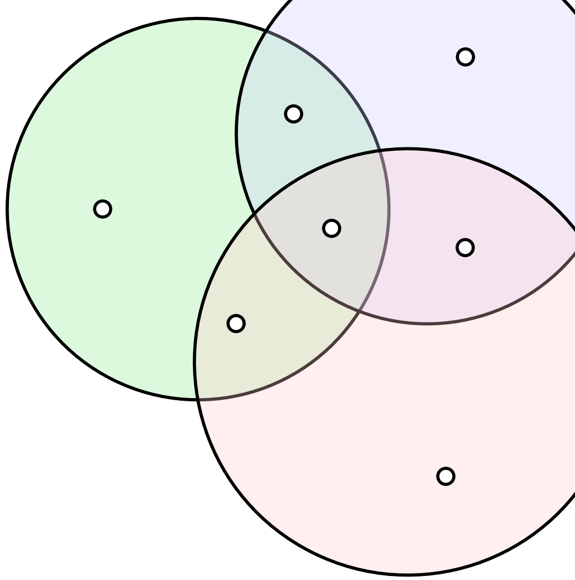
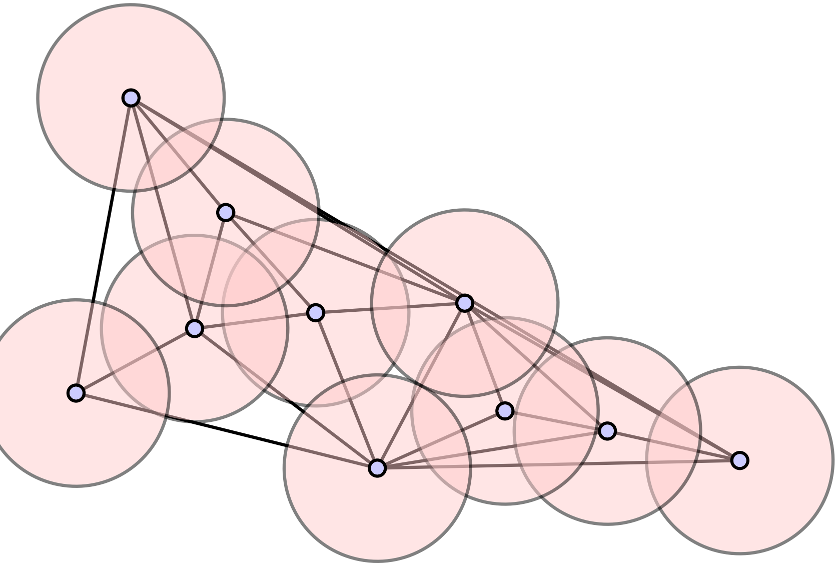
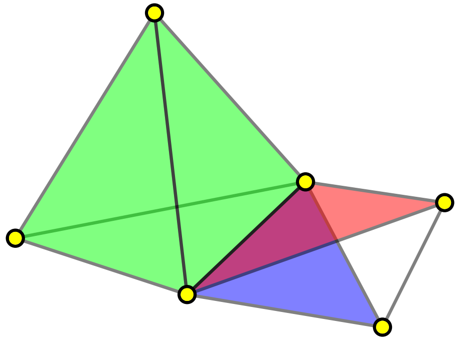


A zest of combinatorial topology applied to the simplification of inclusion-exclusion formulas



Xavier Goaoc   Équipe GAMBLE (LORIA/INRIA)

MidiCombi – Janvier 2025



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Let $F = \{b_1, b_2, \ldots, b_n\}$ be a family of equal radius balls in $\mathbb{R}^d$. Letting T denote the Delaunay triangulation of the balls' centers, we have

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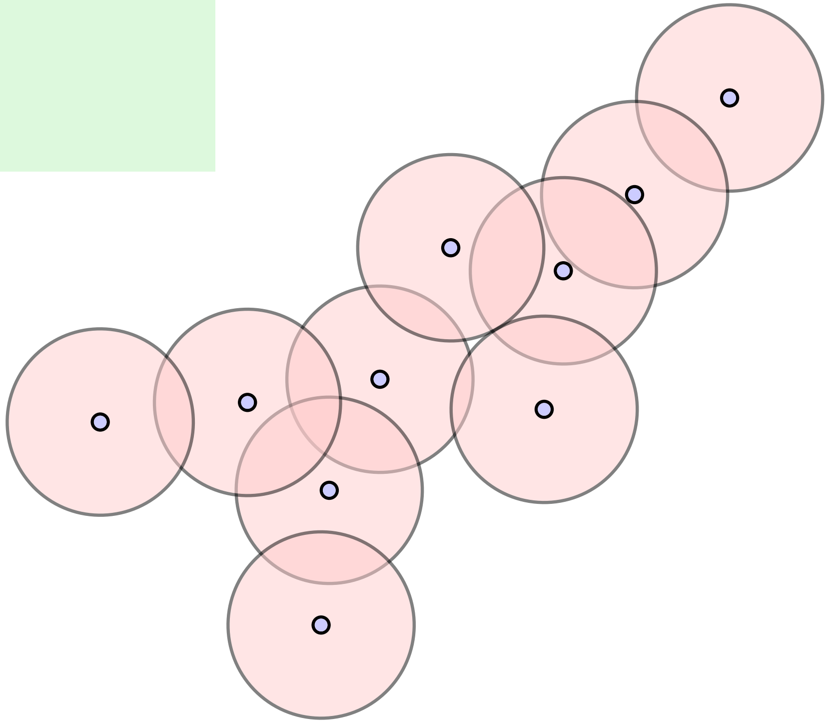

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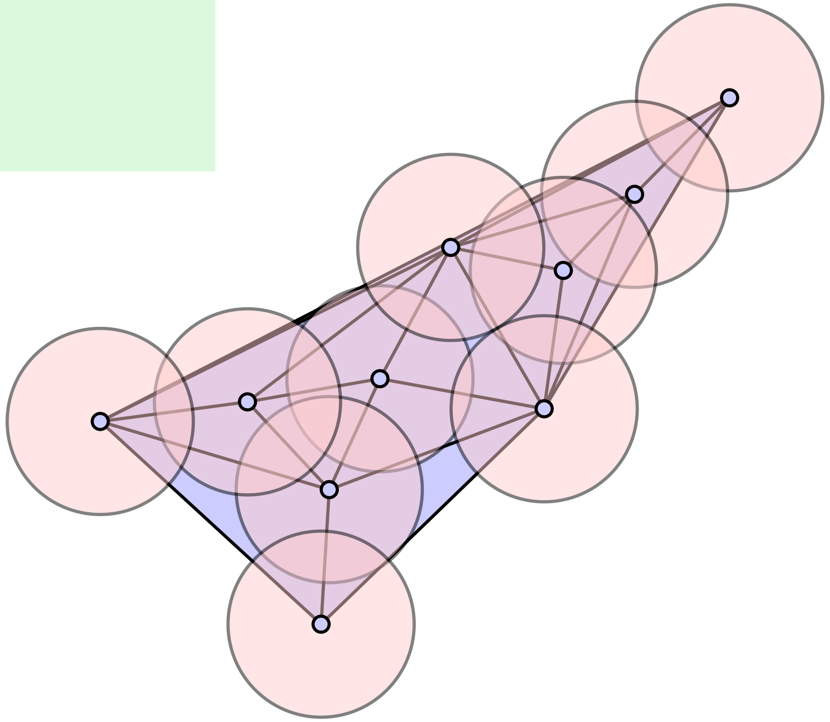

Key ideas...

- ▷ describe inclusion-exclusion formulas via **abstract simplicial complexes**,
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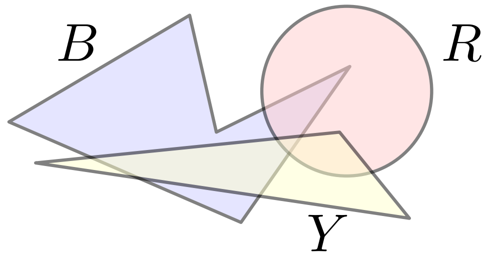
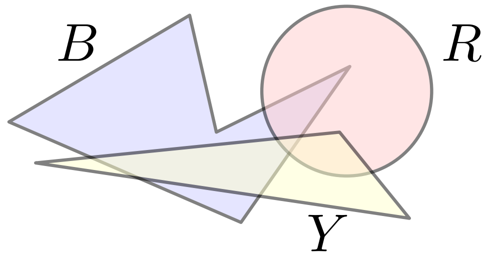
This talk...

1. inclusion-exclusion formulas and their simplifications
2. abstract simplicial complexes and inclusion-exclusion
3. topological space of a graph
4. which topological spaces are we talking about?
5. geometric realization of an abstract simplicial complex
6. nerve complexes and the nerve theorem
7. Delaunay triangulations and Voronoi diagrams
8. proof of the formula for balls

#1. inclusion-exclusion formulas and their simplifications

How can you compute the measure of...  ?

How can you compute the measure of... 

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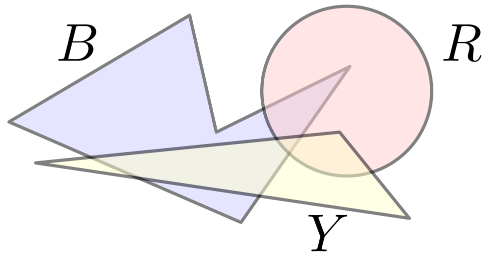
How can you compute the measure of...  ?

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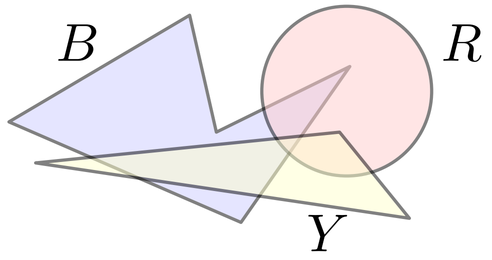

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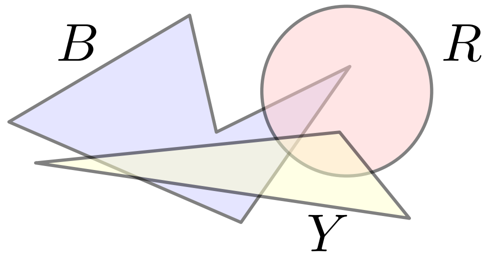
How can you compute the measure of...  ?

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More generally, the **inclusion-exclusion** principle states that for any *measurable sets* $a_1, a_2, \ldots, a_n$

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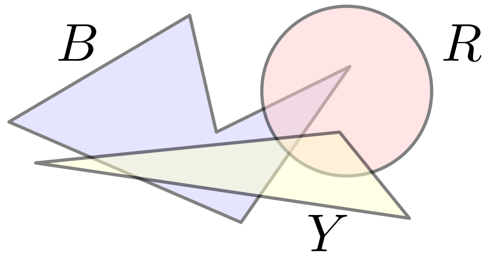

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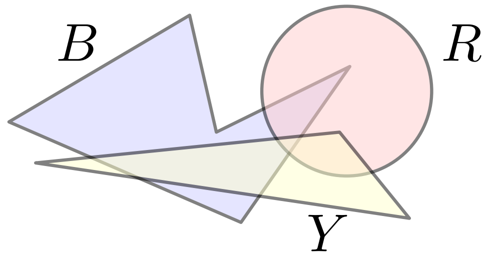
How can you compute the measure of...  ?

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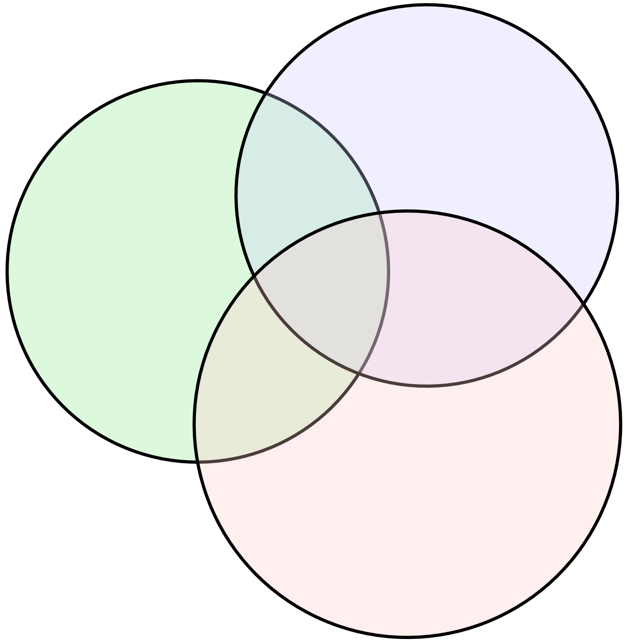

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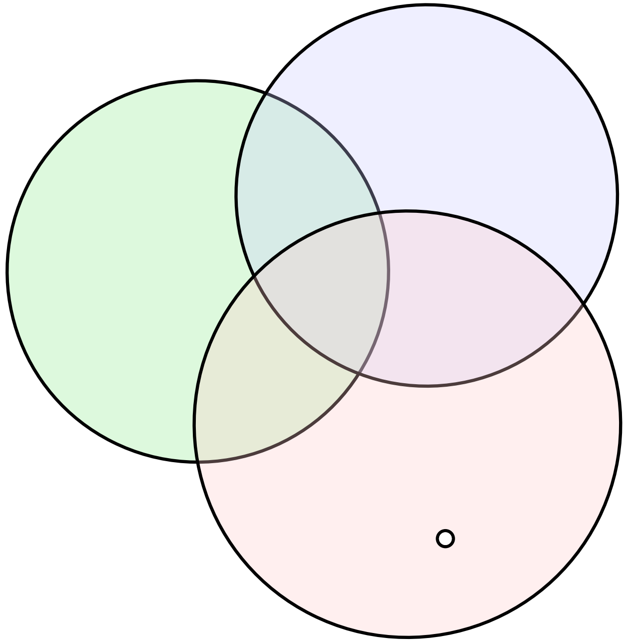

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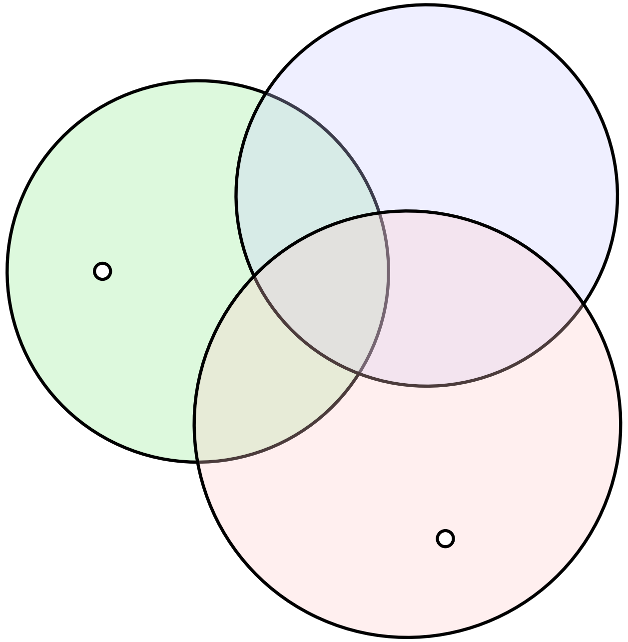

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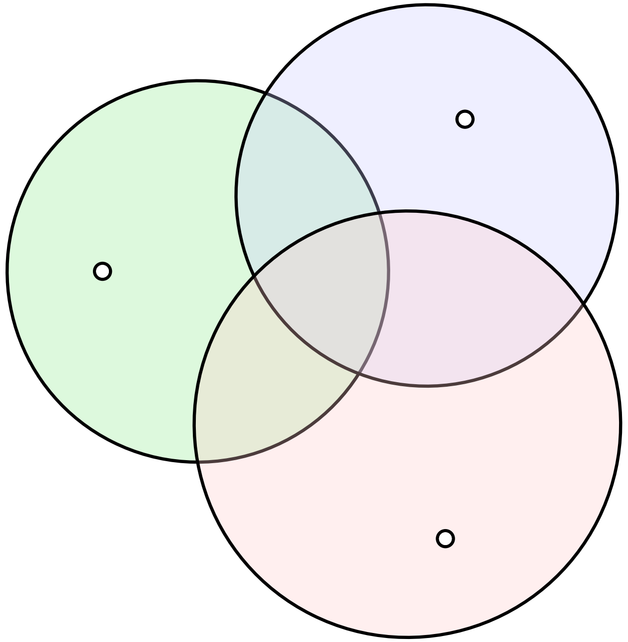

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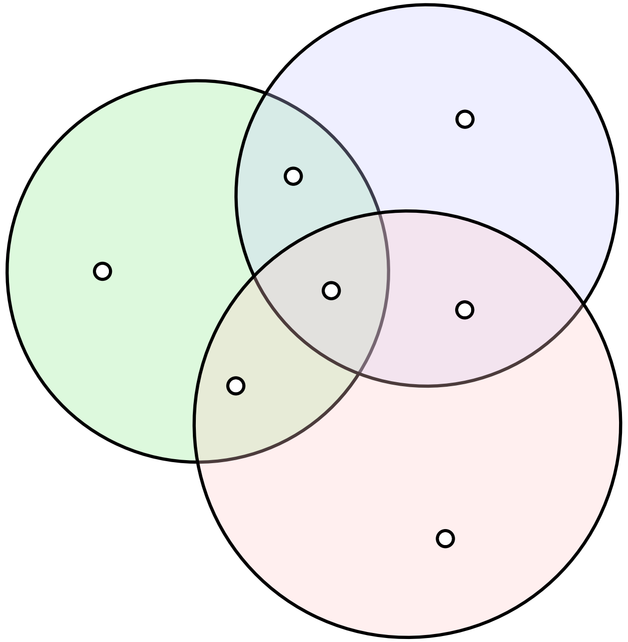

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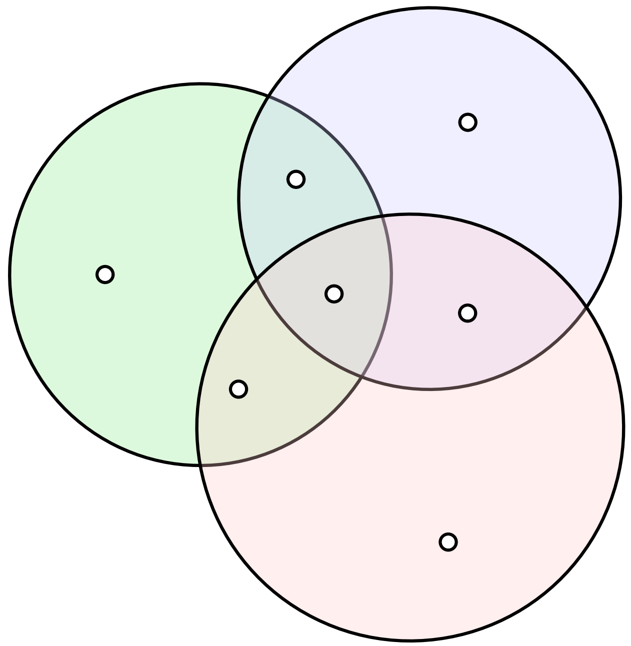

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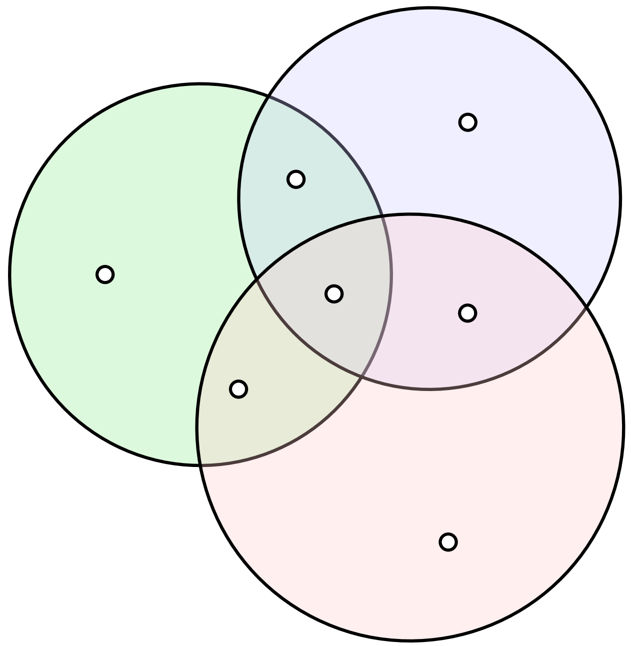

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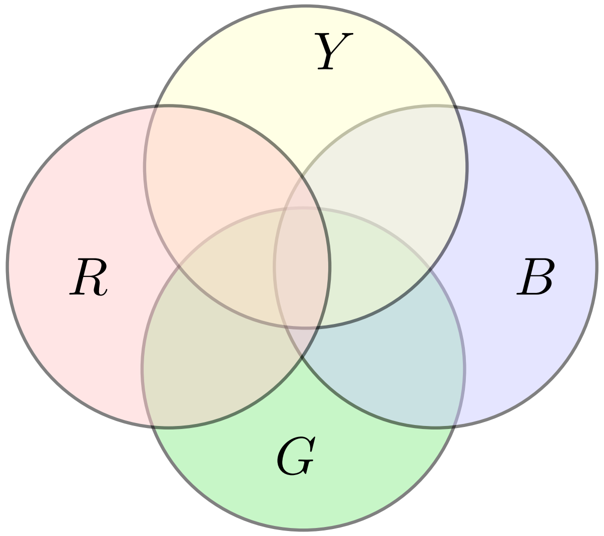

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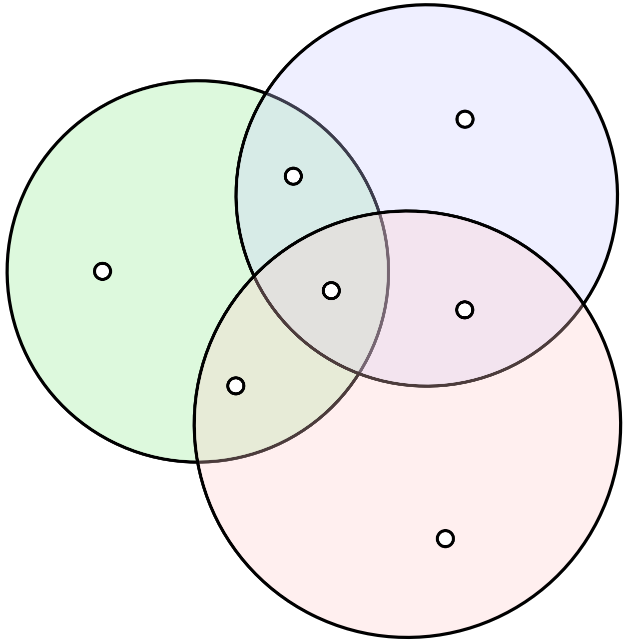

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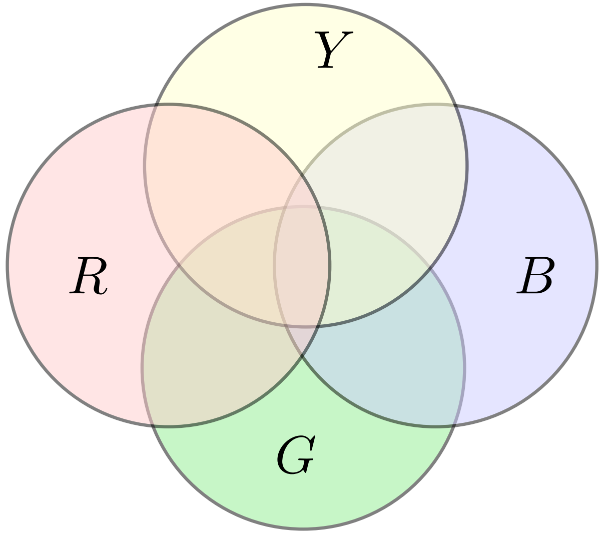

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(NB: also work on **approximate** universal inclusion-exclusion formulas) [Linial-Nisan'93] [Kahn-Linial-Samorodnitsky'96]

#2. abstract simplicial complexes and inclusion-exclusion

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$P_k \stackrel{\text{def}}{=}$ all sets S of primes such that $\prod_{p \in S} p \leq k$.

$$P_{30} = \{\emptyset, \{2\}, \{3\}, \{5\}, \{7\}, \{11\}, \{13\}, \{17\}, \{19\}, \{23\}, \{29\}, \{2,3\}, \{2,5\}, \{2,7\}, \{2,11\}, \{2,13\}, \{3,5\}, \{3,7\}, \{2,3,5\}\}$$

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▷ The formula is true for p iff $\quad 1 = \sum_{\sigma \in K; \sigma \subseteq F_p} (-1)^{|\sigma|-1} = \chi(K[F_p])$.

