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Pattern-Avoiding Inversion Sequences

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Abstract

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The aim of this dissertation is to study pattern avoidance in inversion sequences. I present some enumerative results and focus on the methods used to prove them. Wherever possible, I did some missing proofs and completed the existing ones with some parts omitted in the original papers. The mentioned methods are the following: the one using only the combinatorial characterization of the family of interest, the inductive proof, the proof by recursive construction, the bijective proof, the one using generating functions, the one using generating trees, the kernel method and the obstinate kernel method. For each of these methods some examples and a list of cases falling in the considered type of proof are provided.

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Introduction

1.1 Combinatorics and its history

Combinatorics is an area of mathematics dealing with discrete structures -such as graphs, matroids, partially ordered sets, permutation groups, partitions- and characterized by its methods. It consists of studying some properties of these objects and counting them.

First, let us introduce some history (following [6]). Combinatorics had its origins in the antiquity in the oriental civilization. The first ideas came from the Hindus.

In 1150 CE (common era), Bhaskara -an Indian mathematician- knew the formula for the number of permutations of n and the formula for the number of k-element subsets of an n-element set, and maybe other mathematicians knew them before him. In the first century CE, Chinese documents attest the knowledge of the magic square. From 900 to 1300, magic squares were studied in great depth by Chinese and Islamic cultures. From 1200 to 1300, these two cultures exchanged a body of learning and this knowledge was passed down to the West by Moschopolous, a Byzantine Greek mathematician. Pascal's triangle, as it is known today, was yet another interesting object at those times, when it was named triangle of binomial numbers (the entry in the n-th row and k-th column is the number of k-element subsets of an n-element set, where rows and columns are indexed starting from 0). The earliest attested knowledge of Pascal's triangle dates back to some texts written between 1200 and 1300, but it is likely that this result was already known at least a century before given these texts' references to earlier lost material featuring this result. Therefore, Pascal's 1665 treatise was not the earliest work on the subject. But Pascal was motivated by the problem of foreseeing the results of games of chance and this is a relevant bridge between medieval and modern mathematicians.

In 1665 Pascal presented his treatise. Notice that at the same time Leibniz had written documents on partitions of integers and other topics, which remain unpublished. Between 1600 and 1700, the connection between algebra and combinatorics became more transparent. In 1697, De Moivre proved the multinomial theorem, discovered the principle of inclusion-exclusion and from this he found the formula for the number of derangements of n objects. In 1736, Euler solved the problem of Königsberg bridges, which consists in finding a road passing through the 7 bridges only once, and proved that this road was not possible. The problem of Königsberg bridges gave rise to the graph theory. Moreover, this Swiss mathematician studied partitions, where he made notable improvements, and Latin squares.

In the 19th century, Cauchy, Galois and Lagrange studied the groups of permutations. Moreover, in 1812 Binet and Cauchy worked on the functions permanents. In 1852, Francis Guthrie presented the four-color problem, which states that four colors are enough to color a map in such a way that any two adjacent regions have different colors. In the 20th century, MacMahon improved a lot the subfield of enumerative

combinatorics.
In the second half of the 20th century, combinatorics has developed conspicuously and the study of pattern avoidance in permutations had its origins.

1.2 Pattern avoidance in permutations

The study of patterns in permutations had its origins in the 1970s and early 1980s (see [9]). The following definitions introduce the main concept of the topic.

Definition 1.2.1. *A permutation of* $\{1, 2, \ldots, n\}$ *is a word* $\pi_1\pi_2\ldots\pi_n$ *with no repeated letters and such that* $\pi_i \in \{1, 2, \ldots, n\}$ *for all* $i = 1, \ldots, n$. *The set of all permutations of* $\{1, \ldots, n\}$ *of size* n *is denoted by* S_n*, where the size is defined as* $|\pi| = n$ *for* $\pi \in S_n$.

Definition 1.2.2. *A permutation* $\pi = \pi_1\pi_2\ldots\pi_n \in S_n$ *contains the pattern* $\sigma = \sigma_1\sigma_2\ldots\sigma_k \in S_k$ *if there are* $i_1 < i_2 < \ldots < i_k$ *such that* $\pi_{i_a} < \pi_{i_b}$ *if and only if* $\sigma_a < \sigma_b$ *for all* $a, b \in \{1, 2, \ldots, k\}$. *In other words,* $\pi_{i_1}\pi_{i_2}\ldots\pi_{i_k}$ *is order isomorphic to* σ. *Otherwise,* π *avoids the pattern* σ.

Example 1.2.3. *The permutation* $\pi = 31542 \in S_5$ *contains the pattern* $\sigma = 132 \in S_3$ *since* $\pi_2\pi_3\pi_4 = 154$ *has the same relative order as* 132*, but it avoids the pattern* $\sigma' = 123$.

The first systematic study of pattern-avoiding permutations appeared in 1985 in the paper [13] published by Simion and Schmidt, where they proved different results concerning the enumeration of $S_n(\beta)$ for $\beta \subseteq S_3$. Many problems have been studied, for example how to count permutations that avoid one or more patterns or how to discover the generating function for the number of occurrences of one or more patterns. Pattern avoidance in permutations has been studied in full and has proved to have connections with other fields of mathematics, computational biology and theoretical physics. Nowadays there is a considerable collection of results (see, for example, the second and the third chapter of [9]). The following introduces the concept of enumeration.

Definition 1.2.4. *We define* $S_n(\sigma) = \{\pi \in S_n \mid \pi \text{ avoids } \sigma\}$ *to be the avoidance set of* σ *in size* n. *The avoidance sequence of* σ *is the integer sequence* $|S_1(\sigma)|, |S_2(\sigma)|, |S_3(\sigma)|, \ldots$. *Let* β *be a set of patterns. Similarly, we define* $S_n(\beta) = \{\pi \in S_n \mid \pi \text{ avoids } \sigma, \sigma \in \beta\}$ *to be the avoidance set of* β *in size* n. *The avoidance sequence of* β *is the integer sequence* $|S_1(\beta)|, |S_2(\beta)|, |S_3(\beta)|, \ldots$.

From a combinatorial point of view, permutation patterns have been useful as an interpretation that establishes connections between many combinatorial structures. The avoidance sequences count a very big number of well-known combinatorial structures, explaining the interpretation mentioned before.

1.3 Pattern avoidance in inversion sequences

There is a more recent study in pattern avoidance, but this time concerning inversion sequences. I start to define them and another concept that relates to their name.

Definition 1.3.1. *An inversion sequence of length* n *is any integer sequence* $(e_1, e_2, \ldots, e_n)$ *such that* $0 \le e_i < i$ *for all* $i \in \{1, \ldots, n\}$.

Definition 1.3.2. *An inversion of a permutation* π *is a pair* (i, j) *satisfying* $i > j$ *and* $\pi_i < \pi_j$.

There are several encodings of permutations. A very common one is the bijection $T : S_n \to I_n$ from the set of permutations of size n to the set of inversion sequences of length n, where $T(\pi) = (e_1, e_2, \ldots, e_n)$ and $e_i =| \{j < i : \pi_j > \pi_i\} |$. Note that if you sum the entries of the image of π you obtain the number of inversion of π and this gives rise to the name inversion sequences.
Two other examples of codings of S_n are the Lehmer code L and the *invcode* (the reverse of Lehmer code). Let $e = (e_1, e_2, \ldots, e_n) \in I_n$ and E be a set of sequences, then we define $e^R = (e_n, e_{n-1}, \ldots, e_1)$ and $E^R = \{e^R : e \in E\}$. Moreover, let $\pi = (\pi_1, \pi_2, \ldots, \pi_n) \in S_n$ and P be a set of permutations, then we define $\pi^C = (n+1-\pi_1, n+1-\pi_2, \ldots, n+1-\pi_n)$ and $P^C = \{\pi^C \mid \pi \in P\}$. The Lehmer code sends $\pi \in S_n$ to $e \in I_n^R$ where $e_j =| \{i > j : \pi_i < \pi_j\} |$. The *invcode* sends $\pi \in S_n$ to $e \in I_n$ where $e^R = L(\pi)$. We can observe that $invcode(\pi) = e$ if and only if $e = T((\pi^C)^R)$. Another observation is that if $T(\pi) = e$, then $\pi_i > \pi_{i+1}$ (that is, i is a descent of π) if and only if $e_i < e_{i+1}$ (that is, i is an ascent of e).
These correspondences led to studying pattern avoidance in inversion sequences much in the same way as pattern avoidance in permutations. So these connections were a natural bridge between the two interesting topics of pattern avoidance. However, the translation from pattern containment in permutations to pattern containment in inversion sequences is not immediate.

In the sequel I give some definitions.

Definition 1.3.3. *A pattern is an integer sequence $p = p_1 p_2 \ldots p_k$ with $p_i \in \{0, 1, \ldots, k-1\}$ for all i, where p_i can take the value j only if $j-1$ appears in p.*

Definition 1.3.4. *The reduction of a word $p = p_1 p_2 \ldots p_k \in \{0, 1, \ldots, k-1\}^k$ is the word obtained by replacing all the occurrences of the ith smallest entry of p with $i-1$.*

Example 1.3.5. *The reduction of* 31140 *is* 21130*.*

Definition 1.3.6. *An inversion sequence e of length n contains the pattern $p = p_1 p_2 \ldots p_k$ if there are $i_1 < i_2 < \ldots < i_k$ such that the reduction of the subsequence $e_{i_1} e_{i_2} \ldots e_{i_k}$ is p. Otherwise, e avoids the pattern p.*

Example 1.3.7. *The inversion sequence $e = (0, 0, 2, 1, 4) \in I_5$ contains the pattern $p = 012$ since the reduction of $e_2 e_4 e_5 = 014$ is 012, but it avoids the pattern $p' = 210$.*

While pattern-avoiding permutations have been extensively studied in the literature, the systematic study of pattern-avoiding inversion sequences appeared for the first time around 2015 in the papers published by Corteel-Martinez-Savage-Weselcouch [4] and Mansour-Shattuck [11] independently. Both were motivated to bring the study of patterns in inversion sequences at the same level as the study of patterns in permutations. Permutations and inversion sequences are considered as words over $\mathbb{N}$ and while patterns in permutations can have only distinct entries, patterns in inversion sequences can have repeated values. The first authors studied inversion sequences that avoid permutations and words of length 3, the second ones studied inversion sequences that avoid permutations of length 3: together they contributed to the almost completeness of the enumeration of inversion sequences that avoid one pattern of length 3.

In the following two more definitions.

Definition 1.3.8. *We define $I_n(p) = \{e \in I_n \mid e \text{ avoids } p\}$ to be the avoidance set of p in length n. The avoidance sequence of p is the integer sequence $|I_1(p)|, |I_2(p)|, |I_3(p)|, \ldots$, with the convention that $|I_0(p)| = 1$.*
Let β be a set of patterns. Similarly, we define $I_n(\beta) = \{e \in I_n \mid e \text{ avoids } p, p \in \beta\}$ to

be the avoidance set of β in length n. The avoidance sequence of β is the integer sequence $|I_1(\beta)|, |I_2(\beta)|, |I_3(\beta)|, \ldots$, with the convention that $|I_0(\beta)| = 1$.

Definition 1.3.9. *A refinement of the avoidance set $I_n(p)$ is $I_{n,k}(p) = I_{n,k} \cap I_n(p)$, where $I_{n,k} = \{e \in I_n : e_n = k\}$.*
Let β be a set of patterns. Similarly, we define $I_{n,k}(\beta) = I_{n,k} \cap I_n(\beta)$, where $I_{n,k} = \{e \in I_n : e_n = k\}$, to be a refinement of the avoidance set $I_n(\beta)$.

Definition 1.3.10. *Two patterns are equivalent if they provide the same avoidance sets; two patterns are Wilf-equivalent if their avoidance sequences are the same.*

Remark 1.3.11. *The origin of the name "Wilf-equivalent" is to attribute to the American mathematician Herbert Wilf. The concept was defined for permutation classes: two permutation classes are Wilf-equivalent if they have the same number of permutations of size n, for every n.*

The involvement and the study of patterns in inversion sequences have increased since the publication of these papers and hence many enumeration results and many connections to well-known combinatorial families have been discovered. But there were not only connections between some avoidance sequences and some well-known combinatorial sequences: some pattern-avoiding inversion sequences create some sequences that were not in the On-Line Encyclopedia of Integer Sequences (OEIS) before. Moreover, pattern-avoiding inversion sequences are simpler interpretations of some combinatorial sequences than known interpretations as pattern-avoiding permutations or other discrete structures. Duncan and Steingrímsson studied one of the most relevant subset of inversion sequences: the ascent sequences (see [5]). Surprisingly, this happened a little before the work on inversion sequences.

Now I quickly present the different types of pattern that have been investigated after the classical words of length 3.

The first type I present are the triples of relations, proposed by Martinez-Savage [12].

Definition 1.3.12. *A triple of relations is a triple (ρ_1, ρ_2, ρ_3) of binary relations, where the relations are in the set $\{<, >, \leq, \geq, =, \neq, -\}$ and the relation "$-$" on a set S means that $x - y$ for all $x, y \in S \times S$.*

Example 1.3.13. $(=, \leq, -)$ *is a pattern of relation triples.*

Definition 1.3.14. *An inversion sequence e of length n contains the triple of relations (ρ_1, ρ_2, ρ_3) if there are $i < j < k$ such that $e_i \rho_1 e_j$, $e_j \rho_2 e_k$ and $e_i \rho_3 e_k$. Otherwise, e avoids (ρ_1, ρ_2, ρ_3).*

Example 1.3.15. *The inversion sequence $e = (0, 0, 0, 1, 4) \in I_5$ contains the pattern $(=, \leq, -)$ since for example $e_1 = e_2 \leq e_5$, but the inversion sequence $e' = (0, 1, 2, 2, 1)$ avoids it.*

Remark 1.3.16. *Generally this type of pattern represents some multiple classical patterns of length 3.*

Example 1.3.17. $I_n(\geq, \neq, \geq) = I_n(101, 110, 201, 210)$.

The number of possible triples of relations is 343 and these 343 patterns subdivide into 98 equivalence classes (which are labeled by A, B, C, etc in a Wilf-equivalence class) and 63 conjectured Wilf-equivalence classes (which are labeled by $a_7 := |I_7(p)|$, the number of inversion sequences of length 7 that avoid a pattern p in the Wilf-equivalence class) (see [12]). Moreover, 5 Wilf-equivalence classes have avoidance sequences that are ultimately constant.

Many inversion sequences that avoid a triple of binary relations have been enumerated.

The second type are the consecutive patterns, proposed by Auli-Elizalde [1].

Definition 1.3.18. *A consecutive pattern is a pattern where the entries are underlined.*

Example 1.3.19. $\underline{021}$ *is a consecutive pattern.*

The notion of containment is defined differently than in the case of classical pattern.

Definition 1.3.20. *An inversion sequence e of length n contains the consecutive pattern $p = \underline{p_1 p_2 \dots p_k}$ if there exists a consecutive subsequence $e_i e_{i+1} \dots e_{i+k-1}$ of e such that its reduction is equal to p. In this case, we call $e_i e_{i+1} \dots e_{i+k-1}$ an occurrence of p in position i. Otherwise, e avoids p.*
Moreover, $Em(p,e) = \{i : e_i e_{i+1} \dots e_{i+k-1}$ is an occurrence of $p\}$.

Example 1.3.21. *The inversion sequence $e = (0,1,1,3,2) \in I_5$ contains the pattern $p = \underline{021}$ since the reduction of $e_3 e_4 e_5 = 132$ is 021, but the inversion sequence $e' = (0,1,1,3,1) \in I_5$ avoids it (however, it contains the classical pattern 021). Since there are no other occurrences of $\underline{021}$ in e we have that $Em(\underline{021}, e) = \{3\}$.*

Definition 1.3.22. *Two consecutive patterns are Wilf-equivalent if their avoidance sequences are the same; two consecutive patterns are strongly Wilf-equivalent if the number of inversion sequences in I_n containing m occurrences of the first pattern is the same as for the second pattern for all n and m; two consecutive patterns, say p and p', are super-strongly Wilf-equivalent ($p \overset{ss}{\sim} p'$) if $| \{e \in I_n : Em(p,e) = S\} | = | \{e \in I_n : Em(p',e) = S\} |$ for all n and all $S \subseteq \{1, \dots, n\}$.*

Several inversion sequences that avoid consecutive patterns of length 3 have been enumerated and other results involving patterns of length 4 have been founded. Some results have been generalized to patterns of arbitrary length.

The third type are the vincular patterns, proposed by Lin-Yan [10].

Definition 1.3.23. *A vincular pattern is a pattern where some consecutive entries are underlined.*

Remark 1.3.24. *Equivalently, a vincular pattern is a pattern containing dashes showing the entries that don't need to occur consecutively (in Definition 1.3.6, no entries need to occur consecutively).*

Example 1.3.25. $0\underline{21}$ *(or equivalently* 0-21*) is a vincular pattern.*

Definition 1.3.26. *An inversion sequence e of length n contains the vincular pattern p of length k if there are $i_1 < i_2 < \dots < i_k$ such that the reduction of the subsequence $e_{i_1} e_{i_2} \dots e_{i_k}$ is p (disregarding which values are underlined) and such that the underlined entries occur consecutively. Otherwise, e avoids p.*

Example 1.3.27. *The inversion sequence $e = (0,1,2,3,2) \in I_5$ contains the pattern $p = 0\underline{21}$ since for example the reduction of $e_2 e_4 e_5 = 132$ is 021, but the inversion sequence $e' = (0,1,2,0,1) \in I_5$ avoids it (however, it contains the classical pattern 021).*

Many inversion sequences that avoid vincular patterns of length 3, where two of the three entries are required to occur consecutively, have been enumerated.

Lastly, I present some operations and statistics on inversion sequences. These tools will be useful later as they will appear in some proofs.

Definition 1.3.28. *The operation σ_k on an inversion sequence e adds k to the positive elements of e (k can be positive or negative). The result of this operation is not always an inversion sequence.*
The concatenation is another operation on inversion sequences that appends an element at the beginning or at the end of an inversion sequence: $0 \cdot e = (0, e_1, \ldots, e_n)$ (at the beginning you can append only the 0) and $e \cdot i = (e_1, e_2, \ldots, e_n, i)$ for all $i \in \{1, \ldots, n\}$ (e_{n+1} can be at most n). With these constraints the result is an inversion sequence, but this doesn't hold in general for concatenation.

The following definition is needed in order to introduce the concept of statistic.

Definition 1.3.29. *A combinatorial class $\mathcal{C}$ is a set of (discrete) objects equipped with a notion of size, a function from $\mathcal{C}$ to $\mathbb{N}$, such that there is a finite number of objects of size n for all natural numbers n. We denote $\mathcal{C}_n$ the set of objects of size n and $c_n = |\mathcal{C}_n|$, for all $n \geq 0$.*

Definition 1.3.30. *A statistic is a function from a combinatorial class $\mathcal{C}$ to the natural numbers $\mathbb{N}$ (or to the real numbers $\mathbb{R}$).*

Definition 1.3.31. *Let $e = (e_1, e_2, \ldots, e_n) \in I_n$. The following are statistics on e:*

- $zeros(e) = |\{i \in \{1, \ldots, n\} : e_i = 0\}|$ = *number of zero entries in e;*
- $dist(e) = |\{e_1, e_2, \ldots, e_n\}|$ = *number of distinct entries in e;*
- $repeats(e) = |\{i \in \{1, \ldots, n-1\} : e_i \in \{e_{i+1}, \ldots, e_n\}| = n - dist(e)$ = *number of repeated entries in e;*
- $asc(e) = |\{i \in \{1, \ldots, n-1\} : e_i < e_{i+1}\}|$ = *number of ascents in e;*
- $maxim(e) = |\{i \in \{1, \ldots, n\} : e_i = i - 1\}|$ = *number of maximal entries in e (that is, the entries of e having the maximal possible value at their position);*
- $maxx(e) = max\{e_1, e_2, \ldots, e_n\}$ = *maximum value occurring in e;*
- $last(e) = e_n$ = *last entry of e.*

1.4 Objective

My master thesis is about a recent development in the study of pattern avoidance in permutations: pattern-avoiding inversion sequences. In particular, I present some enumeration results and some Wilf-equivalence results, stressing the various methods used to obtain them.

In the second chapter, I explain the methods for enumerating families of pattern-avoiding inversion sequences. The first type of proof will be the one using the combinatorial characterization of the considered family of inversion sequences. The second type will be the inductive proof. The third type will be the proof by recursive construction. The fourth type will be the bijective proof. The fifth type will be the one using generating functions. The sixth type will be the one using generating trees. The last two types will be the kernel method and the obstinate kernel method, respectively. For each of these methods I give a description and I provide some examples as well as a list of cases falling in this type of proof. This chapter is organized in sections, one for each method of proof I identified, and subsections, one for each example I present.

