UNIVERSITÀ DI SIENA 1240
DEPARTMENT OF INFORMATION ENGINEERING AND MATHEMATICS

On enumeration sequences generalising Catalan and Baxter numbers

Author:
Veronica Guerrini

Advisor:
Prof. Simone Rinaldi,
University of Siena

Co-advisor:
Dr. Mathilde Bouvel,
University of Zurich

A thesis submitted in fulfilment of the requirements for the degree of Doctor of Philosophy

December 13, 2017

Abstract

The study carried out along this dissertation fits into the field of enumerative combinatorics. The structures that have been investigated are families of discrete objects which display combinatorial properties. They are mainly subfamilies of well-known combinatorial structures, such as lattice paths, pattern-avoiding permutations, polyominoes (more specifically, parallelogram polyominoes), or inversion sequences. Furthermore, these combinatorial families are closely related to two famous number sequences known in the literature as the Catalan and Baxter numbers.

We start defining and studying the properties of a combinatorial class whose elements appear as tilings of parallelogram polyominoes, and thus are called *slicings of parallelogram polyominoes*. This first combinatorial class results to be enumerated by Baxter numbers, and shows relations with Catalan and Schröder numbers.

Then, we turn to two families of pattern-avoiding permutations. First, we study the semi-Baxter permutations, which reveal a natural generalisation of the Baxter numbers. Next, we deal with the strong-Baxter permutations, which are defined by restricting the Baxter numbers.

Thereafter, some generalisations of Catalan numbers are presented. A first generalisation involves families of inversion sequences and lattice paths among the combinatorial structures enumerated, and is shown to be related to the sequence of Baxter numbers as well. Meanwhile, the family of *fighting fish* provide another generalisation of the Catalan numbers that appears to be independent from the Baxter numbers. These objects generalise the family of parallelogram polyominoes, and display remarkable probabilistic and enumerative properties.

In this dissertation we tackle the problem of enumerating these combinatorial classes and exhibiting their combinatorial properties, resolving some conjectures and open problems. The methods used are rather diverse: for instance, establishing one-to-one correspondences with other structures, or combining the use of generating functions and succession rules. A succession rule is a powerful tool for counting discrete objects, and moreover it allows to generate them exhaustively. Owing to this remarkable fact succession rules, and equivalently generating trees, are largely used in our study of combinatorial structures.

Riassunto

Il presente lavoro si inserisce nell'ambito della combinatoria, e più precisamente nel ramo della combinatoria enumerativa. L'oggetto di studio sono classi di oggetti discreti caratterizzabili per mezzo di proprietà combinatorie. In particolare, le strutture trattate sono sottofamiglie di note classi combinatorie quali i cammini nel piano, o le permutazioni a motivo escluso, o i poliomini parallelogrammi. Inoltre, le strutture studiate sono strettamente legate a due sequenze di numeri ben note in letteratura: i numeri di Catalan e i numeri di Baxter.

La prima struttura combinatoria che abbiamo definito e di cui abbiamo studiato le proprietà è una particolare tassellatura dei poliomini parallelogrammi, che chiamiamo *slicings of parallelogram polyominoes*. Questa famiglia risulta essere contata dai numeri di Baxter, e legata ai numeri di Catalan e di Schröder. Nel seguito presentiamo due famiglie di permutazioni a motivo escluso. Dapprima studiamo le *semi-Baxter permutations*, che rivelano una naturale generalizzazione dei numeri di Baxter; poi passiamo alle *strong-Baxter permutations* che si propongono, invece, come una loro naturale restrizione. Successivamente definiamo due diverse generalizzazioni dei numeri di Catalan. La prima generalizzazione coinvolge alcune famiglie di *inversion sequences* e di cammini nel piano, e presenta relazioni anche con i numeri di Baxter. La seconda, invece, si configura come una generalizzazione della famiglia dei poliomini parallelogrammi, e resta indipendente dai numeri di Baxter. Gli oggetti combinatori definiti nella seconda generalizzazione sono chiamati *fighting fish* e mostrano notevoli proprietà sia probabilistiche, che combinatorie.

I problemi di enumerazione affrontati nella presente tesi dottorale utilizzano per la loro risoluzione diversi approcci. Ad esempio, alcuni risultati sono ottenuti stabilendo corrispondenze biunivoche con altre strutture note, altri combinando l'uso di funzioni generatrici con quello di regole di successione. Le regole di successione sono un potente strumento enumerativo, che permette di generare esaustivamente gli oggetti di una data classe combinatoria. Per tale motivo, gli alberi di generazione, e la loro formulazione come regole di successione, si configurano in questo lavoro come il principale strumento di enumerazione utilizzato.

Ringraziamenti

Giunta al termine di questi tre anni di dottorato, ho il desiderio di ringraziare tutti coloro che mi hanno accompagnato in questo percorso di ricerca scientifica e crescita personale, e senza i quali la realizzazione di questo lavoro non sarebbe stata possibile.

Innanzitutto voglio ringraziare il mio tutor, Prof. Simone Rinaldi, per avermi fatto scoprire circa quattro anni fa questo ramo meno noto della matematica, chiamato combinatoria. Lo ringrazio per avermi dato l'opportunità di fare ricerca in questo campo, e per essersi adoperato affinché io potessi fare esperienze costruttive e formative. Lo ringrazio, inoltre, per aver messo a mia disposizione il suo tempo e le sue conoscenze, per avermi donato i suoi preziosi consigli, e per avermi offerto il suo supporto nei momenti meno gratificanti che caratterizzano il lavoro di ricerca.

Un altro speciale ringraziamento voglio rivolgerlo a Mathilde Bouvel, che come mia co-tutor, ha diretto assieme a Simone il mio programma di ricerca. La ringrazio per essersi sempre dimostrata disponibile anche nei periodi più impegnativi, e per essersi prodigata affiché potessi intraprendere esperienze produttive e utili alla mia crescita come dottoranda di ricerca. La ringrazio per il lavoro che abbiamo svolto a Zurigo e a Siena, e per il suo contributo teorico e metodologico, nonché per la sua dedizione nel seguire le mie ricerche. La ringrazio, in particolare, per essere stata una persona comprensiva e gentile, e sulla quale fare completo affidamento.

Voglio ringraziare, poi, sia Simone che Mathilde, perché hanno speso parte del loro tempo per leggere e discutere con me non solo le bozze di questo elaborato, ma anche gli altri progetti realizzati durante il dottorato.

Ringrazio tutti coloro con i quali ho stabilito delle collaborazioni scientifiche, e i quali mi hanno arricchito trasferendomi parte delle loro conoscenze. In particolare, ringrazio Robert Cori, Nick R. Beaton, Andrew Rechnitzer, Enrica Duchi e Gilles Schaeffer. Devo un particolare ringraziamento ad Andrew per avermi gentilmente ospitato al Dipartimento di Matematica dell'Università della Columbia Britannica, e per avermi mostrato l'utilizzo del metodo del kernel nella variante "obstinate". Ringrazio poi Enrica e Gilles che nei due anni trascorsi a Siena hanno condiviso con me non solo il loro sapere e l'amore per la ricerca, ma anche moltissimi piacevoli momenti di vita quotidiana.

Inoltre, voglio ringraziare tutti i colleghi e amici, che ho avuto il piacere di incontrare e di conoscere in questi anni. Con loro ho condiviso non solo le occasioni di lavoro e gli impegni del dottorato, ma anche le gradite pause caffé e altri numerosi momenti che ricorderò sempre con il sorriso.

Un ringraziamento speciale è rivolto ai miei amici più cari, che specialmente nel periodo di stesura di questa tesi, mi hanno dimostrato il loro affetto supportandomi con pazienza.

Voglio ringraziare, infine, ma in modo particolare, la mia famiglia per il loro sostegno e i loro consigli, per l'amore e la fiducia che mi sono stati dimostrati in ogni occasione.

Veronica

Contents

Introduction

This dissertation fits into the field of combinatorics, also referred to as combinatorial theory or combinatorial analysis (G. Rota [95]).

> COMBINATORIAL ANALYSIS - or, as it is coming to be called, combinatorial theory - is both the oldest and one of the least developed branches of mathematics.
>
> Gian-Carlo Rota - 1969

The above statement dates back to almost 50 years ago, and since then a large amount of research has been carried out in this field.

Combinatorics is the branch of discrete mathematics that studies discrete systems of objects and their properties. It defines and exploits operations on these objects, as well as ways of selection and arrangement of them. In particular, some real phenomena or physical problems can suitably be modelled as discrete objects. Their study by means of combinatorial techniques allows to show that they display peculiar properties, which in turn might improve the understanding of their physical behaviour.

Moreover, there has been an increasing interest in combinatorial issues and combinatorial arguments especially after the introduction of computers: computers need programs with a higher and higher level of efficiency, and of a great help is the combinatorial study of the algorithms they are based on. Then, the research activity in combinatorics has been enhanced by the recent major concern for computer science and artificial intelligence, as well as for applied mathematics and information technology, where combinatorial results often find applications.

Nonetheless, combinatorial problems arise not only in application areas, such as computational biology, or statistical physics, or more recently data mining, but also in many areas of pure mathematics. Algebra, probability theory, topology, and geometry, for instance, have many problems of combinatorial flavour. Combinatorics can thus be subdivided into many subfields according to the topics studied and the methods used. From above, we can cite algebraic combinatorics, probabilistic combinatorics, topological combinatorics, geometric combinatorics, without completing the list. This dissertation falls within the subfield of *enumerative combinatorics*.

Enumerative Combinatorics and combinatorial classes

Enumeration, or counting, is probably the oldest question that mathematicians have asked. The problem of counting how many elements are in a given set $\mathcal{C}$ is formalised by

$$|\mathcal{C}| = \sum_{o \in \mathcal{C}} 1 \,.$$

Enumerative combinatorics is the branch of combinatorics that faces the problem of counting the elements of $\mathcal{C}$ when they are described by combinatorial properties. To be more precise, the object of study is an infinite sequence of sets $\{\mathcal{C}_n\}_{n=0}^{\infty}$, where each set $\mathcal{C}_n$ consists of combinatorial objects and n is the value that a parameter, called *size*, assumes on these objects. Enumerative combinatorics is aimed at tackling the question "for each n, how many objects are in $\mathcal{C}_n$?". The class of objects $\mathcal{C}$ is defined as the union of all the subsets $\mathcal{C}_n$, and is referred to as *combinatorial class*. The enumerative number sequence of $\mathcal{C}$ is the infinite sequence given by $\{|\mathcal{C}_n|\}_{n=0}^{\infty}$.

Studying a combinatorial class from an enumerative point of view consists not only in counting its elements numerically, but also among other problems in generating them exhaustively, or in establishing size-preserving bijective correspondences with other classes. Moreover, discovering new properties, such as finding the distribution of specific parameters, or providing equivalent definitions for the same class, also fits into the field of enumerative combinatorics.

Some methods for enumerating a combinatorial class

In order to answer a counting problem in enumerative combinatorics, many methods have been developed and some others are still a work in progress. Some of these methods owe their validity to analytical studies, while some others are closer to algebraic approaches. Among the huge variety of methods existing in literature, we choose to introduce here only those used along this work.

The more naive method we can refer to is the bijective one, which establishes a size-preserving one-to-one correspondence between two classes. Yet it is also the most preferable since it displays evidently that the enumeration sequences of the two classes are the same. Moreover, the bijective method is often useful to recognise the distribution of some statistics on the two classes of combinatorial objects involved.

Another good strategy to count the objects of a combinatorial class is to make use of their generating function. As the first two lines of H. S. Wilf's book [146] state, "a generating function is a clothesline on which we hang up a sequence of numbers for display". In fact, generating functions are (extremely useful) tools for treating the sequence of numbers enumerating a combinatorial class as a formal power series of type $\sum_n |\mathcal{C}_n| x^n$. In the generating function approach both analytical and algebraic methods can be used, and this is probably the richness of this approach.

Another remarkable tool, which often recurs along this dissertation, is known in the literature with the name of *generating tree*. The origin of this approach is due to J. West [144] who introduced generating trees in 1995 to generate all the permutations avoiding particular patterns. Then, in 1999 the generating tree approach was defined in a more systematic way by the Florentine school of combinatorics. They introduced the notion of ECO operator and of ECO method [13], thus formally explaining how to generate exhaustively the objects of a combinatorial class by means of generating trees. A thorough analysis of generating trees and their structural properties was then carried out in 2002 by C. Banderier et al. [8], in which the definition of generating tree merges into the one of *succession rule*. Not only are generating trees and succession rules valid tools for generating all the objects of a given combinatorial class, but they are also extremely useful for enumeration purposes, especially when combined with generating functions.

Our combinatorial classes

The combinatorial classes considered in our study are rather diverse. Along this dissertation we encounter different combinatorial structures that display disparate properties. The guideline we follow in presenting them is related to their interconnections with other combinatorial structures broadly studied in the literature, which are proved to have well-known enumerative number sequences.

On the one hand, the combinatorial objects involved in our study fall into mainly four well-known families of objects (precise definitions of them will be provided later):

- some lattice paths, which hold an incredibly rich tradition of works, for instance [10, 33, 62, 66, 85, 102, 115];
- pattern-avoiding permutations, whose massive study began in 1968 when D. Knuth published a first landmark in this topic [99], and which have since then been the subject of much subsequent research [6, 7, 22, 24, 98, 131, 137, 138];
- polyominoes, which arose in the antiquity as mathemathical recreations and games, and later have been found useful to model physical phenomena or to approach tiling problems [34, 51, 60, 61, 91, 117, 123, 140];
- inversion sequences, whose definition is closely related to permutations [99, 127, 134], and whose study from a pattern-avoiding standpoint has only recently been proposed [58, 97, 108, 110].

On the other hand, there are mainly two number sequences recurring along our study:

- the rather ubiquitous sequence of Catalan numbers, which raised recent and past interest [21, 52, 64, 66, 88, 101, 128, 133, 135];
- and the intriguing and fascinating sequence of Baxter numbers, which despite being younger than the Catalan sequence has been extensively studied in recent times [27, 46, 48, 53, 71, 75, 142].

Detailed plan of the dissertation

In this dissertation we investigate combinatorial structures and tackle enumerative problems closely related to the sequences of Catalan and Baxter numbers.

Chapter 1 provides basic and formal definitions of the main concepts: what is a combinatorial class of objects and how to approach a counting problem. Then, we recall some of the combinatorial methods used in enumerative combinatorics, precisely those used along this dissertation. These have been briefly summed up earlier in this introduction without any formal details, which instead are provided in this first chapter. Finally, the chapter introduces the two aforementioned sequences of numbers: Catalan and Baxter. Some historical facts are recounted in addition to their enumerative properties and their combinatorial interpretations in terms of discrete structures. These two sections about Catalan and Baxter numbers illustrate that although both sequences have been known to combinatorialists for a long time, the interest in their study has never ceased.

In **Chapter 2** we start studying a new family of combinatorial objects called *slicings of parallelogram polyominoes*, which has been introduced in [G1] and then further studied in [G2]. This family is proved to be enumerated by the sequence of Baxter numbers and reveals the inclusions "Catalan in Schröder in Baxter". Schröder numbers are introduced in this chapter, combined with well-known results, such as formulas and combinatorial structures they enumerate, including a new one we defined in [G3]. Baxter structures are often visualised as generalisations either of Catalan objects, or of Schröder objects. Then, the aim of this chapter is to reconcile at the abstract level of generating trees and succession rules the aforementioned inclusions "Catalan in Schröder in Baxter", and slicings of parallelogram polyominoes allow us to reach this goal.

In **Chapter 3** we turn to the study of a family of pattern-avoiding permutations that naturally generalise the well-known family of Baxter permutations, and thus called *semi-Baxter permutations*. Their definition has been provided in [G4, G5]. Since the name Baxter numbers is actually owed to the combinatorial family of Baxter permutations, we call semi-Baxter numbers those enumerating semi-Baxter permutations. This chapter thus focuses on a sequence of numbers which is pointwise larger than the one of Baxter numbers. We provide a generating tree for semi-Baxter permutations that in turn permits us to acquire a full knowledge of semi-Baxter numbers. By using the generating function approach, both explicit expressions and a recurrence relation are shown. We also derive the behaviour of semi-Baxter numbers as they become larger and larger. In addition, we show other combinatorial interpretations of semi-Baxter numbers.

In **Chapter 4** we study another family of pattern-avoiding permutations that was introduced in [G4, G5]. It is a restriction of the family of Baxter permutations as natural as the generalisation of Chapter 3. We call the permutations of this family *strong-Baxter permutations*, and the numbers enumerating them strong-Baxter numbers. The main result of this chapter is the succession rule for the exhaustive generation of strong-Baxter permutations that reveals interesting properties. Indeed, this succession rule seems to be the intersection of two different Baxter succession rules, thus motivating the name. Furthermore, we show other combinatorial interpretations of these numbers in terms of lattice paths and walks in the quarter plane. The generating function of these walks has been studied in literature, and this allows to derive that the strong-Baxter generating function is non D-finite. This result is remarkable especially in the pattern-avoiding permutations framework, where non D-finite generating functions are quite rare in the literature.

