Rough Diagrams

Laurent Vigneron
LORIA – Université Nancy 2,
Campus Scientifique, B.P. 239
54506 Vandoeuvre-lès-Nancy Cedex, France.
E-mail: vigneron@loria.fr

Anita Wasilewska
Department of Computer Science,
State University of New York,
Stony Brook, NY 11794-4400, USA.
E-mail: anita@cs.sunysb.edu

The study of algebras may be a hard task, even with powerful theorem provers. We present here an approach we used for proving important properties of rough Boolean algebras, combining theorem provers with a graphical technique, called Rough Diagrams. Rough Diagrams correspond to a classical notion of Ven Diagrams and can also serve as a new and very intuitive way of constructing counter models for formulas that are not tautologies of modal logic S5.

1 Generalized Rough Sets

The idea of rough set was proposed by Pawlak in 1982 [5]. The starting point of the rough set theory is the assumption that we have initially some information (knowledge) about the elements of the universe and that some elements can be indiscernible (equivalent) in the context of the information about them, that they can be "seen" as the same.

In [11], we have defined the notion of *generalized rough sets*: any topological space (X, I) with a clopen sets property.
A *Topological space* is a tuple (X, I), where X is a set and I a function from X to X, associating what is called the *interior of a set*, such that the following conditions are satisfied for any $A, B \subset X$: $I(A \cap B) = IA \cap IB$, $IA \subset A$, $IIA = IA$, and $IX = X$. For every $A \subset X$, the set $-I-A$, where $-$ is the complement function over X, is called the *closure of* A and denoted by CA. Rough sets with operators I and C are often presented by the diagram of Figure 1.

A topological space satisfies the *Clopen sets property* if, for any $A \subset X$, $CIA = IA$ and $ICA = CA$.
In 1988, Wiweger proved the following theorem:
Wiweger theorem [12]: *Let X be a topological space with an interior operation I, having the clopen sets property. Then there exists an equivalence relation R in X such that the interior operation I', induced by R, coincides with I.*

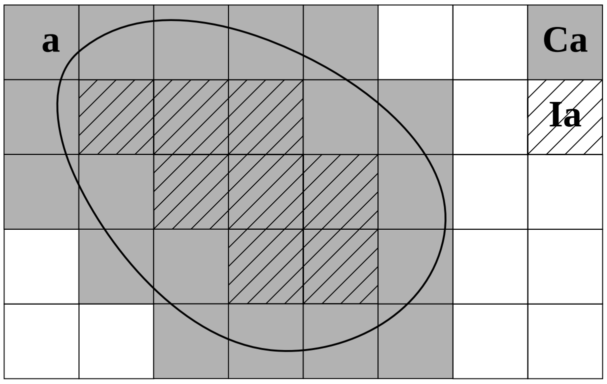

Figure 1: Rough Sets

This means that rough sets represent any approximation topological space, i.e. any triple (X, I, R), where (X, I) is a topological space, R is an equivalence relation, and I is induced by R.

2 Topological and Rough Boolean Algebras

Let (X, I, R) be a generalized rough set, then a field $\mathbf{P}(\mathbf{X})$ of all subsets of X with set-theoretic operations $\cup, \cap, -, I, C$ is called a rough field of sets. It is well known that fields of sets are typical examples of Boolean algebras, and that, by Stone [7] representation theorem: *every Boolean algebra is isomorphic to a field of sets.*

The topological fields of sets are typical examples of *topological Boolean algebras* [4, 6] defined by an abstract algebra $(A, \cap, \cup, \neg, I, C)$, where $(A, \cap, \cup, \neg)$ is a Boolean algebra and the following conditions hold: $I(a \cap b) = Ia \cap Ib$, $Ia \cap a = Ia$, $IIa = Ia$, $I1 = 1$, and $Ca = \neg I \neg a$ for any $a, b \in A$.
A topological Boolean algebra is called a *rough Boolean algebra* iff for any element $a \in A$, $ICa = Ca$ and

$CIa = Ia$, i.e. if it is clopen.

