the same. In addition, for any new neighbour z that v might gain when lifting u, the multiplicity of the edge vz in G_u cannot exceed the multiplicity of the edge vu in G, and hence is at most $\frac{d}{2}$. It follows that v can be completely lifted in G_u, as required.

Now suppose that there is a vertex x such that $\{u, v, x\}$ forms a triangle T in G. Since G does not contain a diamond by Corollary 2.3, the vertices u and v have no common neighbour other than x. Recall also that by Lemma 2.9, if $\{u, v, x\}$ induces more than d edges, then the triangle T must be pendant. As a pendant triangle contains at most one vertex not incident to an edge of multiplicity greater than $\frac{d}{2}$, we deduce that $w_G(u, v) + w_G(u, x) + w_G(v, x) \leq d$. Since $d = w_G(u, v) + w_G(u, x) + \sum_{y \neq v,x} w_G(u, y)$, we infer that $w_G(v, x) \leq \sum_{y \neq v,x} w_G(u, y)$. Consequently, when lifting u, we may chose to pair at least $\min\{w_G(u, v), w_G(v, x)\}$ edges between u and v with edges incident to u but not to x. Then, after the complete lift of u, the multiplicity of the edge vx in the obtained graph G_u is at most $\max\{w(u, v), w(v, x)\} \leq d/2$, and therefore we can completely lift v in G_u. $\square$

4.2 Proof of Theorem 1.3

We prove Theorem 1.3 by induction on the number n of vertices. Throughout this section, the integer n is assumed to be even.

A key element of our inductive proof is the observation that, for $n \geq 6$,

$$\frac{\tau(PC_{d,n})}{\tau(PC_{d,n-2})} = \frac{n(d-1)}{n-2} \leq \frac{3}{2}(d-1).$$

Then, whenever $PC_{d,n-2}$ minimises the number of spanning trees among connected d-regular multigraphs on $n-2$ vertices, we can argue that a connected d-regular multigraph G on n vertices cannot be optimal if $\tau(G) > \frac{3}{2}(d-1)\tau(G')$ for some connected d-regular multigraph G' on $n-2$ vertices.

Fix $n \geq 6$ and suppose that $\tau(G') \geq \tau(PC_{d,n-2})$ for every connected d-regular multigraph G' on $n-2$ vertices. Let G be a connected d-regular n-vertex multigraph minimising the number of spanning trees.

Lemma 4.7. *Let $u \sim_G v$. If $d/2 + 1 \leq w_G(u, v) \leq d - 2$, then there exists a vertex z such that*

$$w_G(u, v) + w_G(u, z) = d = w_G(v, u) + w_G(v, z).$$

In other words, u and v are terminal vertices of a (same) pendant triangle.

Proof. First note that the hypothesis of the lemma cannot hold for $d = 4$. So suppose that $d \geq 6$ and set $m = w_G(u, v)$. Suppose there is no vertex z such that $w(u, z)+w(v, z) \geq d-m+1$. Let G' be a connected d-regular multigraph on $n-2$ vertices obtained from G by deleting all m edges between u and v and identifying the two vertices u and v into a new vertex x. Notice that $\tau(G) \geq m \cdot \tau(G')$. Now, since x has degree $2(d-m)$ in G' and $w_{G'}(x, z) = w_G(u, z) + w_G(v, z) \leq d-m$ for every neighbour z of x in $V(G')$, we can completely lift the vertex x in G', which produces a connected d-regular multigraph G'_x on $n-2$ vertices. It follows from Theorem 2.4 and Proposition 2.7 that

$$\begin{aligned}
\tau(G) &\geq m \cdot c_{d-m} \cdot \tau(G'_x) \geq \frac{2m(d-m+1)}{3} \cdot \tau(G'_x) \\
&\geq 2(d-2) \cdot \tau(G'_x) \geq 2(d-2) \cdot \tau(PC_{d,n-2}) \\
&> \frac{3}{2}(d-1) \cdot \tau(PC_{d,n-2}) \geq \tau(PC_{d,n}),
\end{aligned}$$

where we used that $d \geq 6$ for the third line and $m \mapsto m(d-m+1)$ is decreasing over $[d/2+1\ , d-2]$ for the second line. Therefore, G fails to be optimal unless there is a vertex z such that $w_G(u, z)+w_G(v, z) \geq d-m+1$. Consequently, the set $\{u, v, z\}$ induces at least $d+1$ edges in G, and hence Lemma 2.9 yields that $\{u, v, z\}$ induces a pendant triangle in G. As $w_G(u, v) \geq d/2+1$, the vertex z must be the articulation point. This concludes the proof. □

It follows from Lemma 4.7 that every vertex of G that is not a terminal vertex of a pendant triangle either is incident to an edge of multiplicity $d-1$, or is incident only to edges of multiplicity at most $d/2$. We call a vertex *exceptional* if all its incident edges have multiplicity at most $d/2$.

Lemma 4.8. *If $d \geq 6$, then G has at most one exceptional vertex.*

Proof. Suppose, for the sake of contradiction, that u and v are distinct exceptional vertices. By Proposition 4.6, it is possible to produce a connected d-regular $(n-2)$-vertex multigraph G' by sequentially lifting u and v. Theorem 2.4 ensures that $\tau(G) \geq (c_{d/2})^2 \cdot \tau(G')$, and hence $\tau(G) > \frac{d^2}{e^2} \cdot \tau(G')$ by Proposition 2.7.