In **Chapter 5** we focus on two different families of combinatorial structures: on the one hand inversion sequences defined by pattern avoidance, on the other hand *steady paths*. The reason for studying these specific families of inversion sequences is that they display a discrete continuity from Catalan numbers to *powered Catalan numbers*.

Powered Catalan numbers, as the name suggests, are a generalisation of Catalan numbers: indeed they admit a succession rule which naturally generalises the most famous one for Catalan numbers. The above discrete continuity involves moreover semi-Baxter and Baxter numbers, as well as an intermediate number sequence between Catalan and Baxter.

Meanwhile, steady paths display an interpretation of the powered Catalan numbers essentially different from the one existing in the literature, in terms of lattice paths. Indeed, it appears as if the powered Catalan structures were divided into two different groups which differ according to the succession rules that generate and enumerate them.

Finally, in **Chapter 6** we define and study the family of fighting fish, whose first definition was in [G6]. These combinatorial objects are namesake of the tropical fish famous for their multiple tails, whose appearance these objects resemble. Despite their misleading name, fighting fish arose to generalise the well-known Catalan family of parallelogram polyominoes, and display remarkable probabilistic and enumerative properties.

Overall, also in this final chapter we deal with a generalisation of Catalan numbers. Yet this time, the generalisation appears to be independent from the Baxter structures, and rather involves some families of plane trees and their enumeration sequences. The combinatorial properties of fighting fish have subsequently been explored in [G7], yet bijective interpretations of them are still missing in spite of underpinned conjectures.

Table of sequences

In the following table we summarise the number sequences involved in our study of combinatorial structures. The first column provides a precise reference for the number sequence according to the On-line Encyclopedia of Integer Sequences (OEIS [132], for brevity); the last column specifies in which chapter of this dissertation it appears for the first time.

Number sequence	**First terms**	**Chapter**
A000108 (*Catalan*)	$1, 2, 5, 14, 42, 132, \dots$	1. *Introductory notions*
A108307	$1, 2, 5, 15, 51, 191, \dots$	5. *Inversion sequences and steady paths*
A281784 (*Strong-Baxter*)	$1, 2, 6, 21, 82, 346, \dots$	4. *Strong-Baxter permutations*
A006318 (*Schröder*)	$1, 2, 6, 22, 90, 394, \dots$	2. *Slicings of parallelogram polyominoes*
A000139	$1, 2, 6, 22, 91, 408, \dots$	6. *Fighting fish*
A001181 (*Baxter*)	$1, 2, 6, 22, 92, 422, \dots$	1. *Introductory notions*
A117106 (*Semi-Baxter*)	$1, 2, 6, 23, 104, 530, \dots$	3. *Semi-Baxter permutations*
A113227 (*powered Catalan*)	$1, 2, 6, 23, 105, 549, \dots$	5. *Inversion sequences and steady paths*
A006013	$1, 2, 7, 30, 143, 728, \dots$	6. *Fighting fish*

Chapter 1

Introductory notions

1.1 Getting started

The aim of this chapter is to provide tools and notions needed for the whole comprehension of this dissertation. First, we provide concrete examples of classes of discrete objects and related counting problems: the two general families introduced - lattice paths and pattern-avoiding permutations - present a rather simple combinatorial description, yet in specific cases their enumeration appears to be complex. We choose to define these families of objects at the very beginning since occurrences of them will frequently return along our study. Then, the next subsection illustrates some powerful methods that are useful to count objects and a large use of them will be made in the main body of this dissertation. Finally, since all the combinatorial objects defined and enumerated in this work are somehow related to Catalan and Baxter numbers, the last two sections of this first chapter are to gather known results and properties about these two number sequences and, to define the combinatorial structures that will be recalled in the following chapters.

1.1.1 Well-posed counting problems

A *class of combinatorial objects*, or combinatorial class, is any set $\mathcal{C}$ satisfying the following property: $\mathcal{C}$ can be equipped with an enumerative parameter $p : \mathcal{C} \to \mathbb{N}$, called usually *size* or *length*, such that the cardinality of the set

$$\mathcal{C}_n = \{o \in \mathcal{C} : \, p(o) = n\}$$

is finite, for every $n \in \mathbb{N}$. It must be stressed that the cardinality of $\mathcal{C} = \cup_{n\geq 0} \mathcal{C}_n$ is not required to be finite; on the contrary, it is generally infinite.

A well-posed counting problem for a class $\mathcal{C}$ consists in finding the number sequence $\{c_n\}_{n\geq 0}$, where $c_n = |\mathcal{C}_n|$, and providing information about these numbers c_n. As we will see later, enumerative problems involve finding formulas to express c_n (for instance, closed or recursive formulas), or studying their asymptotic behaviour for large n.

1.1.2 Lattice paths

As a first example of combinatorial class we exhibit the hugely studied case of lattice paths. An accurate survey on this topic is provided in [25, Chapter 10] and for the main results cited in this part we address to it.

A lattice path is intuitively defined as its name suggests: a path (or walk) in a lattice in some d-dimensional Euclidean space. Our definition of lattice paths sets $d = 2$ and, formally, it reads as follows.

Definition 1.1.1. A *lattice path* (*path* for short) P in the Cartesian plane starting at (x_0, y_0) is a sequence $P = (P_1, \dots, P_n)$ of vectors $P_i \in \mathbb{Z}^2$, called *steps*, such that each P_i joins two points (x_{i-1}, y_{i-1}) and (x_i, y_i) in $\mathbb{Z}^2$. The point (x_0, y_0) is said *starting point* and (x_n, y_n) *ending point*. The *length* n of P is the number of its steps.

Figure 1.1(a) shows an example of lattice path in the positive quarter plane.

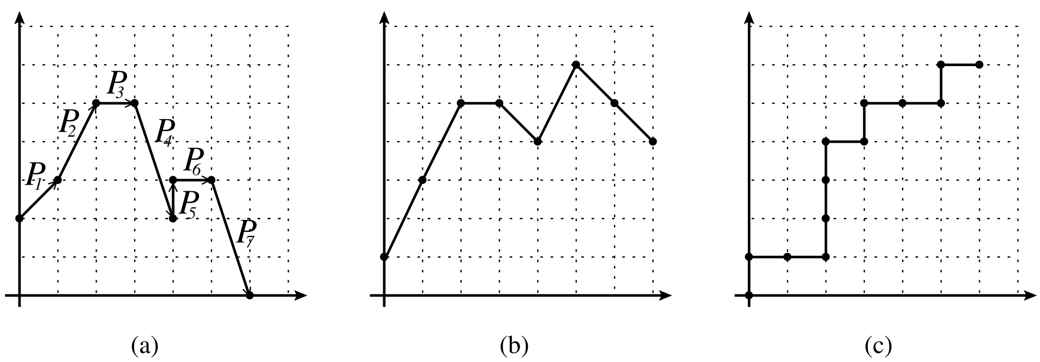


Figure 1.1: (a) A path of length 7 starting at $(0, 2)$ and ending at $(6, 0)$; (b) a path of length 7 made of steps in $\mathfrak{S} = \{(1, 2), (1, 0), (1, -1)\}$; (c) a path with only east and north steps starting at $(0, 0)$ and ending at $(6, 6)$.

Supposing we restrict the step set $\mathfrak{S}$ to be finite, then the number of all paths having n steps is finite as well, for any $n \in \mathbb{N}$. Thus, the length $n \in \mathbb{N}$ of a path P is an enumerative parameter for the family of lattice paths.

Although steps are formally vectors, we could omit arrows in the graphical representation of a path if there can be no misinterpretations: we can draw a segment instead of a vector, provided that the set $\mathfrak{S}$ of steps does not contain both a vector and its opposite. Figure 1.1(b) depicts a path with fixed step set $\mathfrak{S} = \{(1, 2), (1, 0), (1, -1)\}$.

A well-posed counting problem for paths with a given step set $\mathfrak{S}$ could be to determine the number of all paths starting at $(0, 0)$ and ending at a certain point (x_f, y_f) or obeying to certain constraints. For instance, a classical example to be cited is $\mathfrak{S} = \{(1, 0), (0, 1)\}$, namely the step set is made of only two unit steps called *east step* and *north step* respectively - see Figure 1.1(c). It is a standard combinatorial result that the number of paths starting at $(0, 0)$ and ending at $(2n, 2n)$ is determined by the central binomial coefficient $\binom{2n}{n}$.

An insight into the study of generic lattice paths and its close link with Probability and Statistics is reported in [102] and in [115], as well as a discussion about the basic methods for counting lattice paths. In literature (see [10]), precise computable estimates are given for the number of lattice paths under various constraints: with ending point lying on the x-axis (*bridges*) or, constrained to remain in the positive quarter plane (*meanders*) or, both conditions at the same time (*excursions*). Moreover, lattice paths confined to a particular region of the plane have been extensively studied [16, 39, 94]. In particular, much attention has recently been paid to a comprehensive classification of random walks in the quarter plane with step set contained in $\{0, +1, -1\}^2$ (see [33, 37, 112, 114]).

Families of lattice paths are frequently involved in our study: precisely, Table 1.1 briefly sums up our families of paths with their enumeration sequences and precise section references.

Family of paths	**Number sequence**	**Section**	**K/N**
Dyck paths	A000108 (Catalan)	1.3.2	K
Schröder paths	A006318 (Schröder)	2.2.1	K
Baxter paths	A001181 (Baxter)	2.1.4	N
Semi-Baxter paths	A117106	3.2.3	N
Strong-Baxter paths	A281784	4.2	N
Valley-marked Dyck paths	A113227	5.2.1	K
Steady paths	A113227	5.3.1	N

Table 1.1: Families of lattice paths defined along this dissertation; the last column specifies if the family of paths was already known in literature (K) or if it forms a new combinatorial interpretation of the corresponding number sequence (N).

1.1.3 Pattern-avoiding permutations

Our second example of families of combinatorial objects is the case of pattern-avoiding permutations. Recall that any permutation can be thought in one-line notation: a permutation of length n is simply an ordering $\pi = \pi_1 \ldots \pi_n$ of the set of integers $\{1, \ldots, n\}$. The enumerative parameter on permutations is known to be the length n so that there are $n!$ permutations, for any length $n \in \mathbb{N}$ (factorial number sequence A000142 [132]).

Definition 1.1.2. A permutation π of length n *contains* the permutation τ, called *pattern*, of length $k \leq n$ (equivalently, $\tau \preccurlyeq \pi$), if π has a subsequence of length k which is order-isomorphic to τ, namely it has the same pairwise comparisons as τ. Else if there exists no subsequence of π order-isomorphic to τ, the permutation π is said to *avoid* the pattern τ (equivalently, $\tau \not\preccurlyeq \pi$).

For example, the subsequence 79254 of $\pi = 371925846$ is order-isomorphic to $\tau = 45132$, thus $\tau \preccurlyeq \pi$; whereas $\sigma = 4321$ is avoided by π owing to the absence of a subsequence order-isomorphic to σ.

Permutation containment is generally best seen by plotting the elements of a permutation into a grid: let π be a permutation of length n, the set of points $\{(i, \pi_i)\}$, for any $1 \leq i \leq n$, forms the graphical representation of π - as shown in Figure 1.2.

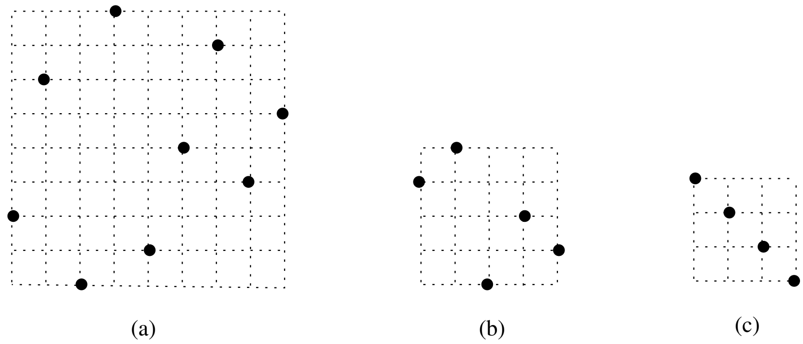


Figure 1.2: (a) The graphical representation of $\pi = 371925846$; (b) the graphical representation of the pattern $\tau = 45132$ contained in π; (c) the graphical representation of the pattern $\sigma = 4321$ avoided by π.

Permutation containment $\preccurlyeq$ is a partial order on the set of all finite permutations $\mathcal{S}$. Indeed, if a permutation π contains τ as pattern, every permutation σ containing π contains τ as well. On the other hand, by definition every permutation contained in a permutation π that avoids a pattern τ still avoids the pattern τ. Properties of the poset $(\mathcal{S}, \preccurlyeq)$ have been described in [139], such as, for instance, the fact that it does not contain infinite descending chains, or that it contains infinite anti-chains (*i.e.* sets of pairwise incomparable elements).

According to [25, Chapter 12] pattern avoidance in permutations was first introduced by D. Knuth [99] in his dissertation about sorting sequences by means of stacks and double-ended queues. Then, this study gained interest and inspired many subsequent papers; among them we cite two monographs [24, 98] and recent survey articles [137, 138].

The most natural question to address in this context is how many permutations of length n avoid a given set of permutations. More formally, if $\mathfrak{P}$ is a set of permutations, we define the family

$$AV(\mathfrak{P}) = \{\pi : \tau \not\preccurlyeq \pi, \text{ for all } \tau \in \mathfrak{P}\}.$$

Then, let $AV_n(\mathfrak{P})$ denote the set of all permutations of length n that avoid any permutation in $\mathfrak{P}$. The question becomes which number sequence $\{a_n\}_{n\geq 0}$ enumerates $AV(\mathfrak{P})$ where $a_n = |AV_n(\mathfrak{P})|$, for every n.

Note that according to the properties of $(\mathcal{S}, \preccurlyeq)$, $AV(\mathfrak{P})$ is a downward-closed set for the containment order: for all $\pi \in AV(\mathfrak{P})$, if $\sigma \preccurlyeq \pi$, then $\sigma \in AV(\mathfrak{P})$. In literature [25, Chapter 12], any family of permutations which is a downward-closed set for the containment order is called *permutation class*, or briefly *class*. Thus, the family $AV(\mathfrak{P})$ is a permutation class, for any set $\mathfrak{P}$ of permutations, and the converse statement holds as well.

Proposition 1.1.3 ([24]). *For every permutation class $\mathcal{C}$, there is a unique antichain $\mathfrak{P}$*

such that $\mathcal{C}$ coincides with $AV(\mathfrak{P})$. The set $\mathfrak{P}$ consists of all the minimal permutations (with respect to containment order) that do not belong to $\mathcal{C}$ and may be infinite.

The case where $\mathfrak{P}$ is a singleton has received considerable attention: for any $\tau \in \mathcal{S}$, the permutation class $AV(\tau)$ has been called *principal class* [25, Chapter 12]. In 1980, R. P. Stanley and H. S. Wilf (independently) asked for determining the behaviour of $|AV_n(\tau)|$, for a general permutation τ of length k and large n. They both formulated a conjecture stating that, for every permutation τ of length k, there is a finite number $L(\tau)$ such that $\lim_{n\to\infty} |AV_n(\tau)|^{1/n} = L(\tau)$.

In 2004, A. Marcus and G. Tardos [109] proved the following general enumerative result.

Theorem 1.1.4 (Stanley-Wilf Conjecture, Marcus-Tardos Theorem [109]). *For any permutation τ, there exists a constant C depending only on τ such that, for every n,*

$$|AV_n(\tau)| \leq C^n .$$

From that moment on, much attention has been focused on the limit $L(\tau)$, for any permutation τ, which is then called *Stanley-Wilf limit* - on the problem of establishing general bounds to Stanley-Wilf limits see [2, 26, 54, 55, 81].

To this regard, a particular result first stated by D. Knuth [99], and then proved in many different ways in the literature [56], is the following well-known fact.

Theorem 1.1.5 ([99, 56]). *For any permutation τ of length 3,*

$$|AV_n(\tau)| = \frac{1}{n+1}\binom{2n}{n}, \quad \textit{for every } n .$$

In 2000, E. Babson and E. Steingrímsson [6] introduced the notion of *generalised pattens*, showing that this definition is intimately related to the distributions of Mahonian permutation statistics. Generalised patterns, also known as *vincular patterns*, differ from the classical ones by some additional adjacency constraints: precisely, a vincular pattern requires the adjacency of certain elements in any occurrence of the pattern itself. Thus, an occurrence that does not respect the adjacency rules is not valid as occurrence of the vincular pattern: for example, 13254 does not contain the adjacent pattern $\underline{123}$, since there are no three consecutive elements in increasing order. We will provide in this introductory chapter (Section 1.4.2) the formal definition of vincular pattern, since throughout this dissertation there will be several occurrences of families of permutations avoiding vincular patterns, as Table 1.2 summarises.

Moreover, we specify that in order not to create potential misinterpretations of classical patterns, we prefer to adopt the notation "␣" to indicate which elements are required to be adjacent, instead of the historical dashed notation that separates with the symbol "$-$" those elements that can be non-adjacent in an occurrence of the pattern.

