Solving the Vlasov equation using the Semi-Lagrangian scheme on a 2D hexagonal mesh

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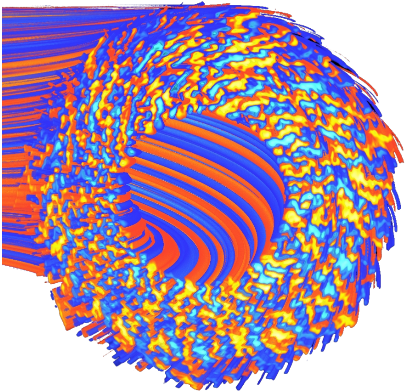
Outline

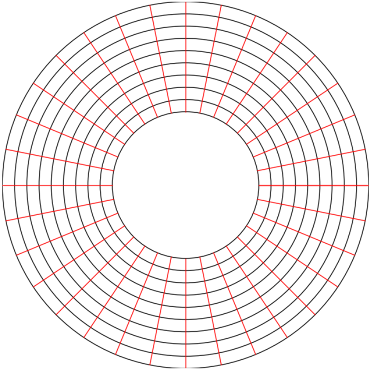
Motivation

The Gyrokinetic Semi-Lagrangian (**GYSELA**) code:



- **Gyrokinetic model**: 5D kinetic equation on the charged particules distribution
- 5 Dimensions: 2 in velocity space, 3 in configuration space
- **Simplified geometry**: concentric toroidal magnetic flux surfaces with circular cross-sections
- Based on the **Semi-Lagrangian** scheme

Motivation



Current representation of the poloidal plane:

- Annular geometry
- **Polar mesh** (r, θ)

Some limitations of this choice:

- Geometric (and numeric) **singular point** at origin of mesh
- Unrepresented area and very costly to minimize that area
- Impossible to represent **complex geometries**

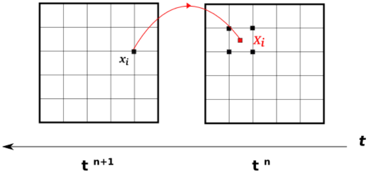
The Backwards Semi-Lagrangian Scheme

We consider the simplest form of the Vlasov equation, the advection equation

$$\frac{\partial f}{\partial t} + \mathbf{a} \cdot \nabla_{\mathbf{x}} f = 0$$

The Semi-Lagrangian scheme:

- Initial distribution known on all mesh points
- **Method of characteristics:** gives the origin of the trajectory of the particle at previous time step (density conserved along characteristics)
- Interpolate on the origin using known values of previous step at mesh points (usually **cubic B-spline interpolation**)



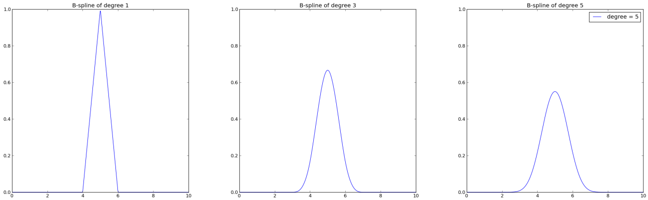
B(asis)-Splines basics*

B-Splines of degree d are defined by the **recursion** formula:

$$B_j^{d+1}(x) = \frac{x - x_j}{x_{j+d} - x_j} B_j^d(x) + \frac{x_{j+1} - x}{x_{j+d+1} - x_{j+1}} B_{j+1}^d(x)$$

Some important properties about B-splines

- Piecewise polynomials of degree $d \Longrightarrow$ **smoothness**
- Compact support $\Longrightarrow$ **sparse matrix system**
- Partition of unity $\sum_j B_j(x) = 1, \ \forall x \Longrightarrow$ **conservation laws**



Interpolating with cubic B-Splines

Initial data:

- Uniform mesh
- Initial distribution function $f(t_0, x)$ known at all mesh points x_i

The interpolant f_h is **exact on the mesh points** and is defined by

$$f_h(x) = \sum_{j=0}^{N} c_j B^3(x - x_j) \tag{1}$$

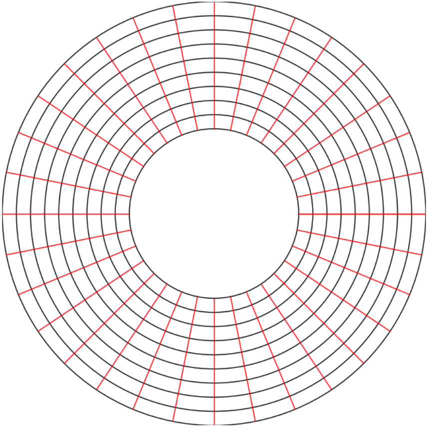
where the coefficients c_j are computed using the property

$$f_h(x_i) = \sum_{j=0}^{N} c_i B^3(x_i - x_j) = f(x_i) \tag{2}$$

Which can be written as a **sparse matrix system**, where boundary conditions intervene.