If $d \geq 10$ then $\frac{d^2}{e^2} > \frac{3(d-1)}{2}$, and it thus follows that

$$\tau(G) > \frac{3}{2}(d-1)\tau(G') \geq \frac{3}{2}(d-1)\tau(PC_{d,n-2}) \geq \tau(PC_{d,n}).$$

If $d = 8$, then we use the exact value of c_4 to deduce the following:

$$\tau(G) \geq (c_4)^2\tau(G') = \frac{18^2}{5^2}\tau(G') > \frac{3 \cdot 7}{2}\tau(PC_{8,n-2}) \geq \tau(PC_{8,n}).$$

For $d = 6$, as $c_3 = 8/3$ we have $c_3^2 = 64/9 < 15/2 = 3(d-1)/2$. However, $64/9 > 4(d-1)/3 \geq \tau(PC_{6,n})/\tau(PC_{6,n-2})$ for $n \geq 8$ and our assertion will hold if we show that $PC_{6,6}$ is optimal for $n = 6$. Indeed, let G_u be a connected 6-regular multigraph obtained from G by completely lifting u. As $\tau(F_{6,5}) = 220 \leq \tau(G_u)$ by Lemma 4.4, it follows that

$$\tau(G) \geq c_3 \cdot \tau(G_u) \geq \frac{8}{3} \cdot 220 > 450 = \tau(PC_{6,6}),$$

which conclude the proof. □

As mentioned earlier, it follows from Lemma 4.7 that every non-exceptional vertex of G either has an edge of multiplicity $d-1$ or belongs to a pendant triangle. If all vertices are of the former type, then G must be the padded cycle $PC_{d,n}$. Otherwise, since G is connected, edges of alternating multiplicities 1 and $d-1$ must form odd cycles. Consequently, if $d \geq 6$, then Lemma 4.8 implies that G must be a collection of odd cycles (and pendant triangles) all sharing one vertex. However, this requires G to have an odd number of vertices, a contradiction. This establishes the statement of Theorem 1.3 whenever $d \geq 6$.

4.3 Proof of Theorem 1.4 and Proposition 4.3

In this section, the integer n is assumed to be odd. We will simultaneously prove Theorem 1.4 and Proposition 4.3 by first identifying a family of graphs which includes the stated extremal structures for the two statements. Next, we show that, within this family, the graphs $F_{d,n}$ and $F^*_{d,n}$ are the minimisers of τ for the stated values of d and n. We then establish the desired result by proving that, for $d \geq 6$, any minimiser of τ over the whole class of connected d-regular multigraphs must belong to the above-identified family.

Let $\mathcal{H}_{d,n}$ be the class of all connected d-regular n-vertex multigraphs that consist of pendant triangles and padded odd cycles (i.e., containing only edges of multiplicity 1 or $d-1$) all sharing the same vertex. This vertex is called the *centre* of the graph. Since the centre is a cut-vertex, the number of spanning trees of such a graph is the product of the number of spanning trees of the odd cycles and triangles composing it. Let $H_{d,n}$ be a minimiser of τ over the class $\mathcal{H}_{d,n}$.

4.3.1 Optimising within $\mathcal{H}_{d,n}$.

We start by analyzing edge multiplicities within the pendant triangles.

Lemma 4.9. *All but at most one pendant triangle in $H_{d,n}$ contain edges of multiplicities 1 and $d-1$.*

Proof. Suppose, on the contrary, that $T_1 = \{u, v, w\}$ and $T_2 = \{u, x, y\}$ are two pendant triangles in $H_{d,n}$ sharing the vertex u and containing no edges of multiplicity 1. Consider an even walk $W = uvwuxyu$ that starts with u then traverses both triangles (arbitrarily choosing any of the (at least two) edges between consecutive vertices) and let $M_1 = \{uv, wu, xy\}$ and $M_2 = \{vw, ux, yu\}$ be the complementary sets of alternating edges in W. We note that both $H_1 = H_{d,n} - M_1 + M_2$ and $H_2 = H_{d,n} + M_1 - M_2$ are connected, since every edge induced by $T_1 \cup T_2$ has multiplicity at least 2. Furthermore, $H_1, H_2 \in \mathcal{H}_{d,n}$ and $2H_{d,n} = H_1 + H_2$. Proposition 2.2 then applies and contradicts the optimality of $H_{d,n}$. $\square$

Lemma 4.10. *There is at most one cycle in $H_{d,n}$ that is not a triangle.*

Proof. Suppose, on the contrary, that C_1 and C_2 are cycles in $H_{d,n}$ of lengths $2k+1$ and $2\ell+1$, respectively, where $2 \leq k \leq \ell$. Recall that each edge in $C_1 \cup C_2$ has multiplicity either 1 or $d-1$ so

$$\tau(C_1 \cup C_2) = (d-1)^{k-1}(d(k+1)-1)(d-1)^{\ell-1}(d(\ell+1)-1).$$

Consider the graph F' where C_1 and C_2 are replaced by C_1' and C_2' where C_1' is a triangle and C_2' has length $2(k+\ell)-1$. Then $F' \in \mathcal{H}_{d,n}$ and

$$\tau(C_1' \cup C_2') = (2d-1)(d-1)^{k+\ell-2}(d(k+\ell)-1).$$

As $2 < k+1 \leq \ell+1 < k+\ell$, it follows by convexity that $\tau(C_1' \cup C_2') < \tau(C_1 \cup C_2)$, contradicting the optimality of $H_{d,n}$ in $\mathcal{H}_{d,n}$. $\square$