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The subcomplex of K induced by F_p.

▷ Pick $p \in \cup_{i=1}^n a_i$ and let $F_p \overset{\text{def}}{=} \{i \colon a_i \ni p\}$.

▷ The formula is true for p iff $\quad 1 = \sum_{\sigma \in K; \sigma \subseteq F_p} (-1)^{|\sigma|-1} = \chi(K[F_p])$.

Abstract simplicial complex

$\stackrel{\text{def}}{=}$ a collection of sets that is closed under taking subsets.

Let $F = \{a_1, a_2, \ldots, a_n\}$ be a set system.

Let K be some abstract simplicial complex with vertices $\{1\}, \{2\}, \ldots, \{n\}$.

When do we have

$$\mathbb{1}_{\bigcup_{i=1}^n a_i} = \sum_{\sigma \in K} (-1)^{|\sigma|-1} \mathbb{1}_{\bigcap_{i \in \sigma} a_i} \qquad ?$$

▷ Always true for every $p \notin \cup_{i=1}^n a_i$.

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Abstract simplicial complex

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When do we have

$$\mathbb{1}_{\bigcup_{i=1}^n a_i} = \underbrace{\sum_{\sigma \in K} (-1)^{|\sigma|-1} \mathbb{1}_{\bigcap_{i\in\sigma} a_i}}_{(1)} \quad ?$$

Let $F = \{a_1, a_2, \ldots, a_n\}$ be a set system.

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▷ Always true for every $p \notin \cup_{i=1}^n a_i$.

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The subcomplex of K induced by F_p.

$\chi(K) \overset{\text{def}}{=} \sum_{\sigma \in K} (-1)^{|\sigma|-1}$.

K induces an IE-formula for F in the sense of (1)
$\Leftrightarrow$ for any $p \in \cup F$, the subcomplex $K[F_p]$ has Euler characteristic 1.

Let's associate to every abstract simplicial complex
some topological space that helps analyze its Euler characteristic.

#3. topological space of a graph

- start with any graph $G = (V, E)$ with V finite

$K_5 = (V, E)$ with

$V = \{1, 2, 3, 4, 5\}$

and

$E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}.$

- start with any graph $G = (V, E)$ with V finite
- fix some map $f : V \to \mathbb{R}^3$ such that no four images are coplanar.

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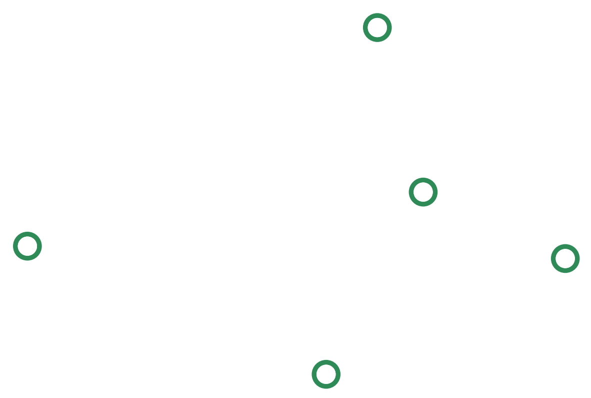

- ▷ start with any graph $G = (V, E)$ with V finite
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- ▷ $\Gamma_{G,f} \stackrel{\text{def}}{=} \left(\bigcup_{u \in V} \{f(u)\} \right) \cup \left(\bigcup_{\{u,v\} \in E} \operatorname{conv}(\{f(u), f(v)\}) \right)$

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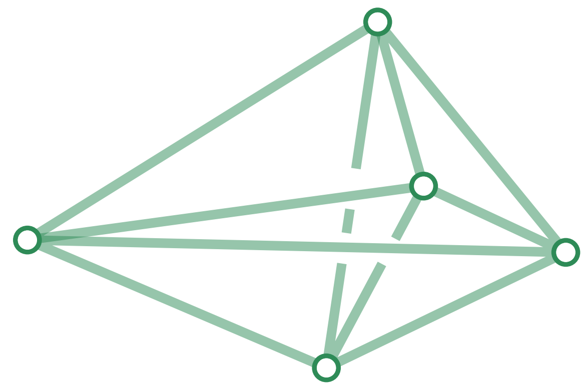

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- The **topological space** $\Gamma_{G,f}$ is independent of f $\rightsquigarrow |G|$.

For any maps f_1, f_2, Γ_{G,f_1} is homeomorphic to Γ_{G,f_2}.

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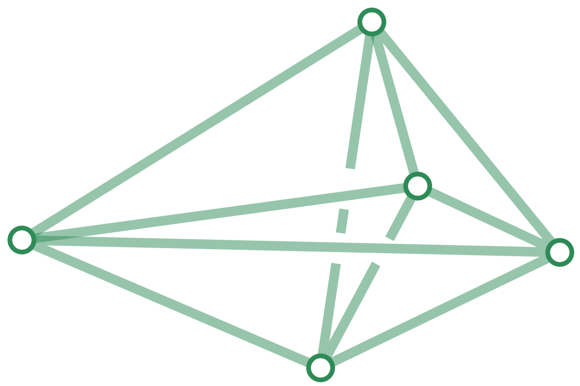

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> G is planar $\Leftrightarrow |G| \hookrightarrow \mathbb{R}^2$.
>
> Genus of $G \overset{\text{def}}{=}$ min. g s.t. $|G| \hookrightarrow T_g$.
>
> Crossing number of $G \overset{\text{def}}{=}$ min # crossings in a map $|G| \to \mathbb{R}^2$.
>
> ...

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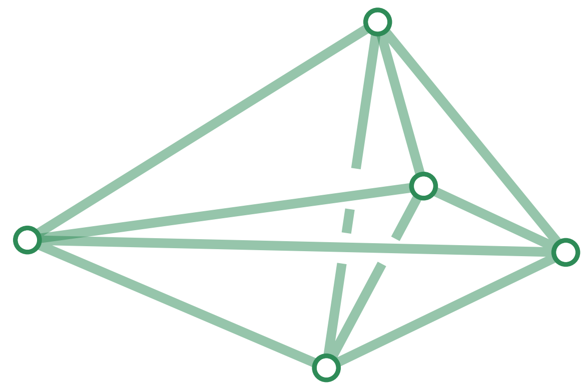
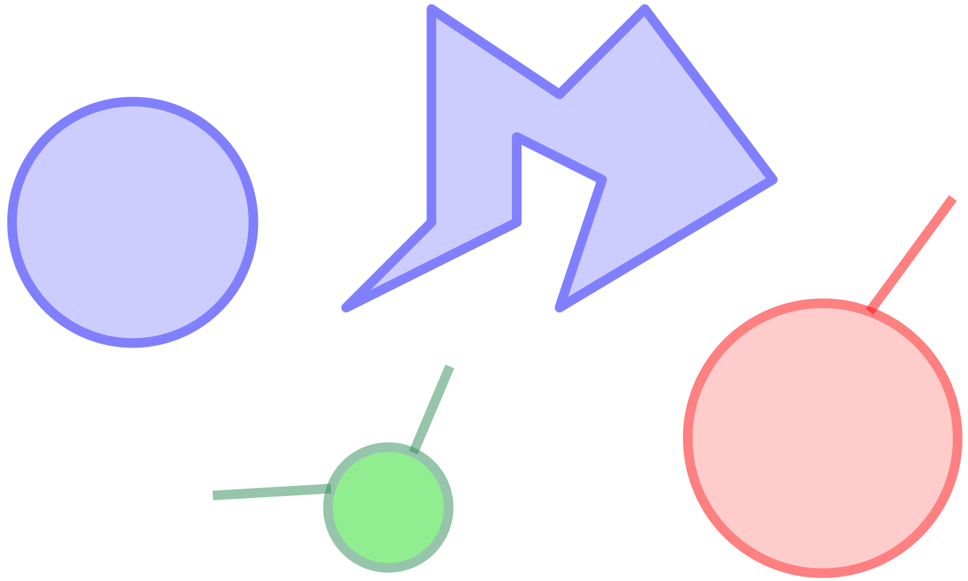


#4. which topological spaces are we talking about?

▷ our **topological spaces** are subsets of $\mathbb{R}^d$, $d < \infty$,
$+$ the topology induced by open balls of $\mathbb{R}^d$.



▷ our **topological spaces** are subsets of $\mathbb{R}^d$, $d < \infty$,
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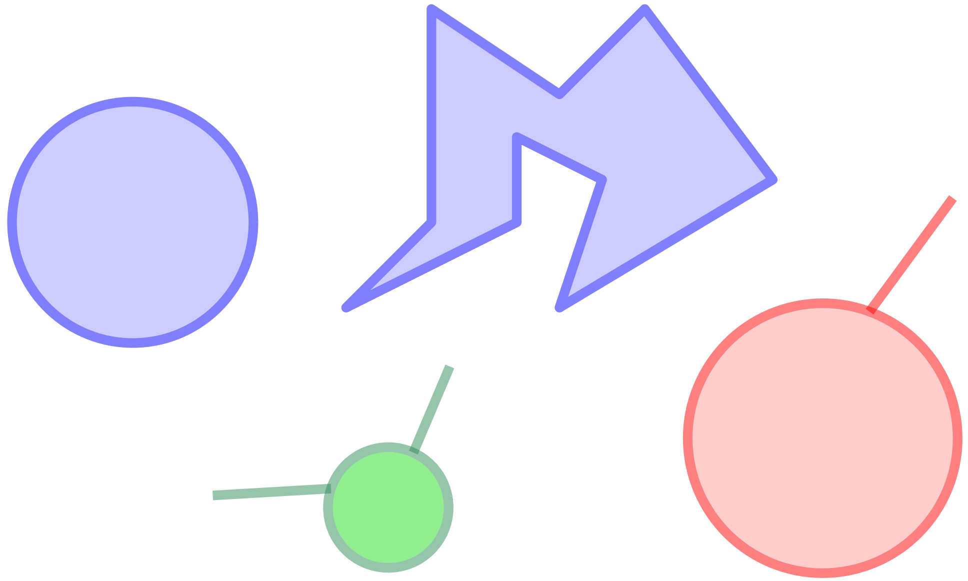
▷ $f : X \to Y$ is a **homeomorphism** iff
f is bijective, and f and f^{-1} are continuous.



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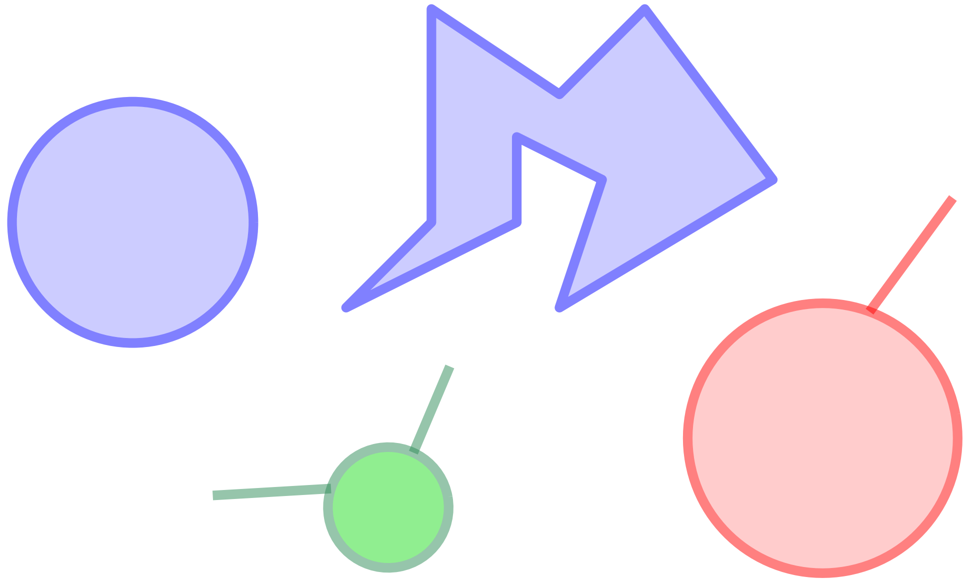

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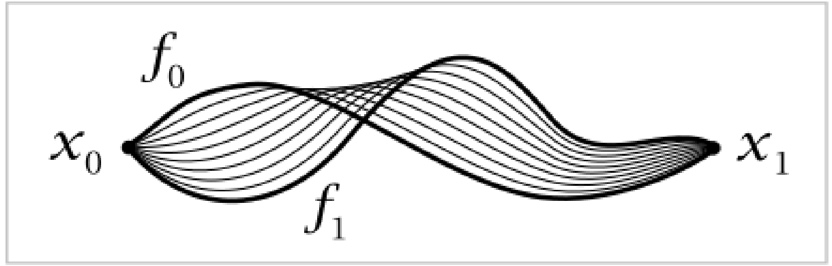

[Hatcher]

▷ $f_0 : X \to Y$ and $f_1 : X \to Y$ are **homotopic** $(f_0 \simeq f_1)$
iff there exist $f : X \times [0,1] \to Y$ continuous
s.t. $f(\cdot, 0) = f_0$ and $f(\cdot, 1) = f_1$.

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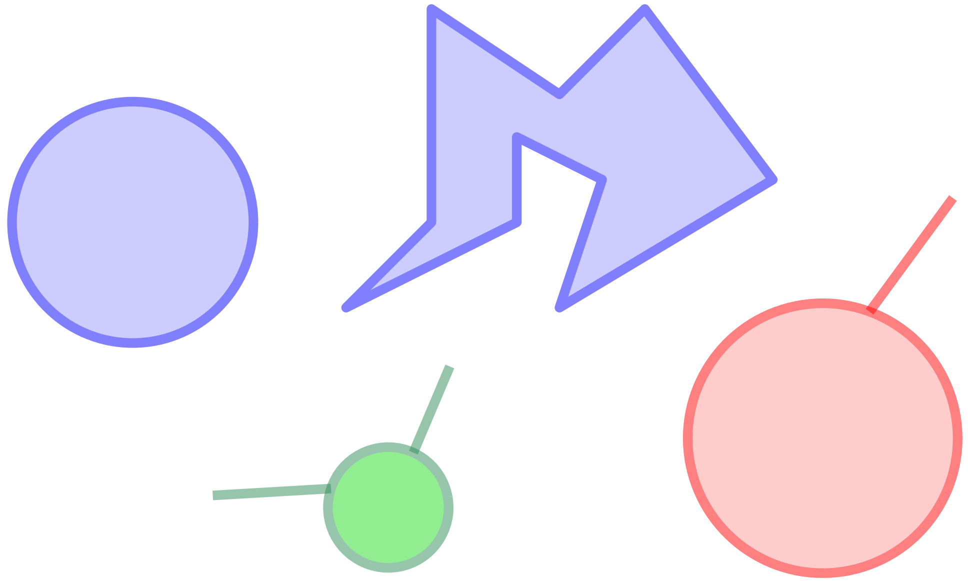

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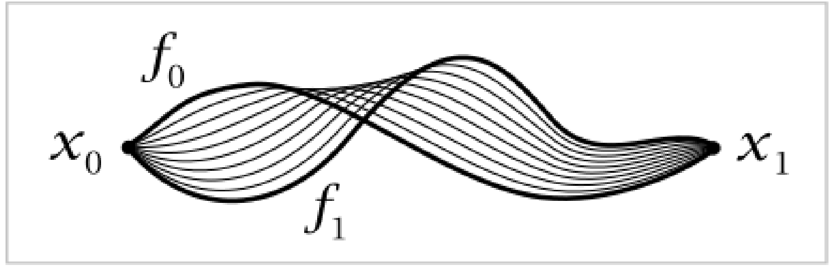

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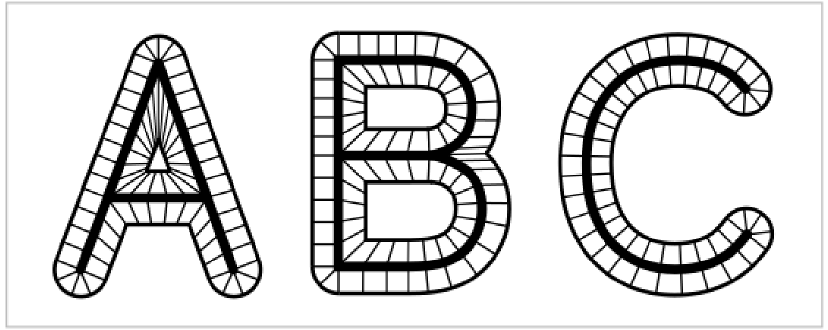

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- X and Y are **homotopic spaces** iff there exist
  maps $g : X \to Y$ and $h : Y \to X$
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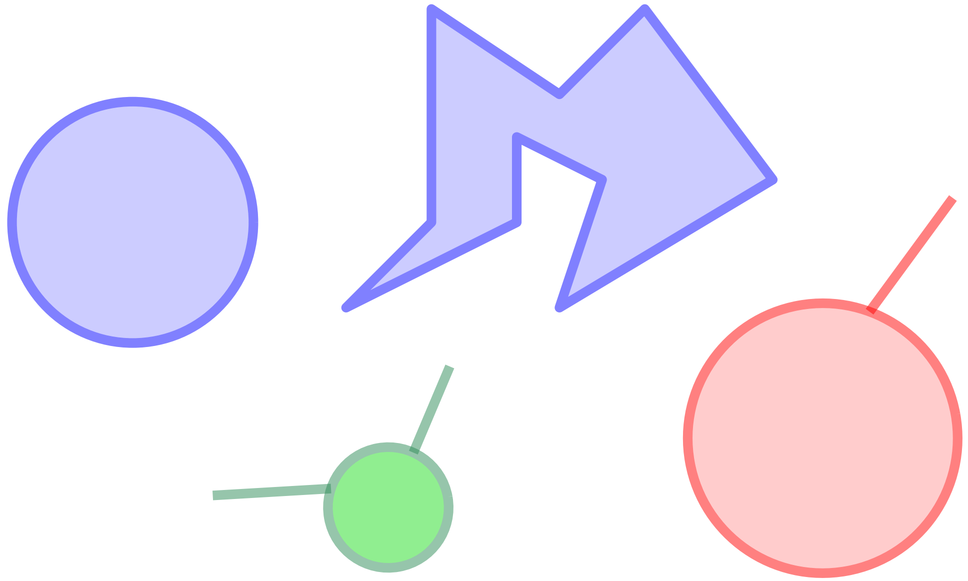

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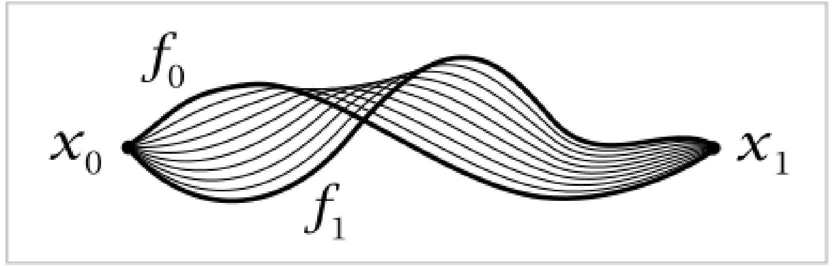

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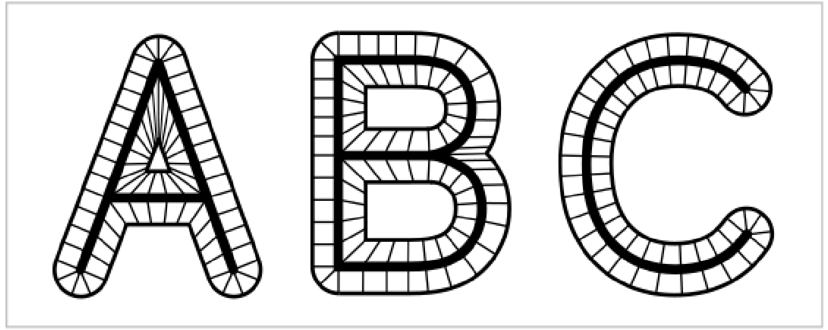

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▷ X is **contractible** if X is homotopic to a point.