The representation theorem for topological Boolean algebras [4] can be easily extended to the rough Boolean algebras (called also clopen topological algebras) as follows:

Representation theorem: *For every topological (rough) Boolean algebra there exists a monomorphism into a topological (rough) field of all subsets of a topological space (generalized rough set).*

3 Topological Equalities as Congruences

The rough equality [5] was recently examined from the logical point of view by Banerjee [1], from algebraic point of view by Wasilewska [10] and from algebraic-automated theorem proving angle by Wasilewska-Vigneron [9].

In [11], we have extended the notion of rough equality to general topological spaces (X, I) and generalized rough sets (X, I) and consequently, to topological and rough Boolean algebras, respectively:

Topological equality $\approx_r$: Given a generalized rough set (X, I), for any two sets $A, B \subset X$ we say that they are *topologically equivalent*, and denote $A \approx_r B$ if and only if $IA = IB$ and $CA = CB$.

Given a rough Boolean algebra $(A, \cap, \cup, \neg, I, C)$, for any $a, b \in A$, $a \approx_r b$ if and only if $Ia = Ib$ and $Ca = Cb$.

Given a rough Boolean algebra $\mathcal{A} = (A, \cap, \cup, \neg, I, C)$ and a topological equality $\approx_r$, the following natural question arises: *can we define an algebra $\mathcal{B}$ similar to $\mathcal{A}$, such that topological equality is a congruence with respect to all its operations?*

The definition of a new algebra $\mathcal{B}$ has been considered by defining new intersection and union operators. The reason is that the classical operators do not satisfy the two important properties $C(a \cap b) = Ca \cap Cb$ and $I(a \cup b) = Ia \cup Ib$, as illustrated in Figures 2 and 3.

The first step in this direction has been achieved by Banerjee [2]. She has defined two new operations $\sqcup$ and $\sqcap$ in the modal S5 logic using the Lindenbaum-Tarski construction. However, the definition of these operators, that we translate into equivalent algebraic form as follows

$$a \sqcup b = (a \cup b) \cap (a \cup Ia \cup Ib \cup \neg I(a \cup b))$$
$$a \sqcap b = (a \cap b) \cup (a \cap Ca \cap Cb \cap \neg C(a \cap b)),$$

has never been motivated, nor explained. She has defined a binary relation on the set of formulas of the

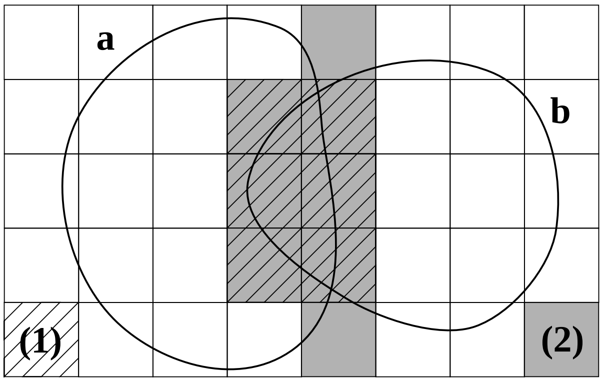

Figure 2: (1): $C(a \cap b)$, (2): $Ca \cap Cb$

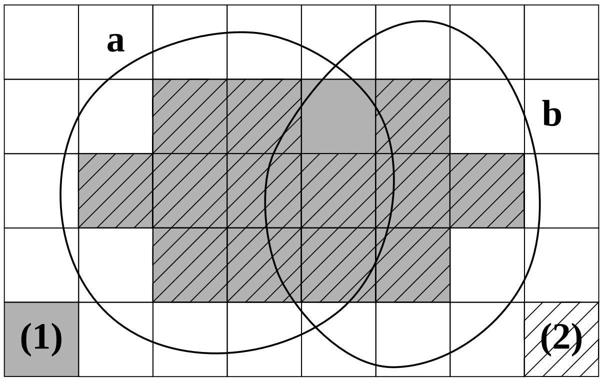

Figure 3: (1): $I(a \cup b)$, (2): $Ia \cup Ib$

modal S5 logic, and has claimed that it was a congruence with respect to the logical connectives of S5. But the semantical proofs are only sketched and impossible to understand.

These operators have been translated into expressions of a rough Boolean algebra $\mathcal{A}$ [10] and, using the theorem prover daTac [8], we have tried to get formal proofs of all the properties claimed in Banerjee [2]. But we have failed to obtain proofs of some of them, and then we started to doubt their truthfulness.