For every integer t such that $1 \leq t \leq \min\{d/2-1, (n-3)/2\}$, let us define the d-regular multigraph $H_{d,n}^t \in \mathcal{H}_{d,n}$ as one having $t-1$ triangles with edge multiplicities 1 and $d-1$ one odd cycle of length $n-2t$ and one triangle with edge multiplicities at the centre v both being $(d-2t)/2$. Note that $H_{d,n}^1$ is isomorphic to the d-regular fish graph $F_{d,n}$. Lemmas 4.9 and 4.10 show that $H_{d,n}$ must be isomorphic to $H_{d,n}^t$ for some integer t. In fact, we can make the following stronger statement.

Lemma 4.11. *We have $H_{d,n} \in \{H_{d,n}^1, H_{d,n}^{t^*}\}$ where $t^* = \min\{d/2-1, (n-3)/2\})$.*

Proof. Observe that

$$\tau(H_{d,n}^t) = \frac{1}{8}(2d-1)^{t-1}(d-2t)(3d+2t)(d(n-2t+1)-2)(d-1)^{(n-2t-3)/2}.$$

If $2 \leq t \leq t^*-1$, we compare the number of spanning trees of $H_{d,n}^t$ with the number of spanning trees of $H_{d,n}^{t-1}$ and $H_{d,n}^{t+1}$ to obtain

$$\begin{aligned}\frac{\tau(H_{d,n}^{t+1})\tau(H_{d,n}^{t-1})}{\tau(H_{d,n}^t)^2} &= \frac{(d-2t-2)(d-2t+2)}{(d-2t)^2} \times \frac{(3d+2t+2)(3d+2t-2)}{(3d+2t)^2} \\ &\quad \times \frac{(d(n-2t-1)-2)(d(n-2t+3)-2)}{(d(n-2t+1)-2)^2} \\ &< 1,\end{aligned}$$

since each of the three fractions is less than 1 by convexity. Consequently, $\tau(H_{d,n}^t)$ is minimised either at $t=1$ or at $t=t^*=\min\{d/2-1,(n-3)/2\}$. $\square$

All that remains is to compare these two configurations. Note that for $d=4$, the two configurations are identical and therefore optimal within $\mathcal{H}_{d,n}$. Observe also that if $n=5$, then necessarily $t^*=1$, and hence $H_{d,5}^1 = H_{d,5}^{t^*}$. We now consider the case where $n \geq 7$.

Lemma 4.12. *For all even $d \geq 6$ and odd $n \geq 7$ we have*

- $\tau(H_{d,n}^1) < \tau(H_{d,n}^{t*})$ *if* $(d,n) \notin \{(6,7),(6,9),(6,11),(8,9)\}$.
- $\tau(H_{d,n}^{t^*}) < \tau(H_{d,n}^1)$ *if* $(d,n) \in \{(6,7),(6,9),(6,11),(8,9)\}$.

Proof. First note that

$$\tau(H_{d,n}^1) = \tau(F_{d,n}) = \frac{1}{8}(d-2)(3d+2)(d(n-1)-2)(d-1)^{(n-5)/2}$$

and

$$\tau(H_{d,n}^{t^*}) = \begin{cases} \frac{1}{2}(2d-1)^{d/2-1}(d(n-d+3)-2)(d-1)^{(n-d-1)/2} & \text{if } n \geq d+1 \text{ and} \\ \frac{1}{4}(2d-1)^{(n-3)/2}(d-n+3)(3d+n-3) & \text{if } n \leq d+1. \end{cases}$$

Consider the function

$$f_d \colon n \mapsto \frac{\tau(H_{d,n}^1)}{\tau(H_{d,n}^{t^*})} = \begin{cases} \frac{(d-2)(3d+2)(d-1)^{d/2-2}}{4(2d-1)^{d/2-1}} \frac{(d(n-1)-2)}{(d(n-d+3)-2)} & \text{if } n \geq d+1 \text{ and} \\ \left(\frac{d-1}{2d-1}\right)^{(n-5)/2} \cdot \frac{1}{2} \cdot \frac{d-2}{2d-1} \cdot \frac{(3d+2)(d(n-1)-2)}{(d-n+3)(3d+n-3)} & \text{if } n \leq d+1. \end{cases}$$

Let us first look at the cases $d = 6$ and $d = 8$. We can simplify the above expressions to obtain

$$f_6(n) = \frac{100(3n-4)}{121(3n-10)} \quad \text{for } n \geq 7; \qquad f_8(7) = \frac{299}{300} \quad \text{and} \quad f_8(n) = \frac{637(4n-5)}{1125(4n-21)} \quad \text{for } n \geq 9.$$

It follows that $f_6(n) \geq 1$ if and only if $n \in \{7, 9, 11\}$ and $f_8(n) \geq 1$ if and only if $n = 9$.