Vincular patterns have been studied in the last years [18, 19, 73] together with a new notion of pattern (*mesh pattern*) introduced by P. Brändén and A. Claesson [43] to provide expansions for certain permutation statistics results.

Moreover, it is well worth noticing that in case of a set $\mathfrak{P}$ of vincular or mesh patterns, we cannot use the name permutation class for $AV(\mathfrak{P})$, since in general the property of being a downward-closed set of $\mathcal{S}$ for the containment order vanishes.

Family of permutations	**Name**	**Sequence**	**Section**	**Y/N**
$AV(\tau)$, $\tau \in \mathcal{S}_3$		A000108	1.3.2	Y
$AV(2\,\underline{41}\,3, 3\,\underline{14}\,2)$	*Baxter permutations*	A001181	1.4.2	Y
$AV(2\,\underline{41}\,3, 3\,\underline{41}\,2)$	*Twisted Baxter permutations*	A001181	1.4.2	Y
$AV(2413, 3142)$	*Separable permutations*	A006318	2.2.1	Y
$AV\,(2\,\underline{41}\,3, 3\,\underline{14}\,2, 41323^+, 42313^+)$		A006318	2.5.2	N
$AV(2\,\underline{41}\,3)$	*Semi-Baxter permutations*	A117106	3.1.1	N
$AV(2\,\underline{14}\,3)$	*Plane permutations*	A117106	3.2.1	N
$AV(2\,\underline{41}\,3, 3\,\underline{14}\,2, 3\,\underline{41}\,2)$	*Strong-Baxter permutations*	A281784	4.1.1	N
$AV(1\,\underline{23}\,4)$		A113227	5.2.1	Y
$AV(1\,\underline{34}\,2)$		A113227	5.3.1	Y
$AV(\underline{23}\,14)$		A113227*	5.3.1	open

Table 1.2: Families of pattern-avoiding permutations treated along this dissertation; the last column specifies whether their enumeration problem was already solved in literature (Y) or not (N), or if it is still open, and thus their enumerative number sequence (with a superscript $*$) is only conjectured.

1.2 Methodology

In order to find a solution for a counting problem different strategies might be applied. This rather encyclopaedic section helps to sum up some known methods we will make use of throughout this dissertation. Many other methods though exist in literature, see for intances the books [79, 134, 135, 146], as well as [25, Part I].

1.2.1 Bijective method

Given two families of combinatorial objects $\mathcal{A}$ and $\mathcal{C}$, suppose that only $\mathcal{C}$ has been enumerated with respect to an enumerative parameter $p : \mathcal{C} \to \mathbb{N}$ and the number $c_n = |\mathcal{C}_n|$ is known, for every n. One way to show that $\mathcal{A}$ is also enumerated by the number sequence $\{c_n\}_{n\geq 0}$ is first to define an enumerative parameter $a : \mathcal{A} \to \mathbb{N}$ and a mapping β such that for every n, it associates an element of $\mathcal{A}_n = \{o \in \mathcal{A} : a(o) = n\}$ to an element of $\mathcal{C}_n$. Then, show that the mapping β is onto and injective.

Not only has this method the advantage of providing a clear proof of the fact that the two families of objects are equinumerous, but also function β may suggest (and prove) the equidistribution of other parameters on the same families of objects. For instance, regarding the two permutation classes $AV(321)$ and $AV(132)$, in [56] the authors classify all the bijections existing in literature between these two permutation classes and manage to extend known results about the equidistribution of some parameters, by studying the statistics preserved by these bijections.

1.2.2 ECO method

The ECO (Enumerating Combinatorial Objects) method was introduced in 1999 by the Florentine school of Combinatorics [12, 13, 116] and appeared as a formalisation of an earlier enumerative approach used by J. West in [143, 144]. It inspired many subsequent papers, such as [51, 67, 69, 76].

This method provides a recursive construction of the objects of a combinatorial class $\mathcal{C}$ according to a finite parameter p, generally called *size*. According to the ECO method, starting from a unique object of minimum size, we can build up all the other objects by performing some local expansions that increase the objects size. These possible expansions are formally described as operations performed by an operator $\vartheta : \mathcal{C} \to 2^{\mathcal{C}}$, which goes from $\mathcal{C}$ to its power set and associates to an object of $\mathcal{C}$ all the objects resulting from those expansions. The following result from [13] helps us to provide a formal definition of an *ECO operator* ϑ.

Proposition 1.2.1. *Let $\mathcal{C}$ be a class of combinatorial objects. For $n \geq 0$, if $\vartheta : \mathcal{C} \to 2^{\mathcal{C}}$ satisfies*

1. *for each $o \in \mathcal{C}_{n+1}$, there exists $o' \in \mathcal{C}_n$ such that $o \in \vartheta(o')$,*

2. *for every $o, o' \in \mathcal{C}_n$, $\vartheta(o) \cap \vartheta(o') = \emptyset$ whenever $o \neq o'$,*

then the family of sets $\{\vartheta(o) : o \in \mathcal{C}_n\}$ is a partition of $\mathcal{C}_{n+1}$.

Definition 1.2.2. An operator ϑ satisfying conditions *1.* and *2.* above is said to be an *ECO operator.*

Thus, an ECO operator generates all the objects of $\mathcal{C}$ in such a way that each object $o \in \mathcal{C}_{n+1}$ is uniquely obtained starting from a unique object $o' \in \mathcal{C}_n$. This process of generation associated with ϑ, which is unambiguous by definition, is generally called *growth of* $\mathcal{C}$, owing to the fact that it causes an increasing of size. So, along this dissertation, we use equivalently the expressions "to define an ECO operator" and "to define a growth" for a given combinatorial class $\mathcal{C}$.

1.2.3 Generating trees and succession rules

The growth performed by an ECO operator $\vartheta : \mathcal{C} \to 2^{\mathcal{C}}$ can be described by means of a *generating tree*: an infinite rooted tree whose nodes are decorated with objects of $\mathcal{C}$ each one appearing exactly once. The root of the tree corresponds to the (unique) object of $\mathcal{C}$ with minimum size. The children of a node carrying $o \in \mathcal{C}_n$ are as many as the objects belonging to $\vartheta(o)$ and carry exactly all the objects in $\vartheta(o)$. Thus, all the objects of size n in the decorated generating tree lie at level n - with the convention that the root is at level 1. Consequently, the enumerative sequence $\{c_n\}_{n\geq 0}$ of the combinatorial class $\mathcal{C}$ is visible by looking at the shape of the generating tree: for every $n \geq 0$, c_n is the number of nodes at level n.

Of importance for enumeration purposes is the general shape of a generating tree, not the specific objects that its nodes carry. So, in our study each time we refer to a generating tree, we intend only its shape, without the objects of $\mathcal{C}$ that decorate its nodes. In this sense generating trees become substantially useful as they could be described in an abstract way discarding the details of the combinatorial objects.

Generating trees were rigorously defined together with the ECO method introduction [13], although the basic idea they are founded on appeared occasionally in literature, expecially in the context of permutations with forbidden patterns [53, 70, 143, 144]. A thorough analysis of generating trees and their structural properties was carried out in [8], where the definition of generating trees matches the concept of *succession rule*.

In case the growth for a combinatorial class $\mathcal{C}$ is particularly regular, namely there exists a parameter $s : \mathcal{C} \to \mathbb{Z}$ whose values control the number of objects generated by each object of $\mathcal{C}$, then the corresponding generating tree can be encoded in a more compact way by what is called a succession rule. More precisely, we can label each node of the generating tree with the value $s(o)$, where $o \in \mathcal{C}$ is the object decorating that node. Then, the values of the parameter $s : \mathcal{C} \to \mathbb{Z}$ determine uniquely the number of children of each node in the generating tree of $\mathcal{C}$ and the (shape of the) generating tree depends only on how the values of the statistics s evolve from an object to its children.

Definition 1.2.3. Given a growth for a combinatorial class $\mathcal{C}$, let $s : \mathcal{C} \to \mathbb{Z}$ be a parameter whose values determine uniquely the number of objects generated by each object of $\mathcal{C}$. A *succession rule* is the system $\Omega_{\mathcal{C}} = ((r), \mathfrak{R})$,

$$\Omega_{\mathcal{C}} = \begin{cases} (r) \\ (h) \rightsquigarrow (h_1), (h_2), \ldots, (h_j), \end{cases}$$

where the value r, called *axiom*, is the value that s assumes on the minimum size object of $\mathcal{C}$ and the set $\mathfrak{R}$, called *production*, describes all the values $h_1, \ldots, h_j$ that s assumes on the objects generated by any object $o \in \mathcal{C}$ such that $s(o) = h$.

Therefore, the axiom is the label of the root in the labelled generating tree, and the production set explains for each node which are the labels of its children.

In general, the statistics that control the growth for a combinatorial class are likely to be more than one. Precisely, we need to label the nodes of the corresponding generating tree with arrays of integers, whose entries evaluate those statistics each once. Then, Definition 1.2.3 generalises as follows.

Definition 1.2.4. Given a growth for a combinatorial class $\mathcal{C}$, let $s_1, \dots, s_t$ be t different parameters whose values determine uniquely the number of objects of $\mathcal{C}$ generated by each object. A *succession rule* is a system $\Omega_{\mathcal{C}} = (\mathbf{r}, \mathfrak{R})$,

$$\Omega_{\mathcal{C}} = \begin{cases} \mathbf{r} \\ \\ \mathbf{k} \rightsquigarrow \mathbf{k_1},\, \mathbf{k_2}, \dots, \mathbf{k_j}, \end{cases}$$

where the axiom $\mathbf{r} = (r_1, \dots, r_t)$ is an array given by considering the value r_i that s_i assumes on the minimum size object of $\mathcal{C}$, and the set of productions $\mathfrak{R}$ describes for each array $\mathbf{k} = (s_1(o), \dots, s_t(o))$, with $o \in \mathcal{C}$, all the arrays $\mathbf{k_i} = (s_1(o_i), \dots, s_t(o_i))$, for $1 \leq i \leq j$, where $o_1, \dots, o_j$ are the objects generated by o.

Thus, as described for generating trees, the enumerative sequence $\{c_n\}_{n\geq 0}$ of the combinatorial class $\mathcal{C}$ can be deduced directly from a succession rule $\Omega_{\mathcal{C}}$ by iterating its productions and recording the total number of labels at each iteration.

As a direct consequence, two families of objects having the same succession rule are enumerated by the same number sequence and moreover, they are trivially in bijection (see [70, 145]). The bijection is easily established by putting in correspondence objects of the two different combinatorial families in accordance with their position in the associated generating tree. This bijection is clearly recursive: it follows the way along each object is built starting from the smallest one.

On the other hand, we will notice further that there could exist different generating trees and succession rules associated with a combinatorial class $\mathcal{C}$ and with a number sequence $\{c_n\}_{n\geq 0}$. Therefore, given different combinatorial classes, it is not obvious to define growths for them that correspond to a common succession rule Ω. To fix notation, we say that a class $\mathcal{C}$ of combinatorial objects can be generated by a given succession rule Ω, if can be found a growth for $\mathcal{C}$ that defines the succession rule Ω.

Succession rules result to be a very powerful tool for enumeration purposes: they raised interest in the last two decades [32, 44, 47], and have been extremely useful to define algorithms for generating uniformly and randomly words of a given language [14]. Consequently, their study led to some generalisations and classifications according to shape and features.

First, two succession rules are *equivalent* if they correspond to the same generating tree. Equivalent succession rules have been studied in [44] as well as *finite* succession rules, namely succession rules with a finite number of labels and finite productions.

According to the technique of *coloured labels* [57, 77], if some labels of a succession rule are allowed to have different productions, then in that case we distinguish those labels with different productions by using some colours and the corresponding succession rule is said *coloured*.

Another slight generalization of the concept of succession rule is provided by *jumping* succession rules [78]. Roughly speaking, the idea is to consider a growth for the objects of a class $\mathcal{C}$ which involves elements at different levels (not only at the next level).

In addition, the notion of succession rule has been translated in terms of matrices [67], *production matrices*, and operators, which form its algebraic counterpart [77], in order to encode known operations for matrices and operators on the corresponding number sequences.

1.2.4 Generating functions

In this section we report the basic notions of generating functions, which are probably the most powerful tool to tackle counting problems. The majority of definitions and standard results listed here are taken from P. Flajolet and R. Sedgewick's book *Analytic Combinatorics* [79] and part from the Chapter 1 of [25] developed by F. Ardila.

Definition of ordinary generating function

Given a combinatorial class of objects $\mathcal{C}$ equipped with a notion of size $p : \mathcal{C} \to \mathbb{N}$, the (ordinary) generating function of the objects of $\mathcal{C}$, counted by their size, is the following formal power series in the indeterminate x

$$F(x) := \sum_{o \in \mathcal{C}} x^{p(o)} = \sum_{n \geq 0} c_n \, x^n,$$

where c_n is the number of objects of $\mathcal{C}$ such that $p(o) = n$.

There are two approaches toward power series: the analytic attitude, which treats $F(x)$ as an honest analytic function of x, and the algebraic approach, which treats $F(x)$ as a formal algebraic expression, without any concern for convergence. The last simple idea is extremely powerful since the most common algebraic operations on power series correspond to some of the most common operations on combinatorial classes. Indeed, as formal power series, generating functions belong to the ring $\mathbb{C}[[x]]$ of formal power series in x with coefficients in the field $\mathbb{C}$, where sum and product of $A(x) = \sum_{n \geq 0} a_n x^n$ and $B(x) = \sum_{n \geq 0} b_n x^n$ are defined by

$$A(x) + B(x) = \sum_{n \geq 0} (a_n + b_n) x^n, \qquad A(x) \cdot B(x) = \sum_{n \geq 0} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n.$$

Operators on classes and on their generating functions

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be combinatorial classes. The following operations are defined:

1. *Disjoint union* ($\mathcal{C} = \mathcal{A} + \mathcal{B}$): any object of size n of $\mathcal{C}$ is either an object of size n of $\mathcal{A}$ or an object of size n of $\mathcal{B}$.

2. *Product* ($\mathcal{C} = \mathcal{A} \times \mathcal{B}$): any object of size n of $\mathcal{C}$ is obtained by choosing an object of size k of $\mathcal{A}$ and an object of size $n-k$ of $\mathcal{B}$, for some k.

3. *Sequence* ($\mathcal{C} = \mathsf{Seq}(\mathcal{B})$): assume $|\mathcal{B}_0| = 0$, any object of size n of $\mathcal{C}$ is obtained by choosing a sequence of objects of $\mathcal{B}$ such that their total size is n.

4. *Composition* ($\mathcal{C} = \mathcal{A} \circ \mathcal{B}$): assume $|\mathcal{B}_0| = 0$, any object of size n of $\mathcal{C}$ is obtained by choosing a sequence of k objects of $\mathcal{B}$ having total size equal to n and inserting them into an object of size k of $\mathcal{A}$.

There operations on combinatorial classes have an algebraic counterpart.

Proposition 1.2.5. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be combinatorial classes and let $A(x)$, $B(x)$ and $C(x)$ be their respective generating functions.*

1. *If $\mathcal{C} = \mathcal{A} + \mathcal{B}$,*
$$C(x) = A(x) + B(x)\,.$$

2. *If $\mathcal{C} = \mathcal{A} \times \mathcal{B}$,*
$$C(x) = A(x) \cdot B(x)\,.$$

3. *If $\mathcal{C} = \mathsf{Seq}(\mathcal{B})$,*
$$C(x) = \frac{1}{1 - B(x)}\,.$$

4. *If $\mathcal{C} = \mathcal{A} \circ \mathcal{B}$,*
$$C(x) = \sum_{n \geq 0} a_n \left(\sum_{m \geq 0} b_m x^m \right)^n = A(B(x))\,.$$

Coefficients of generating functions

For $F(x) = \sum_{n \geq 0} c_n x^n$, we write

$$[x^n]F(x) := c_n \qquad \text{and,} \qquad F(0) := [x^0]F(x) = c_0\,.$$

For the sake of completeness, we report here some formal power series inspired by series from analysis that will help us to extract coefficients from generating functions. The formal power series in the following occur frequently in applications, as well as along this dissertation, but they are not the only: a more accurate list of these series can be found in [146, Section 2.5]).

Let $\binom{\alpha}{n} := \alpha(\alpha-1)\cdots(\alpha-n+1)/n!$, with $\alpha \in \mathbb{C}$,

$$(1+x)^\alpha = \sum_{n \geq 0} \binom{\alpha}{n} x^n, \qquad \text{and for } k \in \mathbb{N}, \qquad \frac{1}{(1-x)^{k+1}} = \sum_{n \geq 0} \binom{n+k}{n} x^n.$$

A highly important tool that permits to extract the nth coefficient from a generating function, is the well-known Lagrange inversion formula, which reads as follows.