In the third chapter, I conclude with some remarks about the results and about the topic of pattern-avoiding inversion sequences in general.

Chapter 2

Methods to enumerate pattern-avoiding inversion sequences

2.1 Combinatorial characterization

The first method I present is the one using only the combinatorial characterization of the family of inversion sequences you want to enumerate. Each set of pattern-avoiding inversion sequences can be described by a combinatorial characterization (maybe several), which is very useful in order to keep in mind the properties that the inversion sequences avoiding a certain pattern must have. In these cases, as we will see in the next two examples, we have a "very constrained structure" (implied by the avoidance of the patterns) from which we get the enumeration immediately.

In what follows, I present two examples of this method, first proving the characterization and secondly using it to prove the enumerative statement.

2.1.1 First example

The pattern of the first example (presented by Martinez-Savage [12] in subsection 2.4.1 and where no proof is provided) is the triple of relations $(<, \neq, -)$ and the inversion sequences avoiding it are characterized in the following way:

Proposition 2.1.1. *An inversion sequence $e \in I_n$ has no $i < j < k$ such that $e_i < e_j \neq e_k$ if and only if*

$$0 = e_1 = e_2 = \ldots = e_{t-1} \leq e_t = e_{t+1} = \ldots = e_n \tag{2.1.1}$$

for some t with $1 \leq t \leq n$.

Proof. "$\Rightarrow$" let $e \in I_n$ with no $i < j < k$ such that $e_i < e_j \neq e_k$. Assume e does not satisfy 2.1.1.

Let t be the smallest index such that $e_t > 0$ (if such t doesn't exist then e is the inversion sequence $(0, 0, \ldots, 0)$, which contradicts the assumption).

Then, since e does not satisfy 2.1.1, there exists an integer ℓ, with $t + 1 \leq \ell \leq n$, such that $e_t \neq e_\ell$ $(t > 1)$. But then we have $t - 1 < t < \ell$ and $e_{t-1} < e_t \neq e_\ell$, which contradicts the assumption. Consequently e satisfies 2.1.1.

"$\Leftarrow$" let $e \in I_n$ such that e satisfies 2.1.1.

If we have $i < j$ such that $e_i < e_j$, then all the entries to the right of e_j are equal to e_j. So there is no $i < j < k$ such that $e_i < e_j \neq e_k$. □

From this we obtain the following enumerative result.

Theorem 2.1.2. $|I_n(<, \neq, -)| = \binom{n}{2} + 1$.

Proof. From Proposition 2.1.1 we deduce that e can be the inversion sequence $(0, 0, \ldots, 0)$ (the case $t = 1$) or a sequence composed by $t - 1$ zeros followed by $n - t + 1$ copies of ℓ, where t is such that $2 \leq t \leq n$ and ℓ is such that $1 \leq \ell \leq t - 1$ (since e_t can be at most $t - 1$).

The zero inversion sequence contributes 1 to the cardinality of $I_n(<, \neq, -)$. Moreover, for every t we have $t - 1$ possibilities of choosing ℓ and since t is between 2 and n we obtain

$$\sum_{t=2}^{n}(t-1) = \sum_{t=1}^{n-1} t = \frac{(n-1)n}{2} = \binom{n}{2}.$$

possible inversion sequences in addition to the zero inversion sequence. So in total we have $|I_n(<, \neq, -)| = \binom{n}{2} + 1$. □

2.1.2 Second example

The patterns of the second example (presented by Yan-Lin [15] in Section 2.2 and where no proof is provided) are three pairs of classical patterns of length 3 and the characterizations for each of these pairs are the following:

Proposition 2.1.3. *An inversion sequence $e \in I_n$ avoids $\{001, 110\}$ if and only if*

$$e_1 < e_2 < \ldots < e_t \geq e_{t+1} = e_{t+2} = \ldots = e_n \tag{2.1.2}$$

for some t with $1 \leq t \leq n$.

Proof. "⇒" let $e \in I_n$ such that e avoids $\{001, 110\}$. Suppose e does not satisfy 2.1.2.

Let t be the smallest index such that $e_t \geq e_{t+1}$ (if no such t exists set $t = n$ and then e satisfies 2.1.2, obtaining a contradiction to our assumption). Note that if $t = n - 1$, then e clearly avoids $\{001, 110\}$ since to the right of the two equal values there are no more entries.

Then one possibility (if $t \neq n - 1$) is that there exists an integer $\ell \geq t + 2$ such that $e_\ell < e_{t+1}$. But then we have $j < t + 1 < \ell$, with $1 \leq j \leq t$, and $e_j = e_{t+1} > e_\ell$ (where the equality holds since $e_j = j - 1$ for all $1 \leq j \leq t$ and e_t is the maximum among $\{e_1, \ldots, e_t\}$), which contradicts the fact that e avoids 110.

Another possibility (if $t \neq n - 1$) is that there exists an integer $\ell \geq t + 2$ such that $e_\ell > e_{t+1}$. But then we have $j < t + 1 < \ell$, with $1 \leq j \leq t$, and $e_j = e_{t+1} < e_\ell$ (where the equality holds since $e_j = j - 1$ for all $1 \leq j \leq t$ and e_t is the maximum among $\{e_1, \ldots, e_t\}$), which contradicts the fact that e avoids 001.

Consequently e satisfies 2.1.2.

"⇐" let $e \in I_n$ such that e satisfies 2.1.2.

So, if we have an equality, then we only have equalities and hence neither ascents nor descents. This means that e avoids 001 and 110. □

Proposition 2.1.4. *An inversion sequence $e \in I_n$ avoids $\{001, 021\}$ if and only if*

$$e_1 < e_2 < \ldots < e_t = e_{t+1} = \ldots = e_s > e_{s+1} = e_{s+2} = \ldots = e_n = 0 \tag{2.1.3}$$

for some t with $2 \leq t \leq n - 1$ and $t \leq s \leq n - 1$, or

$$e_1 < e_2 < \ldots < e_t = e_{t+1} = \ldots = e_n \tag{2.1.4}$$

for some $t \leq n$.

Proof. "⇒" let $e \in I_n$ such that e avoids $\{001, 021\}$.

Let's begin with the first case: suppose e does not satisfy 2.1.3. Let t be the smallest index such that $e_{t+1} < t$ (if no such t exists then the entries of e are all consecutive and we are in the second case). Let s be the smallest index among those greater than or equal to t such that $e_s > e_{s+1}$ (if no such s exists then we are in the second case).

Then one possibility is that $e_{s+1} \neq 0$. But then we have $1 < s < s+1$ and $e_1 < e_{s+1} < e_s$, which contradicts the fact that e avoids 021.

Another possibility (if $s \neq n-1$) is that there exists an integer $\ell \geq s+2$ such that $e_\ell \neq 0$ and $e_{s+1} = 0$. But then we have $1 < s+1 < \ell$ and $e_1 = e_{s+1} < e_\ell$, which contradicts the fact that e avoids 001.

Consequently e satisfies 2.1.3.

Let's go on with the second case: suppose e does not satisfy 2.1.4. Let t be the smallest index such that $e_{t+1} < t$ (if no such t exists set $t = n$ and then e satisfies 2.1.4, obtaining a contradiction to our assumption).

Then one possibility is that $e_t > e_{t+1}$. But then we have $1 < t < t+1$ and $e_1 < e_t > e_{t+1}, e_1 \neq e_{t+1}$ (if $e_1 = e_{t+1}$ then we are in the first case), which contradicts the fact that e avoids 021.

Another possibility (if $t \neq n-1$) is that there exists $j \geq t+2$ such that $e_t \neq e_j$. But then a 001 is created, a 021 is created, or we fall in the first case.

Consequently e satisfies 2.1.4.

"$\Leftarrow$" let $e \in I_n$ such that e satisfies 2.1.3.

So, if we have an equality, then we can only have a descent or an equality, hence no ascents. If we have an ascent followed by a descent then the number to the right is zero and hence greater than no elements. This means that e avoids 001 and 021. □

Proposition 2.1.5. *An inversion sequence $e \in I_n$ avoids $\{001, 120\}$ if and only if*

$$e_1 < e_2 < \ldots < e_t = e_{t+1} = \ldots = e_s > e_{s+1} = e_{s+2} = \ldots = e_n = e_{t-1} \tag{2.1.5}$$

for some t with $2 \leq t \leq n-1$ and $t \leq s \leq n-1$, or

$$e_1 < e_2 < \ldots < e_t = e_{t+1} = \ldots = e_n \tag{2.1.6}$$

for some $t \leq n$.

Proof. "$\Rightarrow$" let $e \in I_n$ such that e avoids $\{001, 120\}$.

Let's begin with the first case: suppose e does not satisfy 2.1.5. Let t be the smallest index such that $e_{t+1} < t$ (if no such t exists then all the entries are consecutive and we are in the second case). Let s be the smallest index among those greater than or equal to t such that $e_s > e_{s+1}$ (if no such s exists then we are in the second case).

Then one possibility is that $e_{s+1} < e_{t-1}$. But then we have $t-1 < t < s+1$ and $e_{s+1} < e_{t-1} < e_t$, which contradicts the fact that e avoids 120.

Another possibility is that $e_{s+1} > e_{t-1}$. But then we have $t < t+1 < s+1$ and $e_t = e_{t+1} < e_{s+1}$ ($s \neq t$ in this case, otherwise e satisfies 2.1.5), which contradicts the fact that e avoids 001.

Another possibility (if $s \neq n-1$) is that there exists an integer $\ell \geq s+2$ such that $e_\ell > e_{t-1}$ and $e_{s+1} = e_{t-1}$. But then we have $t-1 < s+1 < \ell$ and $e_{t-1} = e_{s+1} < e_\ell$, which contradicts the fact that e avoids 001.

Another possibility (if $s \neq n-1$) is that there exists an integer $\ell \geq s+2$ such that $e_\ell < e_{t-1}$ and $e_{s+1} = e_{t-1}$. But then we have $t-1 < t < \ell$ and $e_\ell < e_{t-1} < e_t$, which contradicts the fact that e avoids 120.

Consequently e satisfies 2.1.5.

Let's go on with the second case: suppose e does not satisfy 2.1.6. Let t be the smallest index such that $e_{t+1} < t$ (if no such t exists set $t = n$ and then e satisfies 2.1.6, obtaining a contradiction to our assumption).

Then one possibility is that $e_t > e_{t+1}$. But then we have $t-1 < t < t+1$ and $e_{t-1} < e_t > e_{t+1}, e_{t-1} > e_{t+1}$ (if $e_{t-1} = e_{t+1}$ then we are in the first case), which contradicts the fact that e avoids 120.

Another possibility is that there exists $j \geq t+2$ such that $e_t \neq e_j$. But then a 001 is created, a 120 is created, or we fall in the first case.

Consequently e satisfies 2.1.6.

"$\Leftarrow$" let $e \in I_n$ such that e satisfies 2.1.5.

So, if we have an equality, then we don't have ascents. If we have an ascent followed by a descent then the element to the right is not less than the "first" element. This means that e avoids 001 and 120. □

From these characterizations we obtain the following enumerative result.

Theorem 2.1.6. *$|I_n(p)| = \binom{n}{2} + 1$ with $p \in \{(001, 110), (001, 021), (001, 120)\}$.*

Proof. For $p = (001, 110)$: from Proposition 2.1.3 we deduce that if t is such that $1 \leq t \leq n-1$ we have only one possibility for the first t entries since they are consecutive, and we have t possibilities for all the other entries (which are all equal) since $e_t = t-1$. If $t = n$ we have only the inversion sequence $(0, 1, \ldots, n-1)$, hence only one possibility. Summing everything together we obtain

$$1 + \sum_{t=1}^{n-1} t = 1 + \frac{(n-1)n}{2} = \binom{n}{2} + 1.$$

For $p \in \{(001, 021), (001, 120)\}$: if $t = 1$ we have only one possible inversion sequence: $(0, 0, \ldots, 0)$, no matter what is the value of s. If t is such that $2 \leq t \leq n$, we have $n - t + 1$ possibilities of choosing s and then everything is determined because the first t values are consecutive, the following $s - t$ values are equal to $e_t = t-1$ and the remaining values are equal to zero for $p = (001, 021)$ and equal to $e_{t-1} = t-2$ for $p = (001, 120)$. Summing everything together we obtain

$$\begin{aligned}
1 + \sum_{t=2}^{n} (n - t + 1) &= 1 + n(n-1) - \left(\frac{n(n+1)}{2} - 1\right) + n - 1 \\
&= 1 + \frac{2n^2 - 2n - n^2 - n + 2 + 2n - 2}{2} \\
&= 1 + \frac{n^2 - n}{2} \\
&= 1 + \frac{n(n-1)}{2} \\
&= \binom{n}{2} + 1.
\end{aligned}$$

□

Remark 2.1.7. *Note that here and in the previous example we obtain the same number, $\binom{n}{2} + 1$, as cardinality of the avoidance set.*

Table 2.1 shows a list of cases falling in the type of proof presented in this section, where the examples that I have illustrated are marked with * (I will keep this notation for all the other tables).

Pattern	Reference	Section
$(\neq, -, \neq), (\geq, -, \neq)$	[12]	2.2
*$(<, \neq, -)$	[12]	2.4.1
$(-, \geq, \neq)$	[12]	2.7.3
$(\neq, <, \neq)$	[12]	2.9
(001, 010), (001, 011), (001, 012)	[15]	2.1
*(001, 110), (001, 021), (001, 120)	[15]	2.2
(001, 100)	[15]	2.4
(001, 210)	[15]	2.5
(012, 201), (012, 210)	[15]	2.8
(010, 011)	[15]	2.9

TABLE 2.1: Cases falling in the type of proof "Combinatorial characterization".

2.2 Inductive proof

The second method I present is the one where you prove a result by induction. You show that a statement $S(n)$ is true for $n = 0$, that is, you prove the base case. Then assuming that $S(n)$ is true, you prove that $S(n+1)$ is also true and this is called the inductive step.

In the sequel, I present an example (exhibited by Auli-Elizalde [1] in Section 3.1) where the pattern involved is the consecutive pattern $\underline{000}$. The following definition and the related lemma are needed.

Definition 2.2.1. *A derangement of size n is a permutation σ of size n (where the size is defined as $|\sigma| = n$ for $\sigma \in S_n$) that has no fixed points, that is, $\sigma_i \neq i$ for all $1 \leq i \leq n$.*

Lemma 2.2.2. *Let $n \geq 1$ and d_n the number of derangements of size n. It holds that*

$$d_n = (n-1)d_{n-2} + (n-1)d_{n-1} = (n-1)(d_{n-2} + d_{n-1}).$$

Proof. Let σ be a derangement of size n. The entry $\sigma(1)$ cannot be equal to 1, so there are $n-1$ possible choices for this entry. Let us assume that $\sigma(1) = j$ with $j \in \{2, \ldots, n\}$. Now we have to distinguish two cases.

If $\sigma(j) = 1$ then the values $[1, \ldots, n] \setminus \{1, j\}$ must be in one-to-one correspondence with the values $[1, \ldots, n] \setminus \{1, j\}$, without creating fixed points. That is, we have to construct a derangement of size $n-2$. So we have $(n-1)d_{n-2}$ such possible derangements of size n.

If $\sigma(j) \neq 1$ then the values $[1, \ldots, n] \setminus \{1\}$ must be in one-to-one correspondence with the values $[1, \ldots, n] \setminus \{j\}$, without creating fixed points. Each value in $[1, \ldots, n] \setminus \{1\}$ has exactly one forbidden value in $[1, \ldots, n] \setminus \{j\}$. That is, we have to construct a derangement of size $n-1$ where j cannot be assigned to 1. So we have $(n-1)d_{n-1}$ such possible derangements of size n.

Putting everything together we obtain $d_n = (n-1)d_{n-2} + (n-1)d_{n-1} = (n-1)(d_{n-2} + d_{n-1})$. □

Before proving the enumerative result, we show a proposition that we will need in our proof.

Proposition 2.2.3. *Let $n \geq 3$. The following recurrence for $|I_n(\underline{000})|$ holds:*

$$|I_n(\underline{000})| = (n-1)\,|I_{n-1}(\underline{000})| + (n-2)\,|I_{n-2}(\underline{000})|\,,$$

with initial conditions $|I_1(\underline{000})| = 1$ *and* $|I_2(\underline{000})| = 2$.

Proof. It holds that

$$I_1(\underline{000}) = \{(0)\} \text{ and } I_2(\underline{000}) = \{(0,0),(0,1)\},$$

so the initial conditions are satisfied. Let $n \geq 3$. We distinguish two cases, depending on the relation between the last two entries.

If $e_{n-1} \neq e_n$, then the number of possible choices for the first $n-1$ entries is $|I_{n-1}(\underline{000})|$ and the possible choices for the last entry e_n are $\{0,\ldots,n-1\} \setminus \{e_{n-1}\}$, that is, we have $n-1$ possibilities for e_n.

If $e_{n-1} = e_n$, then since e avoids the consecutive pattern $\underline{000}$ the entries e_{n-2} and e_{n-1} cannot be equal. The number of possible choices for the first $n-2$ entries is $|I_{n-2}(\underline{000})|$ and the possibilities for $e_{n-1} = e_n$ are $n-2$, since the possible choices for the last two entries are $\{0,\ldots,n-2\} \setminus \{e_{n-2}\}$.

As a result we obtain the desired recurrence. □

Now we can proceed with the proof of the following proposition.

Proposition 2.2.4. *Let* $n \geq 1$ *and* d_n *the number of derangements of size* n. *It holds* $|I_n(\underline{000})| = \frac{(n+1)!-d_{n+1}}{n}$.

Proof. We start with the base case. For $n = 1$ it holds that

$$|I_1(\underline{000})| = 1$$

and

$$\frac{2!-d_2}{1} = 1,$$

so the base case is proved.

Assume that $|I_k(\underline{000})| = \frac{(k+1)!-d_{k+1}}{k}$, for $k = 1,\ldots,n$. We prove that this holds also for $k = n+1$. By Proposition 2.2.3

$$|I_{n+1}(\underline{000})| = n\,|I_n(\underline{000})| + (n-1)\,|I_{n-1}(\underline{000})|$$

and by the induction hypothesis

$$\begin{aligned}
|I_{n+1}(\underline{000})| &= n \cdot \frac{(n+1)!-d_{n+1}}{n} + (n-1)\frac{n!-d_n}{n-1} \\
&= (n+1)! - d_{n+1} + n! - d_n \\
&= n!(n+1+1) - d_{n+1} - d_n \\
&= \frac{(n+2)!}{n+1} - \frac{(n+1)(d_{n+1}+d_n)}{n+1} \\
&= \frac{(n+2)!-d_{n+2}}{n+1},
\end{aligned}$$

where in the last equality we used Lemma 2.2.2. So the result holds. □

Table 2.2 shows a list of cases falling in the type of proof presented in this section.

Pattern	Reference	Section
021	[4]	4.2
*000	[1]	3.1

TABLE 2.2: Cases falling in the type of proof "Inductive proof".

2.3 Proof by recursive construction

The third method I present is the one in which the proof is by recursive construction. This means that we prove a result regarding an object by determining a method which we use for creating the object. More precisely, we construct an object starting from an object of the same kind but of lower dimension. That is, we construct an object recursively.

In the sequel I present two examples of this method and for both of them we will obtain a recurrence for the avoidance sequence of the considered pattern.

2.3.1 First example

The pattern of the first example (presented by Martinez-Savage [12] in Section 2.11) is the triple of relations $(\leq, \geq, \neq)$ for which the following enumerative result holds.

Theorem 2.3.1. $|I_n(\leq, \geq, \neq)| = (n-1)2^{n-2} + 1$.

Proof. We start by characterizing the inversion sequences that avoid the pattern $(\leq, \geq, \neq)$: an inversion sequence $e \in I_n$ has no $i < j < k$ such that $e_i \leq e_j \geq e_k$ and $e_i \neq e_k$ if and only if

$$1 \leq e_a < e_{n-b+1} < e_{n-b+2} < \ldots < e_n < n, e_1 = \ldots = e_{a-1} = e_{a+1} = \ldots = e_{n-b} = 0 \tag{2.3.1}$$

for some a with $1 < a < n+1$ and $0 \leq b < n-a+1$. That is e takes the form

$$e = (0, \ldots, 0, e_a, 0, \ldots, 0, e_{n-b+1}, e_{n-b+2}, \ldots, e_n),$$

where the first sequence of 0 is non-empty and the sequence $e_{n-b+1}, e_{n-b+2}, \ldots, e_n$ is increasing.

This holds for the following reason. Let $e \in I_n$ with no $i < j < k$ such that $e_i \leq e_j \geq e_k$ and $e_i \neq e_k$. Assume e does not satisfy 2.3.1. Let a be the smallest index such that $e_a > 0$ (if no such a exists then e is the sequence $(0, \ldots, 0)$, which clearly avoids our pattern). Let b be the greatest index such that $e_{n-b+1} > 0, b < n-a+1$ (if no such b exists then e is an inversion sequence with at most one entry different from 0, which clearly avoids our pattern).