We now study the function $x \mapsto f_d(x)$ for $x \in [5\ , \infty)$, for any fixed even value of $d \geq 10$. The typical shape for $f_d(x)$ is given in Figure 6, where $f_d(5) = 1$ and the curve has a local minimum at $x = d-1$ and a local maximum at $x = d+1$. Indeed, we will prove that $f_d(x)$ is decreasing on the two intervals $[5\ , d-1]$ and $[d+1\ , \infty)$. Then, establishing that $f_d(d+1) < 1$ completes the proof of the lemma.

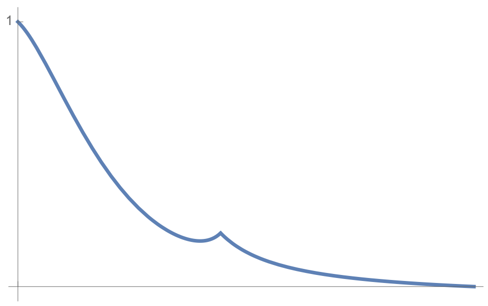


Figure 6: Typical curve for the ratio of $\tau(H_1)$ and $\tau(H_{t*})$.

Let us start with the case where $x \geq d+1$. Observe that

$$\frac{f_d(x+2)}{f_d(x)} = \frac{(d(x+1)-2)(d(x-d+3)-2)}{(d(x-1)-2)(d(x-d+5)-2)}.$$

In the numerator as well as in the denominator, the two terms of the product sum to $2dx - d^2 + 4d - 4$. As $x - d + 3 < x - d + 5 \leq x - 1 < x + 1$, convexity implies that the quotient is less than 1 and, therefore, that $f_d(x+2) < f_d(x)$.

Next assume that $5 \leq x \leq d-3$. Then

$$\begin{aligned}
\frac{f_d(x+2)}{f_d(x)} &= \frac{d-1}{2d-1} \cdot \frac{(d-x+3)(3d+x-3)}{(d-x+1)(3d+x-1)} \cdot \frac{dx+d-2}{dx-d-2} \\
&< \frac{d-1}{2d-1} \cdot 1 \cdot \frac{(d-x+3)}{(d-x+1)} \cdot \frac{dx+d-2}{dx-d-2} \\
&= \frac{d-1}{2d-1} \cdot \left(1 + \frac{2(d^2+2d-2)}{d(d-x+1)(x-1-2/d)}\right) \\
&\leq \frac{d-1}{2d-1} \cdot \left(1 + \frac{2(d^2+2d-2)}{(d-4)(4d-2)}\right) \qquad (5) \\
&= \frac{d-1}{2d-1} \cdot \frac{3d^2-7d+2}{2d^2-9d+4} \\
&= \frac{3d^3-10d^2+9d-2}{4d^3-20d^2+17d-4} \\
&= 1 - \frac{d^3-10d^2+8d-2}{4d^3-20d^2+17d-4} \\
&< 1.
\end{aligned}$$

The last inequality holds because $d \geq 10$, and inequality (5) uses the fact that for every fixed $d \geq 10$, the function $x \mapsto -d(x-d-1)(x-1-2/d)$ is minimised over $[5\ , d-3]$ when $x = 5$.

It remains to deal with the special case where $x = d+1$. Then we can check directly that $f_{10}(11) = \frac{2286144}{2476099} < 1$. Moreover, for $d \geq 12$ we have

$$\begin{aligned}
f_d(d+1) &= \frac{(d-2)(3d+2)(d^2-2)(d-1)^{(d-4)/2}}{8(2d-1)^{d/2}} \\
&< \frac{(d-1)(3d+2)(d+1)(d-1)(d-1)^{d/2-2}}{8(2d-1)^{d/2}} \\
&= \frac{(3d+2)(d+1)(d-1)^{d/2}}{8(2d-1)^{d/2}} \\
&< (3d+2)(d+1)2^{-3-d/2} \\
&< 1.
\end{aligned}$$

This concludes the proof. $\square$

We gather here our observations about the ratios $\tau(H_{d,n})/\tau(H_{d,n-2})$.

Observation 4.13. *For all even $d \geq 6$ and odd $n \geq 7$ we have*

$$\frac{\tau(H_{d,n})}{\tau(H_{d,n-2})} \leq \frac{3d-1}{2d-1} \cdot (d-1).$$

For $d = 6$, the above inequality holds with equality only for $n = 9$. Otherwise, we have the stronger bound

$$\frac{\tau(H_{6,n})}{\tau(H_{6,n-2})} \leq \frac{23 \times 5}{17} < 7 \quad \textit{for odd } n \neq 9.$$

Proof. The statement follows by direct computation, recalling that $H_{d,n} = H_{d,n}^{t^*}$ if $(d,n) \in \{(6,7),(6,9),(6,11),(8,9)\}$ and $H_{d,n} = H_{d,n}^1$ otherwise. One needs to note that for every fixed d, the function $x \mapsto \frac{(d(x-1)-2)}{(d(x-3)-2)}$ is decreasing with x. $\square$

4.3.2 A Minimiser Must Belong to $\mathcal{H}_{d,n}$.

Lemma 4.14. *Suppose G is a minimiser of τ over the class of connected d-regular n-vertex multigraphs, where $d \geq 6$. Then $G \in \mathcal{H}_{d,n}$.*