Theorem 1.2.6 (Lagrange Inversion Theorem, Appendix A.6 [79])**.** *Let* $\Phi(u) = \sum_{k\geq 0} a_k u^k$ *be a power series of* $\mathbb{C}[[x]]$ *with* $\Phi(0) \neq 0$*. Then, the equation* $y = x\,\Phi(y)$ *admits a unique solution in* $\mathbb{C}[[x]]$ *whose coefficients are given by (Lagrange form)*

$$y(x) = \sum_{n=1}^{\infty} y_n x^n, \qquad \textit{where} \qquad y_n = \frac{1}{n}[u^{n-1}]\Phi(u)^n \,. \tag{1.1}$$

Furthermore, one has for $k > 0$ *(Bürmann form)*

$$y(x)^k = \sum_{n=1}^{\infty} y_n^{(k)} x^n, \qquad \textit{where} \qquad y_n^{(k)} = \frac{k}{n}[u^{n-k}]\Phi(u)^n \,. \tag{1.2}$$

By linearity, a form equivalent to Bürmann's, with $g(u)$ *an arbitrary function, is*

$$[x^n]g(y(x)) = \frac{1}{n}[u^{n-1}](g'(u)\Phi(u)^n)\,.$$

The ultimate aim is to provide a closed formula for c_n, or an expression for the generating function $F(x)$. In many cases, it is sufficient to find a functional equation satisfied by the generating function $F(x)$ to prove formulas for the numbers c_n (explicit and/or recursive).

Nature of generating functions

Another aspect of strong interest in studying generating functions is their *nature*. The nature of $F(x)$ can indeed provide information about the numbers c_n and their behaviour when n becomes larger and larger. Moreover, the unsolvability of a combinatorial problem is strictly related to the nature of its generating function solution: in [90], for instance, is provided a numerical method that distinguishes whether a problem is likely to be solvable in terms of simple functions of mathematical physics or not.

A complete classification of the nature of generating functions could be found in [134, 135] as well as [90], yet in the following we report only the definitions of classes of generating functions that will appear onwards.

A formal power series $F(x)$ is *rational* if it can be written in the form

$$F(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x) \neq 0$ are polynomials in x with rational coefficients.

Example 1.1. *The generating function of sequences* $1/(1-x) = \sum_{n\geq 0} x^n$ *and the generating function of subsets* $1/(1-2x) = \sum_{n\geq 0} 2^n\, x^n$ *are clearly rational.*

A series $F(x)$ is said *algebraic* if there exist polynomials $P_0(x), \ldots, P_k(x) \in \mathbb{Q}[x]$, not all 0, such that

$$P_0(x) + P_1(x)F(x) + \ldots + P_k(x)F(x)^k = 0\,,$$

or equivalently, if there exists a bivariate polynomial $\mathcal{P}$ with rational coefficients such that $\mathcal{P}(x, F(x)) = 0$. The smallest positive integer k for which the above equation holds is called the *degree* of F.

Example 1.2. *The generating function* $F(x) = \sum_{n\geq 0} \binom{2n}{n} x^n$ *of central binomial coefficients is algebraic, since*

$$F(x)^2(1-4x) - 1 = 0\,.$$

Finally, the series $F(x)$ is said *D-finite* of order k if there exist polynomials $Q_0(x), \ldots, Q_k(x), Q(x) \in \mathbb{Q}[x]$, with $Q_k(x) \neq 0$, such that

$$Q_0(x)F(x) + Q_1(x)F'(x) + Q_2(x)F''(x) + \ldots + Q_k(x)F^{(k)}(x) = Q(x).$$

Example 1.3. *The generating function* $F(x) = \sum_{n\geq 0} n!\, x^n$ *is D-finite, as it satisfies*

$$x^2F'(x) + (x-1)F(x) + 1 = 0\,.$$

These three classes of formal power series we have defined form a hierarchy: every rational series is also algebraic and every algebraic series is D-finite.

Furthermore, given a number sequence $\{c_n\}_{n\geq 0}$ we make the following definitions that are closely related to the nature of their generating functions.

A sequence $\{c_n\}_{n\geq 0}$ is said *c-recursive* if there are constants $a_0, \ldots, a_d \in \mathbb{C}$ such that for all $n \geq d$,

$$a_0\, c_n + a_1\, c_{n-1} + \ldots + a_d\, c_{n-d} = 0\,.$$

A sequence $\{c_n\}_{n\geq 0}$ is said *P-recursive* if there are complex polynomials $a_0(x), \ldots, a_d(x)$, with $a_d(x) \neq 0$, such that for all $n \geq d$,

$$a_0(n)\, c_n + a_1(n)\, c_{n-1} + \ldots + a_d(n)\, c_{n-d} = 0\,.$$

Note that any number sequence that is c-recursive is trivially P-recursive.

Theorem 1.2.7. *The following implications hold.*

$F(x)$ *is rational*	$\Rightarrow$	$F(x)$ *is algebraic*	$\Rightarrow$	$F(x)$ *is D-finite*
$\Updownarrow$ *[134, Theorem 4.1.1]*				$\Updownarrow$ *[135, Proposition 6.4.3]*
$\{c_n\}_{n\geq 0}$ *is c-recursive*		$\Rightarrow$		$\{c_n\}_{n\geq 0}$ *is P-recursive*

It is not obvious whether a given D-finite power series is algebraic or not, yet some tools are available: Table 1.3, for instance, summarises the behaviour of the nature of generating functions under various key operations. For example, both sum and product of two D-finite functions are D-finite, whereas the composition is not necessarily D-finite. In Table 1.3 the derivative function is $A'(x) = \sum_{n\geq 1} n a_n x^{n-1}$; in the forth column we are assuming $A(0) \neq 0$ so that $1/A(x)$ is well defined, and in the fifth column we are assuming $B(0) = 0$ so that $A(B(x))$ is well defined.

	$aA(x)$	$A(x)+B(x)$	$A(x)\cdot B(x)$	$1/A(x)$	$A(x)\circ B(x)$	$A'(x)$
rational	yes	yes	yes	yes	yes	yes
algebraic	yes	yes	yes	yes	yes	yes
D-finite	yes	yes	yes	no	no	yes

Table 1.3: The nature of formal power series resulting from key operations. Each "yes" entry means that the result preserves the nature, where the column operation is applied to formal power series of nature according to the row.

In addition, for all the negative results showed in Table 1.3 a weaker positive statement holds:

1. If $A(x)$ is D-finite and $A(0) \neq 0$, $1/A(x)$ is D-finite if and only if $A'(x)/A(x)$ is D-finite;
2. If $A(x)$ is D-finite and $B(x)$ is algebraic with $B(0) = 0$, then $A(B(x))$ is D-finite.

The following result is also useful.

Theorem 1.2.8 ([25], Theorem 1.3.11)**.** *If the coefficients of an algebraic power series $F(x) = \sum_{n\geq 0} c_n x^n$ satisfy $c_n \sim c\, n^r\, \alpha^n$, for nonzero c, $\alpha \in \mathbb{C}$ and $r < 0$, then r cannot be a negative integer.*

Generating functions and succession rules

Now, we relate generating functions and generating trees or, equivalently, succession rules. We can refine the generating function $F(x)$ associated with $\mathcal{C}$ so that it takes into account the labels of its corresponding succession rule. Precisely, let $\Omega_{\mathcal{C}}$ be a succession rule with labels $\mathbf{h} = (h_1, \ldots, h_t)$, for some $t \geq 1$. By introducing additional variables, called *catalytic variables* in the sense of D. Zeilberger [151], in order to keep track of the labels $\mathbf{h}$, the univariate generating function $F(x)$ for $\mathcal{C}$ is refined to the function

$$G(x; y_1, \ldots, y_t) = \sum_{\mathbf{h}, n\geq 0} d_{n,\mathbf{h}}\, y_1^{h_1}\, y_2^{h_2} \ldots y_t^{h_t}\, x^n, \tag{1.3}$$

where the sum ranges over all the possible labels $\mathbf{h}$ and $d_{n,\mathbf{h}}$ denotes the number of objects of size n and label $\mathbf{h}$. Depending on the form of the succession rule, we can hopefully write and solve a functional equation whose resolution gives an expression for the univariate function $F(x)$ and the multivariate function $G(x; y_1, \ldots, y_t)$. The next sections about Catalan and Baxter numbers (Sections 1.3.6 and 1.4.5) show examples of translation of succession rules into functional equations.

For multivariate generating functions $G(\mathbf{x}) = G(x_1, \ldots, x_k)$ the basic algebra rules work as for the univariate case. Moreover, the definitions of rational, algebraic and D-finite generating functions can be generalised to multivariate functions $G(\mathbf{x})$ by considering in each definition of page 18 the vector $\mathbf{x} = (x_1, \ldots, x_k)$ instead of a single variable x. To be more precise, in the case of a multivariate D-finite generating function [106]: $G(\mathbf{x})$ is said D-finite if it satisfies a system of linear partial differential equations, one for each $i = 1 \ldots k$, of the form

$$Q_{i,0}(\mathbf{x})G(\mathbf{x}) + Q_{i,1}(\mathbf{x})\frac{\partial}{\partial x_i}G(\mathbf{x}) + Q_{i,2}(\mathbf{x})\frac{\partial^2}{\partial x_i^2}G(\mathbf{x}) + \ldots + Q_{i,r_1}(\mathbf{x})\frac{\partial^{r_1}}{\partial x_i^{r_1}}G(\mathbf{x}) = 0\,.$$

As in the univariate case, the class of multivariate D-finite functions is closed under sum, product, differentiation, algebraic substitutions, and also under setting some variable equal to a constant (specialisations) [79, Theorem B.3].

Nevertheless, Theorem 1.2.7 does not hold for multivariate generating functions: a linear recurrence with constant coefficients does not necessarily yield a rational, algebraic, or even D-finite generating function in 2 or more variables [38].

The methods used along this dissertation to treat and solve functional equations having as solution multivariate generating functions of a given combinatorial class can be summed up in this chart:

- kernel method, discussed in Section 1.3.6;
- obstinate variant of the kernel method, discussed in Section 1.4.5;
- generalisation of the quadratic method, discussed in Section 6.3.2.

1.3 Introduction to Catalan structures

According to R. P. Stanley's monograph [133, Appendix B], Catalan numbers have had a chaotic history and for decades they remained unnamed and rather unknown compared to other famous number sequences, such as Fibonacci numbers (sequence A000045 [132]). In that appendix of [133] the name "Catalan numbers" is attributed to the combinatorialist J. Riordan (1903-1988) who first used it in 1948. Despite the delay in naming them, Catalan numbers cover a large literature - more than 450 references are registered in the research bibliography [88] - and they show up so frequently that their combinatorial interpretations are many and different. Our interest in Catalan numbers is thus justified by the fact that they are "probably the most ubiquitous sequence in Mathematics" (R. P. Stanley [135]) and "the longest entry in the OEIS" (A000108 [132]). Therefore, we summarise in this section all the (well-)known results on Catalan numbers that will be useful to the purposes of our research.

1.3.1 Formulas

The first terms of the sequence A000108 [132] of Catalan numbers C_n are

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, \ldots$$

The most important and transparent recurrence satisfied by C_n is

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \qquad \text{with} \quad C_0 = 1.$$

It explains many of the combinatorial interpretations of Catalan numbers, where the objects being counted have a decomposition into two parts. It was the Belgian-born mathematician E. C. Catalan (1814-1894) in [52] to first provide the well-known formula

$$C_n = \binom{2n}{n} - \binom{2n}{n-1}.$$

Theorem 1.3.1 (Explicit formula)**.** *We have for every $n \geq 0$,*

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}.$$

A combinatorial proof of Theorem 1.3.1 can be found in [133, Section 1.6], where the explicit formula for Catalan numbers is derived by finding an equivalence relation on some sets $\mathcal{E}_n$ with $\binom{2n}{n}$ elements. Another proof that makes use of the so-called cycle lemma is in [63].

Some of the properties of Catalan numbers are in [128] and in [104], as well as a proof of their close relation with the Narayana numbers $N(n,k)$ (sequence A001263 [132]),

$$C_n = \sum_{k=1}^{n} N(n,k), \quad \text{where } N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

1.3.2 Structures

The monographs [133] and [135] have a comprehensive collection of families of objects enumerated by Catalan numbers. Among the 214 different kind of objects presented, we choose to report here the definitions of those that are relevant to our study.

Dyck paths

According to Definition 1.1.1 we define the family of Dyck paths as follows.

Definition 1.3.2. A *Dyck path* of semi-length n is a path P of length $2n$ in the positive quarter plane that uses *up* steps $U = (1,1)$ and *down* steps $D = (1,-1)$ starting at the origin and returning to the x-axis.

Any pair UD (resp. DU) of steps in a Dyck path P is called *peak* (resp. *valley*), and its rightmost sequence of D steps is called *last descent* of P.

Dyck paths are counted by Catalan numbers C_n according to their semi-length $n \geq 0$; for a proof of this result see [66]. We denote by $\mathcal{D}_n$ the set of Dyck paths of semi-length n and by $\mathcal{D} = \cup_n \mathcal{D}_n$.

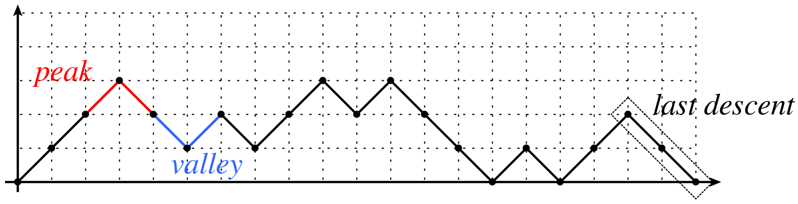


Figure 1.3: A Dyck path of semi-length 10: its first peak is coloured red, its first valley blue and its last descent is squared.

Figure 1.3 shows a Dyck path of $\mathcal{D}_{10}$, with its first peak, and its first valley, and its last descent highlighted.

In addition, a Dyck path P of length $2n$ can be encoded by a word of length $2n$ in the alphabet $\Sigma = \{U, D\}$, so-called *Dyck word.* A Dyck word $w \in \Sigma^*$ is such that for any prefix v, the number of occurrences of U in v is greater than or equal to the number of occurrences of D, and they must be equal if $v = w$. Note that this characterisation of the word w mirrors the constraint of the path P to remain weakly above the x-axis and to end in a point of the x-axis. Onwards we use equivalently Dyck words to denote Dyck paths of Definition 1.3.2. A peak (resp. valley) is thus any UD (resp. DU) factor of P.

Statistics on Dyck paths have been extensively studied in literature - for instance, see [21, 64, 63].

Pattern-avoiding permutations

Recall the notion of pattern avoidance in permutations of Definition 1.1.2. As already stated in Theorem 1.1.5, the principal classes $AV(\tau)$, $\tau \in \mathcal{S}_3 = \{123, 132, 213, 231, 312, 321\}$, are equinumerous and their enumerative sequence is proved to be the Catalan number sequence. Theorem 1.1.5, first established by D. Knuth [99], has several proofs in literature [101, 131].

Parallelogram polyominoes

According to Definition 1.1.1 of path and considering a *cell* the unit square in the Cartesian plane, we define the family of parallelogram polyominoes, as follows.

Definition 1.3.3. A *parallelogram polyomino* is the set S of cells contained between two non-intersecting lattice paths (P, Q). Both paths P and Q must have same length $n \geq 2$, and starting at the origin, by using *north* $(0, 1)$ and *east* $(1, 0)$ steps must end at the same point without intersecting (except at the origin and at the ending point). The length of each path is also called the semi-perimeter, or *size*, of the parallelogram polyomino.

Moreover, paralellogram polyominoes are defined up to translation in the Cartesian plane - see Figure 1.4. Parallelogram polyominoes of size $n+1$ are counted by the Catalan number C_n (see [60, 61, 119]), for $n > 0$. We denote by $\mathcal{PP}_n$ the set of parallelogram polyominoes of size n and by $\mathcal{PP} = \cup_n \mathcal{PP}_n$.

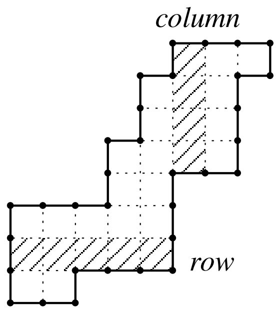


Figure 1.4: A parallelogram polyomino of size 16: a column and a row are striped.

Any parallelogram polyomino as set of cells comprises *columns* and *rows*: a column (resp. row) is the set of cells having same abscissa (resp. ordinate) - see, for instance, Figure 1.4. By definition of parallelogram polyomino rows and columns are connected sets of cells, therefore parallelogram polyominoes are special convex polygons (see [31] or, staircase polygons in [91, Section 7.3]).

Non-decreasing sequences

As 78th entry of the Catalan structures list in [133, 135] it appears the following family of integer sequences.

Definition 1.3.4. A *non-decreasing sequence* is any sequence $a_1 \dots a_n$ of integers such that $a_i < i$ and $a_j \geq a_i$, for all $j > i$.

Figure 1.5 shows the graphical representation of a non-decreasing sequence $a_1 \dots a_n$, obtained by plotting in a grid all the points (i, a_i). The number of non-decreasing sequences of length n is the Catalan number C_n, for every $n \geq 0$.