4 Rough Diagrams

In order to give an intuition to the logical formulas presented by Banerjee, we have used the graphical representation of rough sets, which we call here *Rough Diagrams*.

Drawing sets and doing graphical calculi is much more easy than trying to deal with algebraic formulas. With rough diagrams, we have first tried to find counterexamples of there improved properties. And, unable to find any, we have proved or studied enough examples for being finally convinced of their validity.

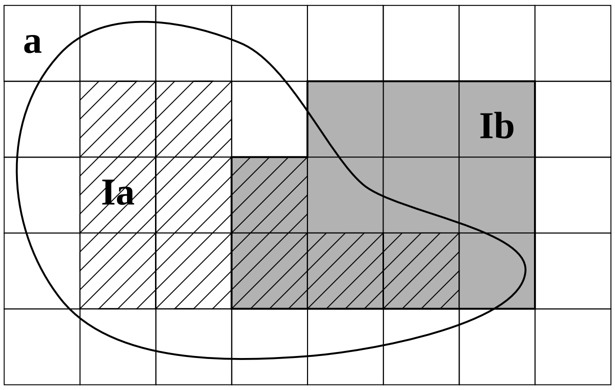

Figure 4: $I(a \cup Ib) = Ia \cup Ib$

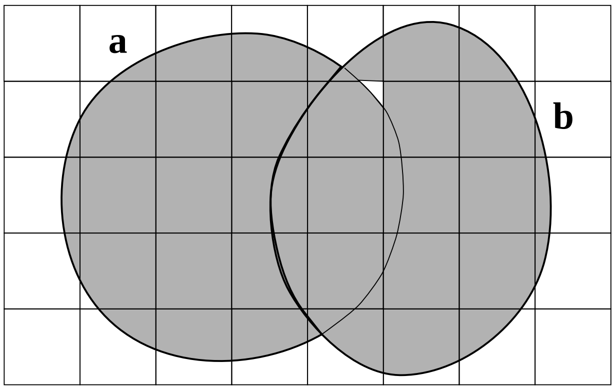

Figure 6: New union ($\sqcup$)

For example, some of these important properties are: $I(a \cup Ib) = Ia \cup Ib$ (see Figure 4), $C(a \cup Cb) = Ca \cup Cb$, $C(a \cup Ib) = Ca \cup Ib$, $I(a \cup Cb) = Ia \cup Cb$.

Encouraged by the results obtained with rough diagrams, we have decided to find what was really needed for defining new union and intersection operators. We did this graphically, starting from the cases setting problems for the classical union and intersection operators, illustrated in Figures 2 and 3. Therefore we have defined the following new operators:

$$a \sqcup b = a \cup Ib \cup \neg I(a \cup b)$$
$$a \sqcap b = (a \cap b) \cup (a \cap Cb \cap \neg C(a \cap b))$$

described in Figures 5 and 6. Our goal has been to take into account the squares setting problems for the classical operators: adding a piece of them for intersection; removing a piece of them for union.
We have then discovered our operators are equivalent to Banerjee's, though more easy to understand.

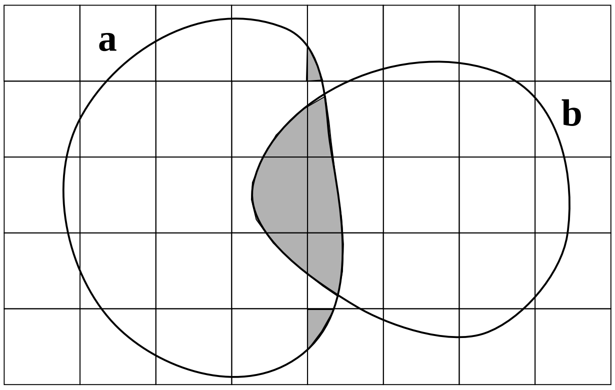

Figure 5: New intersection ($\sqcap$)

Besides this first definition of new operators, we have also tried other definitions, such as a more symmetric version:

$$a \underline{\sqcup} b = (a \cap b) \cup Ia \cup Ib \cup \neg I(a \cup b)$$
$$a \underline{\sqcap} b = (a \cap b) \cup (a \cap Cb \cap \neg C(a \cap b)) \cup (b \cap Ca \cap \neg C(a \cap b))$$