Let $a \neq n-b$. Then, since e does not satisfy 2.3.1, $e_\ell \geq e_{\ell+1}$ for $\ell = n-b+1, \ldots, n-1$. But then if $e_{\ell+1} \neq 0$ we have $1 < \ell < \ell+1$ and $e_1 < e_\ell \geq e_{\ell+1}$, $e_1 \neq e_{\ell+1}$; if $e_{\ell+1} = 0$ we have $a < \ell < \ell+1$ and $e_a < e_\ell \geq e_{\ell+1}$, $e_a \neq e_{\ell+1}$, which both contradict the assumption. Or $e_a \geq e_{n-b+1}$. But then $1 < a < n-b+1$ and $e_1 < e_a \geq e_{n-b+1}$, $e_1 \neq e_{n-b+1}$, which contradicts the assumption.

Let $a = n-b$. Then, since e does not satisfy 2.3.1, $e_\ell \geq e_{\ell+1}$ for $\ell = n-b, \ldots, n-1$. But then we have $1 < \ell < \ell+1$ and $e_1 < e_\ell \geq e_{\ell+1}$, $e_1 \neq e_{\ell+1}$ (since $e_{\ell+1} \neq 0$, otherwise e satisfies 2.3.1) which contradicts the assumption.

Consequently e satisfies 2.3.1.

Conversely, let $e \in I_n$ such that e satisfies 2.3.1. If we have $i < j < k$ such that $e_i \leq e_j \geq e_k$, then e_i and e_k are both equal to 0. So there is no $i < j < k$ such that $e_i \leq e_j \geq e_k$ and $e_i \neq e_k$.

Now that we know how these pattern-avoiding inversion sequences are characterized, we can analyze their construction distinguishing two cases.

If $e_a > 1$, this implies that $a > 2$ and so we have that $e = 0 \cdot \sigma_1(e')$ where $e' \in I_{n-1}(\leq, \geq, \neq)$.

If $e_a = 1$, there are $n - b - 2 + 1 = n - b - 1$ possibilities of choosing where to place the entry $e_a = 1$ since the possible places are $2, 3, \dots, n - b$. Moreover, there are $\binom{n-2}{b}$ possibilities of choosing a b-element subset of $\{2, \dots, n-1\}$ (since $e_{n-b+1} > 1$ and $e_n < n$) for the entries $e_{n-b+1}, \dots, e_n$. Once you have chosen a b-element subset there is only one possibility of placing the values since they are strictly increasing.

Summing everything together we obtain the following for $n \geq 2$:

$$\begin{aligned}
|I_n(\leq, \geq, \neq)| &= |I_{n-1}(\leq, \geq, \neq)| + \sum_{b=0}^{n-2} (n-b-1) \binom{n-2}{b} \\
&= |I_{n-1}(\leq, \geq, \neq)| + \sum_{b=0}^{n-2} (n-1) \binom{n-2}{b} - \sum_{b=0}^{n-2} b \binom{n-2}{b} \\
&= |I_{n-1}(\leq, \geq, \neq)| + (n-1)2^{n-2} - (n-2)2^{n-3} \\
&= |I_{n-1}(\leq, \geq, \neq)| + (2n-2-n+2)2^{n-3} \\
&= |I_{n-1}(\leq, \geq, \neq)| + n2^{n-3},
\end{aligned}$$

where we have used the fact that $\sum_{k=0}^{\alpha} k\binom{\alpha}{k} = \sum_{k=0}^{\alpha} k \frac{\alpha!}{(\alpha-k)!k!} = \sum_{k=0}^{\alpha} \alpha \frac{(\alpha-1)!}{(\alpha-k)!(k-1)!} = \sum_{k=0}^{\alpha} \alpha \binom{\alpha-1}{k-1} = \sum_{k=1}^{\alpha} \alpha \binom{\alpha-1}{k-1} = \alpha \cdot 2^{\alpha-1}$.

This recurrence has initial condition $|I_1(\leq, \geq, \neq)| = 1 = (1-1)2^{1-2} + 1$ and solution $(n-1)2^{n-2} + 1$ since $(n-2)2^{n-3} + 1 + n \cdot 2^{n-3} = (2n-2)2^{n-3} + 1 = (n-1)2^{n-2} + 1$. By induction principle the statement of the theorem holds. □

Remark 2.3.2. *Note that distinguishing the two cases with respect to the value of e_a means that we have partitioned the inversion sequences $e \in I_n(\leq, \geq, \neq)$ into two disjoint sets: $X_n = \{e \in I_n(\leq, \geq, \neq) : e_a > 1\}, Y_n = \{e \in I_n(\leq, \geq, \neq) : e_a = 1\}$. This process is often used, but not always, in proofs by recursive construction.*

2.3.2 Second example

The pattern of the second example (presented by Auli-Elizalde [1] in Section 3.3 and where no proof is provided) is the consecutive pattern $\underline{120}$. In this case the enumerative result is a recurrence that allows to compute the avoidance sequence (more precisely a refinement of it) of our pattern.

Firstly, we recall Definition 1.3.9 of the considered refinement.

Definition 2.3.3. *A refinement of the avoidance set $I_n(p)$ is $I_{n,k}(p) = I_{n,k} \cap I_n(p)$, where $I_{n,k} = \{e \in I_n : e_n = k\}$.*

The following result holds.

Proposition 2.3.4. *Let $n \geq 1$ and $0 \leq k < n$. It holds:*

$$|I_{n,k}(\underline{120})| = |I_{n-1}(\underline{120})| - \sum_{j>k} (n-2-j) \left| I_{n-2,j}(\underline{120}) \right|.$$

Proof. Recall the convention $|I_0(p)| = 1$ in the Definition 1.3.8.
For $n = 1$ we have

$$|I_{1,0}(\underline{120})| = 1 \text{ and } |I_0(\underline{120})| - \sum_{j>0}(-1-j)\left|I_{-1,j}(\underline{120})\right| = 1 - 0 = 1.$$

For $n = 2$ we have

$$|I_{2,0}(\underline{120})| = 1 \text{ and } |I_1(\underline{120})| - \sum_{j>0}(-j)\left|I_{0,j}(\underline{120})\right| = 1 - 0 = 1$$
$$|I_{2,1}(\underline{120})| = 1 \text{ and } |I_1(\underline{120})| - \sum_{j>1}(-j)\left|I_{0,j}(\underline{120})\right| = 1 - 0 = 1.$$

So the result holds for $n = 1$ or 2.

Assume $n \geq 3$. For $\ell \leq n-2$ and $m \leq n-4$ let

$$B_{n-1,j,\ell} := \{e \in I_{n-1,\ell}(\underline{120}) : e_{n-2} < e_{n-1} \text{ and } e_{n-2} = j\}.$$

It holds that if $e = e_1e_2\ldots e_n \in I_{n,k}(\underline{120})$, then $e = e_1e_2\ldots e_{n-1} \in I_{n-1}(\underline{120})$. Moreover, $e \in I_{n,k}(\underline{120})$ if and only if $e_1e_2\ldots e_{n-1} \in I_{n-1}(\underline{120}) \setminus \sqcup_{j>k} \sqcup_{\ell=j+1}^{n-2} B_{n-1,j,\ell}$, since otherwise $e_{n-2}e_{n-1}e_n$ would be an occurrence of $\underline{120}$ by definition of $B_{n-1,j,\ell}$.

Conversely, if we have $e_1e_2\ldots e_{n-1} \in I_{n-1}(\underline{120}) \setminus \sqcup_{\ell=j+1}^{n-2} B_{n-1,j,\ell}$ and we add the entry $e_n = k$, we obtain an inversion sequence $e_1e_2\ldots e_n$ that avoids $\underline{120}$ and whose last entry is k.

From this we obtain the following equality between cardinalities:

$$|I_{n,k}(\underline{120})| = |I_{n-1}(\underline{120})| - \sum_{j>k}\sum_{\ell=j+1}^{n-2}\left|B_{n-1,j,\ell}\right|. \tag{2.3.2}$$

The last thing we have to do is to establish what is $\left|B_{n-1,j,\ell}\right|$. We can see that $e \in B_{n-1,j,\ell}$ if and only if $e_1e_2\ldots e_{n-2} \in I_{n-2,j}(\underline{120})$ where $j > k$ and $e_{n-1} = \ell$. From this consideration equation 2.3.2 becomes

$$\begin{aligned}
|I_{n,k}(\underline{120})| &= |I_{n-1}(\underline{120})| - \sum_{j>k}\sum_{\ell=j+1}^{n-2}\left|I_{n-2,j}(\underline{120})\right| \\
&= |I_{n-1}(\underline{120})| - \sum_{j>k}(n-2-j-1+1)\left|I_{n-2,j}(\underline{120})\right| \\
&= |I_{n-1}(\underline{120})| - \sum_{j>k}(n-2-j)\left|I_{n-2,j}(\underline{120})\right|
\end{aligned}$$

and the statement holds. □

Table 2.3 shows a list of cases falling in the type of proof presented in this section.

Pattern	Reference	Section
012	[4]	2.1
201	[4]	2.3
000	[4]	3.1
011	[4]	3.3
021	[4]	4.1
$(=, \leq, -)$	[12]	2.3
$(\neq, \leq, -)$	[12]	2.5.1
$(<, \leq, -)$	[12]	2.6.2
$(\leq, =, -)$	[12]	2.6.4
$(\neq, \neq, \neq)$	[12]	2.7.2
$(\neq, <, \leq)$	[12]	2.8
*$(\leq, \geq, \neq)$	[12]	2.11
$(-, -, =)$	[12]	2.13
$\underline{000}$	[1]	3.1
$\underline{0^r}$	[1]	3.1
$\underline{110}$	[1]	3.2
$\underline{100}$	[1]	3.2
*$\underline{021}, \underline{102}, \underline{120}, \underline{201}$	[1]	3.3
$\underline{012}$	[1]	3.3
$\underline{210}$	[1]	3.3
$\underline{001}, \underline{010}, \underline{011}, \underline{101}$	[1]	3.3

TABLE 2.3: Cases falling in the type of proof "Constructive proof".

2.4 Bijective proof

The fourth method I present is the one where you prove a result by means of a bijection. This kind of proof is very useful if you want to compare cardinalities of different combinatorial classes and it provides a clear proof. Sometimes you already know the enumeration sequence of one of the two combinatorial classes and so you can prove that the other class is enumerated by the same sequence, but sometimes you don't know how the two classes are enumerated and so you can "just" prove that they are enumerated by the same sequence, without knowing it.

In what follows, I present three examples of this method.

2.4.1 First example

The patterns of the first example (presented by Martinez-Savage [12] in subsections 2.27.3 and 3.2.1, where some parts of the proofs are missing) are the triples of relations $(>, \geq, -), (\geq, >, -), (-, >, >)$ and $(\neq, \geq, >)$. In this case the enumerative result are two Wilf equivalences between these four patterns.

Theorem 2.4.1. *The following holds:*

1. *the patterns* $(>, \geq, -)$ *and* $(\geq, >, -)$ *are Wilf equivalent*
2. *the patterns* $(-, >, >)$ *and* $(\neq, \geq, >)$ *are Wilf equivalent.*

Proof. We first prove item 1. As mentioned in Remark 1.3.16 avoiding the patterns $(>, \geq, -)$ and $(\geq, >, -)$ is the same as avoiding the patterns $(100, 210)$ and

$(110, 210)$, respectively. We construct a bijection between $I_n(110, 210)$ and $I_n(100, 210)$ in order to prove that $|I_n(110, 210)| = |I_n(100, 210)|$.

We define a map

$$\Phi : I_n(110, 210) \to I_n(100, 210) \text{ with } \Phi(e) = e', \text{ where}$$

$$e'_j = \begin{cases} max\{e_1, \dots, e_j\}, & \text{if } e_j = e_k \text{ for some } k > j \\ e_j, & \text{otherwise} \end{cases}$$

for $1 \le j \le n$.

We go on with some considerations. By this definition it holds that

$$e'_j \le max\{e_1, \dots, e_j\} = max\{e'_1, \dots, e'_j\} \tag{2.4.1}$$

for $1 \le j \le n$, where the second equality holds for the following reason. Assume $max\{e_1, \dots, e_j\} = e_k$, for $k \in \{1, \dots, j\}$. If $e'_k \ne e_k$, then $e'_k = max\{e_1, \dots, e_k\} = e_k$. So e_k is still in $\{e'_1, \dots, e'_j\}$.

Moreover, if $e'_j \ne e_j$ then

$$e'_j = max\{e_1, \dots, e_j\} \ge e'_i \tag{2.4.2}$$

for all $1 \le i \le j$, since e'_i can be equal to $e_i \in \{e_1, \dots, e_j\}$ or to $max\{e_1, \dots, e_i\} \le max\{e_1, \dots, e_j\}$.

Now we have to check that $e' \in I_n(100, 210)$. Firstly we prove that e' avoids the pattern 100. Suppose that e' doesn't avoid 100, then there is some $i < j < k$ such that $e'_i > e'_j = e'_k$. But then 2.4.2 implies that $e'_j = e_j$ and $e'_k = e_k$ since both j and k are greater than i. So we have that $e_j = e_k$ with $j < k$ and the definition of Φ implies that $e'_j = max\{e_1, \dots, e_j\} \ge e'_i$, but this contradicts our assumption. Consequently e' avoids 100.

Secondly we prove that e' avoids 210. Suppose that e' doesn't avoid 210, then there is some $i < j < k$ such that $e'_i > e'_j > e'_k$. But then 2.4.2 implies that $e'_j = e_j$ and $e'_k = e_k$ since both j and k are greater than i. We know that e avoids 210 and so we must have that $e'_i \ne e_i$. So, there exists $s \in \{1, \dots, i-1\}$ such that $e'_i = max\{e_1, \dots, e_i\} = e_s$. But then we have $s < j < k$ and $e_k < e_j < e_s$, which contradicts the fact that e avoids 210.

So $e' \in I_n(100, 210)$.

Next we have to check if this map is a bijection.

In order to do this we present some considerations. We take an inversion sequence $e \in I_n(110, 210)$. Consider its image $e' = \Phi(e)$ and assume that there is an entry e_j with $e'_j \ne e_j$. By definition of Φ there is some $k > j$ such that $e_j = e_k = q$, where q is a value less than j. Since $e'_j \ne e_j = q, q < max\{e_1, \dots, e_{j-1}\} = e_t$ for $t \in \{1, \dots, j-1\}$. Moreover we know that e avoids 210 and so the entries after e_j must be at least q, hence $q = min\{e_j, \dots, e_n\}$.

From this observation we deduce that

$$e'_t = e_t = max\{e_1, \dots, e_{j-1}\} = e'_j > e_j = q = min\{e_j, \dots, e_n\} = min\{e'_j, \dots, e'_n\} \tag{2.4.3}$$

where the first equality holds because if $e'_t \ne e_t$ then $e'_t = max\{e_1, \dots, e_t\} = e_t$ (since $e_t = max\{e_1, \dots, e_{j-1}\}$ for $t \in \{1, \dots, j-1\}$). The last equality holds for the following reason. Assume $min\{e_j, \dots, e_n\} = e_k$ for $k \in \{j, \dots, n\}$. If $e'_k \ne e_k$, then there exists an $\ell > k$ such that $e_\ell = e_k$ and $e'_\ell = e_\ell = e_k$. So e_k is still in $\{e'_j, \dots, e'_n\}$.

So now we can define another map and we will prove that it is the inverse of Φ.

$$\Psi : I_n(100, 210) \to I_n(110, 210) \text{ with } \Psi(e') = e, \text{ where}$$

$$e_j = \begin{cases} min\{e'_j, \dots, e'_n\}, & \text{if } e'_i = e'_j \text{ for some } i < j \\ e'_j, & \text{otherwise} \end{cases}$$

for $1 \leq j \leq n$.

Let us make a consideration. If $e'_i \neq e_i$ then

$$e_j \geq min\{e'_j, \dots, e'_n\} \geq min\{e'_i, \dots, e'_n\} = e_i \tag{2.4.4}$$

for $1 \leq i \leq j$.

Now we have to check that $e \in I_n(110, 210)$. Firstly, we prove that e avoids the pattern 110. Suppose that e doesn't avoid 110, then there is some $i < j < k$ such that $e_i = e_j > e_k$. But then 2.4.4 implies that $e_i = e'_i$ and $e_j = e'_j$ since k is greater than both i and j. So we have that $e'_i = e'_j$ with $i < j$ and the definition of Ψ implies that $e_j = min\{e'_j, \dots, e'_n\} \leq min\{e'_k, \dots, e'_n\} \leq e_k$, but this contradicts our assumption.

Secondly, we prove that e avoids 210. Suppose that e doesn't avoid 210, then there is some $i < j < k$ such that $e_i > e_j > e_k$. But then 2.4.4 implies that $e_i = e'_i$ since k is greater than i and $e_j = e'_j$ since k is greater than j. We know that e' avoids 210 and so we must have that $e_k \neq e'_k$. So, there exists $s \in \{k+1, \dots, n\}$ such that $e_k = min\{e'_k, \dots, e'_n\} = e'_s$ for $s \in \{k+1, \dots, n\}$. But then we have $i < j < s$ and $e'_s < e'_j < e'_i$, which contradicts the fact that e' avoids 210.

So $e \in I_n(110, 210)$.

We proceed with the last observation. We take an inversion sequence
$e' \in I_n(100, 210)$. Consider its image $e = \Psi(e')$ and assume that there is an entry e'_j with $e_j \neq e'_j$. By definition of Ψ there is some $i < j$ such that $e'_i = e'_j = q$, where q is a value less than i. Since $e_j \neq e'_j = q, q > min\{e'_{j+1}, \dots, e'_n\} = e'_k$ for $k \in \{j+1, \dots, n\}$. Moreover, we know that e' avoids 210 and so the entries before e'_j must be at most q, hence $q = max\{e'_1, \dots, e'_j\}$.

From this observation we deduce that

$$e_k = e'_k = min\{e'_{j+1}, \dots, e'_n\} = e_j < e'_j = max\{e'_1, \dots, e'_j\} = max\{e_1, \dots, e_j\} \tag{2.4.5}$$

where the first equality holds because if $e_k \neq e'_k$ then $e_k = min\{e'_k, \dots, e'_n\} = e'_k$ (since $e'_k = min\{e'_{j+1}, \dots, e'_n\}$ for $k \in \{j+1, \dots, n\}$). The last equality holds for the following reason. Assume $max\{e'_1, \dots, e'_j\} = e'_k$ for $k \in \{1, \dots, j\}$. If $e_k \neq e'_k$, then there exists an $\ell < k$ such that $e'_\ell = e'_k$ and $e_\ell = e'_\ell = e'_k$. So e'_k is still in $\{e_1, \dots, e_j\}$.

Let $e \in I_n(110, 210)$ and $e' \in I_n(100, 210)$. Then by definition of Φ and Ψ, and by 2.4.3

$$\Psi(\Phi(e))_j = \Psi(e')_j = \begin{cases} min\{e'_j, \dots, e'_n\} = e_j, & \text{if } e'_j \neq e_j \\ e'_j = e_j, & \text{if } e'_j = e_j \end{cases}$$

and by 2.4.5

$$\Phi(\Psi(e'))_j = \Phi(e)_j = \begin{cases} max\{e_1, \dots, e_j\} = e'_j, & \text{if } e_j \neq e'_j \\ e_j = e'_j, & \text{if } e_j = e'_j. \end{cases}$$

Hence $\Psi(\Phi(e)) = e$ and $\Phi(\Psi(e')) = e'$.

We can conclude that Φ is a bijection and so we obtain that $|I_n(110, 210)| = |I_n(100, 210)|$, proving the desired Wilf equivalence.

We now prove item 2. Avoiding the patterns $(-, >, >)$ and $(\neq, \geq, >)$ is the same as avoiding the patterns $(110, 210, 120)$ and $(100, 210, 120)$, respectively. This means that the considered inversion sequences are the ones in $I_n(110, 210)$ and $I_n(100, 210)$, respectively, that avoid in addition 120. So in order to prove statement 2), we have to show that the maps Φ and Ψ preserve 120-avoidance.

Let's begin with the map Φ. Let $e \in I_n(110, 210)$ be an inversion sequence that avoids 120. Suppose that $\Phi(e) = e' \in I_n(100, 210)$ doesn't avoid 120, then there is some $i < j < k$ such that $e'_k < e'_i < e'_j$. But then 2.4.2 implies that $e'_k = e_k$ since k is greater than j. We know that e avoids 120 and hence it cannot be the case that both $e'_i = e_i$ and $e'_j = e_j$. We distinguish three cases.