#4. geometric realization of an abstract simplicial complex

Abstract simplicial complex

$\overset{\text{def}}{=}$ a collection of sets that is
closed under taking subsets.

$\Leftrightarrow$ a "hereditary hypergraph".

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▷ start with any finite abstract simplicial complex K.

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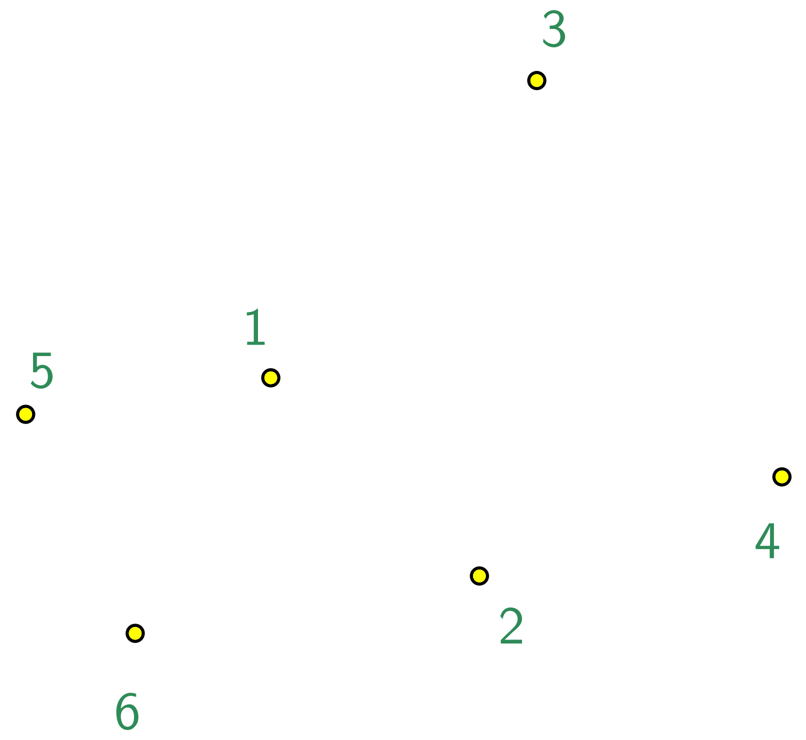

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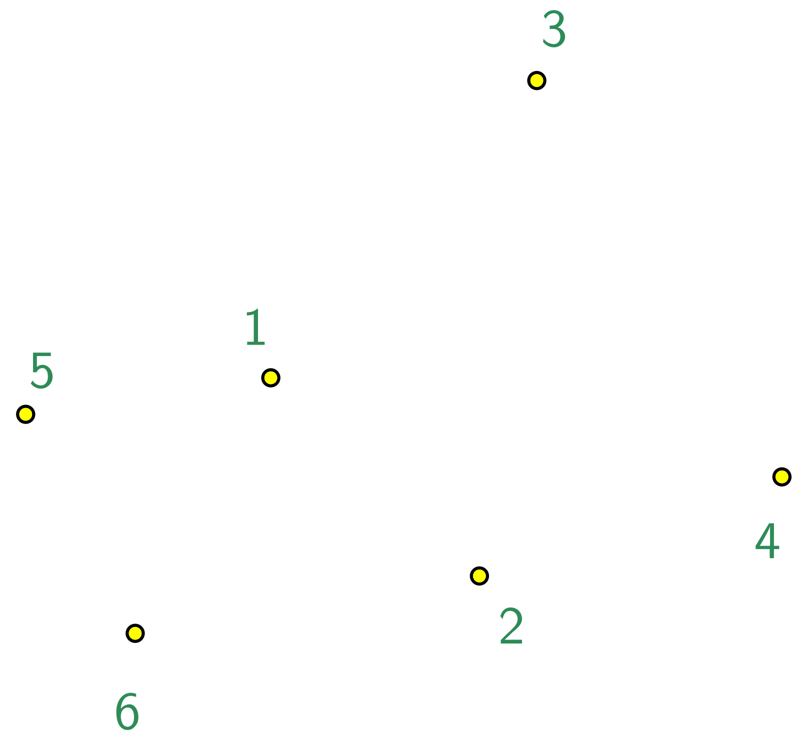

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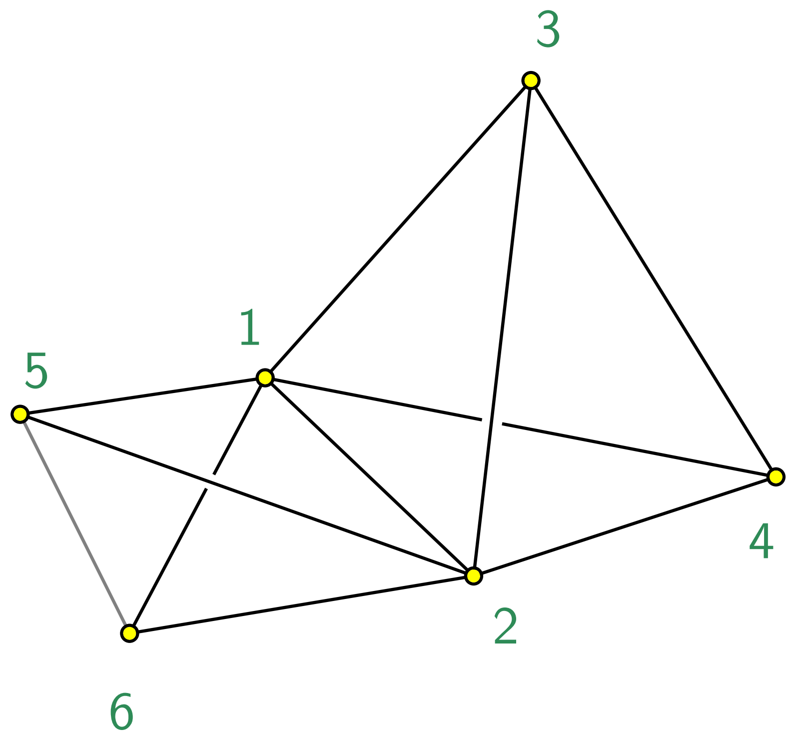

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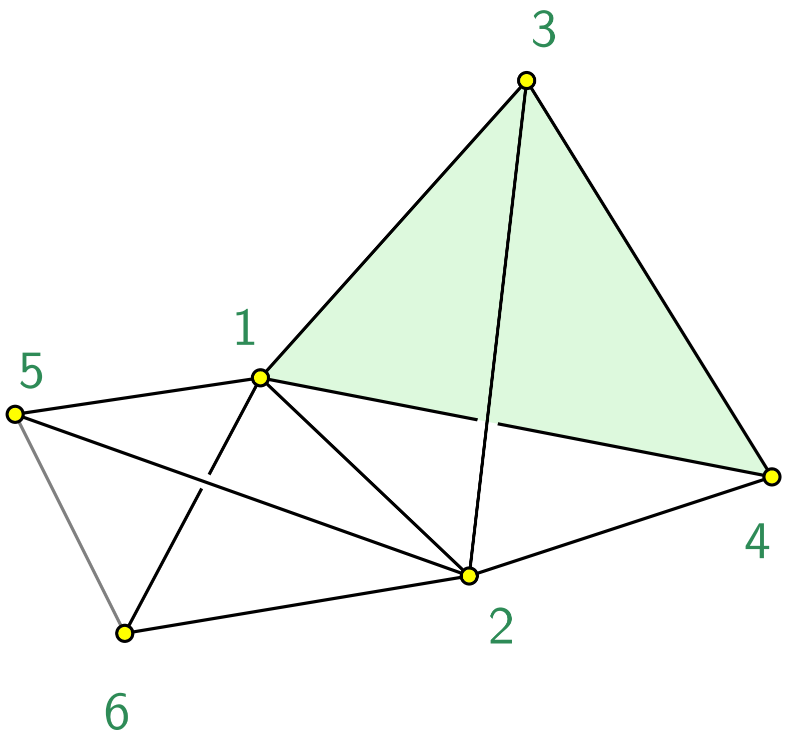

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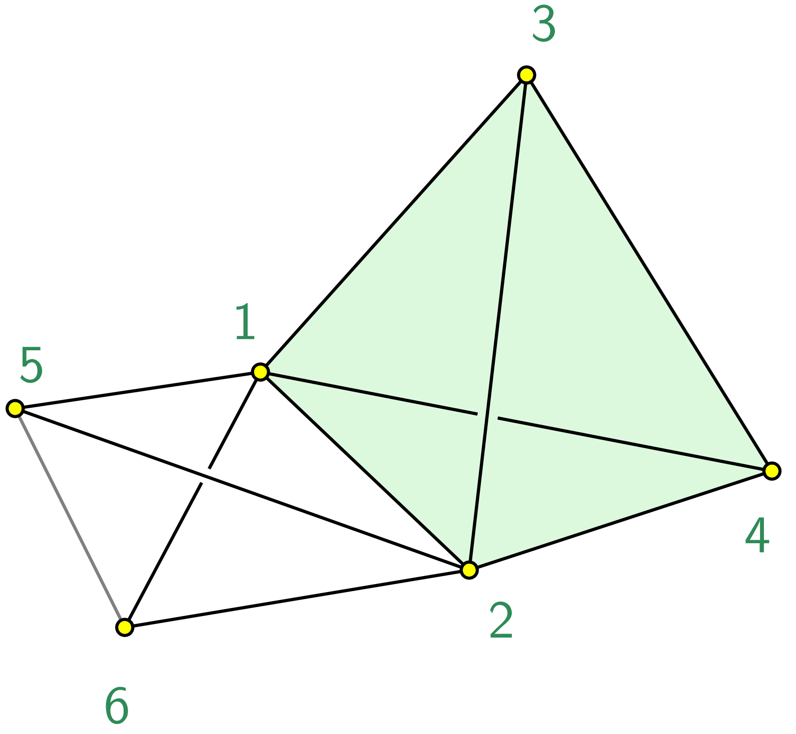

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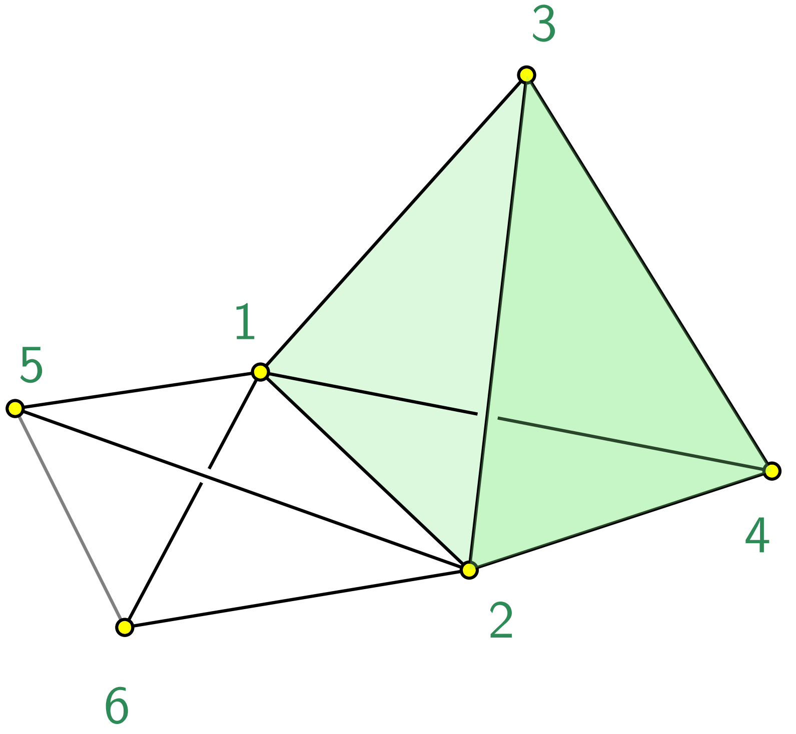

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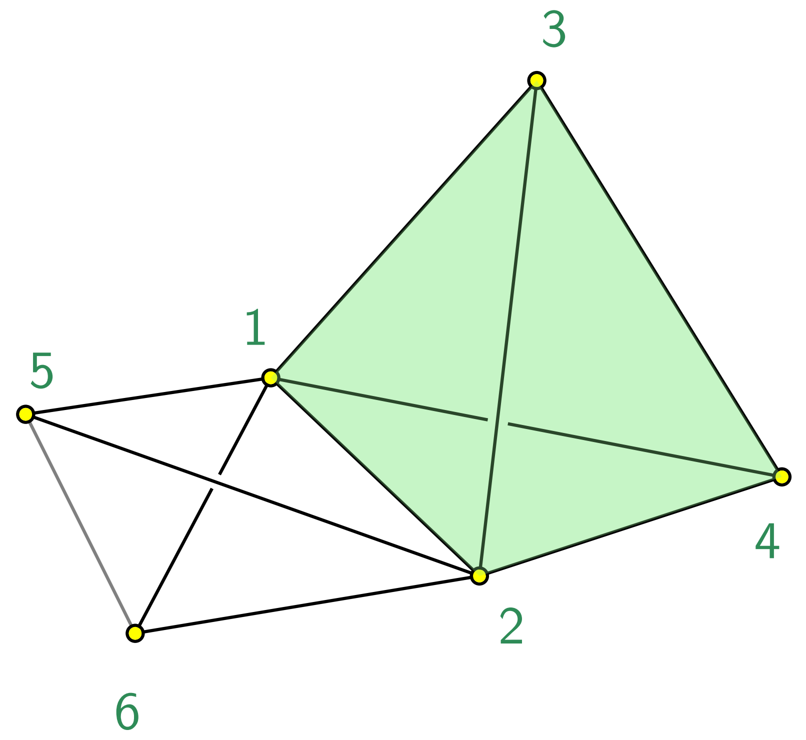

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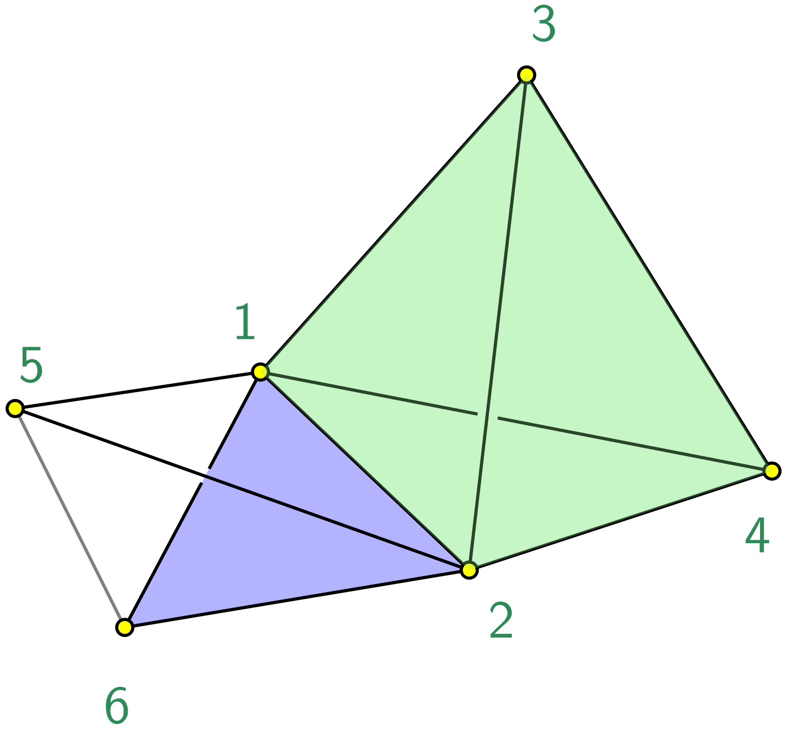

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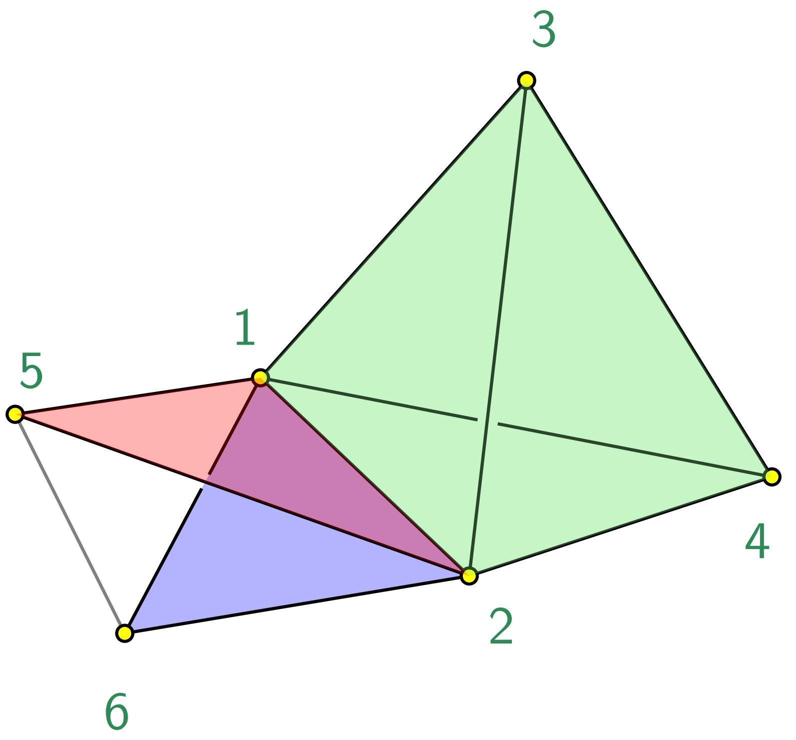

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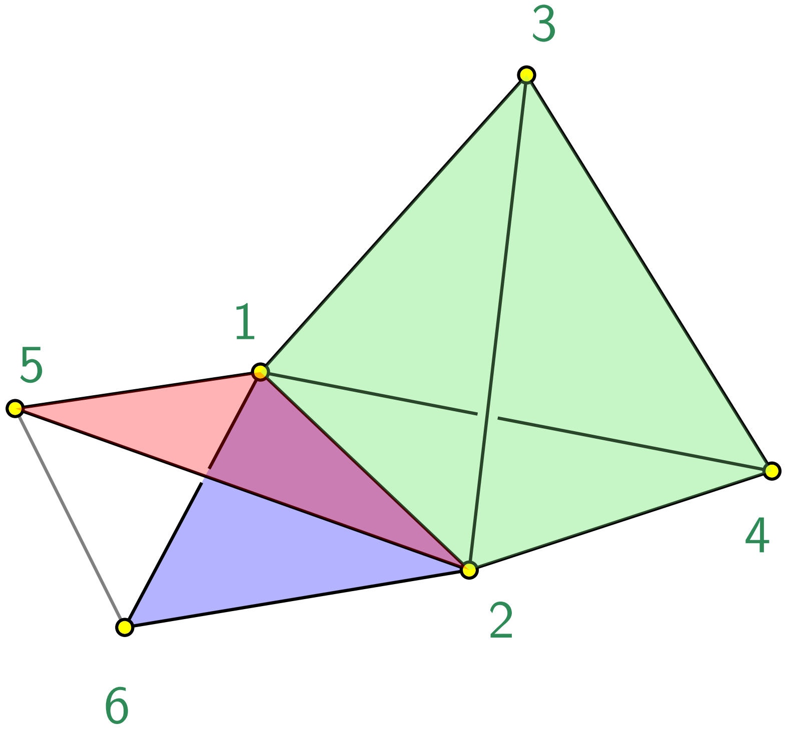
$$\{6\}, \{1,2\}, \{1,3\}, \{1,4\},$$
$$\{1,5\}, \{1,6\}.\{2,3\}, \{2,4\},$$
$$\{2,5\}, \{2,6\}, \{3,4\}, \{5,6\},$$
$$\{1,2,3\}, \{1,2,4\}, \{1,3,4\},$$
$$\{2,3,4\}, \{1,2,5\}, \{1,2,6\},$$
$$\{1,2,3,4\}\}.$$



- ▷ start with any finite abstract simplicial complex K.
- ▷ Let V be the union of all singletons of K.
- ▷ fix some generic map $f : V \to \mathbb{R}^d$ with d large enough
- ▷ $\Gamma_{K,f} = \bigcup_{S \in K} \operatorname{conv}\{f(u) \colon u \in V, \{u\} \subseteq S\}$.

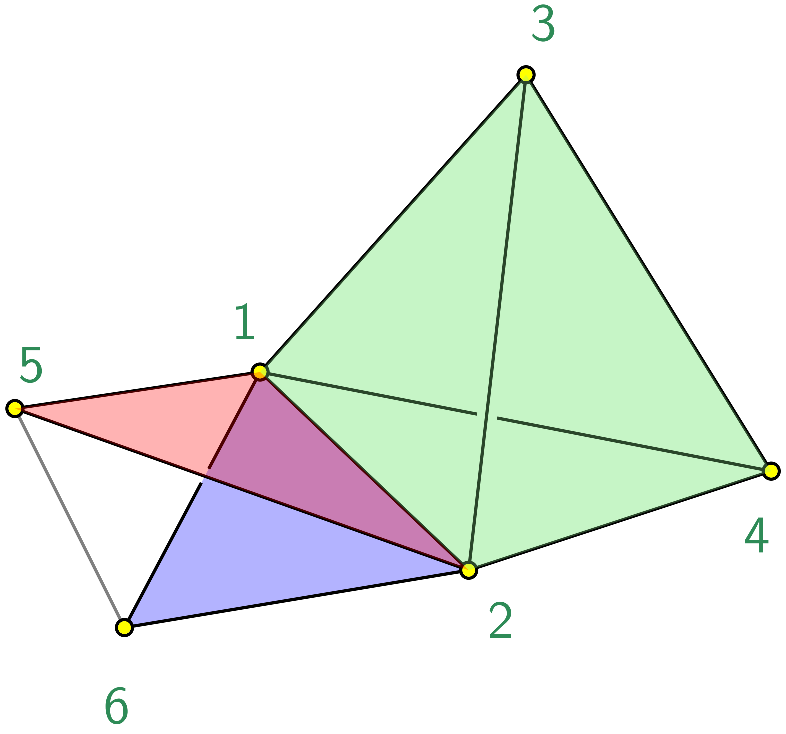
The **topological space** $\Gamma_{K,f}$ is independent of the choice of f.