Proof. Note first that if $n = 5$, then the lemma holds. We now assume that $n \geq 7$ and that the lemma holds for graphs on $n-2$ vertices. We begin by showing that edges of multiplicity m where $3 \leq m \leq d-2$ must be part of pendant triangles. Suppose that $3 \leq m = w_G(u,v) \leq d-2$ and the vertices u and v do not belong to a same pendant triangle. This implies there is no vertex z such that $\{u,v,z\}$ induce a triangle with more than d edges. Let G' be a connected graph obtained by deleting all edges between u and v, identifying the vertices u and v into a new vertex x and completely lifting the vertex x. Then G' is a connected d-regular $(n-2)$-vertex multigraph and

$$\tau(G) \geq m \cdot c_{d-m} \cdot \tau(G') \geq \frac{2m(d-m+1)}{3}\tau(G') \geq 2(d-2)\tau(G') \geq 2(d-2)\tau(H_{d,n-2}).$$

Since $2(d-2) > \frac{3d-1}{2d-1}(d-1)$, we then deduce from Observation 4.13 that $\tau(G) > \tau(H_{d,n})$, a contradiction.

We deduce from the above statement that any vertex that is incident to an edge of multiplicity between $d/2$ and $d-2$ must be a terminal vertex in a pendant triangle — recalling that $d \geq 6$. Now consider vertices that are not terminal vertices in a pendant triangle. It follows that such vertices are either exceptional or incident to an edge of multiplicity $d-1$. We next argue that an optimal graph cannot have more than one exceptional vertex. Before proving this, let us immediately show how this property allows us to conclude the proof. Suppose that the optimal graph must have at most one exceptional vertex and all remaining vertices must either be terminal vertices in a pendant triangle or incident to an edge of multiplicity $d-1$. Then a path with edges of alternating multiplicities 1 and $d-1$ must eventually close and form an odd cycle. Therefore, the optimal graph must be a member of $\mathcal{H}_{d,n}$.

It thus remains to prove the announced property. Suppose, for the sake of contradiction, that u and v are both exceptional vertices. By Proposition 4.6, we can obtain a connected d-regular $(n-2)$-vertex multigraph G' by sequentially lifting the two vertices u and v. Then,

$$\tau(G) \geq (c_{d/2})^2 \cdot \tau(G') \geq (c_{d/2})^2 \cdot \tau(H_{d,n-2}).$$

Therefore, G fails to be optimal if $(c_{d/2})^2 > \tau(H_{d,n})/\tau(H_{d,n-2})$. For $d \geq 12$, we observe, using Proposition 2.7, that

$$(c_{d/2})^2 > \frac{d^2}{e^2} > \frac{3d-1}{2d-1}(d-1) = \max_{\substack{n \geq 7 \\ n \text{ odd}}} \left\{ \frac{\tau(H_{d,n})}{\tau(H_{d,n-2})} \right\}.$$

For $d \in \{8, 10\}$, we use the actual values of c_5 and c_4 and obtain

$$c_5^2 = 81/4 > \frac{9 \cdot 29}{19} = \max_{\substack{n \geq 7 \\ n \text{ odd}}} \left\{ \frac{\tau(H_{10,n,})}{\tau(H_{10,n-2,})} \right\} \quad \text{and} \quad c_4^2 = \frac{324}{25} > \frac{7 \cdot 23}{15} = \max_{\substack{n \geq 7 \\ n \text{ odd}}} \left\{ \frac{\tau(H_{8,n})}{\tau(H_{8,n-2})} \right\}.$$

If $d = 6$, then

$$c_3^2 = \frac{64}{9} > 7 > \frac{23 \cdot 5}{17} \geq \frac{\tau(H_{6,n})}{\tau(H_{6,n-2})} \quad \text{for odd } n \neq 9.$$

These contradictions establish that G has at most one exceptional vertex unless $d = 6$ and $n = 9$.

We now deal with the special case where $d = 6$ and $n = 9$, for which we have to use a different, more structural, approach to prove that G cannot have more than one exceptional vertex. One fact we need is that $H^2_{6,7}$ is the unique connected 6-regular 7-vertex with the fewest number of spanning trees. Indeed, the last two paragraphs do imply that $H_{6,7} \in \mathcal{H}_{6,7}$ and then Lemma 4.12 yields the statement.

We already know that all edges of multiplicity 3 or 4 are part of pendant triangles — actually, edges of multiplicity $d/2$ can never be part of a pendant triangle in a d-regular multigraph on more than 3 vertices, so that G has no edges of multiplicity 3. Now suppose that u and v do not belong to a same pendant triangle, and yet are linked by exactly 2 edges in G. If $G - uv$ is disconnected, then let G_1 and G_2 be the two components of $G - uv$ where G_1 has fewer vertices than G_2. Without loss of generality, assume that $u \in G_1$. Note that G_1 has at most 4 vertices and each of its vertices, except possibly u, has at least 2 neighbours in G_1.

Suppose first that G_1 has 4 vertices. Then u has a unique neighbour u' in G_1 for otherwise G_1, and therefore G, would contain a diamond as a subgraph, which is impossible. Since u and v are linked by 2 edges, It follows that u and u' are linked by 4 edges, and hence u, u' is part of a pendant triangle, which is a contradiction since G_1 has 4 vertices. We deduce that G_1 has 3 vertices, and hence is a triangle with edge multiplicities $2, 2, 4$. In particular, $\tau(G_1) = 20$.