Suppose $e'_i = e_i$. By the 120-avoidance of e, it follows that $e'_j \neq e_j$. So by definition of Φ, there exists $s \in \{1, \ldots, j-1\}$ such that $e_s = e'_j$. If $i < s < k$ then we have that $e_i e_s e_k$ creates a 120 in e. If $s < i < k$ then we have that $e_s e_i e_k$ creates a 210 in e. Both of the options lead to a contradiction. Note that $i = s$ is not possible since $e'_j = max\{e_1, \ldots, e_j\} > e'_i = e_i$.

Suppose $e'_j = e_j$. By the 120-avoidance of e, it follows that $e'_i \neq e_i$. So by definition of Φ, there exists $t \in \{1, \ldots, i-1\}$ such that $e_t = e'_i$. But then we have $t < j < k$ and $e_k < e_t < e_j$, which contradicts the fact that e avoids 120.

For the last case assume both $e'_i \neq e_i$ and $e'_j \neq e_j$. Let s and t be as above. Since from definition of Φ $e'_i = max\{e_1, \ldots, e_i\}$ and $e'_j = max\{e_1, \ldots, e_j\}$ and moreover $e'_i < e'_j$, we deduce that $e_t < e_s$ and $t < s$ (since e_s must be after e_i). Hence $e_t e_s e_k$ forms a 120 in e, which is a contradiction.

We conclude that Φ preserves 120-avoidance.

Now we proceed with the map Ψ. Let $e' \in I_n(100, 210)$ be an inversion sequence that avoids 120. Suppose $\Psi(e') = e \in I_n(110, 210)$ doesn't avoid 120, then there is some $i < j < k$ such that $e_k < e_i < e_j$. But then 2.4.4 implies that $e_i = e'_i$ since k is greater than i and $e_j = e'_j$ since k is greater than j. We know that e' avoids 120 and so it cannot be the case that $e_k = e'_k$. This implies that $e_k \neq e'_k$ and by definition of Ψ there exists $v \in \{k+1, \ldots, n\}$ such that $e'_v = e_k$. But then we have $i < j < v$ and $e'_v < e'_i < e'_j$, which contradicts the fact that e' avoids 120. We conclude that also Ψ preserves 120-avoidance.

Therefore statement 2) holds. □

2.4.2 Second example

The patterns of the second example (presented by Auli-Elizalde [1] in Section 4.3 and where no proof is provided) are the consecutive patterns $\underline{2010}$, $\underline{2110}$ and $\underline{2120}$. In this case the enumerative result is a super-strong Wilf equivalence between these three patterns.

We first recall Definition 1.3.20 of $Em(p, e)$.

Definition 2.4.2. *An inversion sequence e of length n contains the consecutive pattern $p = \underline{p_1 p_2 \ldots p_k}$ if there exist a consecutive subsequence $e_i e_{i+1} \ldots e_{i+k-1}$ of e such that its reduction is equal to p. In this case, we call $e_i e_{i+1} \ldots e_{i+k-1}$ an occurrence of p in position i. Otherwise, e avoids p.*
Moreover, $Em(p, e) = \{i : e_i e_{i+1} \ldots e_{i+k-1}$ is an occurrence of $p\}$.

Now we prove a lemma that we will use in the proof of the Wilf equivalences.

Lemma 2.4.3. *Let p and p' be two consecutive patterns. If for all n and for all $S \subseteq \{1,\dots,n\}$*

$$|\{e \in I_n : Em(p,e) \supseteq S\}| = \left|\{e \in I_n : Em(p',e) \supseteq S\}\right|,$$

then p and p' are super-strongly Wilf-equivalent.

Proof. Let $S \subseteq \{1,\dots,n\}$. We define two functions $f_{eq}, f_{inc} : 2^S \to \mathbb{R}$, where 2^S is the powerset of S, such that

$$f_{eq}(S) = |\{e \in I_n : Em(p,e) = S\}| \text{ and } f_{inc}(S) = |\{e \in I_n : Em(p,e) \supseteq S\}|.$$

From this definition we can see that

$$f_{inc}(T) = \sum_{S \supseteq T} f_{eq}(S)$$

for all $T \subseteq \{1,\dots,n\}$. By the principle of Inclusion-Exclusion ([14]) we obtain that

$$f_{eq}(T) = \sum_{S \supseteq T} (-1)^{|S \setminus T|} f_{inc}(S) \tag{2.4.6}$$

using $f_{inc}(T) = \Phi(f_{eq}(T))$ and $f_{eq}(T) = \Phi^{-1}(f_{inc}(T))$, where Φ is a real linear transformation, in the notation of [14].

Proceeding in the same way, we define two functions $f'_{eq}, f'_{inc} : 2^S \to \mathbb{R}$ such that

$$f'_{eq}(S) = |\{e \in I_n : Em(p',e) = S\}| \text{ and } f'_{inc}(S) = |\{e \in I_n : Em(p',e) \supseteq S\}|.$$

From this we can see that

$$f'_{inc}(T) = \sum_{S \supseteq T} f'_{eq}(S)$$

for all $T \subseteq \{1,\dots,n\}$. By the principle of Inclusion-Exclusion we deduce that

$$f'_{eq}(T) = \sum_{S \supseteq T} (-1)^{|S \setminus T|} f'_{inc}(S). \tag{2.4.7}$$

Our assumption was that $f_{inc}(S) = f'_{inc}(S)$ for all $S \subseteq \{1,\dots,n\}$. Hence from 2.4.6 and 2.4.7 we obtain that $f_{eq}(T) = f'_{eq}(T)$ for all $T \subseteq \{1,\dots,n\}$. Applying Definition 1.3.22, the result follows. □

Now we prove the following result.

Proposition 2.4.4. *The patterns $\underline{2010}$, $\underline{2110}$ and $\underline{2120}$ are super-strongly Wilf equivalent.*

Proof. We construct a bijection between

$$\{e \in I_n : Em(\underline{2010},e) \supseteq S\} \text{ and } \{e \in I_n : Em(\underline{2110},e) \supseteq S\}$$

in order to prove that

$$|\{e \in I_n : Em(\underline{2010},e) \supseteq S\}| = |\{e \in I_n : Em(\underline{2110},e) \supseteq S\}|$$

for all n and for all $S \subseteq \{1,\dots,n\}$.

We start with some considerations about the set S. Since no two occurrences of $\underline{2010}$ and $\underline{2110}$ can overlap in three entries, S doesn't contain consecutive numbers. Moreover, S can be written in a unique way as a disjoint union of maximal subsets whose entries define an arithmetic sequence with difference 2. This means that we write S as $S = \sqcup_{j=1}^{q} B_j$, where $B_j = \{i_j, i_j+2, \ldots, i_j+2l_j\}$ and $i_j + 2(l_j+1) < i_{j+1}$ with $1 \leq j \leq q-1$.

After these considerations about S we can define a map

$\alpha_s : \{e \in I_n : Em(\underline{2010}, e) \supseteq S\} \to \{e \in I_n : Em(\underline{2110}, e) \supseteq S\}$ with $\alpha_s(e) = e'$, where

$$e'_i = \begin{cases} e_{i+1}, & \text{if } i-1 \in S \\ e_i, & \text{otherwise} \end{cases}$$

for $1 \leq i \leq n$. This map transforms occurrences of the pattern $\underline{2010}$ in positions of S into occurrences of the pattern $\underline{2110}$.

For example for $S = \{4,6\}$ $\alpha_s(0,0,1,3,0,2,0,1,0) = (0,0,1,3,2,2,1,1,0)$. It can be noticed that $(0,0,1,3,0,2,0,1,0)$ contains two occurrences of $\underline{2010}$, one in position 4 and the other in position 6; $(0,0,1,3,2,2,1,1,0)$ contains two occurrences of $\underline{2110}$, one in position 4 and the other in position 6 like before.

Now we define another map β_s which is the inverse of α_s (by construction), with $\beta_s(e') = e$, where

$$e_i = \begin{cases} e'_{i_j+2l_j+3}, & \text{if } i-1 \in B_j \\ e'_i, & \text{otherwise} \end{cases}$$

for $1 \leq i \leq n$ and $i_j + 2l_j$ is the last entry of B_j. This map transforms occurrences of of the pattern $\underline{2110}$ in positions of S into occurrences of the pattern $\underline{2010}$.

For example for $S = \{4,6\}, B_1 = \{4,6\}, i_1 = 4$ and $l = 1$, $\beta_s(0,0,1,3,2,2,1,1,0) = (0,0,1,3,0,2,0,1,0)$ since $e'_{i_j+2l_j+3} = e'_9 = 0$. Also in this case it can be noticed that $(0,0,1,3,2,2,1,1,0)$ contains two occurrences of $\underline{2110}$, one in position 4 and the other in position 6; $(0,0,1,3,0,2,0,1,0)$ contains two occurrences of $\underline{2010}$, one in position 4 and the other in position 6 like before.

Since β_s is the inverse of α_s, we can conclude that α_s is a bijection and hence

$$|\{e \in I_n : Em(\underline{2010}, e) \supseteq S\}| = |\{e \in I_n : Em(\underline{2110}, e) \supseteq S\}|$$

for all n and for all $S \subseteq \{1, \ldots, n\}$. Applying Lemma 2.4.3 we obtain the super-strongly Wilf equivalence between $\underline{2010}$ and $\underline{2110}$.

In order to show that all the three patterns $\underline{2010}$, $\underline{2110}$ and $\underline{2120}$ are super-strongly Wilf equivalent, it remains to be shown that the first one and the third one are super-strongly Wilf equivalent. The former considerations about the set S hold also in this case.

So we can define a map

$\gamma_s : \{e \in I_n : Em(\underline{2010}, e) \supseteq S\} \to \{e \in I_n : Em(\underline{2120}, e) \supseteq S\}$ with $\gamma_s(e) = e'$, where

$$e'_i = \begin{cases} e_{i+1}, & \text{if } i-1 \in S \\ e_{i_j}, & \text{if } i-2 \in B_j \\ e_i, & \text{otherwise} \end{cases}$$

for $1 \leq i \leq n$. The first line comes from the fact that the second entry of the pattern changes from 0 to 1. The second line comes from the fact that the third entry of the pattern changes from 1 to 2. The point is that we change two entries of the pattern and not only one, and that two occurrences of the patterns $\underline{2010}$ and $\underline{2120}$ can overlap in two entries. This map transforms occurrences of the pattern $\underline{2010}$ in positions of S into occurrences of the pattern $\underline{2120}$.

For example for $S = \{5,7,9\}, B_1 = \{5,7,9\}, i_1 = 5$ and $l_1 = 2$, $\gamma_s(0,0,1,1,4,0,3,0,2,0,1,0) = (0,0,1,1,4,3,4,2,4,1,4,0)$ since $e_{i_j} = e_5 = 4$. It can be noticed that $(0,0,1,1,4,0,3,0,2,0,1,0)$ contains three occurrences of $\underline{2010}$ in positions $5,7,9$ and $(0,0,1,1,4,3,4,2,4,1,4,0)$ contains three occurrences of $\underline{2120}$ in positions $5,7,9$ like before.

Now we define another map δ_s, which is the inverse of γ_s (by construction), with $\delta_s(e') = e$, where

$$e_i = \begin{cases} e'_{i_j+2l_j+3}, & \text{if } i-1 \in B_j \\ e'_{i-1}, & \text{if } i-2 \in S \\ e'_i, & \text{otherwise} \end{cases}$$

for $1 \leq i \leq n$ and $i_j + 2l_j$ is the last entry of B_j. This map transforms occurrences of the pattern $\underline{2120}$ in positions of S into occurrences of the pattern $\underline{2010}$.

For example for $S = \{5,7,9\}, B_1 = \{5,7,9\}, i_1 = 5$ and $l_1 = 2$, $\delta_s(0,0,1,1,4,3,4,2,4,1,4,0) = (0,0,1,1,4,0,3,0,2,0,1,0)$ since $e'_{i_j+2l_j+3} = e'_{12} = 0$. Also in this case it can be seen that $(0,0,1,1,4,3,4,2,4,1,4,0)$ contains three occurrences of $\underline{2120}$ in positions $5,7,9$ and $(0,0,1,1,4,0,3,0,2,0,1,0)$ contains three occurrences of $\underline{2010}$ in positions $5,7,9$ like before.

Since δ_s is the inverse of γ_s, we can conclude that γ_s is a bijection and hence

$$|\{e \in I_n : Em(\underline{2010}, e) \supseteq S\}| = |\{e \in I_n : Em(\underline{2120}, e) \supseteq S\}|$$

for all n and for all $S \subseteq \{1, \ldots, n\}$. Applying Lemma 2.4.3 we obtain the super-strongly Wilf equivalence between $\underline{2010}$ and $\underline{2120}$.

Therefore we can conclude that $\underline{2010} \overset{s}{\sim} \underline{2110} \overset{s}{\sim} \underline{2120}$. □

2.4.3 Example concerning statistics

I decided to present a third example (exhibited by Yan-Lin [15] in Chapter 5 and where some parts of the proof are missing) in order to illustrate that a bijection can in addition preserve some statistic. In this case the number of preserved statistics is three. The patterns of the this example are two pairs of classical patterns of length 3.

Theorem 2.4.5. *There exists a bijection between $I_n(010, 101)$ and $I_n(010, 100)$ that preserves the triple of statistics (dist, maxim, zeros).*

Proof. We start by characterizing the inversion sequences that avoid the pattern $(010, 101)$: an inversion sequence $e \in I_n$ avoids $(010, 101)$ if and only if $e_i = e_j$ with $i < j$ implies that $e_k = e_i$ for all $i \leq k \leq j$.

This holds for the following reason. Let $e \in I_n$ such that e avoids $(010, 101)$. Assume that $e_i = e_j$ for some $i < j$ and $e_k \neq e_i$ for some k between i and j.

If $e_k < e_i$ then we have $i < k < j$ and $e_i > e_k < e_j, e_i = e_j$, which contradicts the fact that e avoids 101.

If $e_k > e_i$ then we have $i < k < j$ and $e_i < e_k > e_j, e_i = e_j$, which contradicts the fact that e avoids 010.

Consequently $e_i = e_j$ with $i < j$ implies that $e_k = e_i$ for all $i \leq k \leq j$.

Conversely, assume that $e \in I_n$ is such that $e_i = e_j$ with $i < j$ implies that $e_k = e_i$ for all $i \leq k \leq j$. This means that if we have two equal entries, say equal to q, the values between them are neither greater nor less than q. This implies that e avoids 010 and 101.

Now that we know how these pattern-avoiding inversion sequences are characterized, we can define a map

$$\alpha : I_n(010,101) \to I_n(010,100) \text{ with } \alpha(e) = e', \text{ where}$$

$$e'_i = \begin{cases} max\{e_1,\dots,e_{i-2}\}, & \text{if } e_{i-1} = e_i < max\{e_1,\dots,e_{i-2}\} \\ e_i, & \text{otherwise.} \end{cases}$$

For example $\alpha(0,0,1,2,2,5,4,4,4) = (0,0,1,2,2,5,4,5,5)$ since $e_7 = e_8 = e_9 < e_6$.

Next we check if this map is a bijection. By definition of α, if we have $e_{i-1} = e_i < max\{e_1,\dots,e_{i-2}\}$ then e_i changes to the value $max\{e_1,\dots,e_{i-2}\}$ and we don't have anymore the pattern 100. Moreover, let's assume that e avoids 100. If an entry e_i changes, it changes to $max\{e_1,\dots,e_{i-2}\} = e_\ell$ for $\ell \in \{1,\dots,i-2\}$. This doesn't create a 100 because there are no values greater than $max\{e_1,\dots,e_{i-2}\}$ to the left of e_ℓ and there are no two equal entries less than $max\{e_1,\dots,e_{i-2}\}$ to the right of e_i, otherwise e would contain a 100. Hence $e' \in I_n(010,100)$.

We construct the inverse of α

$$\beta : I_n(010,100) \to I_n(010,101) \text{ with } \beta(e') = e, \text{ where}$$

$$e_i = \begin{cases} e'_j, & \text{if } e'_i \in \{e'_1,\dots,e'_{j-1}\} \text{ and } e'_j = \text{rightmost entry to the left of } e'_i \text{ different from } e'_i \\ e'_i, & \text{otherwise.} \end{cases}$$

For example $\beta(0,0,1,2,2,5,4,5,5) = (0,0,1,2,2,5,4,4,4)$ since $e'_8, e'_9 \in \{e'_1,\dots,e'_6\}$.

Let $e \in I_n(010,101)$ and $e' \in I_n(010,100)$. It holds that $\beta(\alpha(e)) = e$ since α changes those values e_i for which $e_{i-1} = e_i < max\{e_1,\dots,e_{i-2}\}$ but then these are exactly the values e'_i for which $e'_i \in \{e'_1,\dots,e'_{j-1}\}$ (and $e'_j = e_i$ since the first entry of the sequence of values equal to e_i and less than $max\{e_1,\dots,e_{i-2}\}$ doesn't change). Moreover, it holds that $\alpha(\beta(e')) = e'$ since β changes those values e'_i for which $e'_i \in \{e'_1,\dots,e'_{j-1}\}$ but then these are exactly the values for which $e_{i-1} = e_i < max\{e_1,\dots,e_{i-2}\}$ (and $max\{e_1,\dots,e_{i-2}\} = e'_i$ since e' avoids 100 and 010). Hence α and β are inverses of each other and so we conclude that α is a bijection.

The last thing we have to check is that α preserves the statistics dist, maxim and zeros.

If we apply the map α to an inversion sequence e, the entries of e remain the same or change to $max\{e_1,\dots,e_{i-2}\}$ and hence no new values are added: the statistic dist is preserved.

If an inversion sequence $e \in I_n(010,101)$ contains $e_i = i-1$ for some i, this implies that $e_i \neq e_{i-1} < i-1$ and hence e_i remains the same; moreover, if you change an entry you add a value which was already used, so you don't add a maximal element: the statistic maxim is preserved.

Let $e \in I_n(010,101)$ be an inversion sequence. Like for all inversion sequences $e_1 = 0$ and so if $e_j = 0$ then $e_i = 0$ for all $1 \le i \le j$, since e avoids 010. This means that the zero entries don't have values $\neq 0$ to the left and hence a zero entry remains the same; moreover, if you change an entry you add the entry $max\{e_1,\dots,e_{i-2}\} > e_i = e_{i-1}$ and so you don't add a zero entry: the statistic zeros is preserved. □

Table 2.4 shows a list of cases falling in the type of proof presented in this section.

Pattern	**Reference**	**Section**
210, 201	[4]	2.3
000	[4]	3.1
001	[4]	3.2
021	[4]	4.1
021	[4]	4.2
$(\geq, \leq, \neq)$	[12]	2.5.2
$(<, \geq, -)$	[12]	2.6.3
$(\neq, <, -)$	[12]	2.7.1
$(\geq, \geq, -), (-, \leq, \geq)$	[12]	2.16
$(\neq, \neq, =)$	[12]	2.17.3
$(>, -, \geq)$	[12]	2.24.2
$(>, -, >), (\geq, \neq, >)$	[12]	2.26
*$(>, \geq, -), (\geq, >, -)$	[12]	2.27.3
$(>, \geq, -), (-, \leq, >)$	[12]	2.27.4
*$(-, >, >), (\neq, \geq, >)$	[12]	3.2.1
$(-, \neq, >), (\neq, -, >)$	[12]	3.2.2
$(-, >, \geq), (\neq, \geq, \geq)$	[12]	3.2.3
$(-, \neq, \geq), (\neq, -, \geq)$	[12]	3.2.4
$(1-23, 2-14-3)$	[2]	2.3
(000, 011), (011, 012), (001, 101), (001, 102), (001, 201), (010, 012)	[15]	2.6
(012, 021), (110, 012)	[15]	2.7
(011, 021), (010, 021)	[15]	2.9
(012, 102), (012, 120), (011, 102)	[15]	3
011	[15]	5
(010, 101)	[15]	5
*(010, 101), (010, 100)	[15]	5
(101, 102), (101, 021)	[15]	6
(011, 201), (011, 210); (000, 201), (000, 210); (100, 021), (110, 021)	[15]	8
(010, 201), (010, 210)	[15]	8
110, 100	[1]	3.2
012, 321	[1]	3.4
021, 1324	[1]	3.4
0021, 0121	[1]	4.1
1002, 1012, 1102	[1]	4.1
0100, 0110	[1]	4.1
2013, 2103	[1]	4.1
1200, 1210, 1220	[1]	4.1
0211, 0221	[1]	4.1
1001, 1101, 1220	[1]	4.1
2001, 2011, 2101, 2201	[1]	4.1
2012, 2101	[1]	4.1
3012, 3102	[1]	4.1
1000, 1110	[1]	4.2
2100, 2210	[1]	4.2
0102, 0112	[1]	4.3
*2010, 2110, 2120	[1]	4.3

TABLE 2.4: Cases falling in the type of proof "Bijective proof".