Abstract simplicial complex

$\overset{\text{def}}{=}$ a collection of sets that is closed under taking subsets.

$\Leftrightarrow$ a "hereditary hypergraph".

$$K = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,6\}.\{2,3\}, \{2,4\}, \{2,5\}, \{2,6\}, \{3,4\}, \{5,6\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,5\}, \{1,2,6\}, \{1,2,3,4\}\}.$$



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The **topological space** $\Gamma_{K,f}$ is independent of the choice of f.

$\rightsquigarrow$ **geometric realization** $|K|$ of K.

Abstract simplicial complex

$\overset{\text{def}}{=}$ a collection of sets that is closed under taking subsets.

$\Leftrightarrow$ a "hereditary hypergraph".

$$K = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}\}$$

$$K' = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

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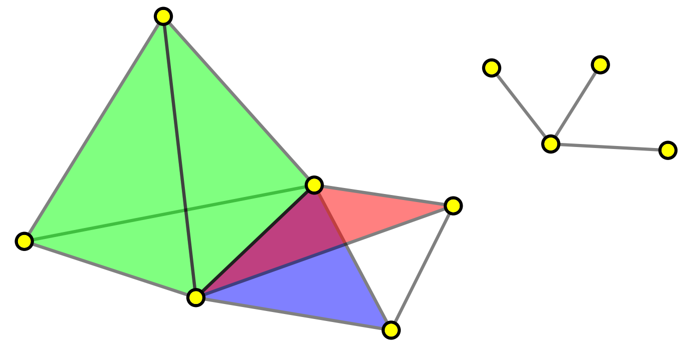
Geometric simplicial complex

$\overset{\text{def}}{=}$ a collection of geometric simplices in $\mathbb{R}^d$ closed by taking faces, and such that any two intersect in a common face.



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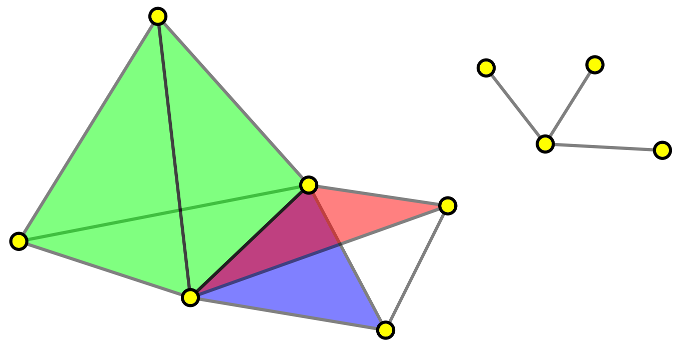
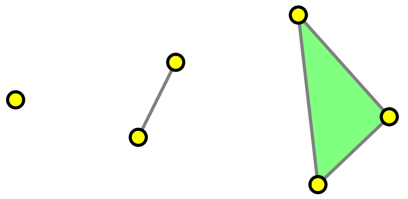


Geometric simplex $\stackrel{\text{def}}{=}$ convex hull of $k \leq d+1$ affinely independent points in $\mathbb{R}^d$



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map vertices to points and take convex hulls

Geometric simplicial complex

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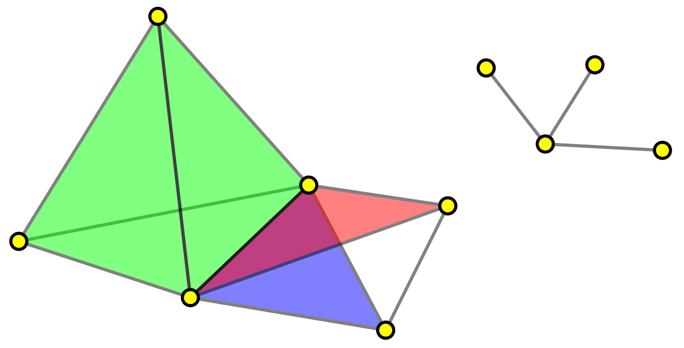

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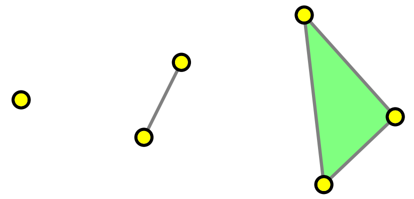

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Collect sets of vertices forming a face

Geometric simplicial complex

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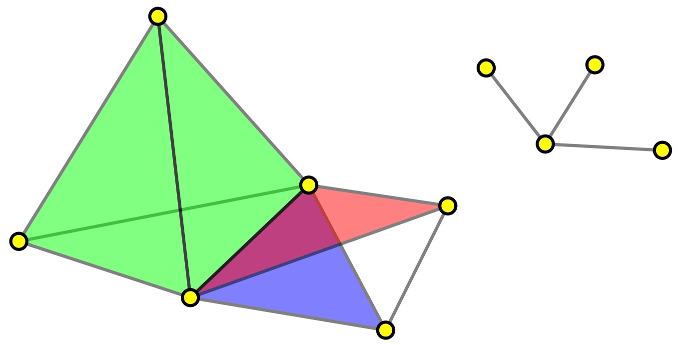

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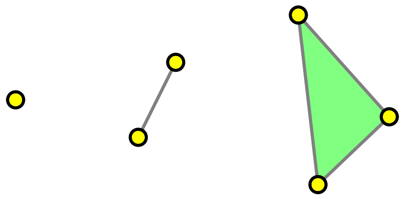
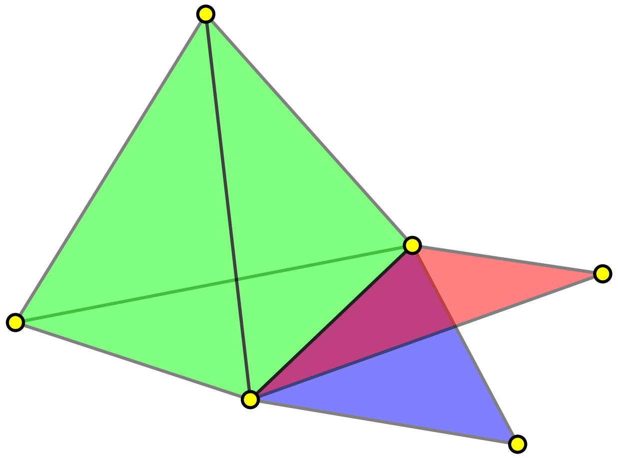


Lemma. Let K be an abstract simplicial complex. If the geometric realization of K is contractible, then

$$\chi(K) \stackrel{\text{def}}{=} \sum_{\sigma \in K} (-1)^{\dim \sigma} = 1.$$

▷ $\dim \sigma \stackrel{\text{def}}{=} |\sigma| - 1.$



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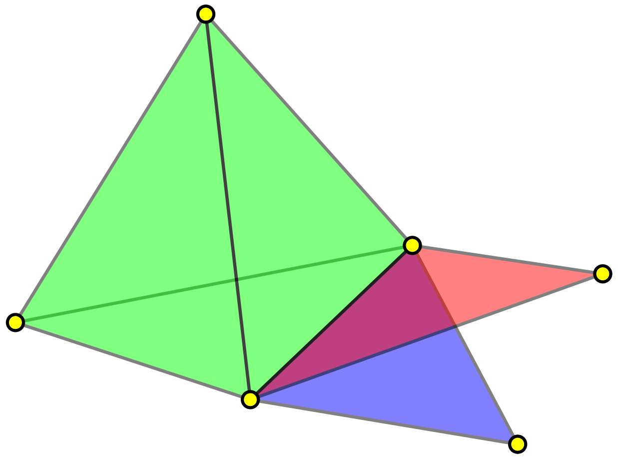


- $\dim \sigma \overset{\text{def}}{=} |\sigma| - 1.$

Ingredients...

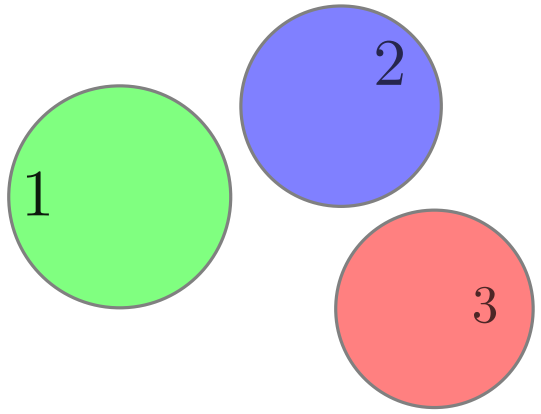
- simplicial and singular homology, Betti numbers $\beta_i(K)$
- $\chi(K) = \beta_0(K) - \beta_1(K) + \beta_2(K) - \dots$
- homology is invariant under homotopy.

#5. nerve complexes and the nerve theorem

Nerve $N(F) \simeq$ intersection **hypergraph** of F

$$N(F) = \{G : G \subseteq F \text{ and } \cap_{A \in G} A \neq \emptyset\}.$$

F



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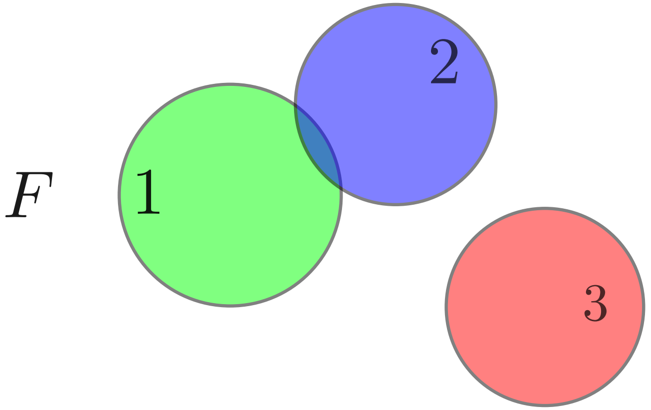

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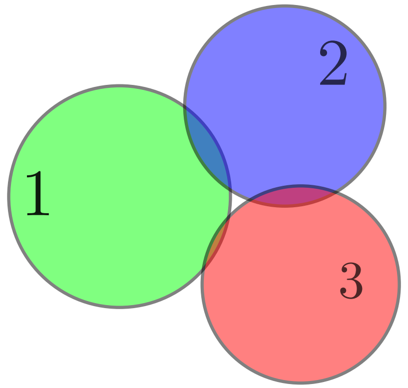

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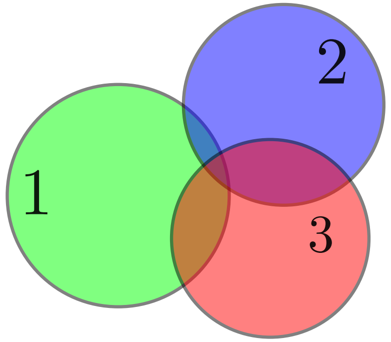

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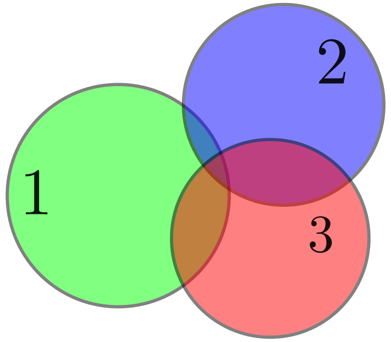

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▷ Nerves are **abstract simplicial complexes**.

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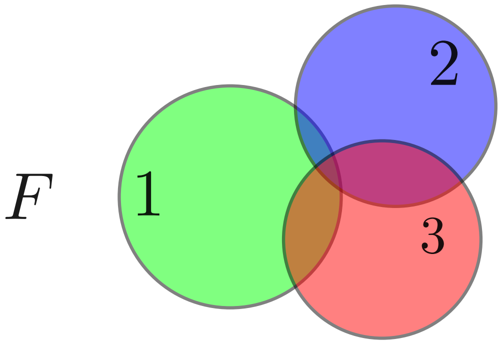

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▷ Nerves are **abstract simplicial complexes**.

Theorem. If all subfamilies of F have empty or **contractible** intersections then $|N(F)|$ and $\cup F$ are homotopy equivalent.

[Borsuk'48] [Leray]

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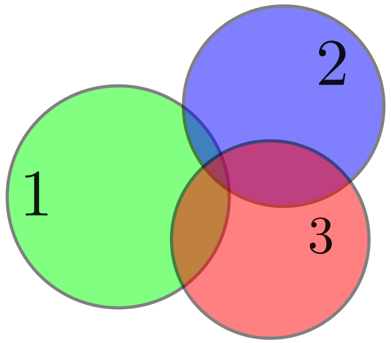

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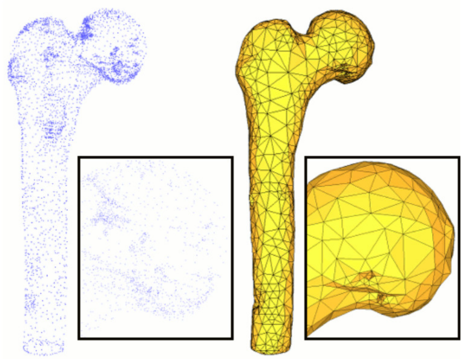
[Borsuk'48] [Leray]

▷ Reconstruction methods.



https://doc.cgal.org/latest/Manual/tuto_reconstruction.html

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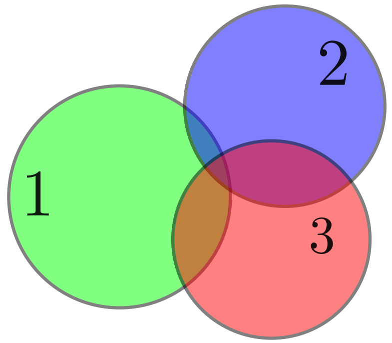

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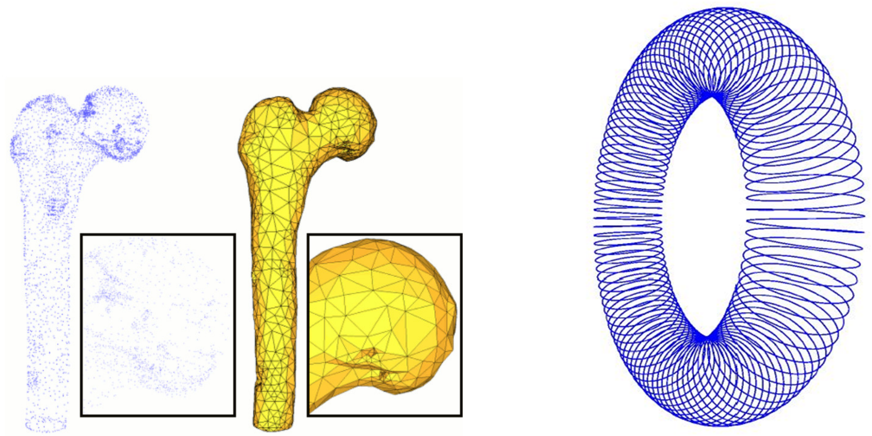
[Borsuk'48] [Leray]

▷ Reconstruction methods.

▷ Topological data analysis.



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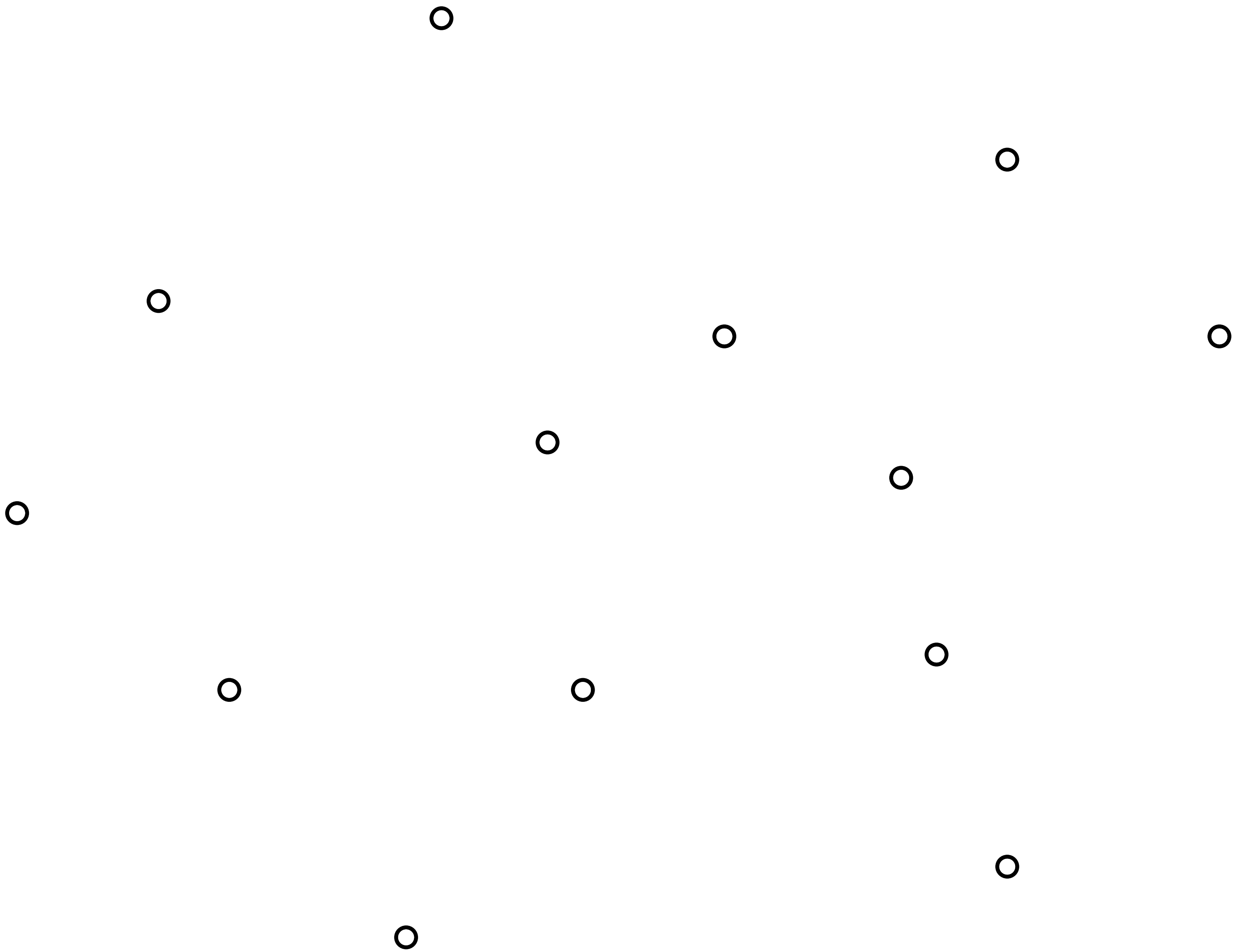
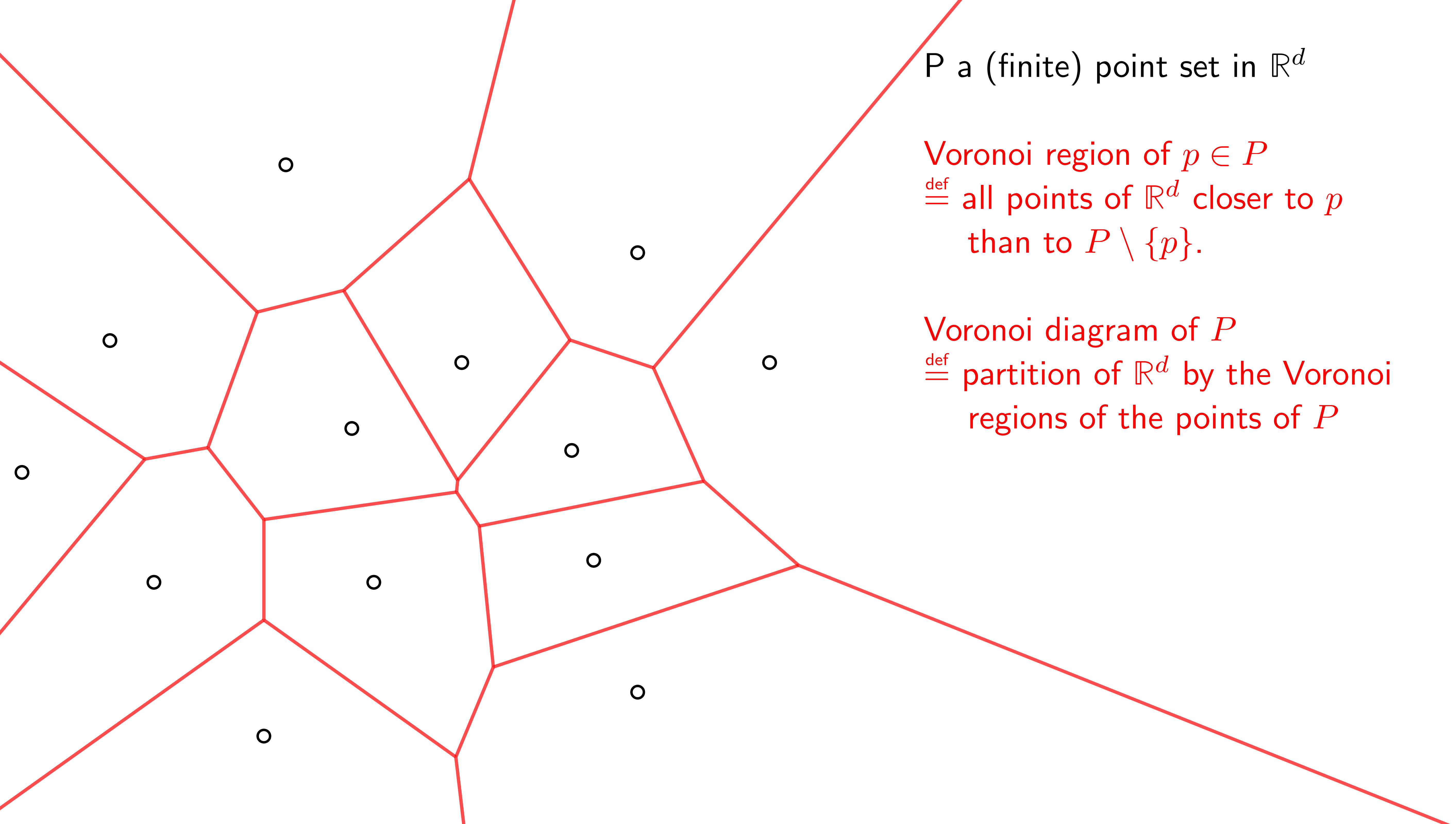
#6. Delaunay triangulations and Voronoi diagrams