In G_2, the vertex v has degree 4 (and no edge of multiplicity more than 2). Let G' be a connected 6-regular 5-vertex multigraph obtained by completely lifting v in G_2. Now,

$$\tau(G) = 2\tau(G_1)\tau(G_2) \geq 40 \cdot c_2 \cdot \tau(G') \geq 80 \cdot \tau(H_{6,5}) = 80 \cdot 220 > 10285 = \tau(H_{6,9}),$$

which is a contradiction.

Consequently, $G - uv$ is connected. In particular, every spanning tree of $G - uv$ is a spanning tree of G that does not use any of the two edges between u and v. Let G'' be the connected 8-vertex multigraph obtained by deleting all edges between u and v and next identifying u and v into a single vertex x. Note that every spanning tree of G'' corresponds in a natural way to two spanning trees of G both containing an edge between u and v, and differing only on this edge. We infer that

$$\tau(G) \geq \tau(G - uv) + 2 \cdot \tau(G''). \tag{6}$$

We now perform some complete lifts in $G - uv$ and in G'' to obtain back connected 6-regular multigraphs. (The complete lifts we are going to make are possible since u and v are exceptional and do not belong to a same pendant triangle.)

Let G' be a connected 6-regular 7-vertex multigraph obtained from $G - uv$ by subsequently lifting u and v. We have

$$\tau(G') \geq c_2^2 \cdot \tau(G - uv) \geq c_2^2 \cdot \tau(H_{6,7}) = 4 \cdot \tau(H^2_{6,7}). \tag{7}$$

Let G''_x be a connected 6-regular 7-vertex multigraph obtained from G'' by completely lifting the new vertex x, so

$$\tau(G'') \geq c_4 \cdot \tau(G''_x) \geq \frac{18}{5} \cdot \tau(H^2_{6,7}). \tag{8}$$

Combining (6), (7) and (8), we obtain

$$\tau(G) \geq \left(4 + \frac{36}{5}\right) \tau(H_{6,7}) = \frac{56}{5} \cdot \tau(H_{6,7}) > \tau(H_{6,9}),$$

where the last inequality follows from Observation 4.13. This contradiction finishes to establish that edges of multiplicity 2 also belong to pendant triangles.

In what follows, most assertions implicitly rely on the fact that every edge of multiplicity greater than 1 and less than 5 is contained in a pendant triangle. Suppose now that G contains an exceptional vertex u that is not a cut-vertex. Then u must have 6 distinct neighbours, all in the same 2-connected block. Otherwise, u would be incident to an edge of multiplicity 2 (recalling that G has no edge of multiplicity 3), which has to be contained in a pendant triangle. Consequently, u would have to be a terminal vertex of this pendant triangle, contradicting that u is exceptional. Similarly, because u is not a cut-vertex, the subgraph H induced in G by the neighbours of u is simple. Moreover, H has maximum degree 1, for otherwise a vertex of degree at least 2 in H along with two of its neighbours and u would induce a subgraph of G containing a diamond, a contradiction to the optimality of G. But then u and all its neighbours induce at most 9 edges, and the two remaining vertices are incident to 12 edges for a total of at most 21 edges in G, a contradiction to the 6-regularity of G.

As a result, every exceptional vertex of G is a cut-vertex. We now establish some properties of the 2-connected blocks of G. First, we note that all edges inside a 2-connected block must have multiplicity 1 or 5. This in particular implies that every 2-connected block of G contains at least 3 vertices, for if a 2-connected block is be composed of only two vertices, then the 1 or 5 edges joining them would form an odd-cut in G, which is impossible in a regular multigraph of even degree.

Second, because G has 9 vertices, no 2-connected block of G can contain more than 3 cut-vertices, since to each cut-vertex x of a block B, we can associate a distinct component of $G-x$ not intersecting B, and such a component contains at least 2 vertices by degree regularity of G. Moreover, if a 2-connected block B of G contains exactly 3 cut-vertices $\{u, v, w\}$, then necessarily G consists of the simple triangle $\{u, v, w\}$, with pendant triangles at all three vertices. It follows that $\tau(G) = 3 \cdot 20^3 > 10285 = \tau(H_{6,9})$, a contradiction. We thus proved that each 2-connected block of G contains at most 2 cut-vertices.

It now follows that a 2-connected block with 2 cut-vertices must contain at least 4 (and at most 5) vertices, for if B is a 2-connected block with 3 vertices containing exactly two cut-vertices of G, then they must induce a triangle that is not pendant and yet contain edges of multiplicities greater than 1 and less than 5, a contradiction.