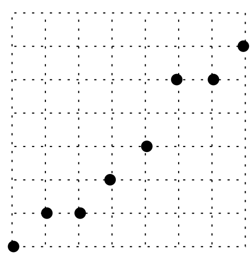


Figure 1.5: The graphical representation of the non-decreasing sequence 01123556.

1.3.3 Bijections

In this part we record two bijections involving Catalan objects rather well-known in literature: these bijections link the above Catalan structures two by two. In particular, we report the bijection between Dyck paths and parallelogram polyominoes as described by M. Delest and G. Viennot [61, Section 4]. This first bijection will be re-used later in Chapter 2. The second bijection presented here is between non-decreasing sequences and permutations avoiding 132. Its proof is folklore and we cannot find a precise reference to address to, except for [135]. Our proof makes use of a general proposition that will be largely used in Chapter 5.

Bijection between Dyck paths and parallelogram polyominoes

First, define a mapping β from $\mathcal{D}_n$, the set of Dyck paths of length $2n$, to $\mathcal{PP}_n$, the set of parallelogram polyominoes of size $n+1$, for $n > 0$, as follows:

- Given a $P \in \mathcal{D}_n$, with $n > 0$, number from left to right its peaks and its valleys. Let $k \geq 1$ be the total number of peaks (obviously, there must be $k-1$ valleys);
- Let $(x_i, y_i) \in \mathbb{N}^2$ pinpoint the ith peak - *i.e.* (x_i, y_i) is the ending (resp. starting) point of the U (resp. D) step of the ith peak. Similarly, let $(u_i, h_i) \in \mathbb{N}^2$ pinpoint the ith valley - *i.e.* (u_i, h_i) is the ending (resp. starting) point of the D (resp. U) step of the ith valley. Call *height* of the ith peak (resp. valley) the ordinate value $y_i > 0$ (resp. $h_i \geq 0$);
- For every i from 1 to k, stack a set of cells in a number equal to the height y_i of the ith peak. Note that each of these vertical bars built has at least one cell. Then they will be used to form the columns of a parallelogram polyomino;
- Glue together all those k bars, as to form a set S of connected cells following the heights of the valleys of P. Precisely, if h_j is the height of the jth valley, with $1 \leq j < k$, then the jth bar is glued to the $(j+1)$st bar so that they are edge-connected by exactly $h_j + 1$ cells - see Figure 1.6.

The property that the height h_j of the jth valley is strictly less than the height y_j of the jth peak allows the glueing process to form a parallelogram polyomino S with k columns. The fact that S has size $n+1$, thus $S \in \mathcal{PP}_n$, can be manually checked as the sum of the peak heights minus the valley heights gives the semi-length n of P. For instance, the Dyck path UD of minimum length is made correspond to the single cell, that is the parallelogram polyomino with minimum size.

Proposition 1.3.5. *Let $\beta : \mathcal{D} \to \mathcal{PP}$ be defined as above so that any Dyck path of length $2n$ and k peaks is associated with a parallelogram polyomino of size $n+1$ and k columns, with $n, k > 0$. Then, the mapping β is a bijection.*

Proof. To show that the mapping β is a bijection we define a mapping $\gamma : \mathcal{PP}_n \to \mathcal{D}_n$ such that $\beta \circ \gamma$ (resp. $\gamma \circ \beta$) is the identity.

Let S be in $\mathcal{PP}_n$ and k be the number of its columns. Define two arrays of positive integers $Y = (y_1, \ldots, y_k)$ and $V = (v_1, \ldots, v_{k-1})$ so that Y records the column heights from left to right and V records the edge contacts between adjacent columns from left to right -

see Figure 1.6. Then, construct a path P in the quarter plane starting at $(0,0)$ and using U and D steps as follows: draw a first sequence of U steps ending at height y_1 followed by a sequence of D steps ending at height $v_1 - 1$, then recursively alternate a sequence of U steps and a sequence of D steps according to the heights y_i and $v_i - 1$, for $1 < i < k$. The rightmost sequence of U steps is drawn ending at height y_k and the rightmost sequence of D steps is uniquely drawn so that it ends on the x-axis. Thus, P is a Dyck path with exactly k peaks.

Moreover, the semi-length of P is given by the sum of the peak heights minus the valley heights,

$$\sum_{i=1}^{k} y_i + \sum_{i=1}^{k-1} (v_i - 1),$$

which is n, since the size $n+1$ of S is given by

$$k + \sum_{i=1}^{k} y_i - \sum_{i=1}^{k-1} v_i \,.$$

□

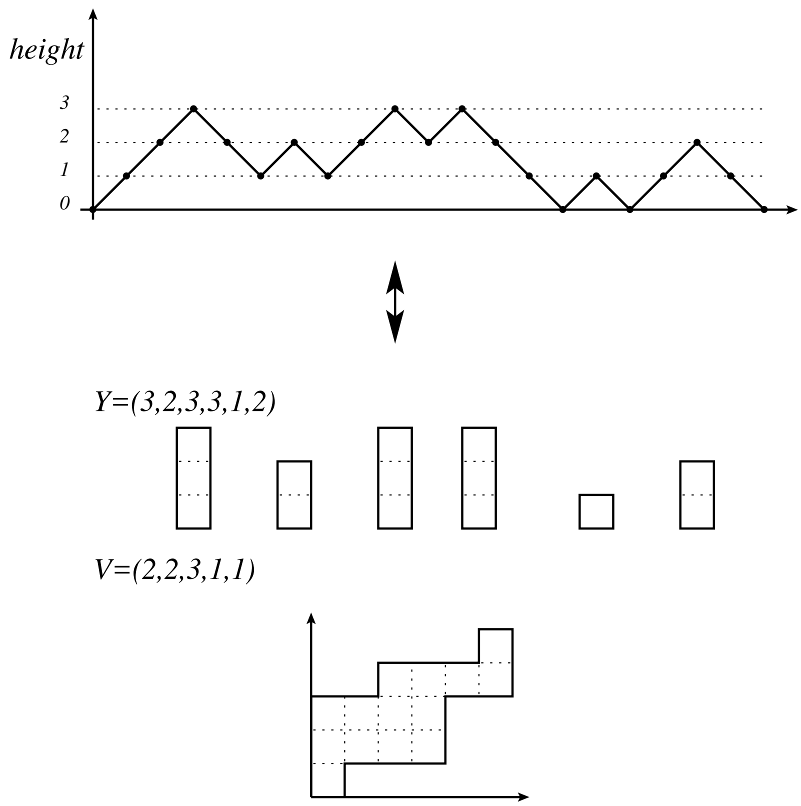


Figure 1.6: An instance of the mapping sending a Dyck path of semi-length 10 and 6 peaks into a parallelogram polyomino of size 11 and 6 columns.

Bijection between non-decreasing sequences and permutations avoiding 132

The second bijection reported here comes out as a restriction of a more general bijection between permutations and particular integer sequences. Then, in order to prove the bijection between non-decreasing sequences and 132-avoiding permutations, we recall here a known bijection denoted by $\mathtt{T}$ between permutations and *left inversion tables*, whose first appearance was in an exercise by D. Knuth [99, Section 5.1.1].

Definition 1.3.6. Let π be a permutation of length n. If $i < j$ and $\pi_i > \pi_j$, then the pair (π_i, π_j) is said an *inversion* of π. For $1 \leq i \leq n$, let t_i be the cardinality of the set $\{\pi_j : j > i \text{ and } (\pi_i, \pi_j) \text{ is an inversion}\}$. Then, $\mathtt{T}(\pi) = (t_1, \ldots, t_n)$ is called *left inversion table* of π.

For instance, the left inversion table of $\pi = 16482753$ is $\mathtt{T}(\pi) = 04240210$. The mapping $\mathtt{T}$ is actually a bijection, since the left inversion table of a permutation π uniquely identifies π [99].

Proposition 1.3.7. *Let* $\mathtt{T}$ *be the mapping that sends a permutation of length* n *in its left inversion table and* $\mathtt{R}$ *be the operation of reverse (or mirror) on sequences. Then, the mapping* $\mathtt{R} \circ \mathtt{T}$ *is a bijection between the family of permutations* $\mathcal{S}$ *and integer sequences* $(e_1, \ldots, e_n)$ *such that* $0 \leq e_i < i$*, for any* i*.*

Restricting this bijection between permutations and sequences of integers yields the following result.

Proposition 1.3.8. *Let* $\mathtt{R} \circ \mathtt{T}$ *be defined as in Proposition 1.3.7. The restriction of* $\mathtt{R} \circ \mathtt{T}$ *to the permutation class* $AV(132)$ *is a bijection between the family of non-decreasing sequences and the permutation class* $AV(132)$*.*

Proof. In order to prove this statement we show that for any n, $\mathtt{T}(\pi)$ is a weakly decreasing sequence of length n if and only if $\pi \in AV(132)$.

$\Rightarrow$) We prove the contrapositive: if $132 \preccurlyeq \pi$, then $\mathtt{T}(\pi) = (t_1, \ldots, t_n)$ is not weakly increasing. If π contains 132, then there are three indices $i < j < k$ such that $\pi_i < \pi_k < \pi_j$. We can suppose without loss of generality that there is no index i', $i < i' < j$, such that $\pi_{i'} < \pi_i$. The pair (π_j, π_k) is an inversion of π, while (π_i, π_j) is not. Thus, the number of inversions of π_i is strictly smaller than the number of inversions of π_j. In other words, there exists two indices $i < j$, such that $t_i < t_j$.

$\Leftarrow$) We prove it again by using the contrapositive. Suppose $\mathtt{T}(\pi) = (t_1, \ldots, t_n)$ is such that there exists an index i, with $t_i < t_{i+1}$. By definition of left inversion table this inequality yields $\pi_i < \pi_{i+1}$: indeed, if $\pi_i > \pi_{i+1}$, then $t_i \geq t_j$ must hold. In addition, since $t_i < t_{i+1}$, there must be a point π_j, $j > i+1$, such that (π_i, π_j) is not an inversion, whereas (π_{i+1}, π_j) is. Therefore, $132 \preccurlyeq \pi$. $\square$

1.3.4 Generating trees

Here is a small collection of ECO operators for Catalan structures - a thorough list can be found in literature [13, 116].

Recall that $\mathcal{D}$ is the combinatorial class of non-empty Dyck paths where the size is the semi-length. As depicted in Figure 1.3, the last descent of any $P \in \mathcal{D}$ has a number of points equal to the number of D steps plus one. Thus, any non-empty Dyck path P has at least two points in its last descent. We define an operator $\vartheta_{\mathcal{D}} : \mathcal{D}_n \to \mathcal{D}_{n+1}$ as follows.

Definition 1.3.9. For $n > 0$, if $P \in \mathcal{D}_n$, then $\vartheta_{\mathcal{D}}(P)$ is the set of Dyck paths obtained from P by inserting a peak in any point of P's last descent.

The set of Dyck paths obtained by performing $\vartheta_{\mathcal{D}}$ on a Dyck path P is depicted in Figure 1.7.

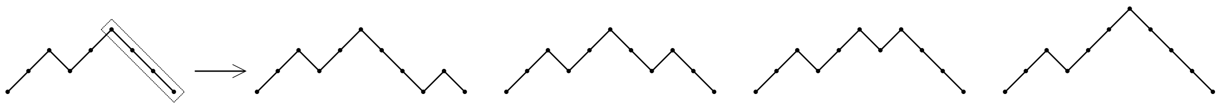


Figure 1.7: The set of paths obtained by operator $\vartheta_{\mathcal{D}}$.

The operator $\vartheta_{\mathcal{D}}$ satisfies both properties of Proposition 1.2.1. Indeed, each Dyck path P of semi-length $n+1$ is produced by a unique Dyck path P' of semi-length n through the application of $\vartheta_{\mathcal{D}}$: the path P' is obtained by removing the rightmost peak of P.

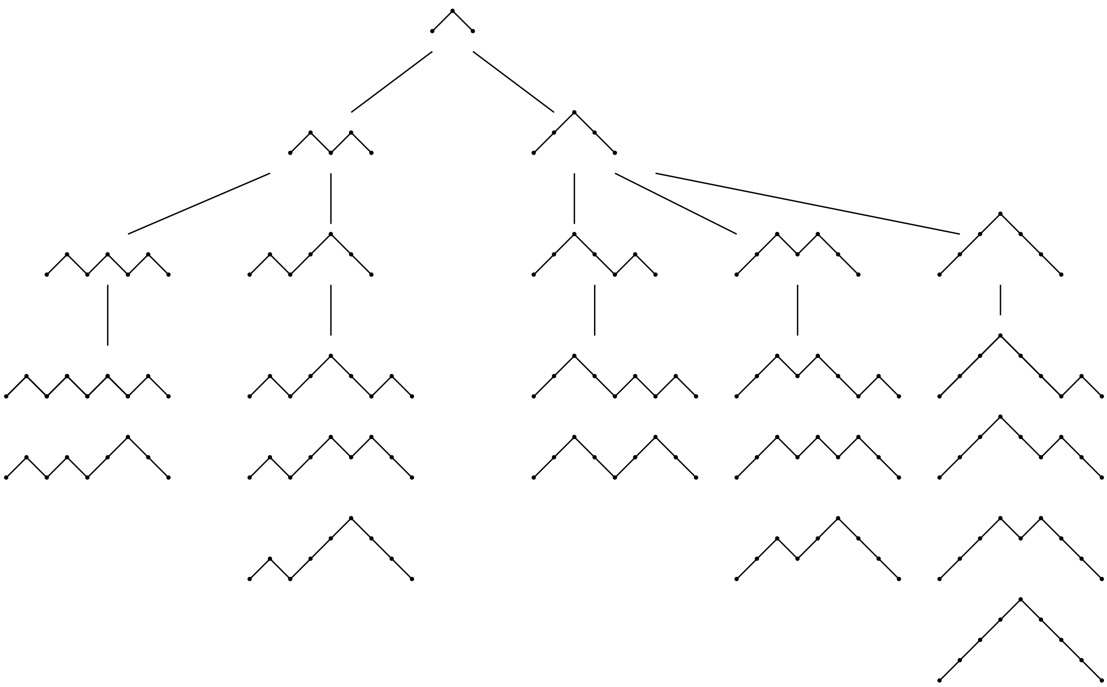


Figure 1.8: The first levels of the decorated generating tree associated with $\vartheta_{\mathcal{D}}$.

Figure 1.8 depicts the first levels of the generating tree induced by the ECO operator $\vartheta_{\mathcal{D}}$ whose nodes are decorated with Dyck paths of semi-length at most 4. The minimum size object labelling the root is the Dyck path UD.

Recall that $\mathcal{PP}$ is the combinatorial class of parallelogram polyominoes where the size is the semi-perimeter. We define an ECO operator $\vartheta_{\mathcal{PP}}$ as follows.

Definition 1.3.10. For $n > 0$, if $S \in \mathcal{PP}_{n+1}$ has the rightmost column of height h, then $\vartheta_{\mathcal{PP}}(S)$ is the set of parallelogram polyominoes obtained from S:

- either by glueing a column of height i to the topmost cells of the rightmost column of S, for any $1 \leq i \leq h$;
- or by glueing a single cell on top of the rightmost column of S.

Figure 1.9 shows the growth of a parallelogram polyomino S according to the definition of $\vartheta_{\mathcal{PP}}$ above.

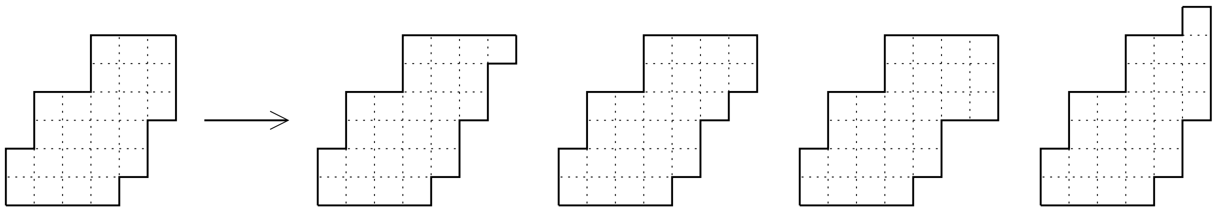


Figure 1.9: The set of parallelogram polyominoes obtained by means of $\vartheta_{\mathcal{PP}}$.

The operator $\vartheta_{\mathcal{PP}}$ is an ECO operator satisfying both properties of Proposition 1.2.1: indeed, each parallelogram polyomino S of size $n+1$ is produced by a unique parallelogram polyomino S' of size n obtained by removing either the topmost row of S, if it consists of only one cell, or the rightmost column of S.

The first levels of the generating tree induced by the ECO operator $\vartheta_{\mathcal{PP}}$ are shown in Figure 1.10. Each node is decorated with a parallelogram polyomino of size at most 5. The single cell is the minimum size object labelling the root.