2.5 Generating function

The fifth method I present is the one where you prove a result by means of generating functions. The following definitions briefly explain the concept of generating function.

Definition 2.5.1. *A formal power series is a sequence* $(f_n)_{n\geq 0}$ *of elements in a field K. It is denoted* $F(x) = \sum_{n\geq 0} f_n x^n$.

Definition 2.5.2. *The ordinary generating function of* $\mathcal{C}$ *is the formal power series (with coefficients in* $\mathbb{Q}$*)*

$$\sum_{n\geq 0} c_n x^n$$

where c_n*= number of objects of* $\mathcal{C}$ *of size n.*

Later we will also use the following concept.

Definition 2.5.3. *Let* $\mathcal{P} : \mathcal{C} \to \mathbb{N}$ *be a parameter on* $\mathcal{C}$*. The ordinary bivariate generating function of* $\mathcal{C}$ *with the parameter* $\mathcal{P}$ *is the formal power series (with coefficients in* $\mathbb{Q}[y]$*)*

$$\sum_{n\geq 0} \sum_{k\geq 0} c_{n,k} x^n y^k$$

where $c_{n,k}$*= number of objects of* $\mathcal{C}$ *of size n and with value k for* $\mathcal{P}$*.*

In this method you use the important fact that two generating functions are equal if and only if their coefficients are also equal. When you want to enumerate a certain family of combinatorial objects you check if its generating function is the same as the generating function for a well-known combinatorial sequence, showing that they are solutions of the same equation on generating functions.

Formal power series can be interpreted as functions on $x \in \mathbb{C}$, defined "closed to $x = 0$"; conversely, the Taylor series expansion of an analytic function $F : \mathbb{C} \to \mathbb{C}$ around the origin defines a formal power series. This is the way we view formal power series in this section.

In the sequel I present two examples of this method, with two different well-known combinatorial sequences.

2.5.1 First example

The pattern of the first example (presented by Corteel et al. [4] in Section 2.2) is the classical pattern 021.

Before proving the enumerative result about the 021-avoiding inversion sequences, we present a certain combinatorial class and its generating function. The combinatorial class in question is the one of Schröder paths. A Schröder path of size n is a path from $(0, 0)$ to $(2n, 0)$, staying weakly above the x-axis and using only the up-step $(1, 1)$, the down-step $(1, -1)$ and the long horizontal step $(2, 0)$. Note that the size of a Schröder path is the half of the x-coordinate of the endpoint. Moreover, for all n the number of Schröder paths of size n is finite. For example $p = UUUDFDFUDDFUFFD$ is a Schröder path of size 15 where U, D, F represent the up-step, the down-step and the long horizontal step.

We denote by $\mathcal{R}$ the combinatorial class of Schröder paths and by $\mathcal{R}_n$ the set of Schröder paths of size n. The ordinary generating function of $\mathcal{R}$ is the formal power series

$$R(x) = \sum_{n\geq 0} r_n x^n \tag{2.5.1}$$

where $r_n = |\mathcal{R}_n|$.

We need to compute $R(x)$ and so we proceed in the following way. A Schröder path can have three different forms. It can be empty and this corresponds to the class $\mathcal{E}$ with one element of size 0. Or it can start with a long horizontal step (followed by a Schröder path): such Schröder paths correspond to pairs (F, p) where F belongs to the class $\mathcal{Z}$ with one element of size 1 and p belongs to the class $\mathcal{R}$ of Schröder paths. Or it can start with an up-step (followed by a Schröder path) that is matched with the down-step (followed by a Schröder path) corresponding to the first return of the path (which is the smallest number $k > 0$ such that $(2k, 0)$ is on the path): such Schröder paths correspond to triples (F, p, p') where F belongs to the class $\mathcal{Z}$ with one element of size 1 and p, p' belong to the class $\mathcal{R}$ of Schröder paths. Hence we obtained that

$$\mathcal{R} = \mathcal{E} + \mathcal{Z} \times \mathcal{R} + \mathcal{Z} \times \mathcal{R} \times \mathcal{R}.$$

Translating this equation on combinatorial classes into an equation on generating functions we obtain

$$R(x) = 1 + xR(x) + xR^2(x)$$

and solving it with respect to x gives

$$R(x) = \frac{1 - x \pm \sqrt{x^2 - 2x + 1 - 4x}}{2x} = \frac{1 - x \pm \sqrt{x^2 - 6x + 1}}{2x}.$$

Since the empty path is in $\mathcal{R}$ it holds that $R(0) = 1$ and hence the formal power series $R(x)$ satisfies

$$R(x) = \frac{1 - x - \sqrt{x^2 - 6x + 1}}{2x}.$$

Now we proceed with the following enumerative result.

Theorem 2.5.4. $|I_n(021)| = r_{n-1}$, *for* $n \geq 1$.

Proof. We start by characterizing the inversion sequences that avoid the pattern 021: an inversion sequence $e \in I_n$ avoids 021 if and only if the positive entries are weakly increasing.

This holds for the following reason. Let $e \in I_n$ such that e avoids 021. Assume there is $i < j$, such that $e_i > e_j$ (both different from 0). But then we have $1 < i < j$ and $0 = e_1 < e_j < e_i$, which contradicts the fact that there is no $i < j < k$ with $e_i < e_k < e_j$. So $e_i \leq e_j$ with $i < j$ for all the entries different from zero implies that the positive entries are weakly increasing.

Conversely, suppose that the positive elements of e are weakly increasing. Assume $e \in I_n$ doesn't avoid 021. Then there is $i < j < k$ with $e_i < e_k < e_j$ (so e_k and e_j are both different from 0). But $e_k < e_j$ (both different from 0) with $j < k$ contradicts the fact that the positive elements of e are weakly increasing. Therefore e avoids 021.

Now that we know how these pattern-avoiding inversion sequences are characterized, we proceed in the following way. We define $G(x) = \sum_{n=1}^{\infty} |I_n(021)| x^n$ and we show that it satisfies the equation

$$G(x) = x + xG(x) + G^2(x). \tag{2.5.2}$$

Solving it with respect to $G(x)$ gives

$$G(x) = \frac{1 - x \pm \sqrt{x^2 - 2x + 1 - 4x}}{2} = \frac{1 - x \pm \sqrt{x^2 - 6x + 1}}{2} = \frac{1 - x - \sqrt{x^2 - 6x + 1}}{2}$$

where the last equality holds because $G(0) = 0$. Hence $G(x) = xR(x) = \sum_{n\geq 0} r_n x^{n+1} = \sum_{n\geq 1} r_{n-1}x^n$ and the claim will hold.

Take an inversion sequence $e \in I_n(021)$ and let $k+1$ be the greatest index such that $e_{k+1} = k$. We distinguish two cases.

If $k = 0$ then $e = 0 \cdot e'$ for some $e' \in I_{n-1}(021)$ since e has no maximal elements except e_1. Conversely, take $g \in I_{n-1}(021)$. If we add a zero as first entry we obtain a sequence in $I_n(021)$ where $e_1 = 0$ and after this entry there are no more maximal elements. This contributes the $x + \sum_{n=2}^{\infty} |I_{n-1}(021)|\, x^n = x + \sum_{n=1}^{\infty} |I_n(021)|\, x^{n+1} = x + xG(x)$ (where the x represents the only object of size 1) to the equation 2.7.

If $k > 0$ we know that $(e_1, \ldots, e_k) \in I_k(021)$ since $e \in I_n(021)$. For the entries after position $k+1$ it holds that either $e_{k+i} = 0$ or $e_{k+1} = k \leq e_{k+i} < k+i-1, i \in \{2, \ldots, n-k\}$, using the above characterization and by definition of k. Then if we subtract $k-1$ from the positive entries of $(e_{k+2}, \ldots, e_n)$ and we add a zero as first entry we obtain a sequence in $I_{n-k}(021)$, in which e_{k+i} is the i-th entry. Conversely, take the inversion sequences $(e_1, \ldots, e_k) \in I_k(021)$ and $f \in I_{n-k}(021)$. If we add $k-1$ to the positive entries of f, we get rid of the initial zero, and we concatenate the resulting sequence after $(e_1, \ldots, e_k, k)$, we obtain an inversion sequence in $I_n(021)$ where $e_{k+1} = k$ and after this entry there are no more maximal elements. This contributes the

$$\begin{aligned}\sum_{n\geq 1}\Big(\sum_{k=1}^{n-1} |I_k(021)| \cdot |I_{n-k}(021)|\Big)x^n &= G(x) \cdot G(x) \\ &= G^2(x)\end{aligned}$$

to the equation 2.7, where in the first equality we used the Cauchy product (with $[x^0]G(x) = 0$ and $[x^n]G(x) = |I_n(021)|$ when $n \geq 1$). □

2.5.2 Second example

The pattern of the second example (presented by Martinez-Savage [12] in subsection 2.14.2) is the triple of relations $(-, \geq, <)$.

Before proving the enumerative result about the $(-, \geq, <)$-avoiding inversion sequences, we show that the generating function for the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ is $\frac{1-\sqrt{1-4x}}{2x}$. We denote $\frac{1-\sqrt{1-4x}}{2x}$ by $C(x)$. Using the formula $(1+z)^\alpha = \sum_{n\geq 0}\binom{\alpha}{n} z^n$ with $z = -4x$ and $\alpha = \frac{1}{2}$ we have

$$\sqrt{1-4x} = \sum_{n\geq 0}\binom{1/2}{n}(-4x)^n.$$

Using the formula for the generalized binomial coefficient for $n \geq 0$

$$\binom{\alpha}{n} = \prod_{k=1}^{n} \frac{\alpha - k + 1}{k} = \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}$$

we compute for $n \geq 0$

$$\begin{aligned}
\binom{1/2}{n} &= \prod_{k=1}^{n} \frac{\frac{1}{2} - k + 1}{k} \\
&= \prod_{k=1}^{n} \frac{1 - 2k + 2}{2k} \\
&= \prod_{k=1}^{n} \frac{-(2k-3)}{2k} \\
&= \frac{(-1)^n}{2^n}(-1) \prod_{k=2}^{n} \frac{2k-3}{k}\frac{2k-2}{2k-2} \\
&= \frac{(-1)^{n+1}}{2^n} \prod_{k=2}^{n} \frac{(2k-3)(2k-2)}{2k(k-1)} \\
&= \frac{(-1)^{n+1}}{2^{n+n-1}} \frac{1}{n!} \frac{1}{(n-1)!}(2n-2)! \frac{(2n-1)\cdot 2n}{(2n-1)\cdot 2n} \\
&= \frac{(-1)^{n+1}}{4^n} \frac{1}{n!} \frac{1}{n!} \frac{1}{2n-1}(2n)! \\
&= \frac{(-1)^{n+1}}{4^n} \frac{1}{2n-1} \binom{2n}{n}
\end{aligned}$$

and substituting it in the expansion of $\sqrt{1-4x}$ we obtain

$$\begin{aligned}
C(x) &= \frac{1}{2x} - \frac{1}{2x} \sum_{n \geq 0} \frac{(-1)^{n+1}}{4^n} \frac{1}{2n-1} \binom{2n}{n} (-4x)^n \\
&= \frac{1}{2x} - \frac{1}{2x} - \frac{1}{2x} \sum_{n \geq 1} \frac{(-1)^{n+1}}{4^n} \frac{1}{2n-1} \binom{2n}{n} (-4x)^n \\
&= \sum_{n \geq 1} \frac{1}{2} \frac{(-1)^n}{4^n} \frac{1}{2n-1} \binom{2n}{n} (-1)^n 4^n x^{n-1} \\
&= \sum_{m \geq 0} \frac{1}{2} \frac{1}{2(m+1)-1} \binom{2(m+1)}{m+1} x^m \\
&= \sum_{m \geq 0} \frac{1}{2} \frac{1}{2m+1} \frac{(2m+2)!}{(m+1)!(m+1)!} x^m \\
&= \sum_{m \geq 0} \frac{1}{2} \frac{1}{2m+1} \frac{(2m+2)(2m+1)(2m)!}{(m+1)(m+1)(m!)^2} x^m \\
&= \sum_{m \geq 0} \frac{1}{m+1} \binom{2m}{m} x^m \\
&= \sum_{m \geq 0} C_m x^m.
\end{aligned}$$

Now we prove the following enumerative result.

Theorem 2.5.5. $|I_n(-, \geq, <)| = C_n$.

Proof. We start by characterizing the inversion sequences that avoid the pattern $(-, \geq, <)$: an inversion sequence $e \in I_n$ has no $i < j < k$ such that $e_j \geq e_k$ and $e_i < e_k$ if and only if the positive entries are strictly increasing.

This holds for the following reason. Let $e \in I_n$ such that there is no $i < j < k$ with $e_j \geq e_k$ and $e_i < e_k$. Assume there is $i < j$ such that $e_i \geq e_j$ (both different from

0). But then we have $1 < i < j$ and $e_1 < e_j \leq e_i$, which is a contradiction. So $e_i < e_j$ with $i < j$ for all the entries different from zero implies that the positive entries are strictly increasing.

Conversely, suppose that the positive elements of e are strictly increasing. But then if we have $j < k$ with $e_j \geq e_k$, this implies that $e_k = 0$ and so there is no $i < k$ such that $e_i < e_k$. Therefore e avoids $(-, \geq, <)$.

Now that we know how these pattern-avoiding inversion sequences are characterized, we proceed in the following way. We define $L(x) = \sum_{n=0}^{\infty} |I_n(-, \geq, <)| \, x^n$ and we show that it satisfies the equation

$$L(x) = 1 + xL^2(x). \tag{2.5.3}$$

Solving it with respect to $L(x)$ gives

$$L(x) = \frac{1 \pm \sqrt{1-4x}}{2x} = \frac{1 - \sqrt{1-4x}}{2x}$$

where the last equality holds because $L(0) = 1$. Since $\frac{1-\sqrt{1-4x}}{2x}$ is the generating function for C_n, the claim will hold.

Take an inversion sequence $e \in I_n(-, \geq, <)$ and let e_m be the last maximal entry (that is, the last entry of e having the maximal possible value at its position). We know that $(e_1, \ldots, e_{m-1}) \in I_{m-1}(-, \geq, <)$ since $e \in I_n(-, \geq, <)$. Moreover, if we subtract $m-1$ from the positive entries of $(e_{m+1}, \ldots, e_n)$ we obtain a sequence in $I_{n-m}(-, \geq, <)$, since e_j is at most $j-2$ for all $m+1 \leq j \leq n$ by definition of m and using the aforementioned characterization.

Conversely, take the inversion sequences $g \in I_{m-1}(-, \geq, <)$ and $h \in I_{n-m}(-, \geq, <)$. If you add $m-1$ to the positive entries of h and we concatenate it after $g \cdot (m-1)$, we obtain an inversion sequence in $I_n(-, \geq, <)$ where $e_m = m-1$ is the last maximal entry.

This contributes the term

$$\begin{aligned}
&\sum_{n\geq 1} \Big(\sum_{m=1}^{n} |I_{m-1}(-, \geq, <)| \cdot |I_{n-m}(-, \geq, <)| \, x^n \Big) \\
=&\sum_{n\geq 1} \Big(\sum_{m=0}^{n-1} |I_{m}(-, \geq, <)| \cdot |I_{n-m-1}(-, \geq, <)| \, x^n \Big) \\
=&\sum_{n\geq 0} \Big(\sum_{m=0}^{n} |I_{m}(-, \geq, <)| \cdot |I_{n-m}(-, \geq, <)| \, x^{n+1} \Big) \\
=&x \sum_{n\geq 0} \Big(\sum_{m=0}^{n} |I_{m}(-, \geq, <)| \cdot |I_{n-m}(-, \geq, <)| \, x^{n} \Big) \\
=&xL(x) \cdot L(x) \\
=&xL^2(x)
\end{aligned}$$

to the equation 2.5.3, where we used the Cauchy product.

The inversion sequence of length $n = 0$, not considered in our process, contributes the term 1 to the equation 2.5.3. □

Table 2.5 shows a list of cases falling in the type of proof presented in this section.

Pattern	Reference	Section
*021	[4]	2.2
*$(-, \geq, <)$	[12]	2.14.2
$(\geq, -, >)$	[12]	2.24.3
$(\leq, >, \neq)$	[12]	3.1.1

TABLE 2.5: Cases falling in the type of proof "Generating function".

2.6 Generating tree

The sixth method I present is the one where you prove a result by means of generating trees. We start by explaining the concept of generating tree, following [2].

Let $\mathcal{C}$ be a combinatorial class such that $|\mathcal{C}_1| = 1$. A generating tree for $\mathcal{C}$ is an infinite rooted tree with the objects of $\mathcal{C}$ being the vertices of the tree, such that every object is represented only once in the tree, and such that at the level n of the tree there are only objects of size n (we index the levels from 1 starting from the root).

You go to the next level of the tree adding an atom, which is a part of the object that increases the size of the object by 1, to an object $c \in \mathcal{C}$ of the previous level: the objects obtained by adding an atom to some object c are called the children of c.

From above we know that every object is represented only once in the tree and therefore we have to define some rules that tell which additions are admitted: the process of adding atoms following these conditions is called the growth of $\mathcal{C}$.

When the number of children of each node in the generating tree is controlled by the value of some statistic, the growth of $\mathcal{C}$ is transferred into a succession rule in which you have a first label, called axiom, that indicates the value of the statistic on the root object, and a set of pairs (label, set of labels), called productions, that indicate how the values of the statistic in the tree evolves.

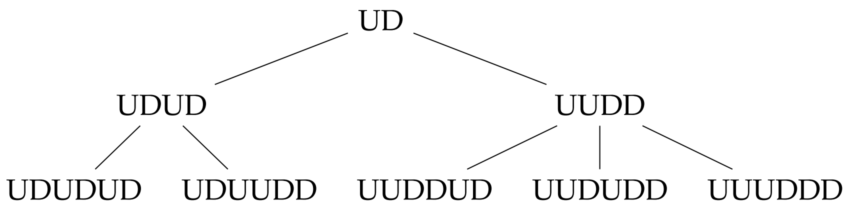
You can find any term of the enumeration sequence of a combinatorial class $\mathcal{C}$ directly from the succession rule: if you count (with repetition) the number of labels that are produced by applying $n-1$ times the set of productions starting from the root, you obtain the n-th term of the enumeration sequence. Another way to find it is from the generating tree: if you count the number of vertices at level n in the tree you obtain the n-th term of the enumeration sequence.

In what follows, I present an example in order to understand better the concepts of generating tree and succession rule. It is about the classical growth for the combinatorial class of Dyck paths.

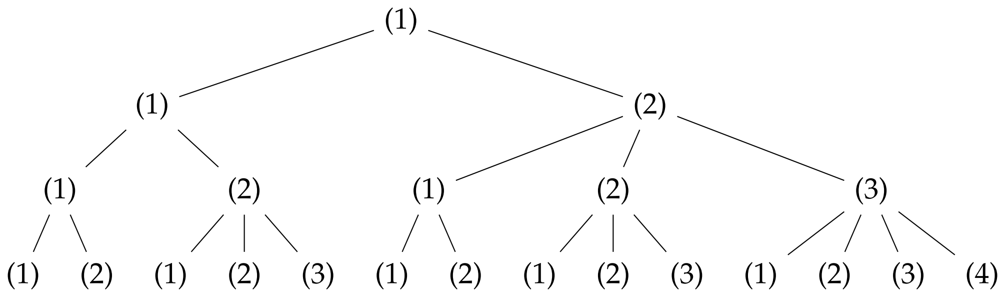
Definition 2.6.1. *A Dyck path of size n is a lattice path from $(0,0)$ to $(2n,0)$, staying weakly above the x-axis and using only the up-step $U = (1,1)$ and the down-step $D = (1,-1)$.*

In this case the atoms are the factors UD. So in order to obtain a child of some Dyck path d we add a factor UD to d. Since every Dyck path is represented only once in the tree we define a rule that says where we can add the atom: we can insert a UD factor only at some point of the last descent, which is the longest suffix without U-steps. Hence if D^k is the last descent, you can insert the UD factor in the following ways: $UDD^k, DUDD^{k-1}, DDUDD^{k-2}, \ldots, D^{k-2}UDDD, D^{k-1}UDD, D^kUD$.

In the next figure we can see the first three levels of the generating tree, with the objects as nodes, for the class of Dyck paths.



In this case the number of children represented in the tree is controlled by the length of the last descent and so the labels in the generating tree indicate the number of D steps in the last descent. In the next figure we can see the first four levels of the generating tree, with the labels as nodes, for the class of Dyck paths.