As a consequence, if G contains at least 2 exceptional vertices, which in particular must be cut-vertices, then we can find two of them, u and v, that belong to the same 2-connected block B. So B has 4 or 5 vertices, and u and v are the only cut-vertices of G in B. Since B contains no pendant triangle, all vertices of B except u and v (which are exceptional) are incident to an edge of multiplicity 5 in B. This is a contradiction, as the only possibility is that B be a path u, u', v', v with edge multiplicities $1, 5, 1$, which is not 2-connected. It follows that G contains at most one exceptional vertex, as needed. This concludes the proof. $\square$

4.4 The Case $d = 4$

The degree-4 case is exceptional in the sense that it is the only one (among regular multigraphs of even degree) for which the padded paddle graph can be optimal, specifically if the number of vertices is either at most 11 or precisely 13. It makes the analysis more tedious if we split it according to the parity of the number of vertices, as we did when d is at least 6. This is why we present this case separately: wrapping the whole argument in a single recurrence avoids some systematic checking of the possibilities for two consecutive lifts yielding a specific graph.

We shall proceed by induction on the number of vertices and, to this end, we first recall that for $n \in \{4, 5\}$, which corresponds to our base cases, Lemma 4.4 ensures the following.

- Every 4-regular connected multigraph G on 4 vertices has at least 24 spanning trees, with equality if and only if G is isomorphic to $PC_{4,4}$.
- Every 4-regular connected multigraph G on 5 vertices has at least 49 spanning trees, with equality if and only if G is isomorphic to $PP_{4,5} = F_{4,5}$.

Proof of Theorem 4.1. We proceed by induction on the number n of vertices. The statement follows from Lemma 4.4 if $n \in \{4, 5\}$. Fix an integer $n \geq 6$ and assume that the conclusion holds for graphs with fewer than n vertices. In particular, for each $n' \in \{4, \ldots, n-1\}$, there exists a unique d-regular n'-vertex multigraph with the fewest number of spanning trees, which we name $G^*_{4,n'}$. Let G be a connected 4-regular n-vertex multigraph with the fewest spanning trees.

We split the analysis regarding whether or not G has an edge of multiplicity 2, and then regarding the range n is in. In each case, we either obtain a contradiction or identify a unique possibility for G.

Suppose first that G has an edge of multiplicity 2 between two vertices u and v. Let G' be obtained from G by first deleting these two edges and then identifying u and v into a single vertex x. It follows that G' is a connected 4-regular $(n-1)$-vertex multigraph and $\tau(G) \geq 2 \cdot \tau(G')$, with equality only if $G - uv$ is disconnected. We now consider three cases regarding the range of n, which reveals whether $G^*_{4,n-1}$ is $PP_{4,n-1}$, or $PC_{4,n-1}$ or $F_{4,n-1}$.

If $n \in \{6, \ldots, 12\} \cup \{14\}$, then the induction hypothesis implies that $G^*_{4,n-1} = PP_{4,n-1}$. As $\tau(PP_{4,n}) = 2 \cdot \tau(PP_{4,n-1})$ we deduce that, necessarily, $\tau(G) = \tau(PP_{4,n})$ and G' is $PP_{4,n-1}$. Since $G - uv$ is disconnected, we deduce that G must be isomorphic to $PP_{4,n}$.

If n is even and at least 16, then $G^*_{4,n-1} = F_{4,n-1}$. For any such value of n, however, $\tau(PC_{4,n}) < 2 \cdot \tau(F_{4,n-1}) \leq \tau(G)$, a contradiction.

Last, if n is odd and at least 13, then by induction $G^*_{4,n-1} = PC_{4,n-1}$. However, this information is not useful because it turns out that no connected d-regular n-vertex multigraph has as few as $2 \cdot \tau(PC_{4,n-1})$ spanning trees. We need to perform a little structural analysis. We want to show that G contains two vertices that can be subsequently completely lifted, thus yielding a connected d-regular $(n-2)$-vertex multigraph G'' such that $\tau(G'') \geq c_2^2 \cdot \tau(G^*_{4,n-2})$. To this end, observe that both u and v are exceptional vertices in G, then they cannot be subsequently completely lifted if and only if they have a common neighbour w such that (up to swapping u and v) there are exactly 2 edges between u and w and 1 edge between v and w.

Let G_u be obtained from G by completely lifing u (which, in this situation, amounts to deleting u and adding 2 edges between v and w). If no vertex of G_u can be completely lifted, then all vertices in G_u are incident to an edge of multiplicity 3, which implies that G_u is $PC_{4,n-1}$. Consequently, G is isomorphic to the graph depicted in Figure 7, which has $(16n-12)\cdot 3^{(n-5)/2}$ spanning trees. This is more than $\tau(F_{4,n})$ and also more than $\tau(PP_{4,13})$ when $n=13$, which is a contradiction. As a result, it is indeed possible to consecutively completely lift two vertices in G, thereby obtaining a connected d-regular $(n-2)$-vertex multigraph G'' with $\tau(G'') \geq 4\cdot\tau(G^*_{4,n-2})$. This implies that $n=13$, as if $n\geq 15$ then $4\cdot\tau(G^*_{4,n-2}) > \tau(F_{4,n})$. Indeed, if $n\geq 17$ then $G^*_{4,n-2}=F_{4,n-2}$ and $4\cdot\tau(F_{4,n-2}) > \tau(F_{4,n})$ if $n\geq 11$, while if $n=15$ then $G^*_{4,n-2}=PP_{4,13}$ and $4\cdot\tau(PP_{4,13}) > \tau(F_{4,15})$. Consequently, $G^*_{4,n-2}=PP_{4,11}$. We thus proved that not only is it possible to subsequently completely lift two vertices in G, but also that every such sequence of two consecutive complete lifts results in the graph $PP_{4,11}$. This is possible if and only if G is isomorphic to $PP_{4,13}$.