The last class of objects considered is the permutation class $AV(132)$. Given a permutation $\pi = \pi_1 \dots \pi_n$ any position between two consecutive points π_i and π_{i+1} is said *site*, for every i. If π avoids a set of patterns $\mathfrak{P}$, we call a site *active*, if the insertion of $n+1$ in that position does not create any occurrence of the forbidden patterns. In the graphical representation it is usual to mark any active site with a diamond and every non-active site with a cross - see, for instance, Figure 1.11 where all the $n+1$ sites have been marked.

In order to define operator ϑ_A performing local expansions on permutations in $AV_n(132)$, it is useful to characterise which sites are active in a permutations of $AV_n(132)$.

Definition 1.3.11. Let $\pi \in \mathcal{S}_n$. A *left-to-right maximum* (resp. *minimum*) of π is any point π_i such that for every $j < i$, $\pi_j < \pi_i$ (resp. $\pi_j > \pi_i$). A *right-to-left maximum* (resp. *minimum*) is defined symmetrically. For brevity, we use the notation LTR maximum (minimum) and RTL maximum (minimum).

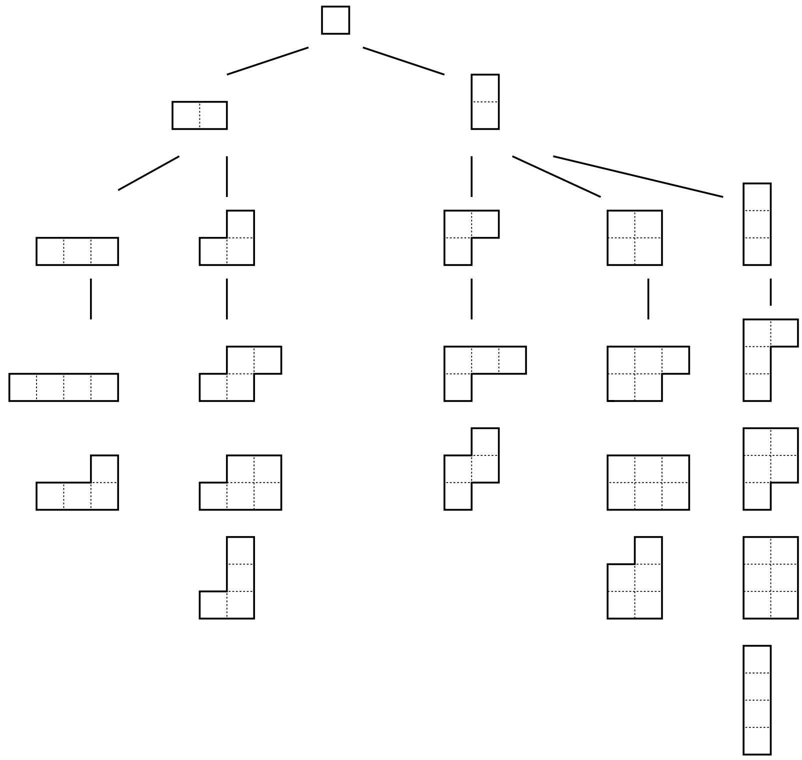


Figure 1.10: The first levels of the decorated generating tree associated with $\vartheta_{\mathcal{PP}}$.

Lemma 1.3.12. *Let π be a permutation of $AV_n(132)$. A site of π is active if and only if it is at the beginning of π or it is immediately after a RTL maximum.*

Proof. In one case there is nothing to prove: π avoids 132 if and only if $\pi' = (n+1)\,\pi$ avoids 132. On the other hand, if π avoids 132 the addition of $n+1$ in a site of π could give rise to an occurrence of 132 provided that $n+1$ plays the role of 3 in that occurrence. Therefore, if $n+1$ is added just after a point which is not a RTL maximum, an occurrence of 132 is generated by definition. Conversely, suppose by contradiction that $n+1$ is added just after a RTL maximum π_i and it gives rise to an occurrence of 132. By definition of RTL maximum, π_i cannot play the role of 1 in that occurrence. Thus, there must be two indices $k < i$ and $j > i$ such that $\pi_k < \pi_j < n+1$. Hence, since π_i is a RTL maximum, $\pi_j < \pi_i$ must hold contradicting the fact that π avoids 132. □

Definition 1.3.13. For $n > 0$, if $\pi \in AV_n(132)$, then $\vartheta_A(\pi)$ is the set of permutations obtained by inserting $n+1$ in any active site of π described by Lemma 1.3.12.

The set of permutations obtained by performing ϑ_A on a permutation $\pi \in AV_5(132)$ is depicted in Figure 1.11.

The operator ϑ_A satisfies both properties of Proposition 1.2.1: each permutation π of length $n+1$ is uniquely obtained from a permutation π' of length n obtained from π by removing its maximum. The first levels of the generating tree associated with ϑ_A are depicted in Figure 1.12

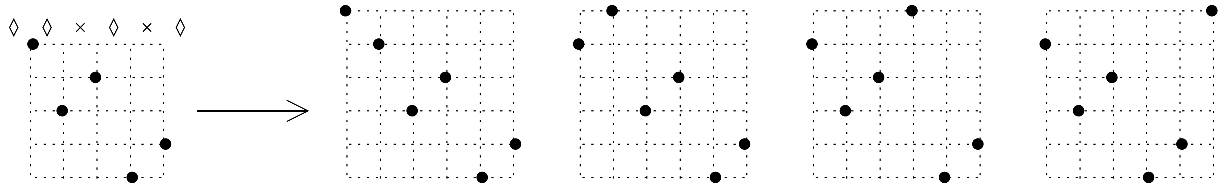


Figure 1.11: The set of permutations obtained from $\pi = 53412$ by means of $\vartheta_{\mathcal{A}}$.

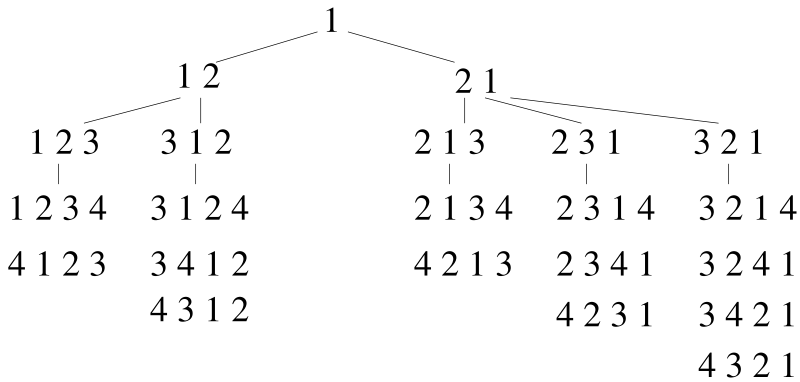


Figure 1.12: The first levels of the decorated generating tree corresponding to $\vartheta_{\mathcal{A}}$.

Remark 1.3.14. *We point out that a growth very similar to the one performed by ϑ_A on the permutation class $AV(132)$ can be provided for the classes $AV(231)$, $AV(312)$, and $AV(213)$. Indeed, symmetrically to $AV(132)$ an active site of a permutation in $AV(231)$ is either immediately before a LTR maximum or at the end of the permutation. Thus, a growth for $AV(231)$ can be defined symmetrically to Lemma 1.3.12 and similarly to Definition 1.3.13.*

On the other hand, the active sites of a permutation $\pi \in AV_n(312)$ (symmetrically, $AV_n(213)$) can be described as those immediately after a RTL maximum (symmetrically, before a LTR maximum) of π or on the left (symmetrically, on the right) of the maximum point n. Thus, also in these cases Definition 1.3.13 can be slightly modified to define a growth for $AV(312)$ as well as $AV(213)$.

In all the three different cases described (Dyck paths, parallelogram polyominoes, and pattern-avoiding permutations), it can be checked that there exists a parameter $s : \mathcal{C} \to \mathbb{N}$ that controls the number of children for each node of the generating tree. When we consider the family of Dyck paths $\mathcal{D}$ the parameter s is the length of the last descent, while for the class $\mathcal{PP}$ it is the height of the rightmost column and for $AV(132)$ it is the number of RTL maxima. As Figure 1.13 shows we can label the nodes of the generating tree using arrays of only one entry that keeps track of these values.

1.3.5 Succession rules

All the growths defined in the previous section define a common generating tree (see Figure 1.13) that can be encoded by a succession rule.

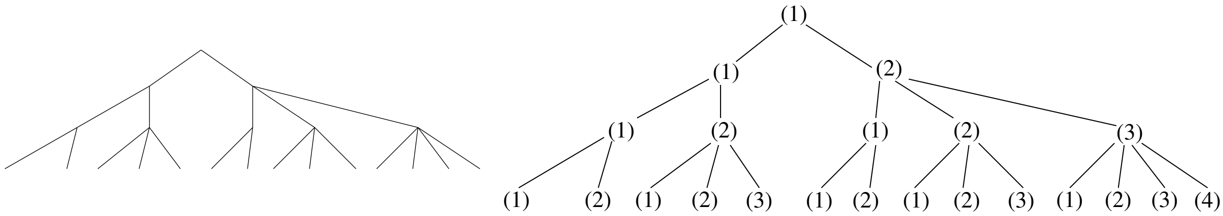


Figure 1.13: On the left the generating tree associated with $\vartheta_{\mathcal{D}}$, $\vartheta_{\mathcal{PP}}$ and ϑ_A up to the fourth level; on the right the same generating tree decorated with labels, $\mathcal{T}_{Cat}$.

Proposition 1.3.15. *The family $\mathcal{D}$ of Dyck paths, and the family $\mathcal{PP}$ of parallelogram polyominoes, and the class $AV(132)$ (as well as $AV(231)$, $AV(312)$, $AV(213)$) can all be generated by*

$$\Omega_{Cat} = \begin{cases} (1) \\ \\ (h) \rightsquigarrow (1), (2), \ldots, (h), (h+1). \end{cases}$$

Proof. For each class, we need to prove that, substituting the objects for their labels in their decorated generating tree, the way in which the label spread is encoded by Ω_{Cat}. Consider, for instance, $\vartheta_{\mathcal{D}}$ of Definition 1.3.9. The label we assign to Dyck paths is the length of the last descent, thus the root label is (1). Then, according to ϑ_D each object having h steps in the last descent produces $h+1$ Dyck paths (as there are $h+1$ points in the last descent). All the Dyck paths produced by inserting a peak have different lengths of their last descent, which vary from 1 to $h+1$. Therefore, any label (h) produces labels $(1), (2), \ldots, (h), (h+1)$. The same reasoning can be repeated for $\mathcal{PP}$ and the height of the rightmost column and, for $AV(132)$ (as well as $AV(312)$) and the number of RTL maxima. The proof is complete noticing that by Remark 1.3.14 a growth of $AV(231)$ (as well as $AV(213)$) is completely controlled by the number of LTR maxima in the same way is the one for $AV(132)$ by RTL maxima. □

For the sake of completeness, it must be said that there are other succession rules known to generate Catalan numbers. For instance,

$$\begin{cases} (1) \\ \\ (1) \rightsquigarrow (2) \\ (2k) \rightsquigarrow (1)^k, (4), (6), \ldots, (2k), (2k+2), \qquad \text{if } k > 0, \end{cases}$$

that has been studied in [44] and,

$$\begin{cases} (1) \\ \\ (2^k) \rightsquigarrow (1)^{2k-1}, (2)^{2k-2}, \ldots, (2^{k-2})^2, (2^{k-1}), (2^{k+1}), \end{cases}$$

whose proof is reported in [17].

1.3.6 Catalan generating function: the kernel method

This part describes the standard technique we use to translate a succession rule associated with a given number sequence into a functional equation whose solution is the generating function of that number sequence.

For the sake of clarity, we call $\mathcal{T}_{Cat}$ the generating tree associated with Ω_{Cat} whose nodes are decorated with labels - see Figure 1.13.

Let y be a catalytic variable introduced to record labels in the generating tree and x be the counting variable keeping track of levels of the nodes (*i.e.* the size of the underlying combinatorial objects). As usual, for a given node $o \in \mathcal{T}_{Cat}$ in the generating tree, we indicate with $p(o)$ its level and with $s(o)$ the value of its label. Then, according to Equation (1.3), the bivariate Catalan generating function is

$$G_{Cat}(x;y) := \sum_{o\in\mathcal{T}_{Cat}} x^{p(o)}\, y^{s(o)} = \sum_{h>0} G_h(x)\, y^h,$$

where $G_h(x) \equiv G_h$ is the size generating function of nodes of $\mathcal{T}_{Cat}$ with label (h). Then, the definition of Ω_{Cat} yields

$$G_{Cat}(x;y) = xy + \sum_{h>0} G_h\, x\,(y + y^2 + \ldots + y^h + y^{h+1}),$$

where the contribute xy corresponds to the axiom (1), and the summation to the production of the rule Ω_{Cat} in which each label (h) at level n produces $h+1$ objects at the next level that contribute for $x^{n+1}y + x^{n+1}y^2 + \cdots + x^{n+1}y^h + x^{n+1}y^{h+1}$. Thus, the following result holds.

Proposition 1.3.16. *The bivariate generating function $G_{Cat}(x;y)$ of Catalan numbers satisfies*

$$G_{Cat}(x;y) = xy + \frac{xy}{1-y}\left(G_{Cat}(x;1) - y\,G_{Cat}(x;y)\right). \tag{1.4}$$

Proof. From the definition of $G_{Cat}(x;y)$ it follows that

$$\begin{aligned} G_{Cat}(x;y) &= xy + x\sum_{h>0} G_h\left(\frac{y}{1-y} - \frac{y^{h+2}}{1-y}\right) \\ &= xy + \frac{xy}{1-y}\sum_{h>0} G_h(1-y^{h+1}) \\ &= xy + \frac{xy}{1-y}\left(G_{Cat}(x;1) - y\,G_{Cat}(x;y)\right). \end{aligned}$$

□

In Equation (1.4), the term $G_{Cat}(x;1)$ is actually our unknown: indeed discarding the label values, it is the generating function of Catalan numbers $\sum_{n>0} C_n x^n$. The nature of $G_{Cat}(x;1)$ is known in the literature to be algebraic and, consequently, the bivariate generating function $G_{Cat}(x;y)$ is algebraic as well.

In the following we apply the so-called *kernel method* to Equation (1.4) mainly in order to show an application of this method and, secondly to provide a proof of the algebraicity of these generating functions.

The kernel method appears in the "mathematical folklore" since 1970's: D. Knuth [99] first introduced it and, then it was rediscovered by several people and turned into a method (see [10, 8, 38]). A collection of examples in which the kernel method is applied for enumerative purposes is in [120] and another application is provided in [113].

The method applied to Equation (1.4) consists in coupling the variables x and y so that the coefficient of the unknown quantity $G_{Cat}(x;y)$ is zero. In particular, the steps to solve Equation (1.4) by using the kernel method are:

- Write (1.4) into its *kernel form* by collecting terms with $G_{Cat}(x;y)$,

$$G_{Cat}(x;y)\,K_{Cat}(x,y) = xy + \frac{xy}{1-y}G_{Cat}(x;1), \tag{1.5}$$

  where the polynomial $K_{Cat}(x,y) = 1 + xy^2/(1-y)$ is called *kernel.*

- Solve $K_{Cat}(x,y) = 0$ with respect to y. Note that $K_{Cat}(x,y)$ is quadratic in y so that there must be two solutions $Y_1(x)$ and $Y_2(x)$ that annihilate $K_{Cat}(x,y)$. Their expansions as power series in 0 are

$$\begin{aligned} Y_1(x) &= \frac{1+\sqrt{1-4x}}{2x} &&= x^{-1} - 1 - x - 2x^2 - 5x^3 - 14x^4 - 42x^5 - 132x^6 + O(x^7),\\ Y_2(x) &= \frac{1-\sqrt{1-4x}}{2x} &&= 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + O(x^7). \end{aligned}$$

  If we substitute in Equation (1.4) y for $Y_1(x) \equiv Y_1$, then $K_{Cat}(x,Y_1) = 0$, but $G_{Cat}(x,Y_1)$ is not a convergent power series in x.

  Indeed, let $G_{Cat} = \sum_n C_n(y)\,x^n$, where $C_n(y)$ is a polynomial in y of degree n (as the maximum possible value for a label of $\mathcal{T}_{Cat}$ at level n is n). The polynomial $C_n(Y_1)$ is a series in x whose lowest power of x is $-n$, so by using the Landau symbols it is $O(x^{-n})$. Then, $C_n(Y_1)\,x^n$ is $O(1)$, for any n, and thus $G_{Cat}(x,Y_1)$ is not convergent.

  On the other hand, if we substitute in Equation (1.4) y for $Y_2 \equiv Y_2(x)$, then not only $K_{Cat}(x,Y_2) = 0$, but also $G_{Cat}(x,Y_2)$ is a convergent power series in x.

  Indeed, contrary to Y_1, $C_n(Y_2)$ is a series in x whose lowest power of x is zero, so it is $O(1)$. Then, for any N, set $G_{Cat}(x,Y_2) = \sum_{n\le N} C_n(Y_2)\,x^n + \sum_{n\ge N+1} C_n(Y_2)\,x^n$, we have that $\sum_{n\ge N+1} C_n(Y_2)\,x^n = O(x^{N+1})$. Thus, $G_{Cat}(x,Y_2)$ converges.