The growth of the class of Dyck paths produces the following succession rule:

$$\Omega_{Cat} = \begin{cases} (1) \\ (h) \rightsquigarrow (1), (2), \ldots, (h), (h+1). \end{cases}$$

As an example, if we look at the third level of the tree, we see 5 objects and if we look at the succession rule we see that the number of labels produced from the root after two applications of the productions is 5. Therefore the third term of the enumeration sequence of Dyck paths is 5 and it is a well-known fact that Dyck paths are counted by Catalan numbers, indeed it holds that $C_3 = 5$.

As shown, generating trees are very useful both for generating the objects of a combinatorial class and for finding any term of its enumeration sequence. In the next section, I show a continuation of the generating tree method to further derive closed formulas.

In what follows, I present two examples of this method and the following definition will be useful for both of them.

Definition 2.6.2. *Let π be a permutation of length n. An entry π_i is a left-to-right maximum if $j < i$ implies that $\pi_j < \pi_i$. An entry π_i is a left-to-right minimum if $j < i$ implies that $\pi_j > \pi_i$. An entry π_i is a right-to-left maximum if $j > i$ implies that $\pi_i > \pi_j$. An entry π_i is a right-to-left minimum if $j > i$ implies that $\pi_i < \pi_j$. The same holds for an inversion sequence e of length n.*

2.6.1 First example

The pattern of the first example (presented by Beaton-Bouvel-Guerrini-Rinaldi [2] in Chapter 4) is the triple of relations $(\geq, \geq, >)$. The family $I(\geq, \geq, >)$ is called the family of Baxter inversion sequences, since (as we shall see) its counting sequence is the sequence of Baxter numbers.

We start by proving two characterizations for this family.

Proposition 2.6.3. *An inversion sequence $e \in I_n$ avoids $(\geq, \geq, >)$ if and only if it avoids* 100, 110 *and* 210.

Proof. "⇒"Let $e \in I_n$ such that there is no $i < j < k$ with $e_i \geq e_j \geq e_k, e_i > e_k$. This means that there is no $i < j < k$ such that $e_i > e_j = e_k$, nor such that $e_i = e_j > e_k$ and nor such that $e_i > e_j > e_k$. Therefore e avoids 100, 110 and 210.
"⇐"Conversely, let $e \in I_n$ such that e avoids 100, 110 and 210. Then there is no $i < j < k$ such that $e_i > e_j = e_k$, nor such that $e_i = e_j > e_k$ and nor such that $e_i > e_j > e_k$, which is equivalent to the absence of $i < j < k$ with $e_i \geq e_j \geq e_k, e_i > e_k$. Therefore e avoids $(\geq, \geq, >)$. □

Proposition 2.6.4. *An inversion sequence $e \in I_n$ avoids $(\geq, \geq, >)$ if and only if for all i and j, with $i < j$ and $e_i > e_j$, both e_i is a LTR maximum and e_j is a RTL minimum.*

Proof. "⇒" Let $e \in I_n$ such that e avoids $(\geq, \geq, >)$ and let $e_i > e_j$ with $i < j$.

If e_i is not a LTR maximum then there exists $k < i$ such that $e_k \geq e_i$. But then $e_k e_i e_j$ forms a 210 or a 110 in e, which is a contradiction.

If e_j is not a RTL minimum then there exists $k > j$ such that $e_k \leq e_j$. But then $e_i e_j e_k$ forms a 210 or a 100 in e, which is a contradiction.
"⇐" Conversely, let $e \in I_n$ such that for all i, j with $i < j$ and $e_i > e_j$ both e_i is a LTR maximum and e_j is a RTL minimum.

If e doesn't avoid 100 then there is $i < j < k$ such that $e_i > e_j = e_k$. But then e_j is not a RTL minimum.

If e doesn't avoid 110 then there is $i < j < k$ such that $e_i = e_j > e_k$. But then e_j is not a LTR maximum.

If e doesn't avoid 210 then there is $i < j < k$ such that $e_i > e_j > e_k$. But then e_j is not a RTL minimum. □

The following result consists in a succession rule for the family $I(\geq, \geq, >)$.

Proposition 2.6.5. *There exists a growth of the Baxter inversion sequences that corresponds to the succession rule*

$$\Omega_{Bax} = \begin{cases} (1,1), \\ (h,k) \rightsquigarrow (h-1,k+1),\dots,(1,k+1), \\ \qquad\qquad (1,k+1), \\ \qquad\qquad (h+1,k),\dots,(h+k,1). \end{cases}$$

Remark 2.6.6. *The succession rule Ω_{Bax} is known to correspond to Baxter numbers (see [8]). I show the proof of this fact in section 2.8.*

Proof. We describe a growth for Baxter inversion sequences in the following way. We start by defining

$$last(e) = \begin{cases} rightmost\ entry\ of\ e\ which\ is\ not\ a\ LTR\ maximum, & \text{if any} \\ smallest\ value\ of\ e, & \text{otherwise} \end{cases}$$

$$= \begin{cases} rightmost\ entry\ of\ e\ which\ is\ not\ a\ LTR\ maximum, & \text{if any} \\ 0, & \text{otherwise.} \end{cases}$$

Firstly, suppose that the rightmost entry of e which is not a LTR maximum is in position k. If e_k is not the largest value that is not a LTR maximum, then there is $i < k$ such that e_i is not a LTR maximum and $e_i > e_k$. But then there is $l < i$ with $e_l \geq e_i$, so we have $l < i < k$ and $e_k < e_i \leq e_l$ which contradicts the fact that e avoids 210 and 110. So, if a rightmost entry of e which is not a LTR maximum exists, it holds that $last(e)$ is also the greatest value that is not a LTR maximum.

Secondly, suppose that the rightmost entry of e which is not a LTR maximum exists and is in position k. This means that there exists $j < k$ with $e_j \geq e_k$. Between these two entries there can be a strict inequality and as a consequence of this $last(e)$ creates an inversion (that is, there is an entry e_ℓ to the left of $last(e)$ with $e_\ell > last(e)$), or there can be an equality and so $last(e)$ doesn't create an inversion.

We want to describe a growth for Baxter inversion sequences by adding a new rightmost entry to e, but in order to do this we have to distinguish two cases according to the previous consideration. We have two possibilities:

1. there doesn't exist an entry which is not a LTR maximum, or there exists one and the rightmost such doesn't create an inversion
2. there exists the rightmost entry of e which is not a LTR maximum and this entry creates an inversion.

The growth in these two cases behaves like this:

1. Claim: the sequence $e' = (e_1, \ldots, e_n, e_{new}) \in I_{n+1}(\geq, \geq, >)$ if and only if $last(e) \leq e_{new} \leq n$.

   Proof of the claim: Let $e' = (e_1, \ldots, e_n, e_{new}) \in I_{n+1}(\geq, \geq, >)$. If all the entries of e are LTR maxima, by Proposition 2.6.4 e_{new} can be whatever you want less than or equal to n and indeed $0 = last(e) \leq e_{new} \leq n$. If the rightmost entry of e which is not a LTR maximum doesn't create an inversion, then we have that if $e_{new} < last(e)$ they create an inversion but $last(e)$ is not a LTR maximum and therefore we obtain a contradiction to Proposition 2.6.4.
   Conversely, assume by contrapositive that $e' = (e_1, \ldots, e_n, e_{new}) \notin I_{n+1}(\geq, \geq, >)$. Then by Proposition 2.6.4 there exists $i < j$ with $e_i > e_j$ such that e_i is not a LTR maximum or e_j is not a RTL minimum. But since $e \in I_n(\geq, \geq, >)$ then $j = n + 1$ and e_i is not a LTR maximum (which in the first possibility of the first case is already a contradiction). So we have $e_{new} < e_i \leq last(e)$. The claim is proved.

   Now we look in which case e' falls, depending on the entry e_{new}: this will be useful for the computation of the productions. One possibility is that $last(e) \leq e_{new} < maxx(e)$ which implies that e_{new} is not a LTR maximum, hence $last(e') = e_{new}$, and $maxx(e)$ and e_{new} create an inversion which leads e' in case 2). Another possibility is that $e_{new} = maxx(e)$ which implies that $last(e') = e_{new}$ but without forming an inversion because before e_{new} there are no elements $> e_{new}$, and this leads e' in case 1). The last possibility is that $maxx(e) < e_{new} \leq n$ which implies that e_{new} is a LTR maximum and that $last(e) = last(e')$, hence $last(e')$ doesn't create an inversion, so this leads e' in case 1).

2. Claim: the sequence $e' = (e_1, \ldots, e_n, e_{new}) \in I_{n+1}(\geq, \geq, >)$ if and only if $last(e) < e_{new} \leq n$.

   Proof of the claim: let $e' = (e_1, \ldots, e_n, e_{new}) \in I_{n+1}(\geq, \geq, >)$. If the rightmost entry of e which is not a LTR maximum creates an inversion, say with e_{inv}, and if you assume that $e_{new} \leq last(e)$, then $e_{inv} last(e) e_{new}$ forms a 100 or a 210 in e,

which contradicts Proposition 2.6.3.
Conversely, assume by contrapositive that $e' = (e_1, \dots, e_n, e_{new}) \notin I_{n+1}(\geq, \geq, >)$. Then by Proposition 2.6.3 e' contains at least one of the patterns 100, 110, 210 and since $e \in I_n(\geq, \geq, >)$ the entry e_{new} plays the role of the last 0. Thus e_{new} must be less than or equal to the entry which plays the role of the second element of the pattern, which in all the three cases is not a LTR maximum. So $e_{new} \leq last(e)$, since $last(e)$ is the greatest value that is not a LTR maximum. The claim is proved.

Now we look in which case e' falls, depending on the entry e_{new}: this will be useful for the computation of the productions. One possibility is that $last(e) < e_{new} < maxx(e)$ which implies that e_{new} is not a LTR maximum, hence $last(e') = e_{new}$, and $maxx(e)$ and e_{new} create an inversion which leads e' in case 2). Another possibility is that $e_{new} = maxx(e)$ which implies that $last(e') = e_{new}$ but without forming an inversion because before e_{new} there are no elements $> e_{new}$, and this leads e' in case 1). The last possibility is that $maxx(e) < e_{new} \leq n$ which implies that e_{new} is a LTR maximum and $last(e') = last(e)$, therefore $last(e')$ creates an inversion, so this leads e' in case 2).

According to this growth, the condition that every $e \in I_n(\geq, \geq, >)$ is represented only once in the generating tree is satisfied, since if you remove the last entry of $e' \in I_{n+1}(\geq, \geq, >)$ you obtain an inversion sequence $e \in I_n(\geq, \geq, >)$ and e' can only be obtained from this e.

The labels for our inversion sequences in $I_n(\geq, \geq, >)$ are (h, k), where

1. $h = maxx(e) - last(e) + 1$ and $k = n - maxx(e)$
2. $h = maxx(e) - last(e)$ and $k = n - maxx(e)$.

The root object is the zero inversion sequence of length 1 and it belongs to case 1), implying that $h = 0 - 0 + 1 = 1$ and $k = 1 - 0 = 1$, so the axiom of our succession rule is $(1, 1)$.

Let $e = (e_1, \dots, e_n) \in I_n(\geq, \geq, >)$ with label (h, k) and $e' = (e_1, \dots, e_n, e_{new}) \in I_{n+1}(\geq, \geq, >)$ with label (h_{new}, k_{new}). Considering our growth (described above in both cases) we obtain the following productions:

1. If $e_{new} = last(e), \dots, maxx(e) - 1$ then $last(e') = e_{new}$ and e' falls in case 2). This implies that $h_{new} = maxx(e') - last(e') = maxx(e) - e_{new} = h + last(e) - 1 - e_{new}$ and $k_{new} = (n+1) - maxx(e') = (n+1) - maxx(e) = k + 1$.
   Hence the labels are $(h-1, k+1), \dots, (1, k+1)$.

   If $e_{new} = maxx(e)$ then $last(e') = e_{new}$ and e' falls in case 1). This implies that $h_{new} = maxx(e') - last(e') + 1 = maxx(e) - e_{new} + 1 = 1$ and $k_{new} = (n+1) - maxx(e') = (n+1) - maxx(e) = k + 1$.
   Hence the label is $(1, k+1)$.

   If $e_{new} = maxx(e) + 1, \dots, n$, that is $e_{new} = maxx(e) + l, l = 1, \dots, k = n - maxx(e)$, then $last(e') = last(e)$ and e' falls in case 1). This implies that $h_{new} = maxx(e') - last(e') + 1 = maxx(e) + l - last(e) + 1 = h + l, l = 1, \dots, k$ and $k_{new} = (n+1) - maxx(e') = (n+1) - (maxx(e) + l) = k + 1 - l, l = 1, \dots, k$.
   Hence the labels are $(h+1, k), (h+2, k-1), \dots, (h+k, 1)$.

2. If $e_{new} = last(e) + 1, \dots, maxx(e) - 1$ then $last(e') = e_{new}$ and e' falls in case 2). This implies that $h_{new} = maxx(e') - last(e') = maxx(e) - e_{new} = h + last(e) - e_{new}$ and $k_{new} = (n+1) - maxx(e') = (n+1) - maxx(e) = k + 1$.
   Hence the labels are $(h-1, k+1), \dots, (1, k+1)$.

If $e_{new} = maxx(e)$ then $last(e') = e_{new}$ and e' falls in case 1). This implies that $h_{new} = maxx(e') - last(e') + 1 = maxx(e) - e_{new} + 1 = 1$ and $k_{new} = (n+1) - maxx(e') = (n+1) - maxx(e) = k+1$.
Hence the label is $(1, k+1)$.

If $e_{new} = maxx(e)+1, \dots, n$, that is $e_{new} = maxx(e) + l, l = 1, \dots, k = n - maxx(e)$, then $last(e') = last(e)$ and e' falls in case 2). This implies that $h_{new} = maxx(e') - last(e') = maxx(e) + l - last(e) = h + l, l = 1, \dots, k$ and $k_{new} = (n+1) - maxx(e') = (n+1) - (maxx(e) + l) = k + 1 - l, l = 1, \dots, k$.
Hence the labels are $(h+1, k), (h+2, k-1), \dots, (h+k, 1)$.

Hence the growth of the Baxter inversion sequences corresponds to the succession rule Ω_{Bax}. □

2.6.2 Second example

This example (presented by Guerrini [7] in subsection 1.3.5 and where some parts of the proof are missing) is a bit different from the others in the sense that the objects considered are permutations and not inversion sequences. But since the topic of pattern avoidance in inversion sequences followed from the one of pattern avoidance in permutations and because of the brief introduction in Section 1.2, I find it interesting to give an example featuring permutations. The pattern we are interested in is 132.

Before proving the main result we need some definitions.

Definition 2.6.7. *Let $\pi = \pi_1\pi_2 \dots \pi_n$ be a permutation of length n. A site is any position between two consecutive elements π_k and π_{k+1}, for all $k = 1, \dots, n-1$.*

Definition 2.6.8. *Let $\pi = \pi_1\pi_2 \dots \pi_n$ be a permutation of length n such that π avoids a set of patterns β . An active site is a site where you can add the value $n+1$ obtaining a permutation π' of length $n+1$ that still avoids the patterns in β.*

The following result provides a succession rule for the family $S_n(132)$.

Proposition 2.6.9. *There exists a growth of the 132-avoiding permutations that corresponds to the succession rule*

$$\Omega_{Cat} = \begin{cases} (1) \\ (h) \rightsquigarrow (1), (2), \dots, (h), (h+1). \end{cases}$$

Proof. We describe a growth for 132-avoiding permutations in the following way.

We have to look at which are the active sites in a permutation that avoids 132. Let $\pi = \pi_1\pi_2 \dots \pi_n$ be a permutation of length n that avoids 132. It holds that π avoids 132 if and only if $\pi' = (n+1)\pi_1\pi_2 \dots \pi_n$ avoids 132, and hence the site before π_1 is active.

But this is not the unique active site of π: a site of π is active also if it is in the position after a RTL maximum, and viceversa. This holds for the following reason. Consider an active site of π and assume it is in the position after an element, say π_i, which is not a RTL maximum. This implies that there exists an index $j > i$ with $\pi_j > \pi_i$. But then we obtain a permutation π' of length $n+1$ with $i < i+1 < j+1$ and $\pi_i = \pi'_i < \pi'_{i+1} = n+1 > \pi_j = \pi'_{j+1}$ and $\pi_i = \pi'_i < \pi_j = \pi'_{j+1}$, hence π' contains 132 and so our active site is in the position after a RTL maximum. Conversely, we add $n+1$ immediately after a RTL maximum, say π_i, and we assume that the resulting permutation π' contains 132. Since π avoids 132, the role of 3 is played by $n+1$. Moreover, π_i being a RTL maximum implies that there is no $j > i$ with $\pi_j > \pi_i$ and so the role of 1 is not played by π_i. From these two observations we deduce that there exist $k < i$ and $j > i$ with $\pi_k < \pi_j$ that together with $n+1$ create a 132 in π'. But then we have $k < i < j$ and $\pi_k < \pi_j < \pi_i$, since π_i is a RTL maximum, which contradicts the 132-avoidance of π.

Now that we know the active sites of a 132-avoiding permutation, we want to describe the growth by adding $n+1$ in the active sites of π. According to this growth, the condition that every permutation in $S_n(132)$ is represented only once in the generating tree is satisfied, since if you take away the maximal value of $\pi' \in S_{n+1}(132)$ you obtain a permutation $\pi \in S_n(132)$ and π' can only be obtained from this π.

The label for our permutations in $S_n(132)$ is (h), where h is the number of RTL maxima. The root object is the permutation 1 of length 1, implying that $h = 1$ and so the axiom of our succession rule is (1).

Let $\pi \in S_n(132)$ with label (h). Considering our growth we obtain the following productions. If a permutation has h RTL maxima, then we can add the value $n+1$ after these h elements and before the first element π_1 and so $h+1$ permutations are produced by π.

If $n+1$ is added before π_1, then $n+1$ becomes a RTL maximum and hence π' has $h+1$ RTL maxima in total, producing the label $(h+1)$.

If $n+1$ is added after a RTL maximum, say π_i, then π_i and all the RTL maxima before π_i are not anymore RTL maxima, since there is the index $i+1 > i$ such that π'_{i+1} is greater than all elements, and $n+1$ becomes a RTL maximum. Hence π' has $h - l + 1$ RTL maxima, where π_i is the l-th RTL maximum of π and $l = 1, \ldots, h$, producing the labels $(h), (h-1), \ldots, (1)$.

We obtain that the growth of the 132-avoiding permutations corresponds to the succession rule Ω_{Cat}. □

Table 2.6 shows a list of cases falling in the type of proof presented in this section.

Pattern	Reference	Section
$(\geq, -, \geq)$	[2]	2.2
$(\geq, -, \geq)$	[2]	2.2
$(\geq, \geq, \geq)$	[2]	3.2
*$(\geq, \geq, >)$	[2]	4.2
$(\geq, >, -)$	[2]	5.2
*132	[7]	1.3.5
$(\neq, \geq, \geq)$	[15]	8.1

TABLE 2.6: Cases falling in the type of proof "Generating tree".

2.7 The kernel method

The seventh method I present is the kernel method. This method is actually a continuation of the previous one. When you know a succession rule for a certain family, you can find a functional equation which has as solution the generating function of this family. At this point the kernel method allows you to solve this equation and consequently to find the desired generating function of the combinatorial class. But this is not always the case, sometimes the kernel method doesn't work.

In this case the mentioned functional equation involves a bivariate generating function where one variable stores the size of the considered object of our combinatorial class and the other stores the label of the considered object in the generating tree. At the end we will arrive at an equation that contains only the first variable.

This method can be used, if we go back to Section 2.6, to find the generating function of the counting sequence that corresponds to the succession rule Ω_{Cat} of the second example since the labels are arrays of length 1. Instead, the labels concerning the first example are arrays of length 2 and this implies that two variables are not enough to store the size, the first entry of the label and the second entry of the label of the considered object. In this case we use another method that I will present in the next section.

In the sequel I give an example (presented by Guerrini [7] in subsection 1.3.6) of this method and more precisely I will go on with the second example of the previous section.

Let $\mathcal{C}$ be the combinatorial class of the 132-avoiding permutations. We start with the bivariate Catalan generating function

$$F_{Cat}(x,y) = \sum_{n,h\geq 1} |\mathcal{C}_{n,h}|\, x^n y^h$$

where $\mathcal{C}_{n,h}$ is the set of objects of $\mathcal{C}$ of size n and with label value h in the generating tree, x stores the size and y stores the value of the label.

Putting the focus on the variable x, the bivariate Catalan generating function takes the form

$$\sum_{n\geq 1} G_n(y)x^n$$

where $G_n(y) = \sum_{h\geq 1} |\mathcal{C}_{n,h}|\, y^h$ is the generating function of the objects appearing in the generating tree associated to Ω_{Cat} that have size n, with y storing the value of the label. $G_n(y)$ is a polynomial in y which has degree n, since an object which has size n stays at level n of the generating tree and here the labels have values at most n. So $G_n(y) = c \cdot y^n + O(y^{n-1})$, for a positive real number c.