We thus established that if G has an edge of multiplicity 2, then necessarily $G=PP_{4,n}$ and $n\in\{6,\ldots,14\}$.

Suppose now that G has no edge of multiplicity 2. Then either G has an exceptional vertex u with 4 different neighbours, or G is $PC_{4,n}$. Suppose the former and let G_u be a connected 4-regular $(n-1)$-vertex multigraph obtained by completely lifting u in G. Then $\tau(G)\geq c_2\cdot\tau(G_u)=2\cdot\tau(G_u)$. We let e_1 and e_2 be the two edges of G_u arising from the complete lift of u, that is, the two edges of G_u that do not belong to G. Observe that e_1 and e_2 must be disjoint since u has 4 different neighbours in G. As before, we split the analysis with respect to the value of n.

If $n\in\{6,\ldots,12\}\cup\{14\}$, then the induction hypothesis implies that $G^*_{n-1}=PP_{4,n-1}$, and therefore $\tau(G)\geq 2\cdot\tau(PP_{4,n-1})$, with equality if and only if G_u is $PP_{4,n-1}$. Since $\tau(PP_{4,n}) = 2\cdot\tau(PP_{4,n-1})$, we deduce that G_u must be $PP_{4,n-1}$. Because G itself has no edge of multiplicity 2, the construction of G_u implies that $n-1\in\{5,6\}$. Moreover, if $n=5$ then by symmetry there are only two choices for e_1 and e_2, both of which yield an edge of multiplicity 2 in G, a contradiction. It follows that G_u cannot be $PP_{4,5}$, and hence $n-1=6$. Then, up to symmetry, there is only one choice for e_1 and e_2 (without loss of generality, e_1 must be an edge between the two exceptional vertices and e_2 is disjoint from e_1), and reversing the operation to recover G must again create an edge of multiplicity 2, a contradiction.

If n is even and at least 16, then $G^*_{4,n-1}=F_{4,n-1}$, and hence $\tau(G)\geq 2\cdot\tau(F_{4,n-1})$. However, $\tau(PC_{4,2k}) < 2\cdot\tau(F_{4,2k-1})$ when $k\geq 4$, which contradicts the optimality of G.

It remains to deal with the case where n is odd and at least 13, in which case $G^*_{4,n-1}=PC_{4,n-1}$ by induction. If u is the unique exceptional vertex of G, then G_u has no exceptional vertex, because completely lifting a vertex cannot create an exceptional vertex (the multiplicities of the edges incident to the other vertices can only increase). Consequently, every vertex of G_u is incident to an edge of multiplicity 3, implying that G_u is isomorphic to $PC_{4,n-1}$. Now, the edges e_1 and e_2 of G_u must be two disjoint edges of multiplicity 1, as G has no edge of multiplicity 2. It follows that G belongs to $\mathcal{H}_{4,n}$, as defined at the beginning of Section 4.3. Therefore, Lemma 4.11 implies that G is isomorphic to $F_{4,n}$.

If G has at least one other exceptional vertex v, then because G has no edge of multiplicity 2 we know that v is still exceptional in G_u. As a result, we can obtain a 4-regular $(n-2)$-vertex multigraph G_{uv} by completely lifting v in G_u. Furthermore, we note that G_{uv} cannot be isomorphic to $PP_{4,n-2}$. Indeed, note that $PP_{4,n-2}$ contains $n-7 \geq 6$ edges of multiplicity 2. However, as each of u and v has 4 different neighbours in G, and G itself has no edge of multiplicity 2, the number of edges of multiplicity 2 in G_{uv} is at most 4. This implies that $n \geq 17$, and it follows that

$$\tau(G) \geq (c_2)^2 \cdot \tau(G_{uv}) = 4 \cdot \tau(G_{uv}) \geq 4 \cdot \tau(G^*_{4,n-2}) = 4 \cdot \tau(F_{4,n-2}).$$

This provides a contradiction, because as reported earlier $\tau(F_{4,n}) < 4 \cdot \tau(F_{4,n-2})$ when $n \geq 11$.

We thus proved that if G contains no edge of multiplicity 2, then either n is even and $G = PC_{4,n}$, or n is odd and at least 13 and then $G = F_{4,n}$.

Summing-up both situations (i.e G has an edge of multiplicity 2 or not), we see that it only remains to compare the cases for which we have found two possibilities: when n is either 13 we need to compare $\tau(PP_{4,13})$ and $\tau(F_{4,13})$, and when n is even and at most 14, we need to compare $PP_{4,n}$ and $PC_{4,n}$. We see that $\tau(PP_{4,13}) < \tau(F_{4,13})$, and also that $\tau(PP_{4,2k}) < \tau(PC_{4,2k})$ if $k \leq 5$ while the inequality is reversed if $k \geq 6$, which concludes the proof. □

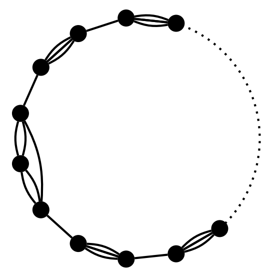


Figure 7: A 4-regular n-vertex multigraph with $(16n-12) \cdot 3^{(n-5)/2}$ spanning trees.

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