P a (finite) point set in $\mathbb{R}^d$

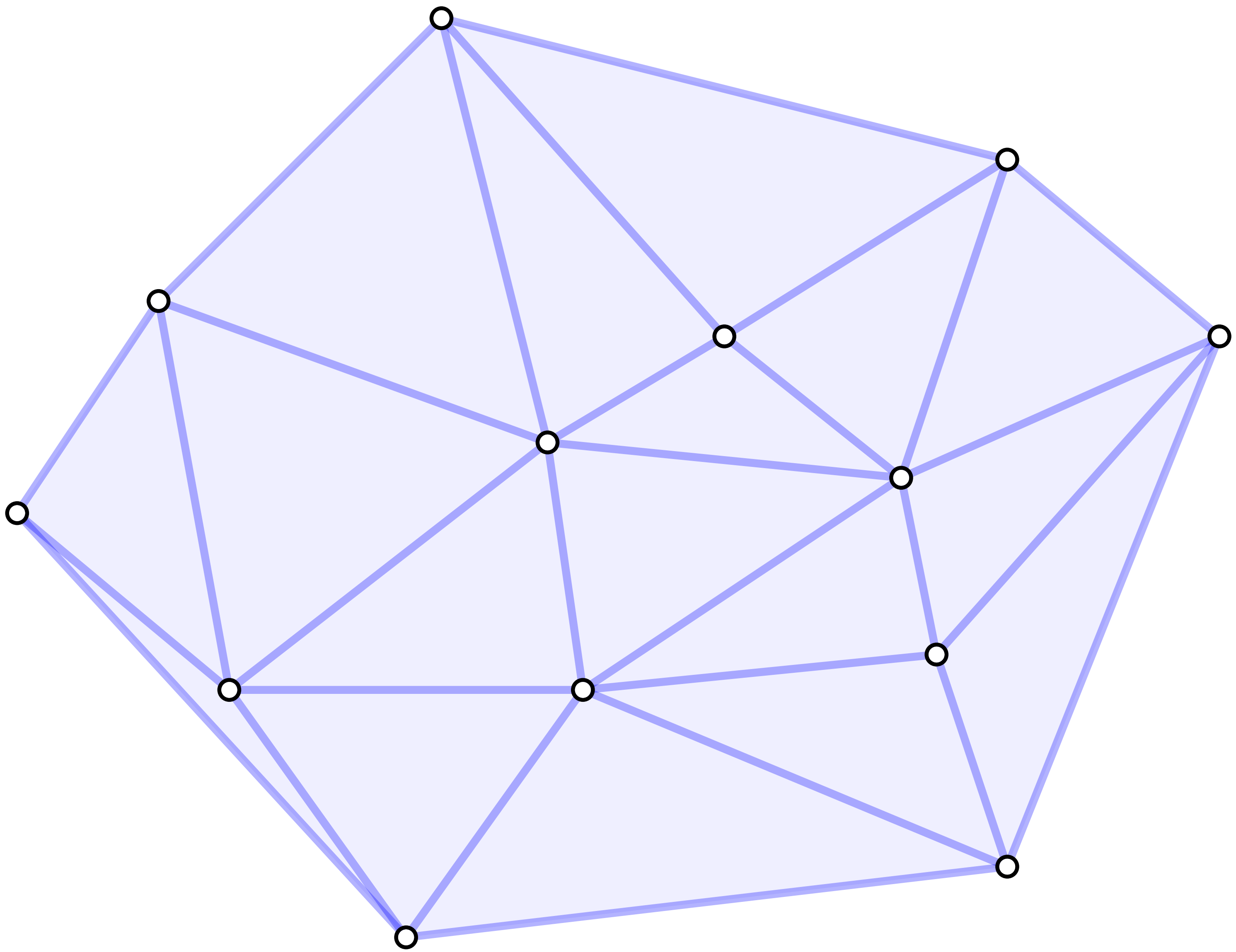




P a (finite) point set in $\mathbb{R}^d$

Voronoi region of $p \in P$
$\stackrel{\text{def}}{=}$ all points of $\mathbb{R}^d$ closer to p
than to $P \setminus \{p\}$.

Voronoi diagram of P
$\stackrel{\text{def}}{=}$ partition of $\mathbb{R}^d$ by the Voronoi
regions of the points of P



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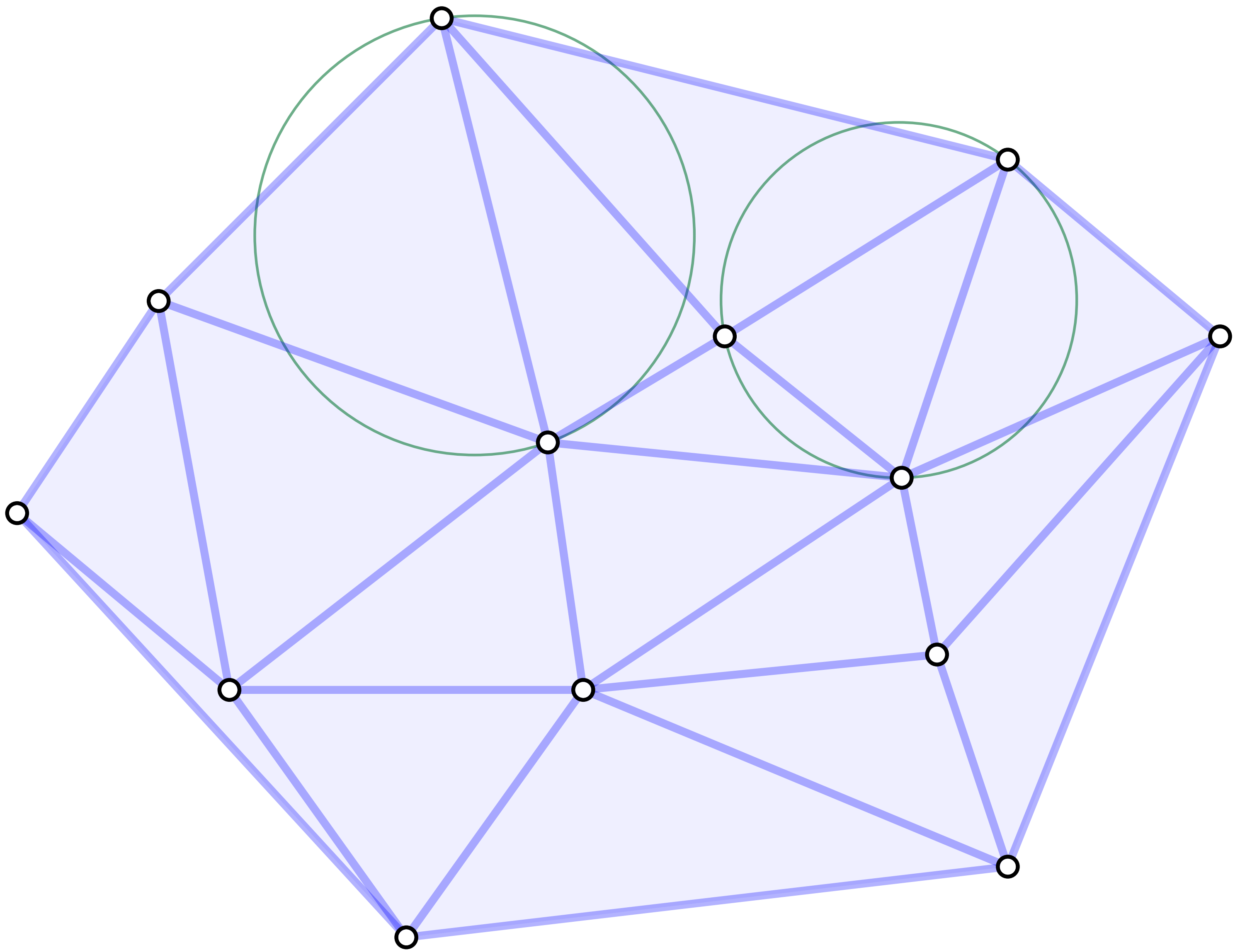
A simplex over P is Delaunay
$\Leftrightarrow$ it is contained in a sphere
enclosing no other point of P.

Delaunay triangulation of P
$\stackrel{\text{def}}{=}$ triangulation of $\operatorname{conv} P$ by the
Delaunay simplices over P.



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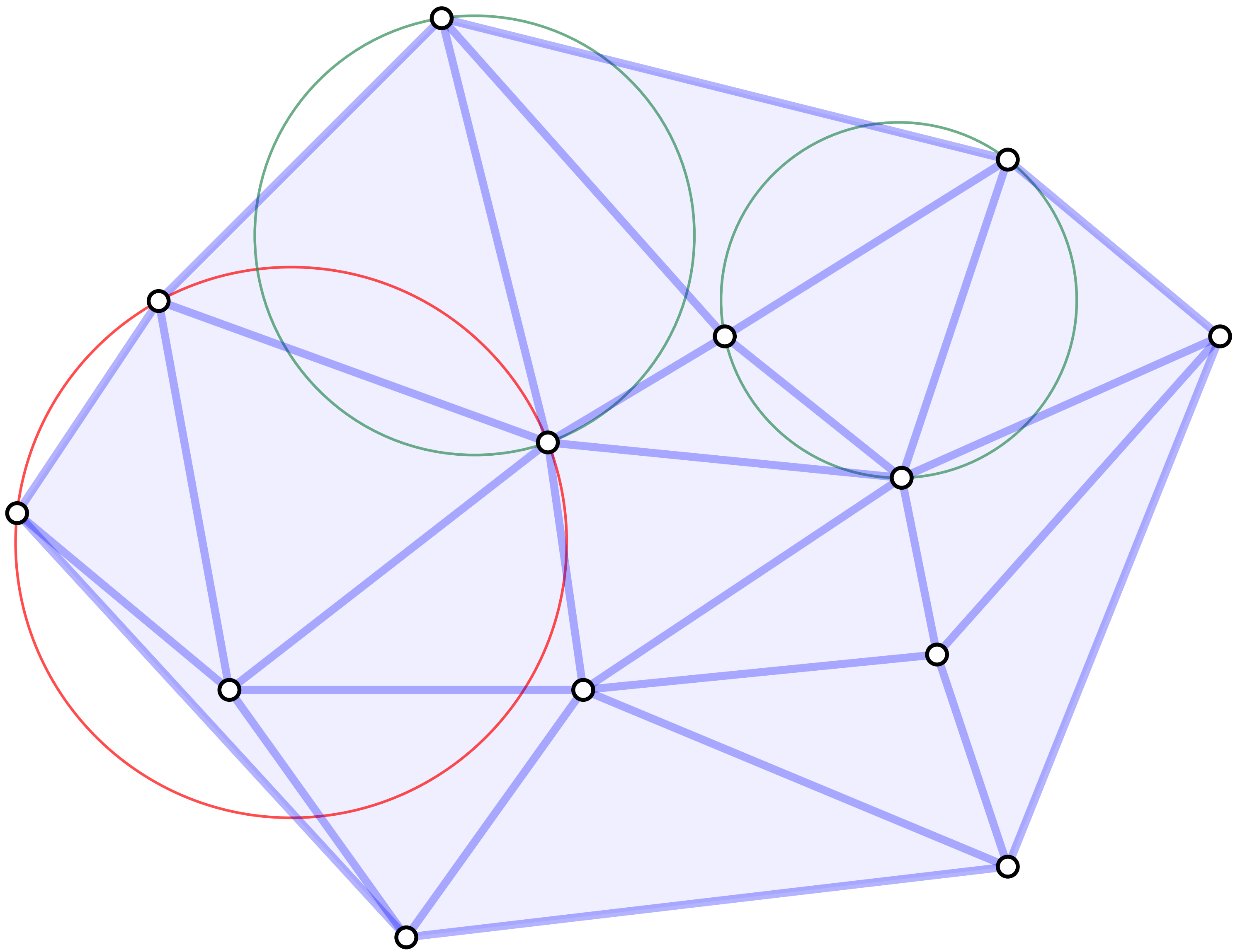

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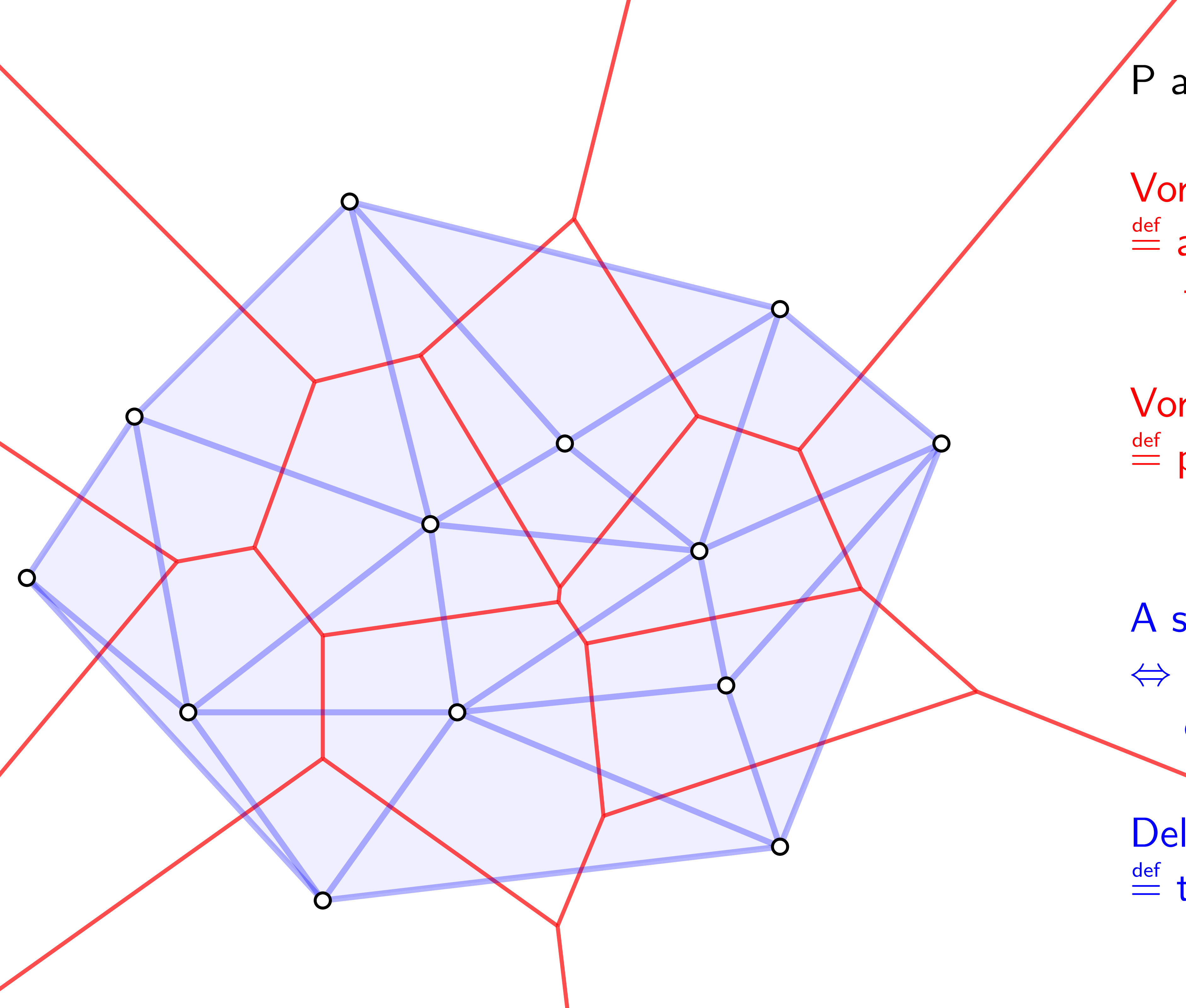

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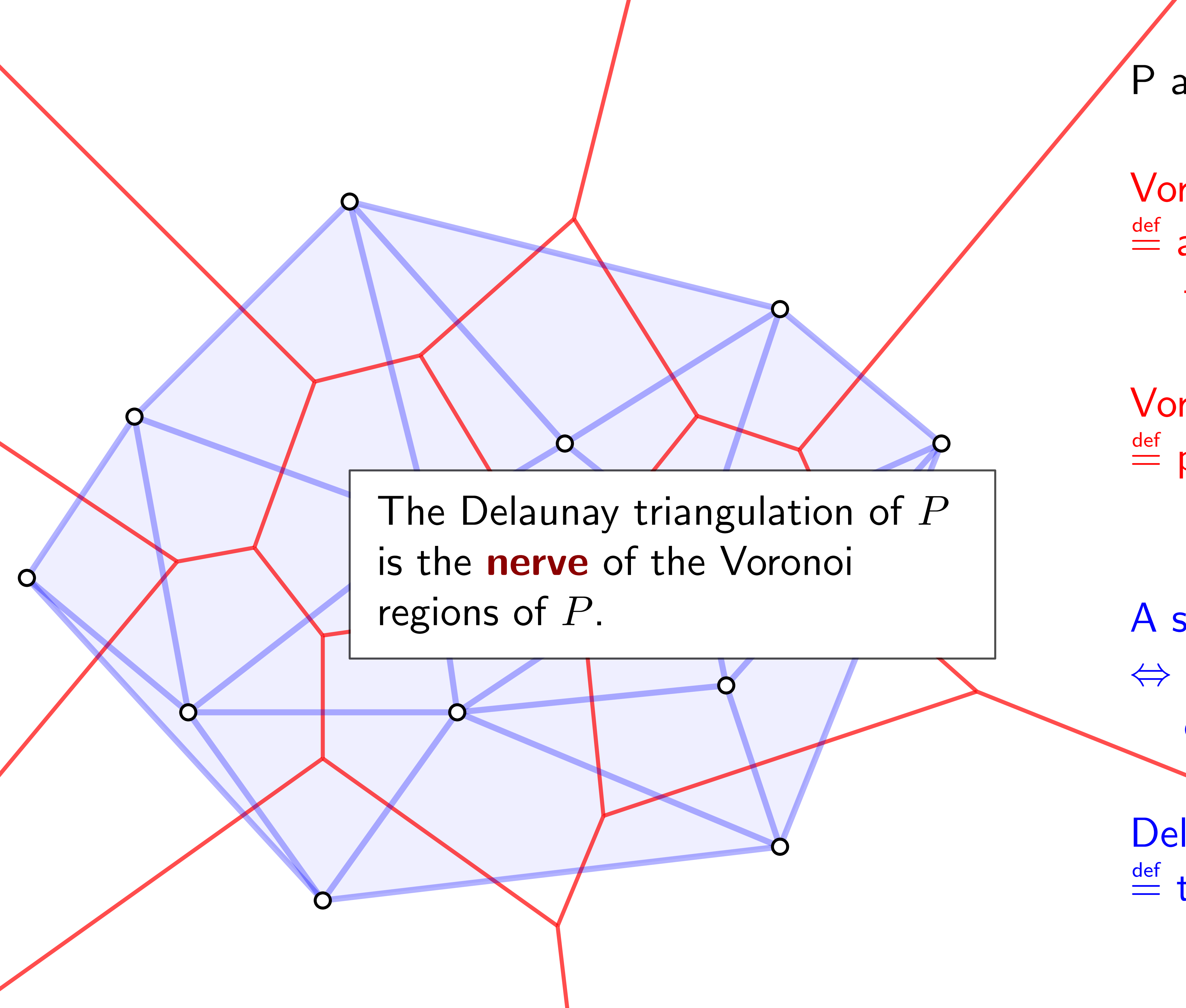

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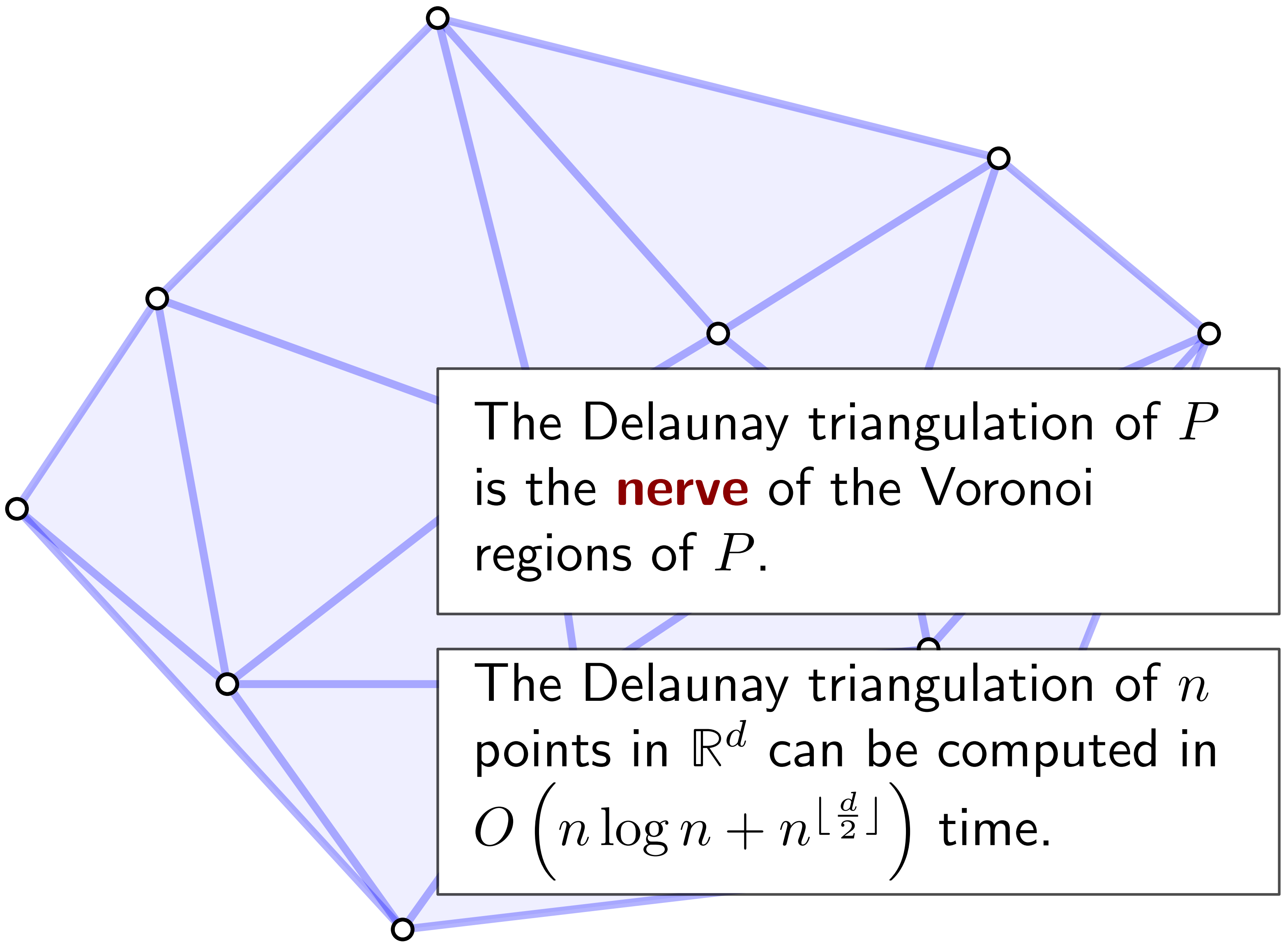

The Delaunay triangulation of P is the **nerve** of the Voronoi regions of P.