Putting the focus on the variable y, the bivariate Catalan generating function takes the form

$$\sum_{h\geq 1} F_h(x)y^h$$

where $F_h(x) = \sum_{n\geq 1} |\mathcal{C}_{n,h}|\, x^n$ is the generating function of the objects appearing in the generating tree associated to Ω_{Cat} that have label (h), with x storing the size.

By looking at the succession rule Ω_{Cat} we can see that the object of size 1 with label (1) contributes the term xy to $F_{Cat}(x,y)$ and an object of size n with label (h) produces $h+1$ objects of size $n+1$ with label $(1),(2),\ldots,(h+1)$, respectively, and these contribute the sum $x^{n+1}y + x^{n+1}y^2 + \ldots + x^{n+1}y^{h+1}$ to $F_{Cat}(x,y)$. The bivariate Catalan generating function satisfies

$$\begin{aligned} F_{Cat}(x,y) &= xy + \sum_{n\geq 1}\sum_{h\geq 1} |\mathcal{C}_{n,h}|\,(x^{n+1}y + x^{n+1}y^2 + \ldots + x^{n+1}y^{h+1}) \\ &= xy + \sum_{h\geq 1} F_h(x)x(y + y^2 + \ldots + y^{h+1}). \end{aligned}$$

We go on with the functional equation satisfied by $F_{Cat}(x,y)$.

Proposition 2.7.1. *The following functional equation for $F_{Cat}(x,y)$ holds:*

$$F_{Cat}(x,y) = xy + \frac{xy}{1-y}(F_{Cat}(x,1) - yF_{Cat}(x,y)). \tag{2.7.1}$$

Proof. Using the above form of $F_{Cat}(x,y)$ and the geometric series we obtain that

$$\begin{aligned}
F_{Cat}(x,y) &= xy + x\sum_{h\geq 1} F_h(x)(y + y^2 + \ldots + y^{h+1}) \\
&= xy + x\sum_{h\geq 1} F_h(x)y(1 + y + \ldots + y^h) \\
&= xy + x\sum_{h\geq 1} F_h(x)\left(\frac{y(1-y^{h+1})}{1-y}\right) \\
&= xy + \frac{xy}{1-y}\sum_{h\geq 1} F_h(x)(1-y^{h+1}) \\
&= xy + \frac{xy}{1-y}\sum_{h\geq 1}(F_h(x) - F_h(x)y^{h+1}) \\
&= xy + \frac{xy}{1-y}(F_{Cat}(x,1) - yF_{Cat}(x,y)).
\end{aligned}$$

□

Now that we have the functional equation for $F_{Cat}(x,y)$, we can use the kernel method in order to solve it and to find the solution $F_{Cat}(x,1)$.

Firstly, we collect the terms containing $F_{Cat}(x,y)$ obtaining

$$\begin{aligned}
F_{Cat}(x,y) + \frac{xy^2}{1-y}F_{Cat}(x,y) &= xy + \frac{xy}{1-y}F_{Cat}(x,1) \\
\text{i.e. } F_{Cat}(x,y)\left(1 + \frac{xy^2}{1-y}\right) &= xy + \frac{xy}{1-y}F_{Cat}(x,1).
\end{aligned}$$

The last equation is called the kernel form of the functional equation 2.7.1 and the coefficient $K_{Cat}(x,y) = 1 + \frac{xy^2}{1-y}$ of $F_{Cat}(x,y)$ is called kernel.

Secondly, we equal the kernel to 0 and we solve the equation $K_{Cat}(x,y) = 0$ with respect to the variable y,

$$\begin{aligned}
K_{Cat}(x,y) = 0 &\iff 1 + \frac{xy^2}{1-y} = 0 \\
&\iff \frac{1 - y + xy^2}{1-y} = 0 \\
&\iff 1 - y + xy^2 = 0
\end{aligned}$$

whose two solutions (since it is a quadratic polynomial in y) are

$$Y_1(x) = \tfrac{1+\sqrt{1-4x}}{2x} \text{ and } Y_2(x) = \tfrac{1-\sqrt{1-4x}}{2x}.$$

Using the formula $(1+z)^\alpha = 1 + \sum_{n\geq 1} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} z^n$ with $z = -4x$ and $\alpha = \frac{1}{2}$ we can compute

$$\begin{aligned}\sqrt{1-4x} &= 1 + \frac{1}{2}(-4x) + \frac{1/2(-1/2)}{2}(-4x)^2 + \frac{1/2(-1/2)(-3/2)}{6}(-4x)^3 \\ &+ \frac{1/2(-1/2)(-3/2)(-5/2)}{24}(-4x)^4 \\ &+ \frac{1/2(-1/2)(-3/2)(-5/2)(-7/2)}{120}(-4x)^5 + O(x^6) \\ &= 1 - 2x - 2x^2 - 4x^3 - 10x^4 - 28x^5 + O(x^6)\end{aligned}$$

and substituting it in $Y_1(x)$ and $Y_2(x)$ we obtain the power series expansions of the two solutions

$$\begin{aligned}Y_1(x) &= \frac{1 + 1 - 2x - 2x^2 - 4x^3 - 10x^4 - 28x^5 + O(x^6)}{2x} \\ &= x^{-1} - 1 - x - 2x^2 - 5x^3 - 14x^4 + O(x^5), \\ Y_2(x) &= \frac{1 - 1 + 2x + 2x^2 + 4x^3 + 10x^4 + 28x^5 + O(x^6)}{2x} \\ &= 1 + x + 2x^2 + 5x^3 + 14x^4 + O(x^5).\end{aligned}$$

Then we analyze the two solutions in order to understand which one we can use in our process.

We know that $K_{Cat}(x, Y_1) = 0$. Moreover, in order to see if $F_{Cat}(x, Y_1)$ is convergent or not, we write $F_{Cat}(x, y)$ as $\sum_{n\geq 0} G_n(y)x^n$ like in Equation 2.7. If we substitute $y = Y_1$ in the polynomial $G_n(y)$ we obtain a series in x where from the term $(x^{-1})^n$ we can see that the lowest power of x is $-n$ and hence $G_n(Y_1) = \Theta(x^{-n})$. Consequently, multiplying $G_n(Y_1)$ by x^n it follows that for all n $G_n(Y_1)x^n = \Theta(1)$. So $\sum_{n\geq 0} G_n(Y_1)x^n = F_{Cat}(x, Y_1)$ is not a convergent power series in x.

For the other solution, we also know that $K_{Cat}(x, Y_2) = 0$. Furthermore, writing $F_{Cat}(x, y)$ as before and substituting $y = Y_2$ in the polynomial $G_n(y)$ we obtain a series in x where from the term 1^n we can see that the lowest power of x is zero and hence $G_n(Y_2) = \Theta(x^0) = \Theta(1)$. Then $\sum_{n\geq N} G_n(Y_2)x^n = \Theta(x^N)$ since in this case the lowest power of x is N and so we obtain that for all N

$$\begin{aligned}F_{Cat}(x, Y_2) &= \sum_{n\geq 0} G_n(Y_2)x^n \\ &= \sum_{n<N} G_n(Y_2)x^n + \sum_{n\geq N} G_n(Y_2)x^n \\ &= \Theta(1) + \Theta(x^N) = \Theta(1)\end{aligned}$$

which implies that $F_{Cat}(x, Y_2)$ is a convergent power series in x.

After these considerations we deduce that substituting $y = Y_1$ in the kernel form of the functional equation gives $K_{Cat}(x, Y_1) = 0$, but $F_{Cat}(x, Y_1)$ not convergent. So we cannot deduce $K_{Cat}(x, Y_1)F_{Cat}(x, Y_1) = 0$. Instead, substituting $y = Y_2$ in the kernel form of the functional equation we can deduce $K_{Cat}(x, Y_2)F_{Cat}(x, Y_2) = 0$, since $K_{Cat}(x, Y_2) = 0$ and $F_{Cat}(x, Y_2)$ convergent.

So substituting $y = Y_2$ in the kernel form of the functional equation leads to an equation with only $F_{Cat}(x, 1)$ as unknown

$$0 = xY_2 + \frac{xY_2}{1 - Y_2} F_{Cat}(x, 1)$$

which then results in

$$F_{Cat}(x,1) = -xY_2\frac{1-Y_2}{xY_2} = Y_2 - 1 = \frac{1-\sqrt{1-4x}}{2x} - 1 = \frac{1-2x-\sqrt{1-4x}}{2x},$$

which is the solution for the enumeration of our objects.
Setting this result in the kernel form of the functional equation we have

$$\begin{aligned}
F_{Cat}(x,y) &= \left(xy + \frac{xy}{1-y}\frac{1-2x-\sqrt{1-4x}}{2x}\right)\frac{1-y}{1-y+xy^2} \\
&= \frac{xy(1-y)}{1-y+xy^2} + \frac{xy(1-2x-\sqrt{1-4x})}{2x(1-y+xy^2)} \\
&= \frac{2x^2y(1-y) + xy(1-2x-\sqrt{1-4x})}{2x(1-y+xy^2)} \\
&= \frac{y(2x(1-y)+1-2x-\sqrt{1-4x})}{2(1-y+xy^2)} \\
&= \frac{y(1-2xy-\sqrt{1-4x})}{2(1-y+xy^2)},
\end{aligned}$$

which provides a refined enumeration of our objects, keeping also track of the value of the label.

Substituting the term $\sqrt{1-4x}$ computed before in $F_{Cat}(x,1)$ and $F_{Cat}(x,y)$, we obtain their power series expansions

$$\begin{aligned}
F_{Cat}(x,1) &= \frac{1-2x-1+2x+2x^2+4x^3+10x^4+28x^5+O(x^6)}{2x} \\
&= x + 2x^2 + 5x^3 + 14x^4 + O(x^5), \\
F_{Cat}(x,y) &= \frac{y(1-2xy-1+2x+2x^2+4x^3+10x^4+28x^5+O(x^6))}{2(1-y+xy^2)} \\
&= \frac{y(-xy+x+x^2+2x^3+5x^4+14x^5+O(x^6))}{1-y+xy^2} \\
&= yx + (y+1)yx^2 + (y^2+2y+2)yx^3 + (y^3+3y^2+5y+5)yx^4 \\
&+ (y^4+4y^3+9y^2+14y+14)yx^5 + O(x^6)
\end{aligned}$$

where in the last equality we performed the Taylor expansion computing

$$F_{Cat}(0,y) = 0,$$
$$F'_{Cat}(x,y) = \frac{((1-y)+2x+6x^2+20x^3+70x^4)(1-y+xy^2)}{(1-y+xy^2)^2} - \frac{(x(1-y)+x^2+2x^3+5x^4+14x^5)y^2}{(1-y+xy^2)^2},$$
$$F'_{Cat}(0,y) = \frac{(1-y)^2}{(1-y)^2} = 1,$$
$$F''_{Cat}(x,y) = ((2+12x+60x^2+280x^3-2y-12xy-60x^2y-280x^3y+2xy^2+12x^2y^2 + 60x^3y^2+280x^4y^2)(1-y+xy^2)^2-(1-2y+2x+6x^2+20x^3+70x^4+y^2 - 2xy-6x^2y-20x^3y-70x^4y+x^2y^2+4x^3y^2+15x^4y^2+56x^5y^2)2(1-y+xy^2)y^2 /(1-y+xy^2)^4,$$
$$F''_{Cat}(0,y) = \frac{(2-2y)(1-y)^2-(1-2y+y^2)2(1-y)y^2}{(1-y)^4} = \frac{2(1-y)(1-y)^2-(1-y)^2 2(1-y)y^2}{(1-y)^4} = \frac{2(1-y)^3(1-y^2)}{(1-y)^4} = \frac{2(1-y)^3(1-y)(1+y)}{(1-y)^4} = 2(1+y),$$
$$\frac{F''_{Cat}(0,y)}{2!} = 1+y$$

and all the other terms $\frac{F^{(k)}_{Cat}(0,y)}{k!}$, where the derivatives are computed with respect to the variable x.

Table 2.7 shows a list of cases falling in the type of proof presented in this section.

Pattern/Class	Reference	Section
*132	[7]	1.3.6
Dyck paths	[7]	1.3.7
$(110, 102)$	[15]	4.2
$(120, 102)$	[15]	4.3

TABLE 2.7: Cases falling in the type of proof "The kernel method".

2.8 The obstinate kernel method

The eighth method I present is the obstinate kernel method. This method is also a continuation of the one of generating trees and it is a variant of the previously presented kernel method. The idea is the same: you solve a functional equation, obtained from the succession rule, which has as solution the generating function of the combinatorial class you are interested in. The obstinate kernel method applies to some cases (not all) where labels are pairs of integers, but sometimes it doesn't work (see for example the fifth chapter of [3]).

But in this case the functional equation involves a trivariate generating function where one variable stores the size of the considered object of our combinatorial class, one variable stores the first entry of the label of the considered object in the generating tree and the other stores the second entry of the label. At the end we will arrive at a system of equations that contain only the first and the second variable.

This method can be used, if we go back to Section 2.6, to find the generating function of the counting sequence that corresponds to the succession rule Ω_{Bax} of the first example since the labels are arrays of length 2.

In what follows, I give an example (presented by Kim-Lin [8] in Chapter 4) of this method, in particular I will continue the first example of Section 2.6. We want to prove that $|I_n(\geq,\geq,>)| = B_n$, where B_n is the n-th Baxter number that is known to count the Baxter permutations, that is the permutations that avoid the vincular patterns $2\underline{41}3$ and $3\underline{14}2$.

We start with the trivariate generating function of $I_n(\geq,\geq,>)$

$$F_{Bax}(x,y,z) = \sum_{n,h,k\geq 1} |\mathcal{B}_{n,h,k}|\, x^n y^h z^k$$

where $\mathcal{B}_{n,h,k}$ is the set of objects of $I_n(\geq,\geq,>)$ of size n and with label's values h and k in the generating tree, x stores the size, y stores the first entry of the label and z stores the second entry of the label.

Putting the focus on the values of the label, the trivariate generating function of $I_n(\geq,\geq,>)$ takes the form

$$\sum_{h,k\geq 1} F_{h,k}(x) y^h z^k$$

where $F_{h,k}(x) = \sum_{n\geq 1} |\mathcal{B}_{n,h,k}|\, x^n$ is the generating function of the objects appearing in the generating tree associated to Ω_{Bax} that have label (h,k), with x storing the size.

By looking at the succession rule Ω_{Bax} we can see that the object of size 1 with label $(1,1)$ contributes the term xyz to $F_{Bax}(x,y,z)$ and an object of size n with label (h,k) produces $h+k$ objects of size $n+1$ with labels

$$(h-1,k+1),\ldots,(1,k+1),(1,k+1),(h+1,k),\ldots,(h+k,1),$$

respectively, and these contribute the sum

$$x^{n+1}y^{h-1}z^{k+1} + \ldots + x^{n+1}yz^{k+1} + x^{n+1}yz^{k+1} + x^{n+1}y^{h+1}z^k + \ldots + x^{n+1}y^{h+k}z$$

to $F_{Bax}(x,y,z)$. The trivariate generating function of $I_n(\geq,\geq,>)$ satisfies

$$\begin{aligned}
F_{Bax}(x,y,z) &= xyz + \sum_{n\geq 1}\sum_{h,k\geq 1} |\mathcal{B}_{n,h,k}|\,(x^{n+1}y^{h-1}z^{k+1} + \ldots + x^{n+1}yz^{k+1} + x^{n+1}yz^{k+1} \\
&\quad + x^{n+1}y^{h+1}z^k + \ldots + x^{n+1}y^{h+k}z) \\
&= xyz + \sum_{h,k\geq 1} F_{h,k}(x)x(y^{h-1}z^{k+1} + \ldots + yz^{k+1} + yz^{k+1} + y^{h+1}z^k + \ldots + y^{h+k}z) \\
&= xyz + \sum_{h,k\geq 1} F_{h,k}(x)x((y^{h-1} + \ldots + y)z^{k+1} + yz^{k+1} + (y^{h+1}z^k + \ldots + y^{h+k}z)).
\end{aligned}$$

We go on with the functional equation satisfied by $F_{Bax}(x,y,z)$.

Proposition 2.8.1. *The following functional equation for $F_{Bax}(x,y,z)$ holds:*

$$\left(1+\frac{xz}{1-y}+\frac{xz}{1-\frac{z}{y}}\right)F_{Bax}(x,y,z)=xyz+xyz\left(1+\frac{1}{1-y}\right)F_{Bax}(x,1,z)+\frac{xz}{1-\frac{z}{y}}F_{Bax}(x,y,y). \tag{2.8.1}$$

Proof. Using the above form of $F_{Bax}(x,y,z)$ and the geometric series we obtain that

$$\begin{aligned}
F_{Bax}(x,y,z) &= xyz+\sum_{h,k\geq 1}F_{h,k}(x)x((y^{h-1}+\ldots+y)z^{k+1}+yz^{k+1}+(y^{h+1}z^k+\ldots+y^{h+k}z))\\
&= xyz+\sum_{h,k\geq 1}F_{h,k}(x)xyz^{k+1}+x\sum_{h,k\geq 1}F_{h,k}(x)(y^{h-1}+\ldots+y)z^{k+1}\\
&+x\sum_{h,k\geq 1}F_{h,k}(x)(y^{h+1}z^k+\ldots+y^{h+k}z)\\
&= xyz+xyzF_{Bax}(x,1,z)+x\sum_{h,k\geq 1}F_{h,k}(x)\left(\frac{1-y^h}{1-y}-1\right)z^{k+1}\\
&+x\sum_{h,k\geq 1}F_{h,k}(x)\left(\frac{1-(\frac{y}{z})^k}{1-\frac{y}{z}}\right)y^{h+1}z^k\\
&= xyz+xyzF_{Bax}(x,1,z)+x\sum_{h,k\geq 1}F_{h,k}(x)\left(\frac{y-y^h}{1-y}\right)z^{k+1}\\
&+\frac{xyz}{z-y}\Big(\sum_{h,k\geq 1}F_{h,k}(x)y^hz^k-\sum_{h,k\geq 1}F_{h,k}(x)y^hy^k\Big)\\
&= xyz+xyzF_{Bax}(x,1,z)+\frac{xz}{1-y}\Big(\sum_{h,k\geq 1}F_{h,k}(x)yz^k-\sum_{h,k\geq 1}F_{h,k}(x)y^hz^k\Big)\\
&+\frac{xyz}{z-y}(F_{Bax}(x,y,z)-F_{Bax}(x,y,y))\\
&= xyz+xyzF_{Bax}(x,1,z)+\frac{xz}{1-y}(yF_{Bax}(x,1,z)-F_{Bax}(x,y,z))\\
&+\frac{xz}{\frac{z}{y}-1}(F_{Bax}(x,y,z)-F_{Bax}(x,y,y)).
\end{aligned}$$

Putting together the terms containing $F_{Bax}(x,y,z)$

$$\left(1+\frac{xz}{1-y}+\frac{xz}{1-\frac{z}{y}}\right)F_{Bax}(x,y,z)=xyz+xyz\left(1+\frac{1}{1-y}\right)F_{Bax}(x,1,z)+\frac{xz}{1-\frac{z}{y}}F_{Bax}(x,y,y)$$

and Equation 2.8.1 holds. □

We move the attention on Baxter permutations.

Definition 2.8.2. *A permutation $\pi=\pi_1\pi_2\ldots\pi_n$ is a Baxter permutation if there are no $i<j<k$ with $1\leq j<j+1<k\leq n$ such that $\pi_{j+1}<\pi_i<\pi_k<\pi_j$ or $\pi_j<\pi_k<\pi_i<\pi_{j+1}$.*

Proposition 2.8.3. *A Baxter permutation is a permutation that avoids the vincular patterns $2\underline{41}3$ and $3\underline{14}2$.*

We consider a growth for Baxter permutations by adding the point $n+1$ either immediately before a LTR maximum of π or immediately after a RTL maximum of π. Moreover, the labels for our objects are (p,q), where p=number of LTR maxima of the Baxter permutation π and q=number of RTL maxima of the Baxter permutation π.

We obtain our enumerative result if we prove the following theorem.

Theorem 2.8.4. *For $n \geq 1$ it holds*

$$\sum_{h,k\geq 1} \sum_{\substack{e\in I_n(\geq,\geq,>) \\ \text{with } label(h,k)}} y^{h+k} = \sum_{\pi\in S_n(2\underline{41}3,3\underline{14}2)} y^{ltrma(\pi)+rtlma(\pi)},$$

where $ltrma(\pi)$ is the number of LTR maxima of π and $rtlma(\pi)$ is the number of RTL maxima of π.