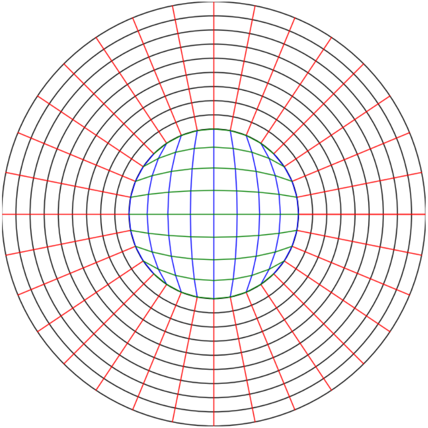
Multi-patch: the general idea

Our original mesh:



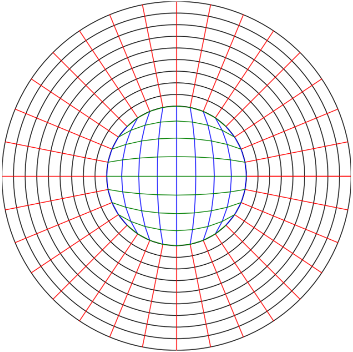
Multi-patch: the general idea

New representation of the poloidal plane:



Multi-patch: the general idea

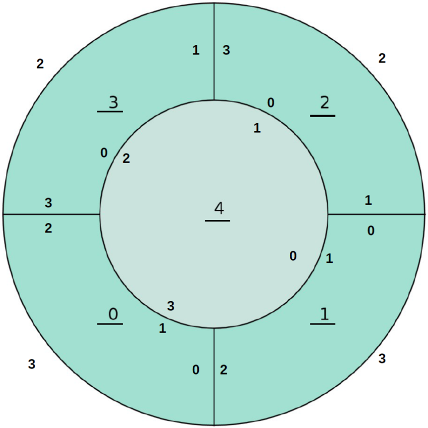
Specificities of the new geometry definition :



- **Additional patch(es)** with no singular point at origin
- Each patch defined as a **transformation** (or mapping) from uniform cartesian grid to new mesh
- Mappings defined with **NURBS** (Non-Uniform Rational B-Splines) $\Longrightarrow$ complex geometries
- **Coupling** between patches defined by boundary condition

The 5 patches configuration

External crown divided into 4 patches and the connectivity is defined as a patch-edge to patch-edge association (creation tool: **CAID**)



Advantages

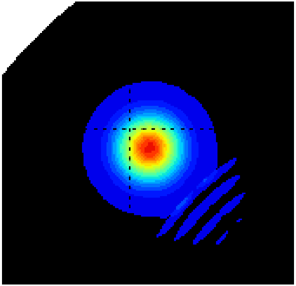
- Flexibility defining complex geometries
- Each patch can be treated separately
- No geometrical singularity

New challenges

- What is the best BC?
- How to treat particules which characteristics' origin are on another patch?
- **4 new numerical singularities**

Multi-patch: Some results

Results always showed instabilities near singular points. What we've tried to avoid them:

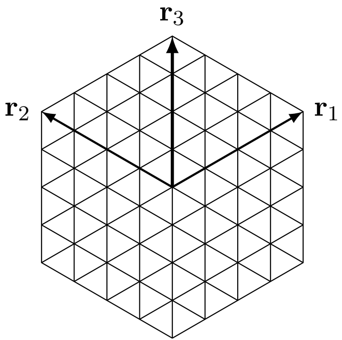


- Boundary conditions tested: strictly interdependent gradients and mean gradients between connecting patches
- Over-lapping: Impossible with interior patch and useless for others
- Squared internal mapping

Problem: Impossible to avoid singular points from mapping from a square to a circle

Second approach: The hexagonal mesh

Idea: Use a new mapping: square $\longrightarrow$ circle.
We define a tiling of triangles of a hexagon as our mesh for a 2D poloidal plane.



Some advantages:

- No singular points
- (Hopefully) no need of multiple patches for the core of the tokamak
- Twelve-fold symmetry $\Longrightarrow$ more efficient programming
- Easy transformation from cartesian to hexagonal coordinates
- Easy mapping to a disk

Box-splines and quasi-interpolation

Box-Splines:

- Generalization of B-Splines
- Depend on the vectors that define the mesh
- Easy to exploit symmetry of the domain

$\Longrightarrow$ More efficient interpolation

Quasi-interpolation:

- Distribution function known at mesh points
- Of order L if perfect reconstruction of a polynomial of order $L-1$
- No exact interpolation at mesh points $f_h(x_i) = f(x_i) + O(\| x_i \|^L)$

$\Longrightarrow$ Additional freedom to choose the coefficients c_j

$$f_h(x) = \sum c_j \chi^L(x - x_j) \quad (3)$$

Computing the spline coefficients using pre-filters

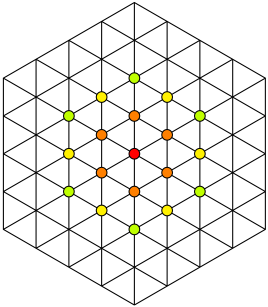
Idea: Coefficients obtained by discrete filtering of sample values $f(x_i)$

$$c = p * f = \sum_i f(x_i) p_i \tag{4}$$

prefilters: Obtained by solving a linear system of L equations (quasi-interpolation conditions)

Example with $L = 2$:

- We use information on two hexagons from point
- Points at same radius have same weight
- Error: $O(\| x \|^2)$



Hexagonal mesh: First results

Using box splines of degree 2, we obtained the following results:

model	**Points**	**a**	**dt**	**loops**	L_2 **error**
On mesh points	17101	$0.$	0.025	1	4.99×10^{-6}
Constant advec.	17101	0.05	0.025	81	4.70×10^{-3}
Circular advec.	17101	$1.$	0.025	81	4.33×10^{-3}

Using the pre-filter $Pfir$ and mirror boundary conditions.
The error increments linearly on time for advections and is of order 4.

Conclusion and perspectives

Hexagonal mesh:

- Results more encouraging than multi-patch results
- No numeric problems due to the mesh
- Efficiency to be compared
- More complex models to be tested
- Results have to be tested on a disk (and not a hexagon)
- Boundary conditions to be defined properly
- Box-MOMS (Maximal order minimal support box splines)

Multi-patch:

- Schwartz iterative method: stabilize singular points
- May still be useful for more complex geometries
- Implementation in the **SELALIB** library

Thank you for you attention

Questions?