The Delaunay triangulation of n points in $\mathbb{R}^d$ can be computed in $O\left(n \log n + n^{\lfloor \frac{d}{2} \rfloor}\right)$ time.

P a (finite) point set in $\mathbb{R}^d$

Voronoi region of $p \in P$
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A simplex over P is Delaunay
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Delaunay triangulation of P
$\stackrel{\text{def}}{=}$ triangulation of $\operatorname{conv} P$ by the Delaunay simplices over P.

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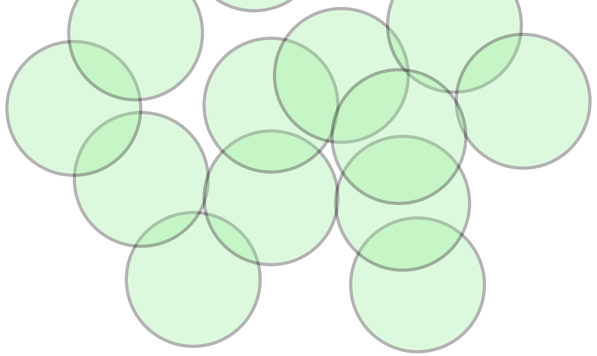
#7. proof of the formula for balls

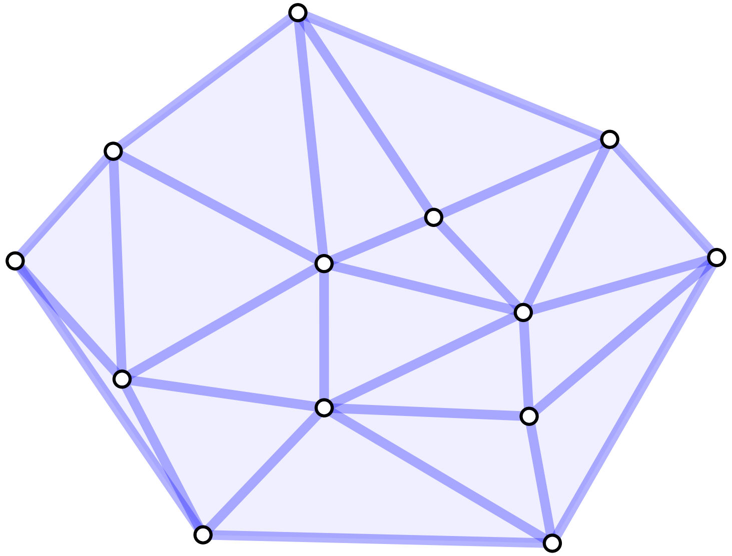
Theorem. [Naiman-Wynn'92]
Let $F = \{b_1, b_2, \ldots, b_n\}$ be a family of equal radius balls in $\mathbb{R}^d$. Letting T denote the Delaunay triangulation of the balls' centers, we have

$$\mathbb{1}_{\bigcup_{i=1}^{n} b_i} = \sum_{\sigma \in T} (-1)^{\dim \sigma} \mathbb{1}_{\bigcap_{i \in \sigma} b_i}$$

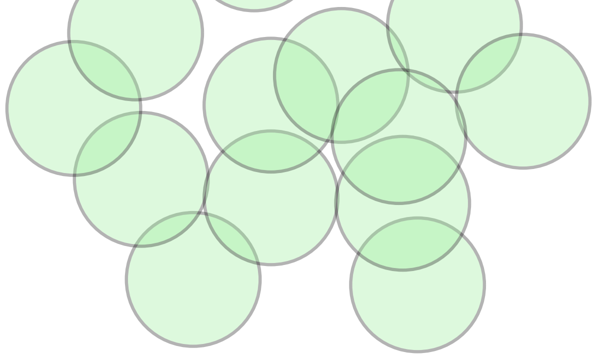
Fast-forward...



Delaunay triangulation induces a correct inclusion-exclusion formula

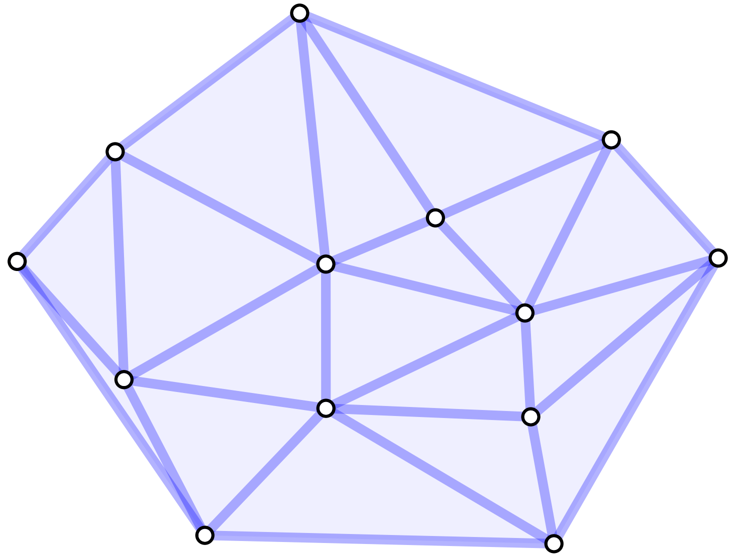


Fast-forward...

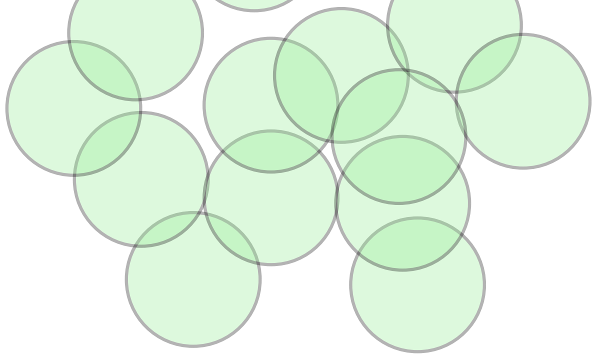


Delaunay triangulation induces a correct inclusion-exclusion formula

(combinatorics) $\Leftrightarrow$ each in a family of subcomplexes of the DT has Euler characteristic 1

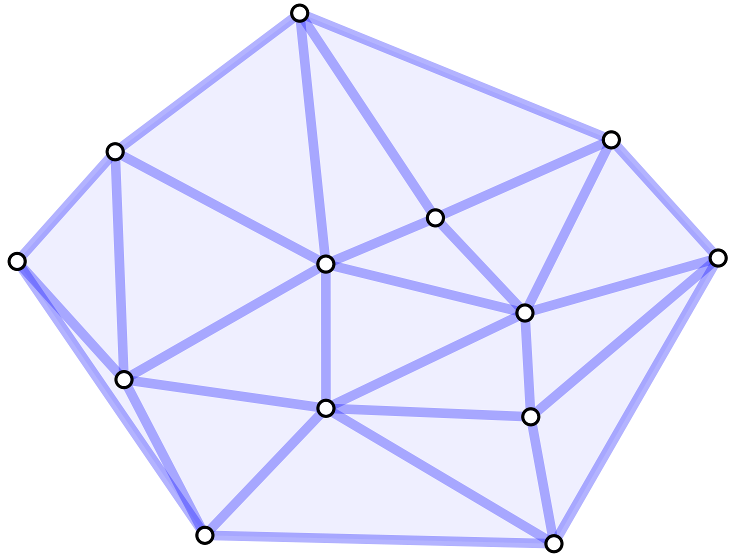


Fast-forward...

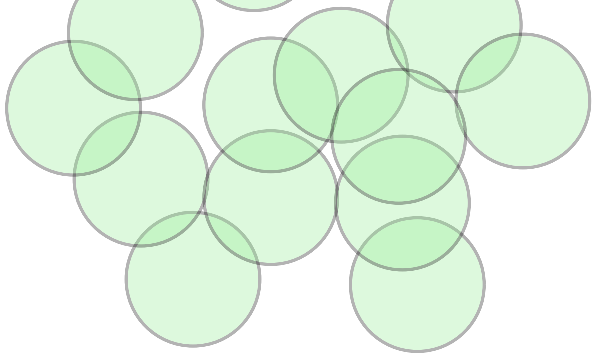


Delaunay triangulation induces a correct inclusion-exclusion formula

(combinatorics)	$\Leftrightarrow$	each in a family of subcomplexes of the DT has Euler characteristic 1
(topology)	$\Leftarrow$	each of these subcomplexes has a contractible geometric realization



Fast-forward...



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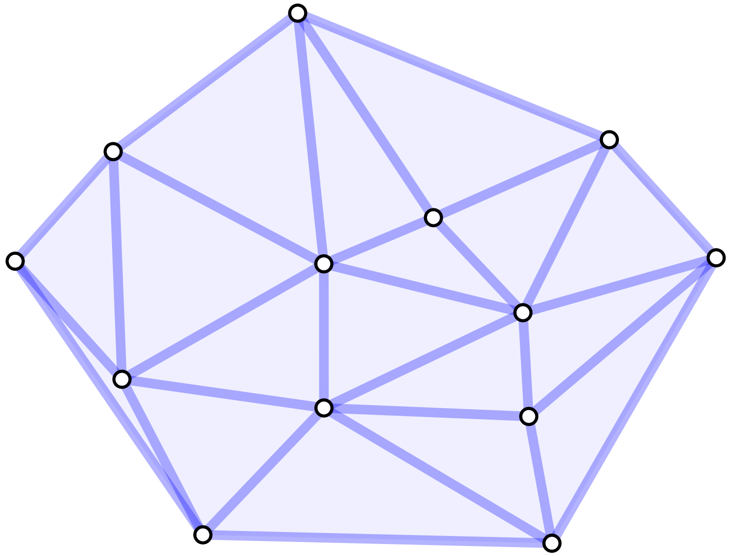
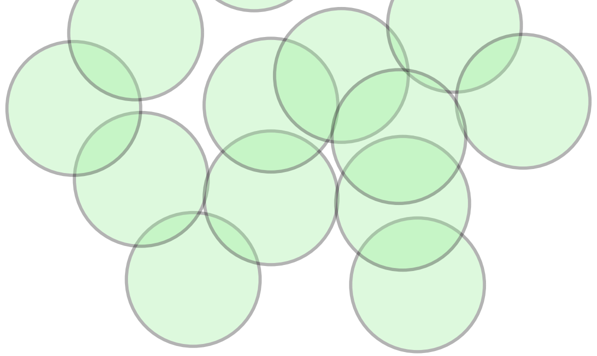
(topology) $\Leftarrow$ each of these subcomplexes has a contractible geometric realization

(nerve theorem) $\Leftrightarrow$ some unions of Voronoi regions are contractible



Fast-forward...



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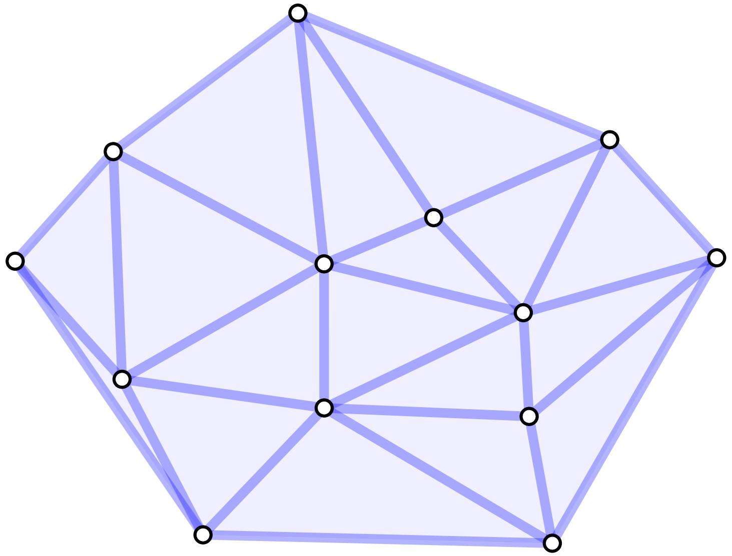

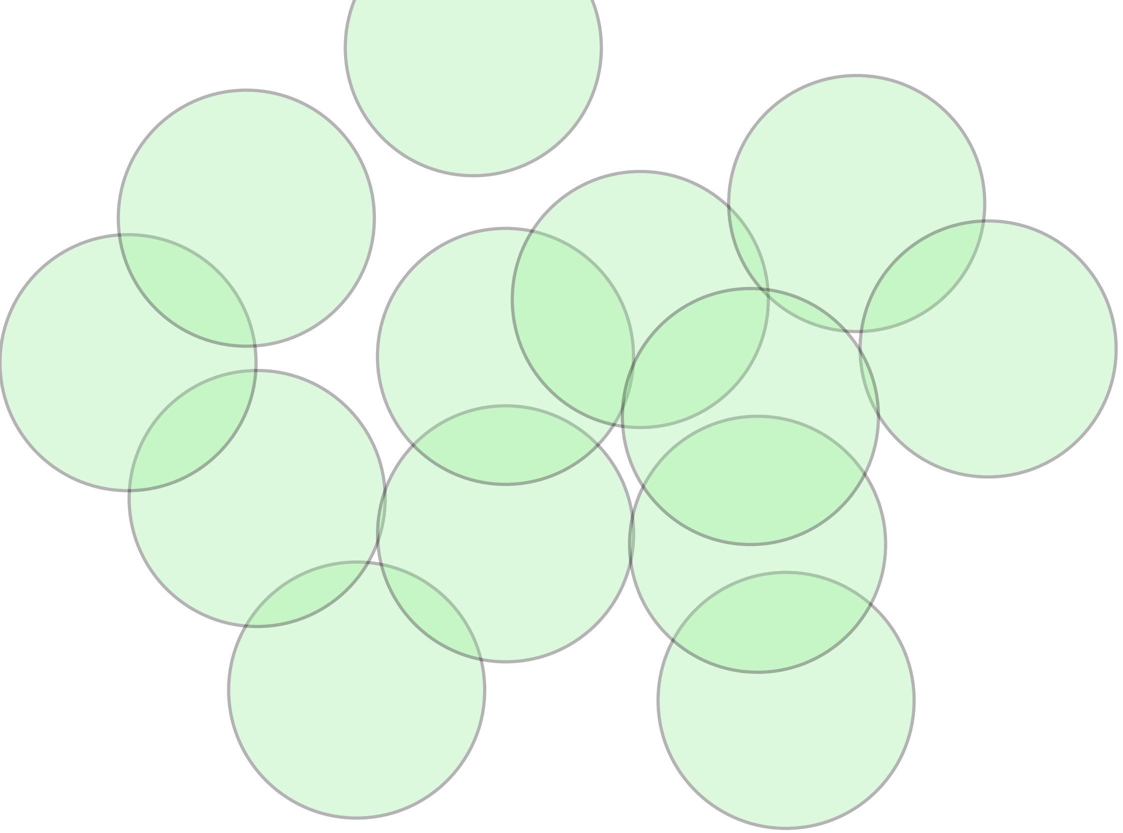

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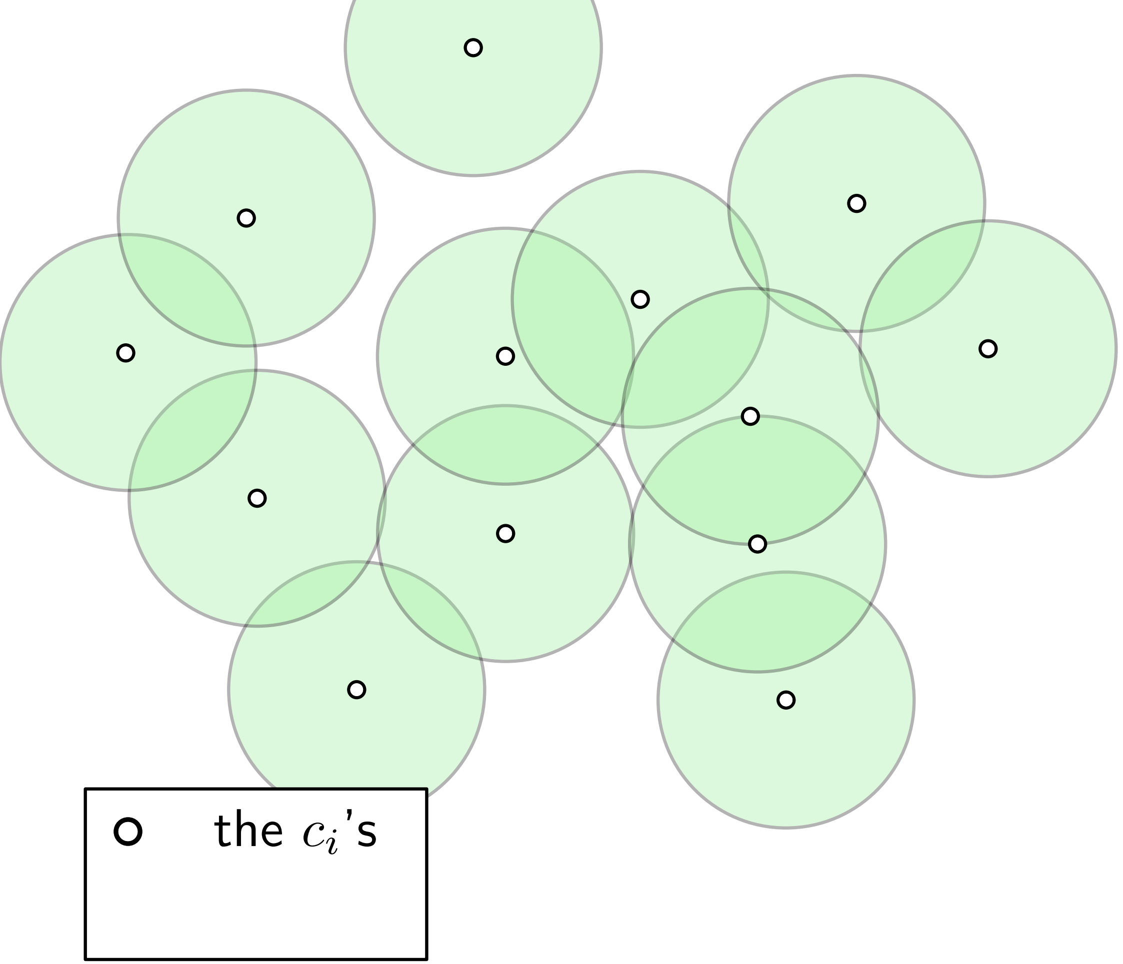