- Set $y = Y_2$ so that the left-hand side of Equation (1.5) vanishes

$$0 = xY_2 + \frac{xY_2}{1-Y_2}G_{Cat}(x;1). \tag{1.6}$$

- Solve Equation (1.6) in $G_{Cat}(x;1)$ obtaining the well-known expressions

$$G_{Cat}(x;1) = \frac{1-2x-\sqrt{1-4x}}{2x}, \quad \text{and} \quad G_{Cat}(x,y) = \frac{y\,(1-2xy-\sqrt{1-4x})}{2\,(1-y+xy^2)}. \qquad (1.7)$$

The series expansions of $G_{Cat}(x;1)$ and $G_{Cat}(x;y)$ are

$$\begin{aligned} G_{Cat}(x;1) &= x+2x^2+5x^3+14x^4+42x^5+132x^6+O(x^7), \\ G_{Cat}(x;y) &= yx+(y+1)yx^2+(y^2+2y+2)yx^3+(y^3+3y^2+5y+5)yx^4+(y^4+4y^3 \\ &\quad +9y^2+14y+14)yx^5+(y^5+5y^4+14y^3+28y^2+42y+42)yx^6+O(x^7)\,. \end{aligned}$$

1.3.7 Alternative use of the kernel method

We report here another functional equation having the generating function of Catalan numbers as solution, which is still susceptible to applying the kernel method. The reason behind this second example is that the following functional equation does not come from any succession rule, but from a combinatorial interpretation of its terms.

To provide this example we introduce the notion of Dyck path prefix: any path in the quarter plane starting at the origin and using U and D steps is a Dyck path prefix. Thus, any Dyck path prefix has *length* given by the number of its steps and *height* given by the ordinate of its ending point.

Let $D(x,s)$ be the generating function of Dyck path prefixes, where x marks their length and the catalytic variable s their height. Then, it holds that

$$D(x,s) = 1 + x(s+\bar{s})D(x,s) - x\bar{s}D(x,0), \quad \text{with } \bar{s} = 1/s\,. \qquad (1.8)$$

Indeed, a Dyck path prefix is either the empty prefix or it is obtained from a Dyck path prefix by adding a new final step, which is either $U=(1,1)$ contributing for xs or $D=(1,-1)$ contributing for $x\bar{s}$. Moreover, we have to eliminate those Dyck path prefixes produced by adding a final D step to a Dyck path prefix of height 0, which corresponds to subtract the term $x\bar{s}D(x;0)$.

Note that the number of Dyck paths of length $2n$ is given by $[s^0x^{2n}]\,D(x,s)$, since we must consider only those Dyck path prefixes of length $2n$ that return to the x-axis.

Therefore, we can obtain a proof of the fact that the number of Dyck path is the Catalan number C_n, for every $n \geq 0$, as follows.

Manipulating Equation (1.8), yields

$$-sD(x,s) + s + x(s^2+1)D(x,s) - xD(x,0) = 0; \qquad (1.9)$$

$$-\bar{s}D(x,\bar{s}) + \bar{s} + x(\bar{s}^2+1)D(x,\bar{s}) - xD(x,0) = 0; \qquad (1.10)$$

$$(xs^2-s+x)(D(x,s)-\bar{s}^2D(x,\bar{s})) + s - \bar{s} = 0; \qquad (1.11)$$

$$sD(x,s) - \bar{s}D(x,\bar{s}) + \frac{s^2-1}{xs^2-s+x} \;=\; 0. \tag{1.12}$$

Equation (1.9) is obtained from (1.8) by multiplying by s, while (1.10) is obtained substituting $\bar{s}$ for s in (1.9). Equation (1.11) is obtained subtracting (1.10) from (1.9). The last equation (1.12) has a handy form: indeed, $sD(x,s)$ is a power series in x with polynomial coefficients in s, in which the lowest power of s is 1, whereas $\bar{s}D(x,\bar{s})$ is a power series in x with polynomial coefficients in $\bar{s}$ whose highest power of s is -1. Hence, in order to retrieve $D(x,0) = [s^0]D(x,s)$, we have to consider the expansion as power series in x of the last term of (1.12) and to take into account (in its polynomial coefficients in s and $\bar{s}$) only the coefficients of s^1. Hence, an explicit expression for the number of Dyck paths of semi-length n can be obtained by

$$\begin{aligned}
[s^0x^{2n}]D(x,s) = [sx^{2n}]\Big(sD(x,s) - \bar{s}D(x,\bar{s})\Big) &= [sx^{2n}]\frac{1-s^2}{xs^2-s+x} = [s\,x^{2n}]\frac{s-\bar{s}}{1-x(s+\bar{s})}\\
&= [s](s-\bar{s})\sum_{i\geq 0}\binom{2n}{i}s^{2(i-n)}\\
&= \binom{2n}{n} - \binom{2n}{n+1} = C_n\,. \qquad (n\geq 0)
\end{aligned}$$

Now, we turn to apply the kernel method to Equation (1.8). Write Equation (1.8) in its kernel form,

$$K(x,s)D(x,s) = 1 - x\bar{s}D(x,0), \qquad \text{where } K(x,s) = (1-x(s+\bar{s}))\,.$$

Solve $K(x,s) = 0$ with respect to s. Since $K(x,s)$ is quadratic in s, there exist two solutions $\sigma_1(x)$ and $\sigma_2(x)$ that annihilate $K(x,s)$. The two solutions $\sigma_1(x)$ and $\sigma_2(x)$ are such that only one of them has non-negative exponent as power series in x,

$$\begin{aligned}
\sigma_1(x) &= \frac{1+\sqrt{1-4x^2}}{2x} = x^{-1} - x - x^3 - 2x^5 - 5x^7 + O(x^9),\\
\sigma_2(x) &= \frac{1-\sqrt{1-4x^2}}{2x} = x + x^3 + 2x^5 + 5x^7 + O(x^9).
\end{aligned}$$

From the same reasoning of the previous section, it holds that $D(x,\sigma_2)$ is a convergent power series, being $\sigma_2(x)$ a well-defined power series in x. Thus, substituting $s = \sigma_2(x)$ implies $K(x,\sigma_2) = 0$ and $D(x,0) = \sigma_2/x$.

Note that the Catalan generating function $D(x,0)$ above is slightly different from the function $G_{Cat}(x;1)$ of Equation (1.7). In fact, one can note that the series $G_{Cat}(x;1)$ does not have the constant term $C_0 = 1$. Thus, $D(x,0) = 1 + G_{Cat}(x^2;1)$.

1.3.8 Asymptotics

The behaviour of Catalan numbers as n goes to infinity is easily provided by Stirling's formula by using the explicit expression for Catalan numbers of Theorem 1.3.1.

Stirling's formula is an important approximation due to the Scottish mathematician J. Stirling (1692-1770), and it reads as

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\left(1 + O\left(\frac{1}{n}\right)\right), \qquad (n \to \infty)\,.$$

Several proofs of this important formula can be found in [79], where its excellent quality as asymptotic estimate is highlighted. By means of a simple calculation, Stirling's formula allows to write the following accurate asymptotic form for the numbers C_n,

$$C_n = \frac{1}{n+1}\frac{(2n)!}{(n!)^2} \sim \frac{1}{n}\frac{(2n)^{2n}e^{-2n}\sqrt{4\pi n}}{n^{2n}e^{-2n}2\pi n}\,.$$

Theorem 1.3.17 (Asymptotic form)**.** *Let C_n be the nth Catalan number. Then, as n goes to infinity,*

$$C_n \sim \frac{4^n}{\sqrt{\pi n^3}}\,.$$

Thus, the growth of Catalan numbers is comparable to an exponential, 4^n, modulated by a subexponential factor, $1/\sqrt{\pi n^3}$.

1.4 Introduction to Baxter structures

Baxter numbers appeared first in order to enumerate *Baxter permutations* in 1977: in [53] F. R. K. Chung et al. succeeded in finding a closed expression for the number of permutations defined in [42] as Baxter permutations (see definition in Section 1.4.2). In fact, these permutations owe their name to G. Baxter, who first used them in the attempt to prove a conjecture about commuting functions (see [20]): if f and g are continuous functions mapping $[0, 1]$ into $[0, 1]$ which commute under composition, then they have a common fixed point. Eventually, this conjecture was shown to be false by W. M. Boyce who, however, proved that Baxter permutations are of greater importance in analysis than had previously been realised. In this section we recollect all the known results about Baxter numbers and Baxter structures that we use along this dissertation.

1.4.1 Formulas

The number sequence enumerating Baxter permutations, which will be defined rigorously in the next section, is known as the sequence A001181 [132] of Baxter numbers B_n, whose first terms are

$$1, 2, 6, 22, 92, 422, 2074, 10754, 58202, 326240, 1882960, 11140560, 67329992, \ldots$$

It was provided by F. R. K. Chung et al. in [53] a first explicit expression for the Baxter numbers B_n,

$$B_n = \frac{2}{n(n+1)^2} \sum_{j=1}^{n} \binom{n+1}{j-1}\binom{n+1}{j}\binom{n+1}{j+1}. \tag{1.13}$$

As reported in [75], the summand of the above equation can be rewritten equivalently as follows.

Theorem 1.4.1 (Explicit formula)**.** *For every $n \geq 1$, we have that*

$$B_n = \sum_{k=0}^{n} \theta_{k,n-k-1}, \qquad \textit{where}$$

$$\theta_{k,\ell} = \frac{2}{(k+1)^2(k+2)} \binom{k+\ell}{k}\binom{k+\ell+1}{k}\binom{k+\ell+2}{k}. \tag{1.14}$$

Equation (1.14) is combinatorially interpreted on several Baxter structures, as the next section illustrates.

Moreover, a linear recurrence satisfied by the numbers B_n is pointed out in the paper of F. R. K. Chung et al. [53] and it is attributed to P. S. Bruckman: for $n \geq 4$,

$$\begin{aligned}(n+1)(n+2)(n+3)(3n-2)B_n &= 2(n+1)(9n^3+3n^2-4n+4)B_{n-1}\\ &\quad +(3n-1)(n-2)(15n^2-5n-14)B_{n-2}\\ &\quad +8(3n+1)(n-2)^2(n-3)B_{n-3},\end{aligned}$$

where $B_1 = 1, B_2 = 2$ and $B_3 = 6$. Another recursive formula satisfied by Baxter numbers has been provided by R. L. Ollerton and inserted on [132, sequence A001181]. It reads as, for $n \geq 2$,

$$B_n = \frac{7n^2+7n-2}{(n+3)(n+2)}B_{n-1} + \frac{8(n-2)(n-1)}{(n+3)(n+2)}B_{n-2}, \text{ with } B_0 = 0, \text{ and } B_1 = 1. \tag{1.15}$$

1.4.2 Structures

Although the sequence of Baxter numbers is less popular than the Catalan one, it has several disparate combinatorial interpretations. A comprehensive list of families of objects enumerated by Baxter numbers can be found in [75], and in [86], as well as their close connections with Hopf algebras. Among the Baxter structures we do not define in the following, but it is well worth mentioning in passing, there are plane bipolar orientations [27, 82], and open partition diagrams with no enhanced 3-nesting, nor future enhanced 3-nesting [48].

Baxter permutations

The definition of Baxter permutations that we adopt is not exactly the original definition of [42] given by W. M. Boyce. In fact, we prefer to use the notion of "reduced" Baxter permutations introduced by C. L. Mallows in [107]. The reason for this choice is that the permutations commonly known as Baxter permutations in the literature nowadays are merely Mallows' reduced Baxter permutations.

Definition 1.4.2. A permutation $\pi = \pi_1 \dots \pi_n$ is a *Baxter permutation* if there are no three indices i, j and k, with $1 \leq i < j < j+1 < k \leq n$, such that

$$\pi_{j+1} < \pi_i < \pi_k < \pi_j, \qquad \text{or} \qquad \pi_j < \pi_k < \pi_i < \pi_{j+1}.$$

For instance, all permutations of length 4 are Baxter permutations, apart from 2413 and 3142.

By using the definition of *generalised patterns* (later called *vicular patterns*) introduced in [6], Baxter permutations can be characterised as a family of pattern-avoiding permutations. Then, with this aim, we provide formal definitions of vincular patterns, and of avoidance of a vincular pattern.

Definition 1.4.3. Let $\tau = \tau_1 \dots \tau_k$ be a pattern of length k. The pattern τ is called *vincular pattern* if certain consecutive elements are marked by $\underbrace{\;}$, *i.e.* $\underbrace{\tau_i\tau_{i+1}\dots\tau_j}$, for some i and $j > i$. A permutation π of length $n \geq k$ *contains* the vincular pattern τ ($\tau \preccurlyeq \pi$), if there exists an occurrence of the pattern τ in π such that the elements marked by $\underbrace{\;}$ in τ are consecutive elements of the permutation $\pi_1 \dots \pi_n$. Otherwise, π *avoids* the vincular pattern τ ($\tau \not\preccurlyeq \pi$).

For instance, the permutation $\pi = 41352$ does not contain the vincular pattern $3\underbrace{14}2$, since the only occurrence of 3142 in π, namely 4152, does not have the entries 1 and 5 adjacent.

The family of Baxter permutations can be characterised as follows.

Proposition 1.4.4 ([87])**.** *The family of Baxter permutations coincides with the family of permutations avoiding the two vincular pattern* $2\underbrace{41}3$ *and* $3\underbrace{14}2$.

Therefore, there are two permutations of length 5 that contain 2413 or 3142, but avoid $2\underbrace{41}3$ and $3\underbrace{14}2$: they are 41352 (as previously shown), and 25314 (by symmetry).

There is another characterisation of Baxter permutations in terms of pattern-avoiding permutations that involves barred patterns [121]. We do not report here the formal definition of barred pattern, since it is not central for our purposes; we rather address to [121] where the avoidance of barred patterns is thoroughly studied. Just to mention, we precise that Baxter permutations are the permutations avoiding two barred patterns, $25\bar{3}14$ and $41\bar{3}52$. In general, avoiding barred patterns is not equivalent to the avoidance of vincular patterns. Yet in our specific case, it holds that avoiding the vincular pattern $2\underbrace{41}3$ (resp. $3\underbrace{14}2$) is equivalent to avoiding the barred pattern $25\bar{3}14$ (resp. $41\bar{3}52$) - see [121].

By definition the number of non-empty Baxter permutations of length n is the Baxter number B_n of Equation 1.13 [53]. Moreover, according to [107], the summand $\theta_{k,\ell}$ of Theorem 1.4.1 is the number of Baxter permutations with k descents and ℓ rises (a *descent* in a permutation π is an element π_i such that $\pi_i > \pi_{i+1}$, similarly a *rise* is an element π_i such that $\pi_i < \pi_{i+1}$).

Twisted Baxter permutations

In [122] N. Reading studying sub-algebras of the Hopf algebra of permutations defined a new family of permutations, and noticed forthwith that it is equinumerous to the family of Baxter permutations through length $n = 15$.

Definition 1.4.5. A *twisted Baxter permutation* π is a permutation that avoids both the vincular patterns $3\,\underline{41}\,2$ and $2\,\underline{41}\,3$.

Nevertheless, the property that twisted Baxter permutations are as many as Baxter permutations was shown to hold by J. West in [142], and later reproved by M. Bouvel and O. Guibert in [41].

Analogously to Baxter permutations, twisted Baxter permutations can be characterised by the avoidance of barred patterns. We do not provide many details, except that avoiding the vincular pattern $3\,\underline{41}\,2$ is equivalent to avoiding $45\bar{3}12$, whereas as above avoiding $2\,\underline{41}\,3$ is equivalent to avoiding $25\bar{3}14$. Thus, twisted Baxter permutations are those permutations avoiding the two barred patterns $45\bar{3}12$ and $25\bar{3}14$.

Permutations depicted in Figure 1.14 provide two examples: a Baxter permutation that is not twisted Baxter, as it contains $3\,\underline{41}\,2$, and a twisted Baxter permutation that is not Baxter, because it contains an occurrence of $3\,\underline{14}\,2$.

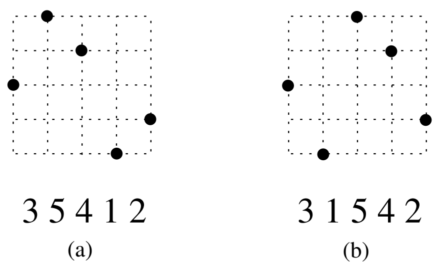


Figure 1.14: (a) A Baxter permutation that is not twisted, because of 3412; (b) a twisted Baxter permutation, which is not Baxter owing to 3152.

Triples of non-intersecting lattice paths

According to Definition 1.1.1 of paths, let (P, Q, R) be a triple of non-intersecting lattice paths (NILPs, for brevity), where P starts at $(0, 2)$, and Q at $(1, 1)$, and R at $(2, 0)$, and P, Q and R all use the same number of north and east steps. Figure 1.15 shows an example of a triple of NILPs.