This definition looked more interesting as the new operators were commutative. However, with rough diagrams, we have proved they were not associative, nor distributive (these properties are not needed, but we hoped they would be satisfied), but more important, we have also shown they do not satisfy the following important properties: $C(a \underline{\sqcup} b) = Ca \underline{\sqcup} Cb$, $I(a \underline{\sqcap} b) = Ia \underline{\sqcap} Ib$. A counter-example of the first property is illustrated in Figure 7, where sets a and b have an empty intersection, but $I(a \cup b)$ is not equal to $Ia \cup Ib$.
The consequence is that the topological equality is not a congruence w.r.t. these operators.

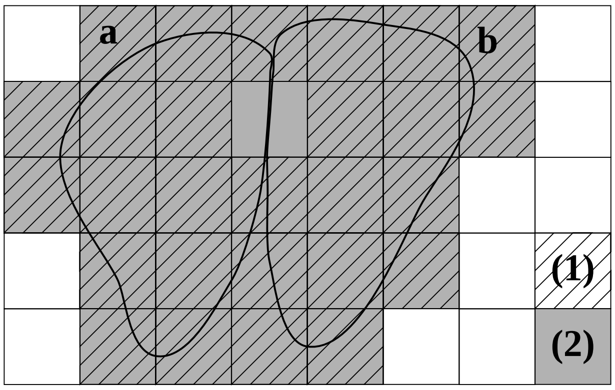

Figure 7: (1): $C(a \underline{\sqcup} b)$, (2): $Ca \underline{\sqcup} Cb$

5 Algebraic Proofs

After the above use of rough diagrams, we have decided to try again to prove that the topological equality is a congruence w.r.t. all the operators of the algebra $\mathcal{B} = (A, \sqcap, \sqcup, \neg, I, C)$, and more particularly w.r.t. our new operators. We have succeeded with the help of rough diagrams and the theorem provers daTac [8] and OTTER [3].

In addition, we have studied the congruent algebra $\mathcal{B}_r = (A/{\approx_r}, \sqcap, \sqcup, \neg, I, C)$ and proved the following properties:

- $(A/{\approx_r}, \sqcup, \sqcap, 0, 1)$ is a distributive lattice with 0 and 1.
- For any $a, b \in A/{\approx_r}$, $\neg(a \sqcup b) = (\neg a \sqcap \neg b)$,
- For any $a \in A/{\approx_r}$, $\neg\neg a = a$.
- The congruent algebra $\mathcal{B}_r$ is not a Boolean algebra, i.e. there is an element $a \in A/{\approx_r}$ such that $\neg a \sqcap a \neq 0$ and $\neg a \sqcup a \neq 1$ (see Figure 8).

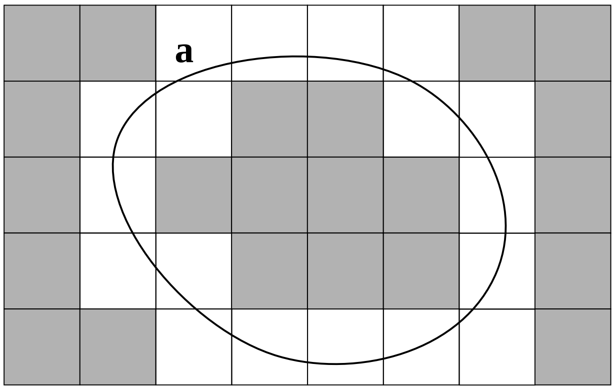

Figure 8: $I(a \sqcup \neg a)$ $(\neq I1 = 1)$

- For any $a, b \in A/{\approx_r}$, $I(a \sqcap b) = Ia \sqcap Ib$, $I(a \sqcup b) = Ia \sqcup Ib$, $Ia \leq a$, $IIa = Ia$, $I1 = 1$, and $CIa = Ia$, where $Ca = \neg I \neg a$ and $\leq$ is a lattice ordering.

6 Conclusion

Thanks to Wiweger's theorem, we know that rough diagrams can be used for studying any rough Boolean algebra. This graphical technique is very natural and very easy to use. Rough diagrams can be used for proving properties, but also for finding counter-examples which is very difficult when manipulating algebraic formulas. The result we have obtained is a very good example of the power of this technique.

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