Let $G_{Bax}(x,y,z) = \sum_{n\geq 1} \sum_{\pi\in S_n(2\underline{41}3,3\underline{14}2)} x^n y^{ltrma(\pi)} z^{rtlma(\pi)}$ be the generating function of Baxter permutations, where $ltrma(\pi)$ and $rtlma(\pi)$ are the two entries of the label of the succession rule associated to the growth of Baxter permutations. Now that we have the functional equation for $F_{Bax}(x,y,z)$, we use the obstinate kernel method in order to solve it. We show that $F_{Bax}(x,y,y) = G_{Bax}(x,y,y)$, and then the theorem is proved since

$$\begin{aligned} G_{Bax}(x,y,y) = F_{Bax}(x,y,y) \Leftrightarrow & \sum_{n\geq 1} x^n \sum_{\pi\in S_n(2\underline{41}3,3\underline{14}2)} y^{ltrma(\pi)+rtlma(\pi)} \\ &= \sum_{n,h,k\geq 1} |\mathcal{B}_{n,h,k}|\, x^n y^{h+k} \\ &= \sum_{n\geq 1} x^n \sum_{e\in I_n(\geq,\geq,>)} y^{h+k}. \end{aligned}$$

Proof. Equation 2.8.1 is already in the kernel form with the kernel $K_{Bax}(x,y,z) = \left(1 + \frac{xz}{1-y} + \frac{xz}{1-\frac{z}{y}}\right)$. Setting $w = \frac{z}{y}$, for convenience, Equation 2.8.1 becomes

$$\left(1 + \frac{xyw}{1-y} + \frac{xyw}{1-w}\right) F_{Bax}(x,y,yw) = xy^2w + xy^2w\left(1 + \frac{1}{1-y}\right) F_{Bax}(x,1,yw) + \frac{xyw}{1-w} F_{Bax}(x,y,y)$$

and setting $y = 1+t$ and $w = 1+u$ in the last equation we obtain

$$\begin{aligned} &\left(1 + \frac{x(1+t)(1+u)}{-t} + \frac{x(1+t)(1+u)}{-u}\right) F_{Bax}(x,1+t,(1+t)(1+u)) \\ =&x(1+t)^2(1+u) + x(1+t)^2(1+u)\left(1 + \frac{1}{-t}\right) F_{Bax}(x,1,(1+t)(1+u)) \\ +&\frac{x(1+t)(1+u)}{-u} F_{Bax}(x,1+t,1+t). \end{aligned}$$

Multiplying by $\frac{tu}{x(1+t)(1+u)}$ the last equation takes the form

$$\begin{aligned} &\frac{tu - x(1+t)(1+u)(t+u)}{x(1+t)(1+u)} F_{Bax}(x,1+t,(1+t)(1+u)) = \\ &tu(1+t) - u(1-t^2) F_{Bax}(x,1,(1+t)(1+u)) - tF_{Bax}(x,1+t,1+t). \end{aligned} \tag{2.8.2}$$

The kernel of the functional equation 2.8.2 is now $K_{Bax}(x,t,u) = \frac{tu - x(1+t)(1+u)(t+u)}{x(1+t)(1+u)}$ (where $K_{NBax}(x,t,u) = tu - x(1+t)(1+u)(t+u)$ is the numerator of the kernel), we equal it to 0 and we solve the equation $K_{Bax}(x,t,u) = 0$ with respect to the variable u,

$$\frac{tu - x(1+t)(1+u)(t+u)}{x(1+t)(1+u)} = 0 \iff tu - x(1+t)(1+u)(t+u) = 0.$$

The quadratic polynomial in u has the form

$$\begin{aligned} tu - x(1+u+t+tu)(t+u) &= tu - x(t+tu+t^2+t^2u+u+u^2+tu+tu^2) \\ &= -x(1+t)u^2 - (x(1+t)^2 - t)u - xt(1+t) \end{aligned}$$

whose two solutions are

$$\begin{aligned} Y_1(x,t) &= \frac{x(1+t)^2 - t - \sqrt{((x(1+t)^2-t)^2 - 4x^2(1+t)^2t}}{-2x(1+t)} \\ &= \frac{1 - x(1+t)(1+\frac{1}{t}) + \sqrt{1 - 2x(1+t)(1+\frac{1}{t}) + \frac{x^2}{t^2}(1+t)^2(1+t)^2 - 4x^2(1+t)(1+\frac{1}{t})}}{2x(1+\frac{1}{t})} \\ &= \frac{1 - x(1+t)(1+\frac{1}{t}) + \sqrt{1 - 2x(1+t)(1+\frac{1}{t}) - x^2(1-t^2)(1-(\frac{1}{t})^2)}}{2x(1+\frac{1}{t})} \end{aligned}$$

and

$$Y_2(x,t) = \frac{1 - x(1+t)(1+\frac{1}{t}) - \sqrt{1 - 2x(1+t)(1+\frac{1}{t}) - x^2(1-t^2)(1-(\frac{1}{t})^2)}}{2x(1+\frac{1}{t})}.$$

Then we analyze the two solutions in order to understand which one we can use in our process. We know that

$$K_{Bax}(x, 1+t, (1+t)(1+Y_1)) = K_{Bax}(x, 1+t, (1+t)(1+Y_2)) = 0.$$

So we would like to substitute $u = Y_1$ or Y_2 in the LHS of 2.8.2, so that its RHS is also 0. But the term $F_{Bax}(x, 1+t, (1+t)(1+Y_1))$ is not a convergent power series in t, because the power series expansion of $Y_1(x,t)$ is not the Taylor expansion in t. Instead, $Y_2(x,t)$ is a well-defined power series in t and so it is convergent. Therefore we can substitute only $u = Y_2$ in 2.8.2, obtaining the equation

$$\begin{aligned} &\frac{tY_2 - x(1+t)(1+Y_2)(t+Y_2)}{x(1+t)(1+Y_2)} F_{Bax}(x, 1+t, (1+t)(1+Y_2)) = \\ &tY_2(1+t) - Y_2(1-t^2)F_{Bax}(x, 1, (1+t)(1+Y_2)) - tF_{Bax}(x, 1+t, 1+t). \qquad (2.8.3) \end{aligned}$$

Until now we have applied the common kernel method. But the resulting equation still has 2 variables and so the kernel method is not enough to solve the problem.

At this point we can use the obstinate kernel method in order to find pairs (t,u) such that $K_{Bax}(x, 1+t, (1+t)(1+u)) = 0$ and such that the substitution is legal. The transformations

$$\alpha : (t,u) \mapsto \left(\frac{u}{t}, u\right), \beta : (t,u) \mapsto \left(\frac{u}{t}, \frac{1}{t}\right)$$

cancel the kernel $K_{Bax}(x,t,u)$ since

$$\begin{aligned}K_{NBax}\left(x,\frac{u}{t},u\right) &= \left[\frac{u}{t}u - x\left(1+\frac{u}{t}\right)(1+u)\left(\frac{u}{t}+u\right)\right]\\ &= \left[\frac{u}{t} - x\left(1+\frac{u}{t}\right)(1+u)\left(\frac{1}{t}+1\right)\right]u\\ &= [tu - x(t+u)(1+u)(1+t)]\frac{u}{t^2}\\ &= K_{NBax}(x,t,u)\cdot\frac{u}{t^2}\end{aligned}$$

and

$$\begin{aligned}K_{NBax}\left(x,\frac{u}{t},\frac{1}{t}\right) &= \frac{u}{t}\cdot\frac{1}{t} - x\left(1+\frac{u}{t}\right)\left(1+\frac{1}{t}\right)\left(\frac{u}{t}+\frac{1}{t}\right)\\ &= [tu - x(t+u)(1+t)(1+u)]\frac{1}{t^3}\\ &= K_{NBax}(x,t,u)\cdot\frac{1}{t^3}.\end{aligned}$$

The transformations

$$\alpha' : (t,u) \mapsto \left(u,\frac{u}{t}\right), \beta' : (t,u) \mapsto \left(\frac{1}{t},\frac{u}{t}\right)$$

do the same, since the kernel is symmetric. So the desired pairs, i.e. the pairs that cancel the kernel, are (t,Y_2), $\left(\frac{Y_2}{t},Y_2\right)$, $\left(\frac{Y_2}{t},\frac{1}{t}\right)$, (Y_2,t), $\left(Y_2,\frac{Y_2}{t}\right)$, $\left(\frac{1}{t},\frac{Y_2}{t}\right)$.
We substitute them in Equation 2.8.2 and we obtain

1. $\frac{tY_2 - x(1+t)(1+Y_2)(t+Y_2)}{x(1+t)(1+Y_2)} F_{Bax}(x,1+t,(1+t)(1+Y_2)) = tY_2(1+t) - Y_2(1-t^2)$ $F_{Bax}(x,1,(1+t)(1+Y_2)) - t\tilde{F}_{Bax}(t)$
2. $\frac{\frac{Y_2}{t}Y_2 - x\left(1+\frac{Y_2}{t}\right)(1+Y_2)\left(\frac{Y_2}{t}+Y_2\right)}{x\left(1+\frac{Y_2}{t}\right)(1+Y_2)} F_{Bax}\left(x,1+\frac{Y_2}{t},\left(1+\frac{Y_2}{t}\right)(1+Y_2)\right) = \frac{Y_2}{t}Y_2\left(1+\frac{Y_2}{t}\right) -$ $Y_2(1-\left(\frac{Y_2}{t}\right)^2)F_{Bax}\left(x,1,\left(1+\frac{Y_2}{t}\right)(1+Y_2)\right) - \frac{Y_2}{t}\tilde{F}_{Bax}(\frac{Y_2}{t})$
3. $\frac{\frac{Y_2}{t}\frac{1}{t} - x\left(1+\frac{Y_2}{t}\right)\left(1+\frac{1}{t}\right)\left(\frac{Y_2}{t}+\frac{1}{t}\right)}{x\left(1+\frac{Y_2}{t}\right)\left(1+\frac{1}{t}\right)} F_{Bax}\left(x,1+\frac{Y_2}{t},\left(1+\frac{Y_2}{t}\right)\left(1+\frac{1}{t}\right)\right) = \frac{Y_2}{t}\frac{1}{t}\left(1+\frac{Y_2}{t}\right)$ $-\frac{1}{t}\left(1-\left(\frac{Y_2}{t}\right)^2\right)F_{Bax}\left(x,1,\left(1+\frac{Y_2}{t}\right)\left(1+\frac{1}{t}\right)\right) - \frac{Y_2}{t}\tilde{F}_{Bax}(\frac{Y_2}{t})$
4. $\frac{Y_2t - x(1+Y_2)(1+t)(Y_2+t)}{x(1+Y_2)(1+t)} F_{Bax}(x,1+Y_2,(1+Y_2)(1+t)) = Y_2t(1+Y_2) - t(1-(Y_2)^2)$ $F_{Bax}(x,1,(1+Y_2)(1+t)) - Y_2\tilde{F}_{Bax}(Y_2)$
5. $\frac{Y_2\frac{Y_2}{t} - x(1+Y_2)\left(1+\frac{Y_2}{t}\right)\left(Y_2+\frac{Y_2}{t}\right)}{x(1+Y_2)\left(1+\frac{Y_2}{t}\right)} F_{Bax}\left(x,1+Y_2,(1+Y_2)\left(1+\frac{Y_2}{t}\right)\right) = Y_2\frac{Y_2}{t}(1+Y_2)$ $-\frac{Y_2}{t}(1-(Y_2)^2)F_{Bax}\left(x,1,(1+Y_2)\left(1+\frac{Y_2}{t}\right)\right) - Y_2\tilde{F}_{Bax}(Y_2)$
6. $\frac{\frac{1}{t}\frac{Y_2}{t} - x\left(1+\frac{1}{t}\right)\left(1+\frac{Y_2}{t}\right)\left(\frac{1}{t}+\frac{Y_2}{t}\right)}{x\left(1+\frac{1}{t}\right)\left(1+\frac{Y_2}{t}\right)} F_{Bax}\left(x,1+\frac{1}{t},\left(1+\frac{1}{t}\right)\left(1+\frac{Y_2}{t}\right)\right) = \frac{1}{t}\frac{Y_2}{t}\left(1+\frac{1}{t}\right)$ $-\frac{Y_2}{t}\left(1-\left(\frac{1}{t}\right)^2\right)F_{Bax}\left(x,1,\left(1+\frac{1}{t}\right)\left(1+\frac{Y_2}{t}\right)\right) - \frac{1}{t}\tilde{F}_{Bax}(\frac{1}{t})$

We multiply 1 by $-(1-Y_2^2)t$, 2 by $-\left(\frac{1}{t}-Y_2^2\frac{1}{t}\right)$, 3 by $-\left(Y_2\frac{1}{t}-Y_2(\frac{1}{t})^3\right)$, 4 by $-(1-t^2)Y_2$, 5 by $-\left(1-Y_2^2(\frac{1}{t})^2\right)$, 6 by $-\left(\frac{1}{t}-Y_2^2(\frac{1}{t})^3\right)$. We subtract, remembering that the left hand sides are equal to 0, equation 4 from equation 1, equation 5 from equation 2 and equation 6 from equation 3. Then we arrive at the following system

$$\begin{cases} (t-tY_2^2)\tilde{F}_{Bax}(t)-(Y_2-t^2Y_2)\tilde{F}_{Bax}(Y_2)=(Y_2-Y_2^3)(t^2+t^3)-(t-t^3)(Y_2^2+Y_2^3) \\ \left(\frac{1}{t}-\frac{Y_2^2}{t}\right)\tilde{F}_{Bax}\left(\frac{Y_2}{t}\right)-\left(1-\frac{Y_2^2}{t^2}\right)\tilde{F}_{Bax}(Y_2)=(Y_2-Y_2^3)\left(\frac{Y_2}{t^2}+\frac{Y_2^2}{t^3}\right)-\left(\frac{1}{t}-\frac{Y_2^2}{t^3}\right)(Y_2^2+Y_2^3) \\ \left(\frac{Y_2}{t}-\frac{Y_2}{t^3}\right)\tilde{F}_{Bax}\left(\frac{Y_2}{t}\right)-\left(\frac{1}{t}-\frac{Y_2^2}{t^3}\right)\tilde{F}_{Bax}\left(\frac{1}{t}\right)=\left(\frac{1}{t}-\frac{1}{t^3}\right)\left(\frac{Y_2^2}{t^2}+\frac{Y_2^3}{t^3}\right)-\left(\frac{Y_2}{t}-\frac{Y_2^3}{t^3}\right)\left(\frac{1}{t^2}+\frac{1}{t^3}\right) \end{cases}$$

where $\tilde{F}_{Bax}(t)=tF_{Bax}(x,1+t,1+t)$. After some computations that permit to cancel the terms $\tilde{F}_{Bax}(Y_2)$ and $\tilde{F}_{Bax}\left(\frac{Y_2}{t}\right)$ we obtain

$$tF_{Bax}(x,1+t,1+t)+\frac{1}{t}F_{Bax}\left(x,1+\frac{1}{t},1+\frac{1}{t}\right)=\frac{Y_2(1+t)(t^4-2Y_2t^3+2Y_2^2t-2Y_2+1)}{t^2(Y_2-1)(Y_2-t)}. \tag{2.8.4}$$

Note that $tF_{Bax}(x,1+t,1+t)$ is a formal power series in x with coefficients in $t\mathbb{N}[t]$ and $\frac{1}{t}F_{Bax}\left(x,1+\frac{1}{t},1+\frac{1}{t}\right)$ is a formal power series in x with coefficients in $\frac{1}{t}\mathbb{N}\left[\frac{1}{t}\right]$. Consequently $tF_{Bax}(x,1+t,1+t)$ is the positive part in t of the right hand side of Equation 2.8.4.

Moreover, moving the attention on the generating function of Baxter permutations we present some facts.

In [7] we can see that the following functional equation for $G_{Bax}(x,y,z)$ holds:

$$G_{Bax}(x,y,z)=xyz+\frac{xyz}{1-y}(G_{Bax}(x,1,z)-G_{Bax}(x,y,z))+\frac{xyz}{1-z}(G_{Bax}(x,y,1)-G_{Bax}(x,y,z)). \tag{2.8.5}$$

If we transform the functional equation into its kernel form and we make some change of variables, we obtain the following equation involving $tG_{Bax}(x,1+t,1)$

$$\frac{t^2-2tx(1+t)^2}{x(1+t)^2}G_{Bax}(x,1+t,1+t)=t^2-2tG_{Bax}(x,1+t,1), \tag{2.8.6}$$

where we used the fact that Equation 2.8.5 is symmetric in y and z. Moreover, following the same method as in [7], we arrive at a system of equations from which we obtain that

$$tG_{Bax}(x,1+t,1)+\frac{1}{t}G_{Bax}(x,1+\frac{1}{t},1)=\frac{1}{t^2}Y_2(t^3-tY_2+1), \tag{2.8.7}$$

where Y_2 is the same solution as before, being $K_{Bax}(x,t,u)$ and the kernel in [7] equal. From 2.8.6 and 2.8.7 we deduce that

$$\begin{aligned}
&tG_{Bax}(x,1+t,1+t)+\frac{1}{t}G_{Bax}\left(x,1+\frac{1}{t},1+\frac{1}{t}\right)\\
&=\frac{x(1+t)^2}{t-2x(1+t)^2}(t^2-2tG_{Bax}(x,1+t,1))+\frac{x\left(1+\frac{1}{t}\right)^2}{\frac{1}{t}-2x\left(1+\frac{1}{t}\right)^2}\left(\frac{1}{t^2}-2\frac{1}{t}G_{Bax}\left(x,1+\frac{1}{t},1\right)\right)\\
&=\frac{x(1+t)^2}{t-2x(1+t)^2}\left(t^2+\frac{1}{t^2}-2\left(tG_{Bax}(x,1+t,1)+\frac{1}{t}G_{Bax}\left(x,1+\frac{1}{t},1\right)\right)\right)\\
&=\frac{x(1+t)^2}{t-2x(1+t)^2}\left(t^2+\frac{1}{t^2}-2\frac{1}{t^2}Y_2(t^3-tY_2+1)\right)
\end{aligned}\tag{2.8.8}$$

and for the same reasoning as before $tG_{Bax}(x,1+t,1+t)$ is the positive part in t of the right hand side of 2.8.8.

By Maple the right hand side of 2.8.4 is equal to the right hand side of 2.8.8, implying the desired equality $F_{Bax}(x,1+t,1+t)=G_{Bax}(x,1+t,1+t)$. This equality provides a refined enumeration of our objects, it keeps also track of the sum of the entries of the label: $F_{Bax}(x,1,1)=G_{Bax}(x,1,1)$ is enough to obtain our enumerative result. □

Table 2.8 shows a list of cases falling in the type of proof presented in this section.

Pattern	**Reference**	**Section**
(2$\underline{41}$3, 3$\underline{14}$2)	[7]	1.4.5
2$\underline{41}$3	[7]	3.3.1
*$(\geq,\geq,>)$	[8]	4

TABLE 2.8: Cases falling in the type of proof "The obstinate kernel method".

Chapter 3

Conclusion

We have seen different methods for proving an enumerative result, each one with its characteristics. We can't do generalizations since, for one statement, there could be more than one proof; however, we can draw some considerations from our results.

The proof that uses only the combinatorial characterization of the family of interest is often used for the triples of relations and for the multiple patterns. Somehow, this is because these avoidance constraints are more restrictive (for example you have a weak inequality instead of a strict inequality, you have more than one pattern, and other restrictions), resulting in a "very constrained structure", as we discussed in Section 2.1.

The inductive proof is not so much used, but it shows that "classical" methods must not be forgotten. The reason of this fact could be that if you want to use this approach you need a "good guess" (like the formula for the number of derangements in Section 2.2), but it is not easy to guess in general.

The proof by recursive construction involves the classical patterns, the triples of relations, and the consecutive patterns. This approach is the most used among the consecutive patterns: in this case the containment of a consecutive pattern p in an inversion sequence e of size n, produced by adding a new entry to an inversion sequence e' of length $n-1$ that avoids p, depends only on the last two entries of e'.

The bijective proof is the most used method and it involves all pattern types. Since this approach is very useful if you want to compare cardinalities of different combinatorial classes, it is used very often for proving Wilf-equivalences. Many Wilf-equivalences can be proven and moreover we can prove them for all pattern types, explaining the previous considerations.

The proof that uses generating functions is a little bit less used and we can see it for the triples of relations and the classical patterns. The same holds for the proof that uses generating trees. In order to use the generating function approach you have to know the generating function of a certain combinatorial class, and this is not always immediate. For the generating tree approach you have to define a growth for the combinatorial class, you have to define the label for your objects, and this is not always an easy task.

The kernel method and the obstinate kernel method are applied when you have a succession rule and this is not something that you find instantly. So there are obviously fewer examples compared for example to the proof by recursive construction, and different types of pattern are involved.

Moreover, for what concerns the topic of pattern-avoiding inversion sequences in general, it would be interesting to see what happens (in general and for the methods of proof) if you consider other pattern types, like longer patterns or longer sets of classical patterns.

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