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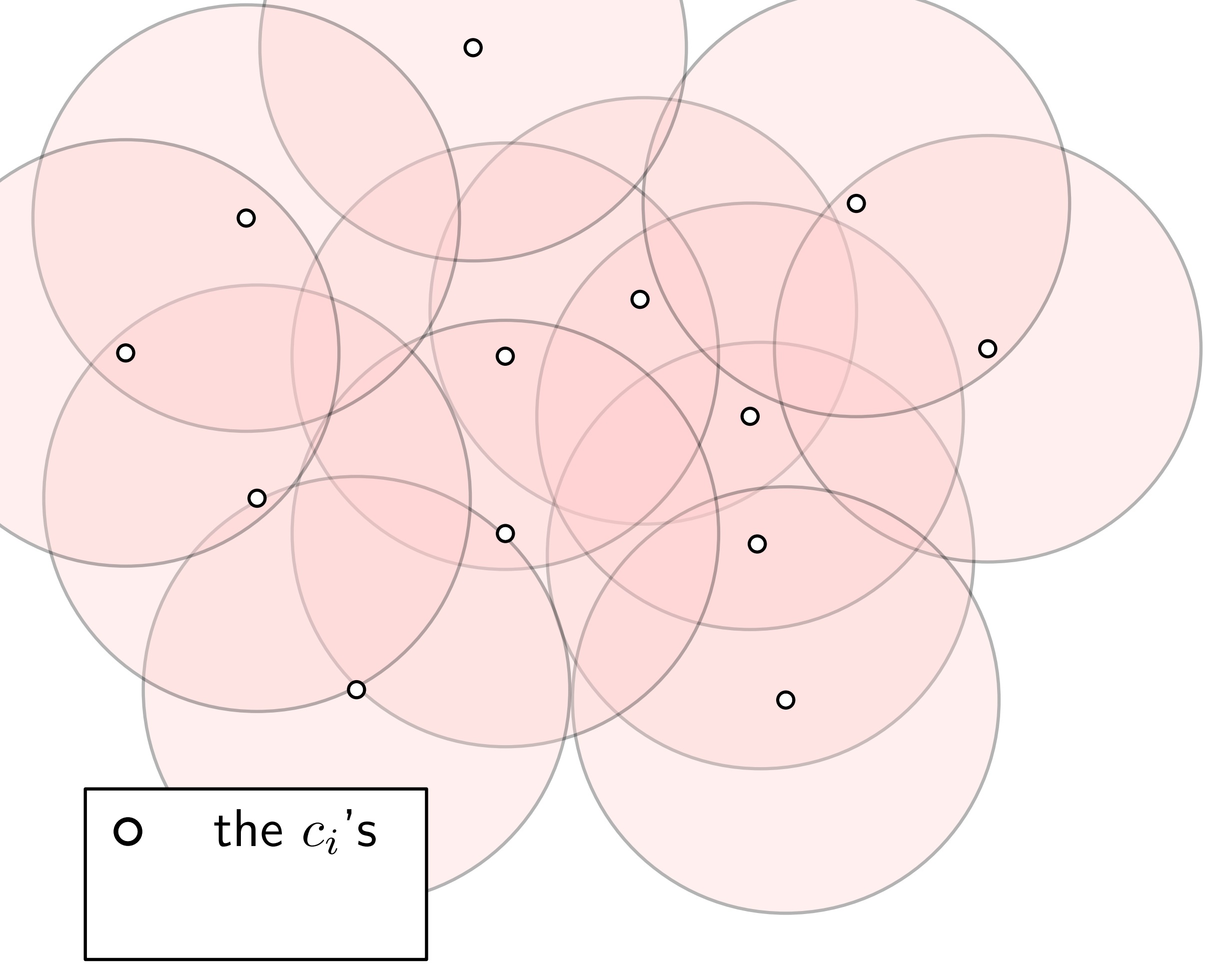

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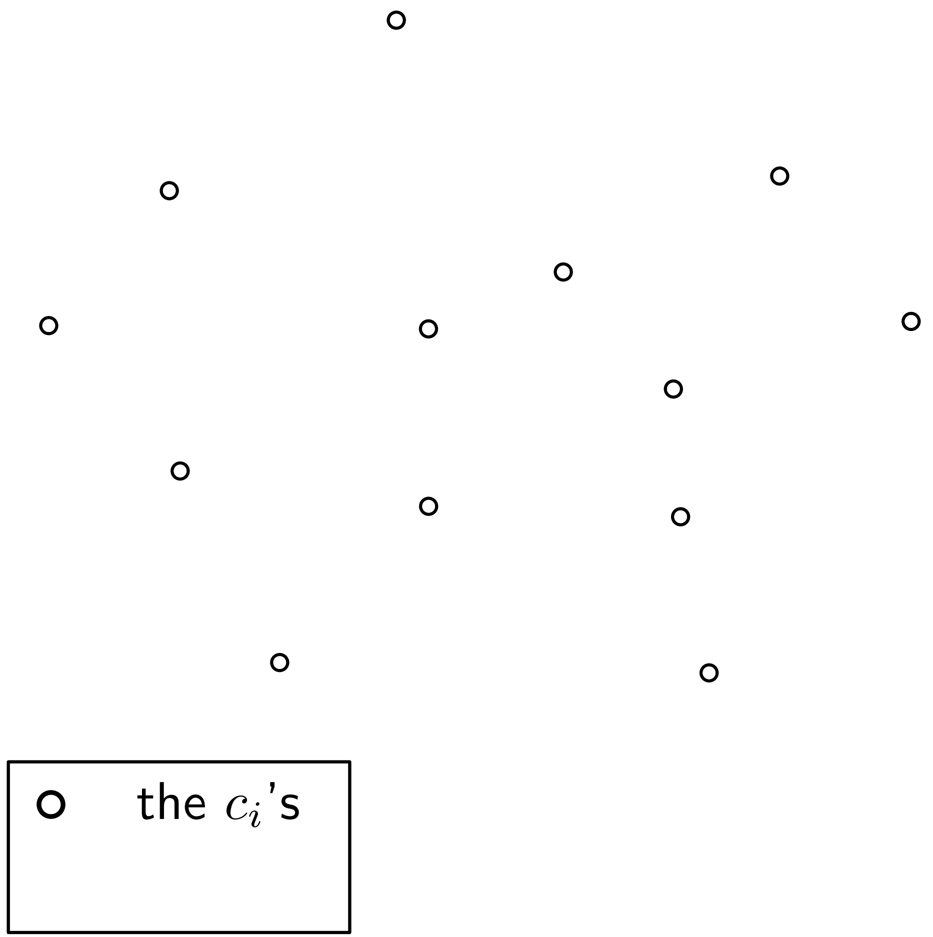

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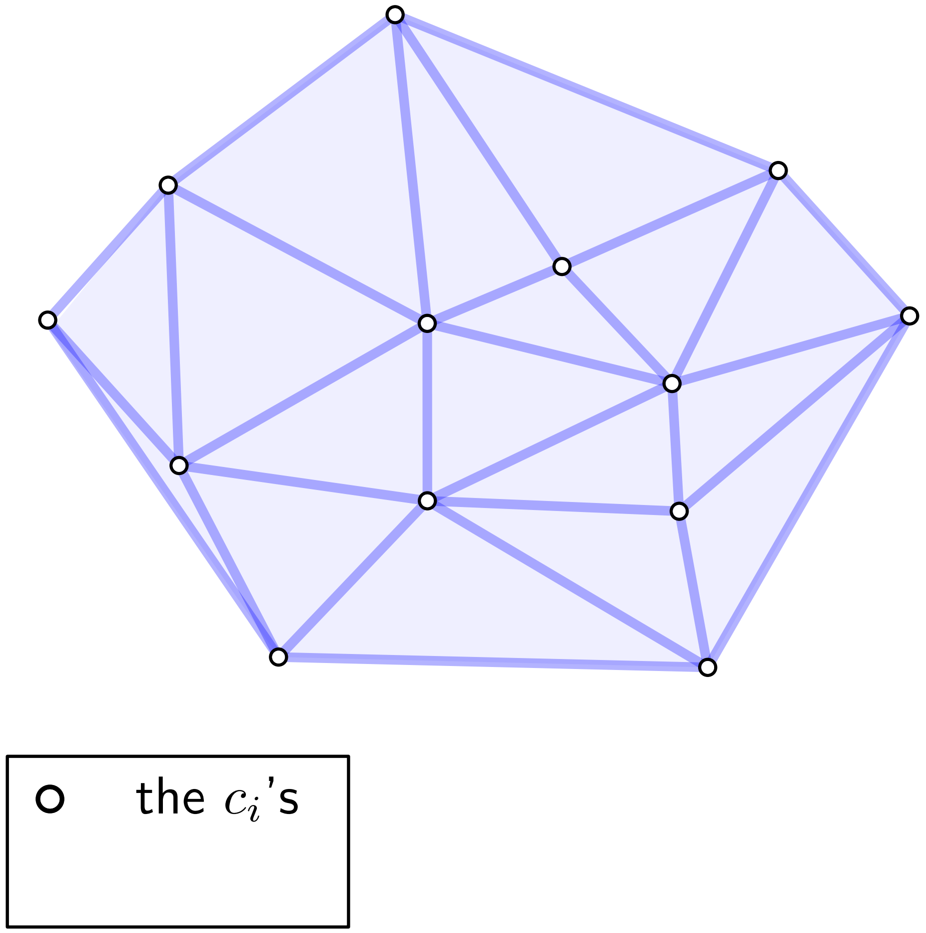

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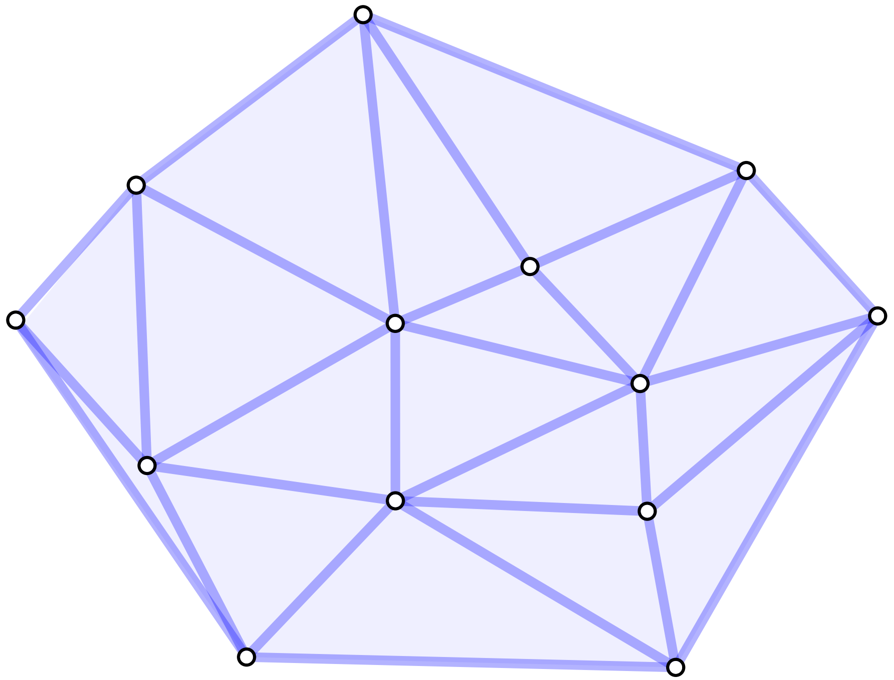

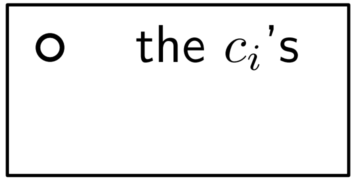

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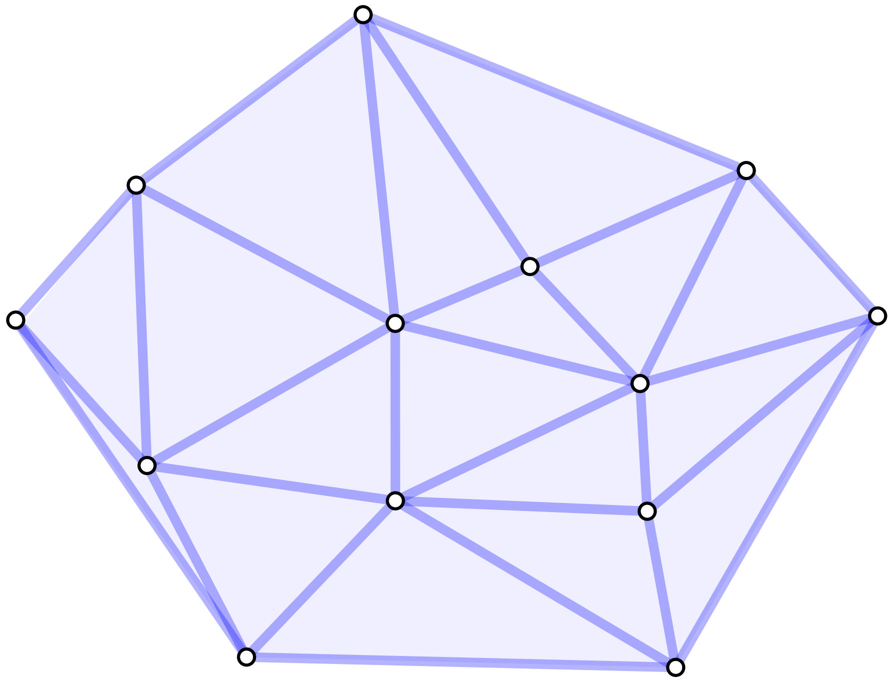

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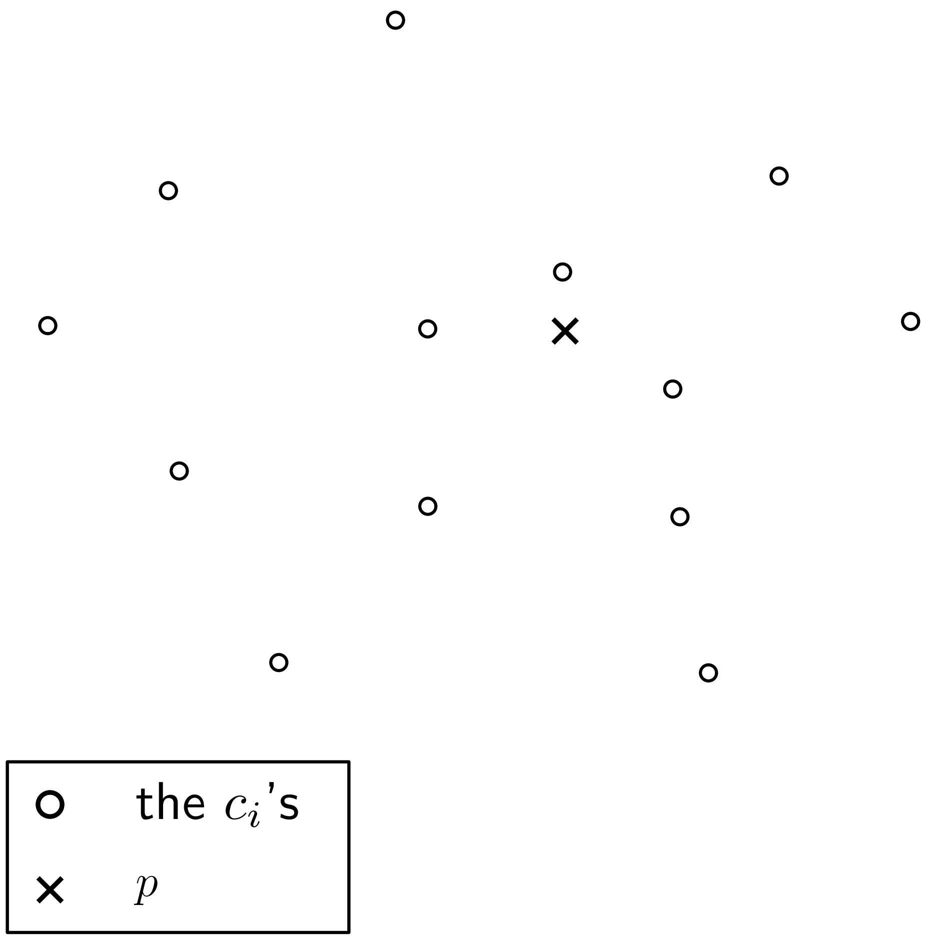

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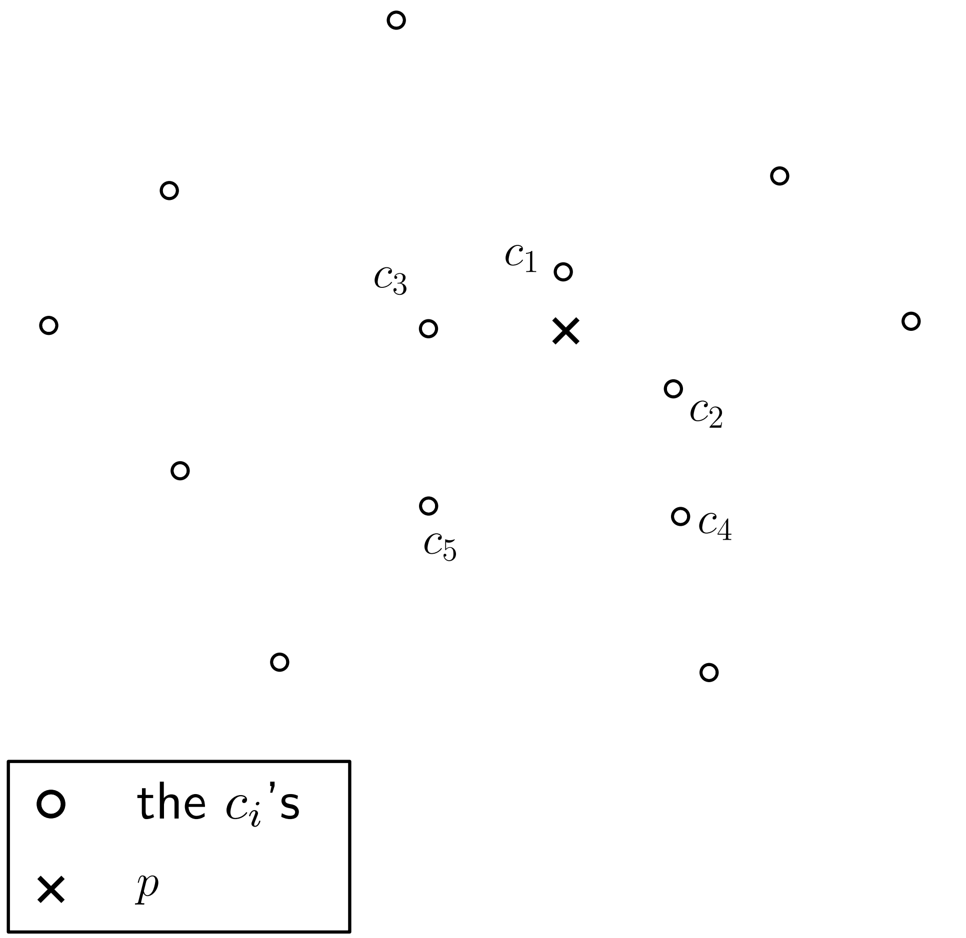

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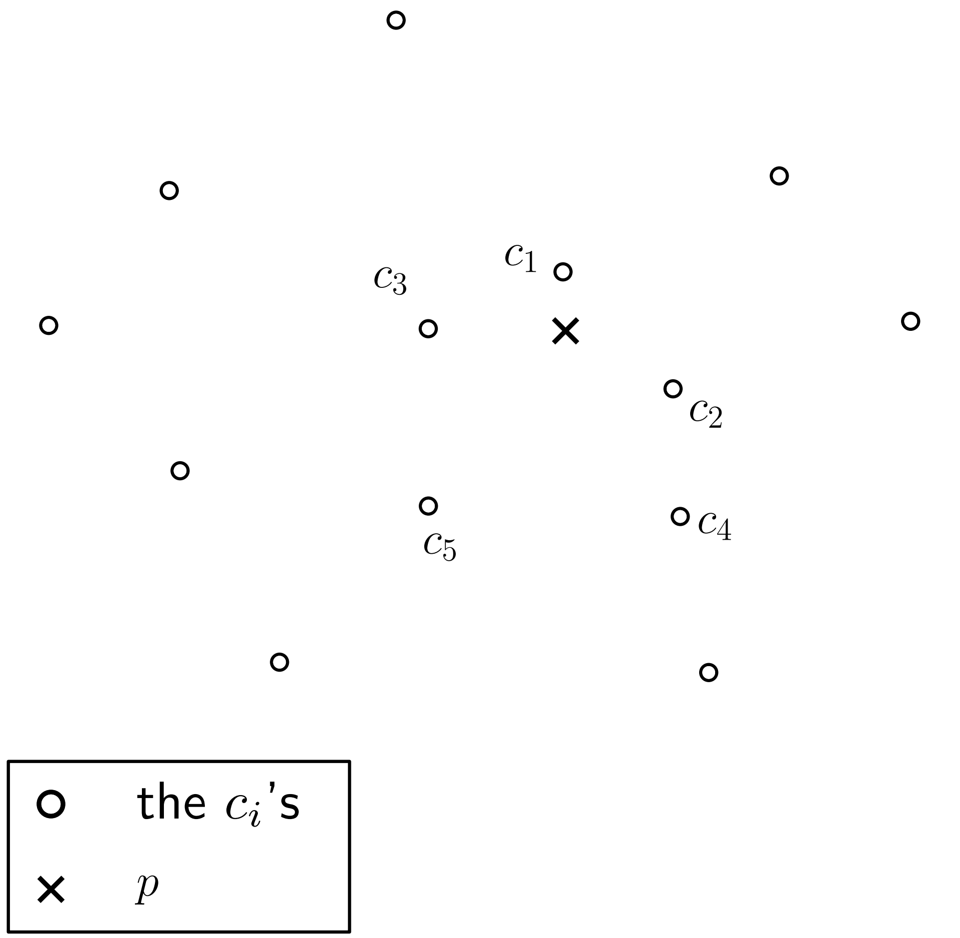

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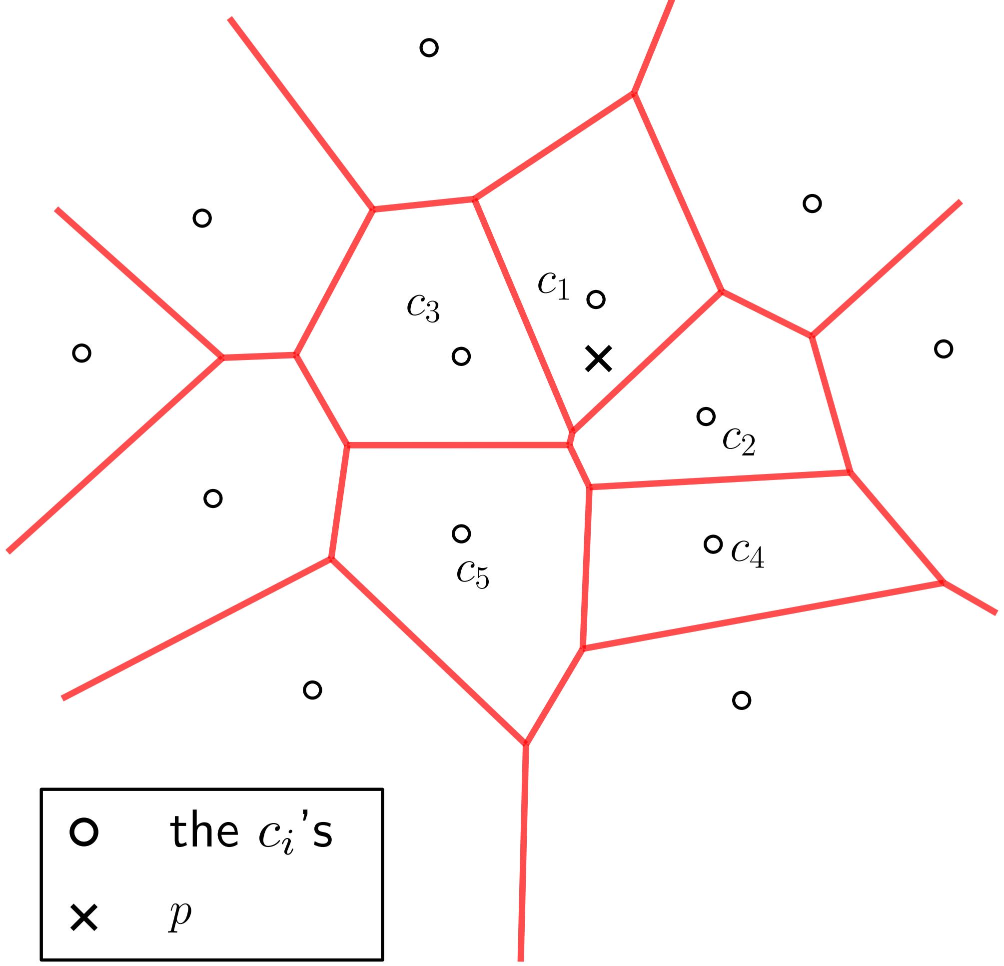

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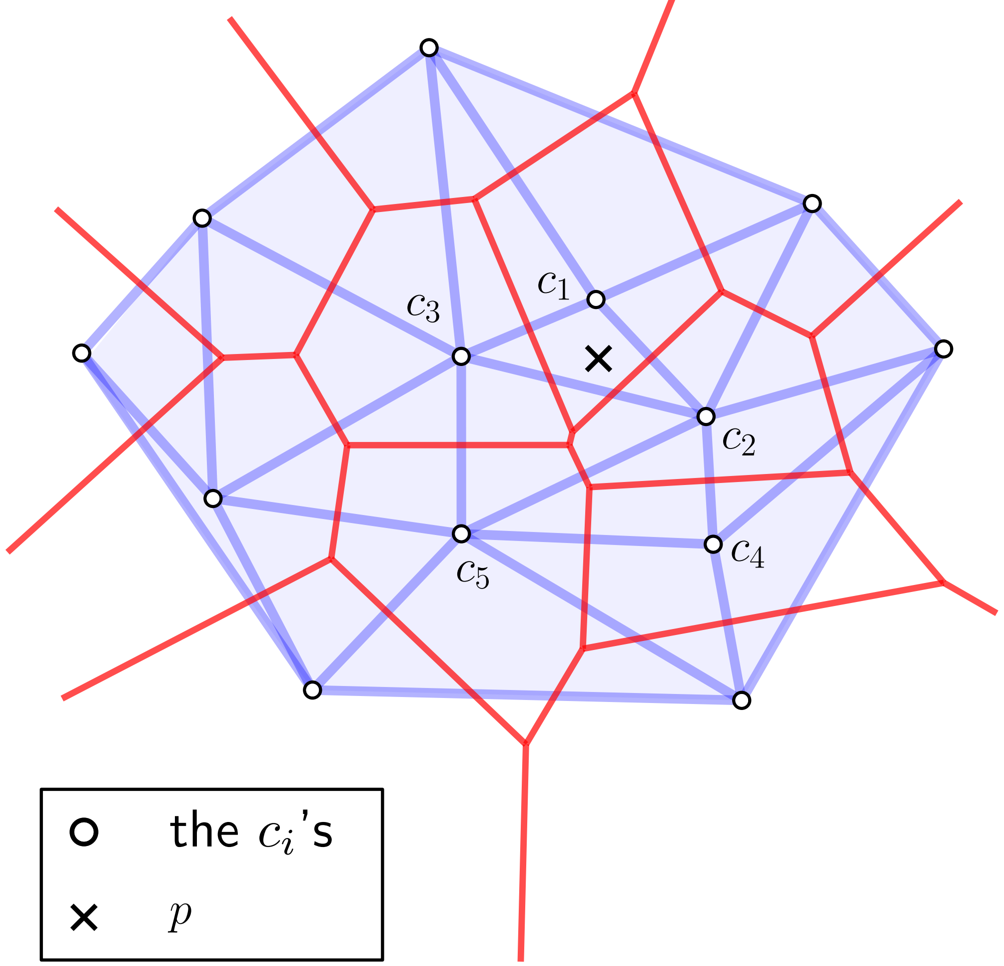

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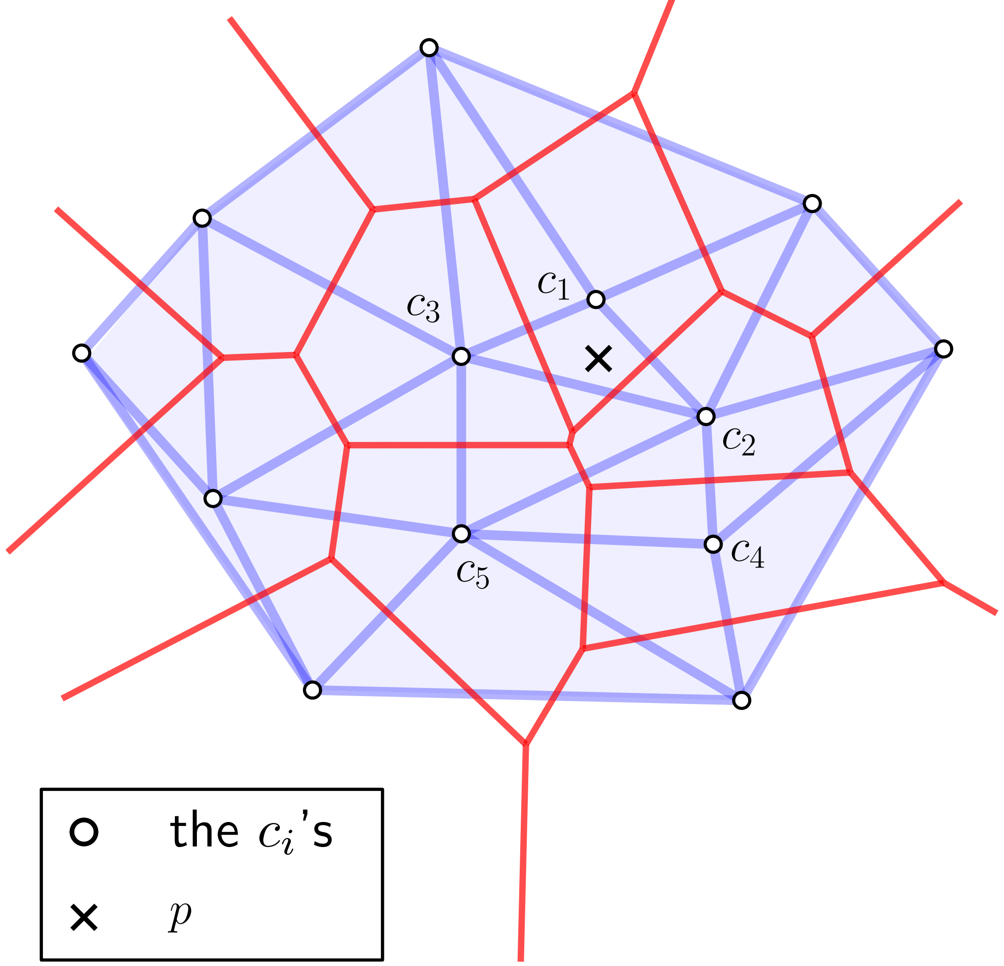

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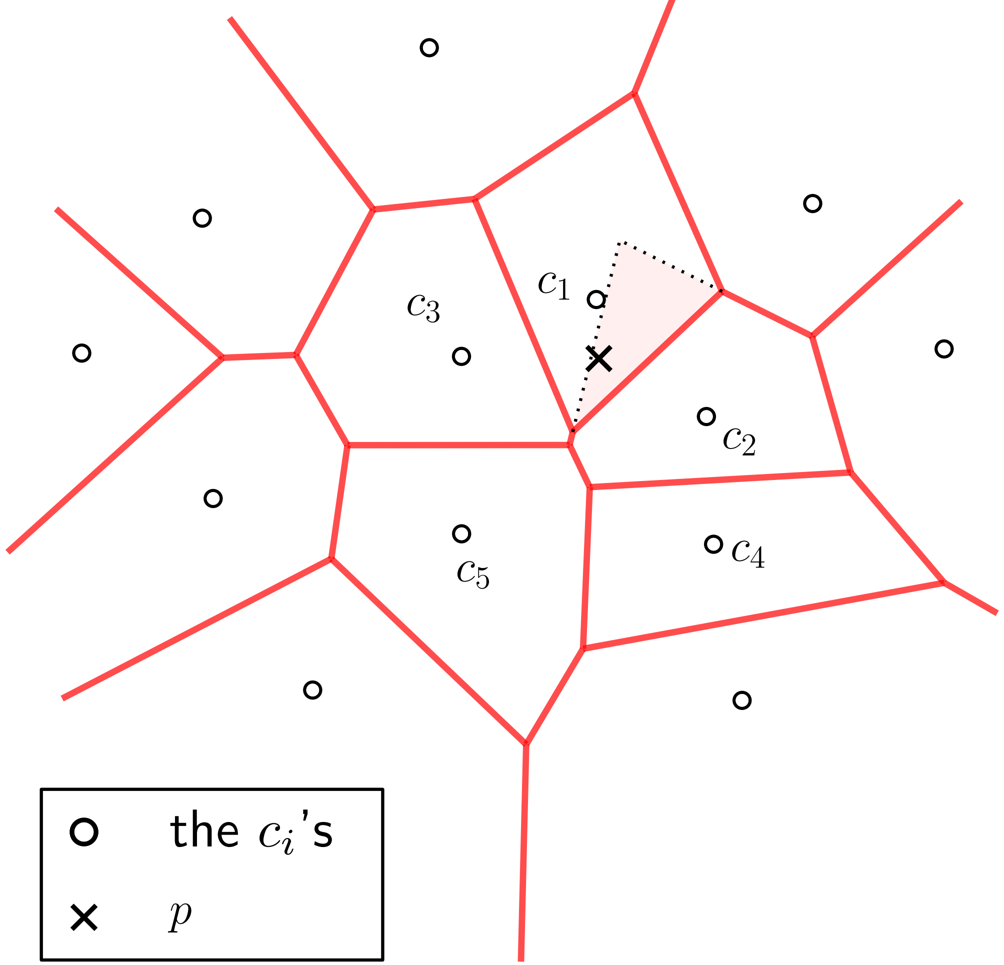

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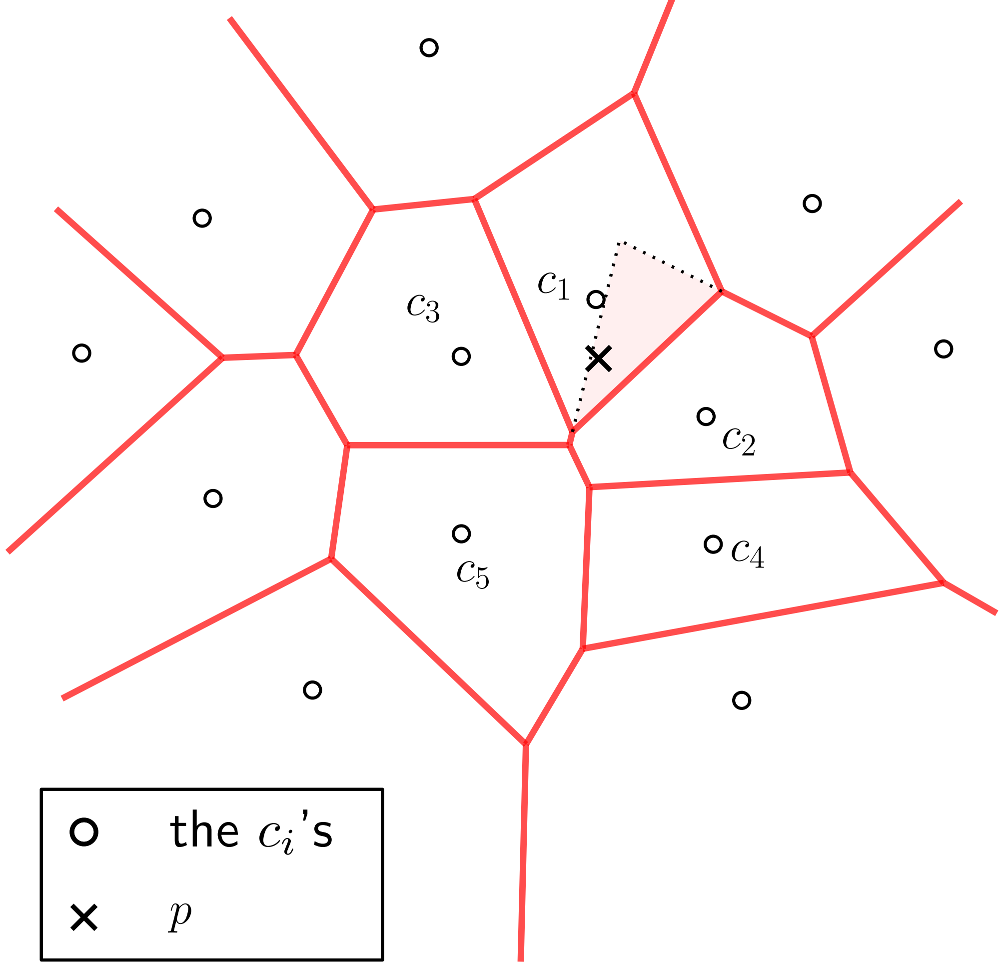

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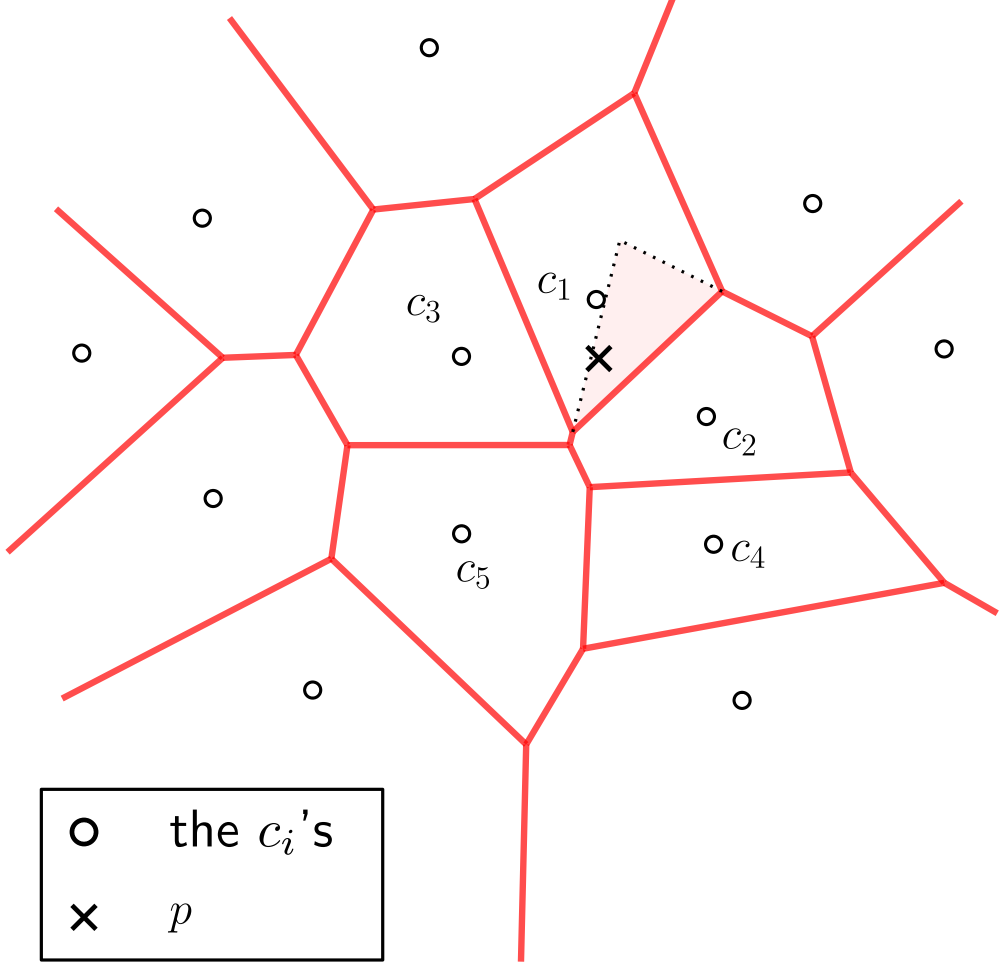

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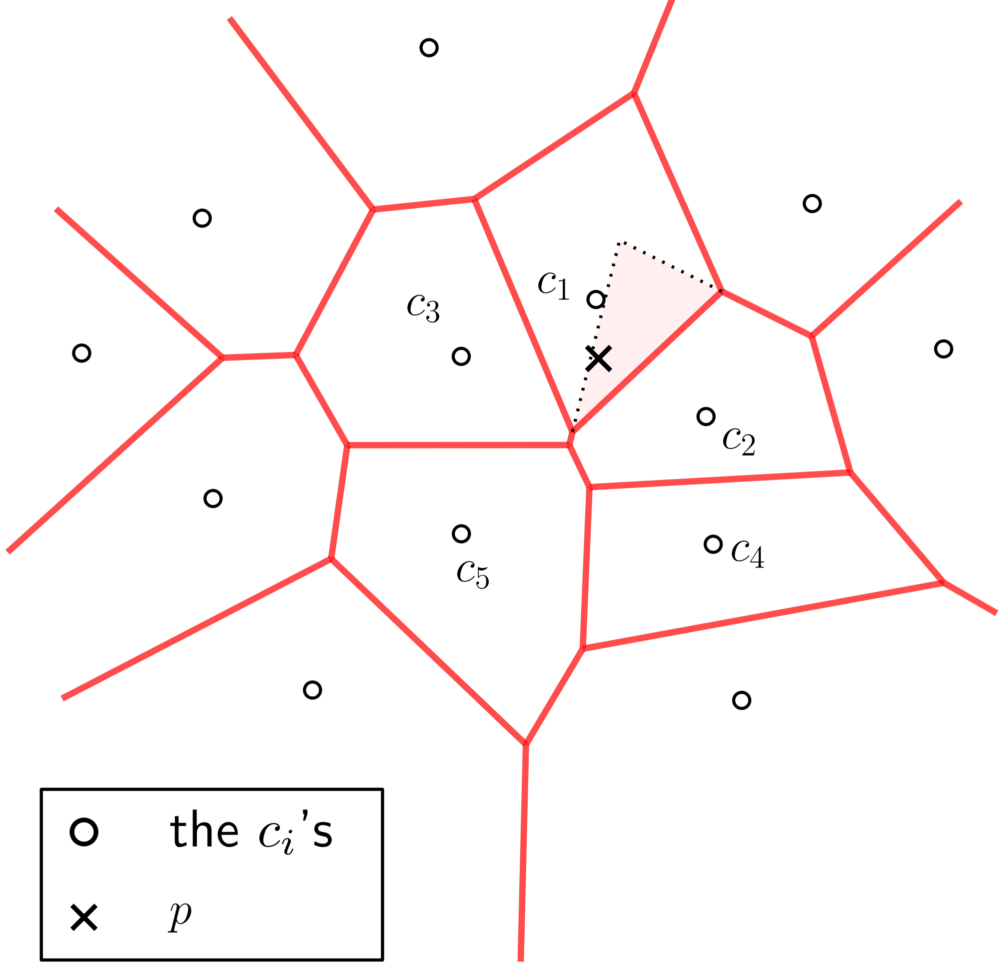

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Some open problems...

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induced by simplicial complexes of **fixed dimension** for...

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- ▷ families of **fixed VC dimension**?

  Arbitrary formula: poly-size support is possible but coefficients blow-up.

Thank you for your attention!

By the way...

$P_k \overset{\text{def}}{=}$ all sets S of primes such that $\prod_{p \in S} p \leq k$.

$$P_{30} = \{\emptyset, \{2\}, \{3\}, \{5\}, \{7\}, \{11\}, \{13\}, \{17\}, \{19\}, \{23\}, \{29\}, \{2,3\}, \{2,5\}, \{2,7\}, \{2,11\}, \{2,13\}, \{3,5\}, \{3,7\}, \{2,3,5\}\}$$

$$\sum_{k=1}^{n} \mu(k) = -\chi(P_n)$$

Prime number theorem $\Leftrightarrow |\chi(P_n)| \leq \epsilon n$ for all $\epsilon > 0$ and sufficiently large n.

Riemann hypothesis $\Leftrightarrow |\chi(P_n)| \leq n^{\frac{1}{2}+\epsilon}$ for all $\epsilon > 0$ and sufficiently large n.

[Björner'12]