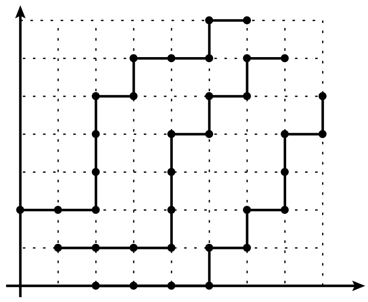


Figure 1.15: A triple of NILPs with 5 north steps and 6 east steps.

Then, by the Gessel-Viennot Lemma [85], it holds that $\theta_{k,\ell}$ defined in Equation (1.14) counts triples of NILPs with k north steps and ℓ east steps (*i.e.* P ends at $(\ell, k+2)$, Q at $(\ell+1, k+1)$, and R at $(\ell+2, k)$).

Mosaic floorplans

Mosaic floorplans are a simplified version of general floorplans defined by X. Hong et al. [92] in the context of chip design.

Definition 1.4.6. A *mosaic floorplan* is a rectangular partition of a rectangle by means of segments that do not properly cross, *i.e.* every pair of segments that intersect forms a T-junction of type $\perp$, $\top$, $\vdash$, or $\dashv$. Mosaic floorplans are generally considered up to equivalence under the action of sliding segments, namely up to translating their internal segments with continuity and without removing any T-junction.

Figure 1.16 shows two mosaic floorplans that are equivalent. Therefore, we write mosaic floorplan to denote an *equivalence class* of mosaic floorplans. So, the two objects of Figure 1.16 are rather two representatives of the same mosaic floorplan. Mosaic floorplans are enumerated according to the number of internal segments: we define the *size* of a mosaic floorplan as the number of its internal blocks, which are $n+1$ if n is the number of its internal segments. B. Yao et al. [148] proved that the number of mosaic floorplan of size n is B_n.

(a) (b)

Figure 1.16: (a) A packed floorplan with 3 internal segments; (b) a (non-packed) mosaic floorplan belonging to the same equivalence class than the one depicted in (a).

In order to escape the inconvenience of dealing with equivalence classes, *packed floorplans* have been introduced in [4]. A packed floorplan is a floorplan whose internal segments do not form configurations of type $\lrcorner\,\ulcorner$. In [4], it is proved that every mosaic floorplan as equivalence class contains exactly one packed floorplan, and thus packed floorplans can be considered as canonical representatives of mosaic floorplans.

It follows from the enumeration of mosaic floorplans [148] that packed floorplans of size n are counted by B_n.

1.4.3 Bijections

There exist in literature many bijections involving Baxter structures. For instance, a correpondence between Baxter permutations and twisted Baxter permutations is established in [142]. Another one between Baxter permutations and triples of NILPs is described in [71]. Another one between triples of NILPs and plane bipolar orientations can be found in [82]. A more extensive collection of bijections for Baxter families is gathered in [75] with accurate bibliographic references. Some of the cited bijections are particularly interesting because they show the equidistribution of some statistics on Baxter objects. The only bijection we do report involves mosaic floorplans and Baxter permutations and was defined by E. Ackerman et al. in [1]. We report in the following the mapping defined in [1], since the restriction of this bijection to a particular family of floorplans yields a bijection with a family of pattern-avoiding permutations contained into Baxter permutations, which will be dealt in Section 2.2.1.

Given a mosaic floorplan of size n, we can obtain a mosaic floorplan of size $n-1$ by using the *block deletion* operation introduced by X. Hong et al. [92].

Let F be a mosaic floorplan with $n > 1$ blocks and let b be the block in the top-left corner of F. We remove the block b according to its bottom-right corner: if the delimiting segments of b give rise to a junction of type $\dashv$ (resp. $\perp$) in its bottom-right corner, then shift the bottom (resp. right) delimiting segment of b upwards (resp. leftwards) pulling all the internal segments attached to it until the boundary. Figure 1.17 shows an example of the block deletion operation.

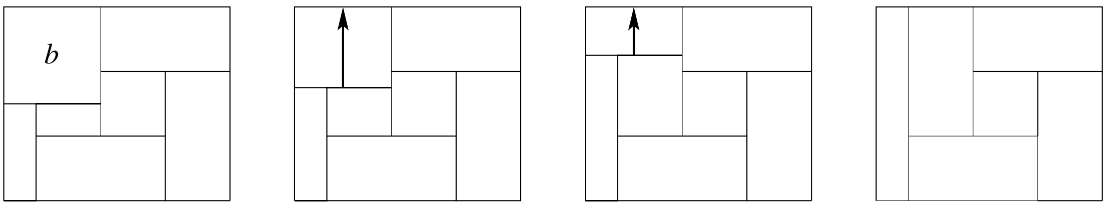


Figure 1.17: Block deletion from the top-left corner.

Now, using the notion of block deletion, we define a mapping ϕ from mosaic floorplans to Baxter permutations. First, note that the block deletion defined for the top-left corner of F, can be performed symmetrically in any other corner of F.

The steps to construct $\phi(F)$, given a mosaic floorplan F with n blocks, are rather simple:

1. label all blocks of F with $\{1, \ldots, n\}$ according to their deletion order from the top-left corner;

2. delete the blocks of F from the bottom-left corner and read the permutation $\phi(F)$ of length n obtained recording their labels.

An example of step 1. of the mapping ϕ is shown in Figure 1.18, thus the permutation corresponding to it through ϕ is 2631574. In [1] it is provided the proof that $\phi(F)$ is a Baxter permutation, for every mosaic floorplan F. Moreover, it is proved that ϕ is effectively a bijection.

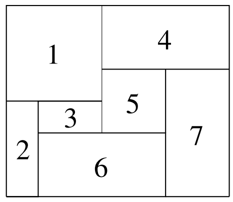


Figure 1.18: Labelling of blocks from the top-left corner.

1.4.4 Generating trees and succession rules

In this section we collect the generating trees known in literature to enumerate Baxter numbers providing for each of them accurate references.

First, we define an ECO operator ϑ_B for the family of Baxter permutations. Analogously to 132-avoiding permutations, the definition of ϑ_B strictly depends on the characterisation of the active sites of Baxter permutations. The fact that they are related to LTR and RTL maxima was first found by S. Gire in her thesis [87] and allows to state the following.

Definition 1.4.7. For $n > 0$, if π is a Baxter permutation of length n, then $\vartheta_B(\pi)$ is the set of Baxter permutations obtained from π by inserting the point $n+1$

- either immediately before a LTR maximum of π;
- or immediately after a RTL maximum of π.

Figure 1.19 shows the set of permutations obtained by performing ϑ_B on a permutation $\pi \in AV_6(2\,\underline{41}\,3, 3\,\underline{14}\,2)$ - as usual diamonds and crosses denote active and non-active sites in the graphical representation of π.

We briefly supply the proof that ϑ_B is in fact an ECO operator. First, by removing n from a Baxter permutation of length $n > 0$ we still obtain a Baxter permutation of length $n-1$, since no occurrences of $2\,\underline{41}\,3$ and $3\,\underline{14}\,2$ can be generated. Then, let π be a non-empty Baxter permutation of length n. Let $\ell_1 < \ldots < \ell_h = n$ be the LTR maxima of π and let $r_1 < \ldots < r_k = n$ be its RTL maxima. Because the Baxter forbidden patterns are one reverse of the other, the situation is symmetric with respect to n, and we can consider only insertions of $n+1$ on the left of n - the situation on the right is symmetric.

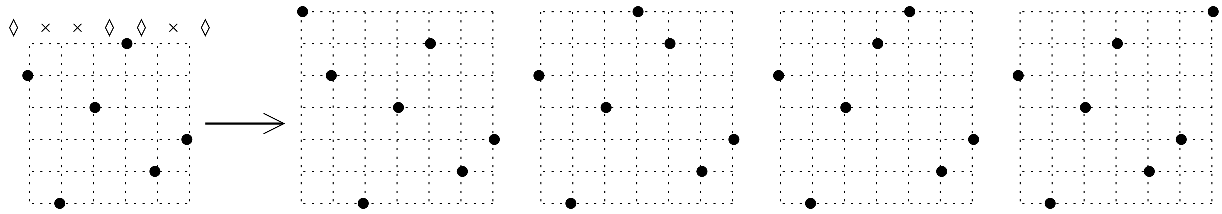


Figure 1.19: The set of Baxter permutations obtained from $\pi = 514623$ by means of ϑ_B.

Suppose $n+1$ is inserted between two LTR maxima ℓ_i and ℓ_{i+1} not immediately before ℓ_{i+1}. Then, let π_s be the point immediately after $n+1$. According to the definition of LTR maximum, an occurrence of $2\underline{41}3$ has been generated by $\ell_i\,(n+1)\pi_s\,\ell_{i+1}$.

Conversely, suppose $n+1$ is inserted immediately before ℓ_{i+1}. Since π contains neither $2\underline{41}3$ nor $3\underline{14}2$, the insertion of $n+1$ can give rise to an occurrence of the forbidden patterns only if it plays the role of 4. First, suppose that $n+1$ plays the role of 4 in an occurrence of $2\underline{41}3$. The point immediately after $n+1$ is ℓ_{i+1}, which is a LTR maximum, and by definition of LTR maximum, ℓ_{i+1} cannot play the role of 1 in $2\underline{41}3$. Therefore, suppose that $n+1$ plays the role of 4 in an occurrence of $3\underline{14}2$. Let π_s be the point immediately before $n+1$. Since they give rise to an occurrence of $3\underline{14}2$, there must be two points π_a and π_b, with $a < s$ and $b > s$, such that $\pi_s < \pi_b < \pi_a$. In addition, it must hold that $\pi_a < \ell_{i+1}$, since the point π_a is on the left of ℓ_{i+1}. The point π_b is on the right of ℓ_{i+1}, since we assume $n+1$ is inserted immediately before ℓ_{i+1}. Then, it follows that $\pi_a\pi_s\ell_{i+1}\pi_b$ forms an occurrence of $3\underline{14}2$ in π, which is a contradiction.

Figure 1.20 depicts the first levels of the generating tree associated with ϑ_B.

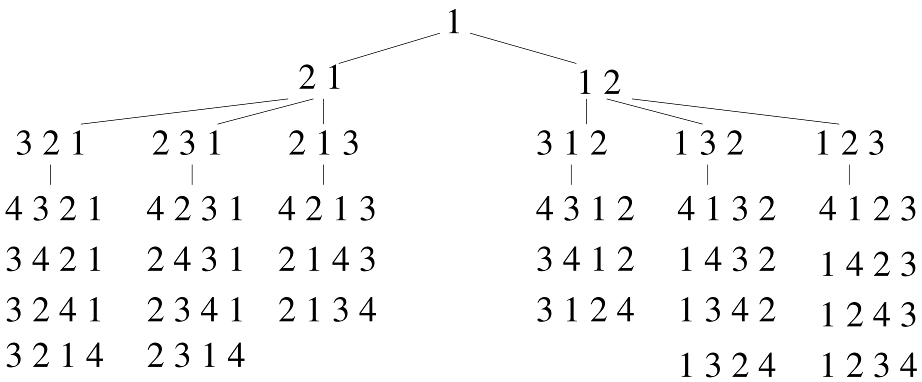


Figure 1.20: The first levels of the decorated generating tree corresponding to $\vartheta_{\mathcal{B}}$.

As Definition 1.4.7 illustrates, the growth of Baxter permutations is controlled by the number of LTR maxima and RTL maxima. Therefore, we assign to each Baxter permutation π of length $n \geq 1$ a label (h, k), where h denotes the number of its LTR maxima and k the number of its RTL maxima. In Figure 1.21, we substitute the objects of Figure 1.20 for their labels - the minimum length permutation has indeed label $(1, 1)$. The generating tree of Baxter permutations decorated with the labels (h, k) in place of permutations is denoted by $\mathcal{T}_{Bax}$.

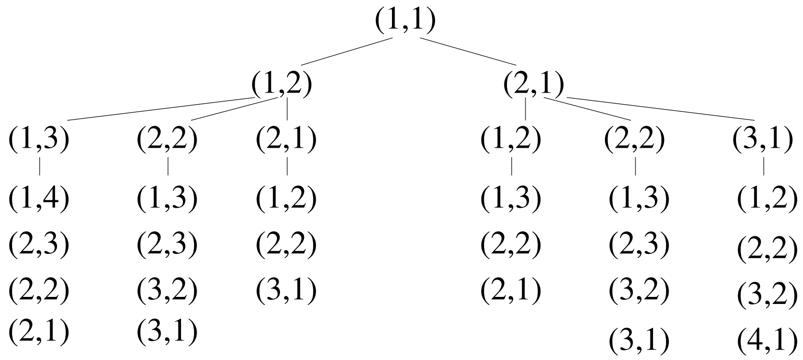


Figure 1.21: The first levels of the generating tree $\mathcal{T}_{Bax}$ associated with ϑ_B decorated with labels (h, k), where h (resp. k) is the number of LTR (resp. RTL) maxima of Baxter permutations.

The set of permutations $\vartheta_B(\pi)$ has cardinality exactly $h + k$. In other words, π has precisely $h+k$ children in the decorated generating tree corresponding to ϑ_B. Moreover, by adding a maximum element in an active site of π, the positions of LTR and RTL maxima change according to the position of the new maximum. Precisely, if $n + 1$ is added just before the ith LTR maximum of π, with $1 \leq i \leq h$, the permutation produced has i LTR maxima and $k + 1$ RTL maxima, and symmetrically, if $n + 1$ is added just after the jth RTL maximum of π, with $1 \leq j \leq k$, the permutation produced has $h + 1$ LTR maxima and j RTL maxima. This observation produced in [87] and reported in [32], allows to write the growth of Baxter permutations in form of a succession rule.

Proposition 1.4.8 ([32], Lemma 2)**.** *Baxter permutations can be generated by*

$$\Omega_{Bax} = \begin{cases} (1,1) \\ \\ (h,k) & \rightsquigarrow (1,k+1),\ (2,k+1),\ \ldots,\ (h,k+1), \\ & \quad (h+1,1),\ (h+1,2),\ \ldots,\ (h+1,k). \end{cases}$$

The rule Ω_{Bax} has an intrinsic symmetry that is reflected by the majority of the known Baxter families - for instance, Baxter permutations [87], triples of NILPs [71], mosaic floorplans [148].

Remark 1.4.9. *One should stress that Definition 1.3.13 of the operator ϑ_A, which provides a growth for the permutation class $AV(132)$, performs asymmetrically what the operator ϑ_B does. Namely, ϑ_A inserts a maximum to the right of any RTL maximum, but not to the left of a LTR maximum apart from the first one. In other words, one can notice that the growth provided above for Baxter permutations is a symmetric version of the growth of* 132*-avoiding permutations.*

This remark that ϑ_B is a symmetric version of ϑ_A extends to their corresponding succession rules as follows.

Observation 1.4.10. *The succession rule Ω_{Bax} can be considered a symmetric version of the Catalan succession rule Ω_{Cat}, and moreover, there are four different ways of restricting the succession rule Ω_{Bax} into the succession rule Ω_{Cat}.*

Proof. Interpreting Remark 1.4.9 on the corresponding succession rules, we easily obtain a first restriction of Ω_{Bax} to Ω_{Cat}. Indeed, by restricting the productions of the label (h, k) according to Ω_{Bax} to labels $(h+1, i)$, for $1 \le i \le k$, and to label $(1, k+1)$, we retrieve the production of (k) according to Ω_{Cat}.

Furthermore, it is well worth considering also the growth of permutations of $AV(312)$, as described in the proof of Proposition 1.3.15. In this case, by restricting the productions of the label (h, k) according to Ω_{Bax} to labels $(h+1, i)$, for $1 \le i \le k$, and to label $(h, k+1)$, we retrieve again the production of (k) according to Ω_{Cat}.

Symmetrically, by Remark 1.3.14, a growth for permutations of $AV(231)$, or $AV(213)$, can be defined according to the positions of their LTR maxima. Both these growths can be considered as restrictions of the growth performed by ϑ_B. Indeed, the restriction for the class $AV(231)$ (resp. $AV(213)$) corresponds to considering, in the label productions of (h, k) according to Ω_{Bax}, only the labels $(i, k+1)$, for $1 \le i \le h$, and the label $(h+1, 1)$ (resp. $(h+1, k)$). This gives back the production of the label (h) according to Ω_{Cat}. □

Therefore, not only can Ω_{Bax} be considered a symmetric version of Ω_{Cat}, but there are essentially four different ways of restricting the Baxter succession rule Ω_{Bax} to the Catalan succession rule Ω_{Cat}.

Moreover, there exist in the literature other succession rules associated with Baxter numbers. We list them in the following, showing in Figure 1.22 that these succession rules are effectively different because their corresponding generating trees are not isomorphic.

Proposition 1.4.11 ([41])**.** *Twisted Baxter permutations can be generated by*

$$\Omega_{TBax} = \begin{cases} (2,0) \\ \\ (r,s) \rightsquigarrow (2, r+s-1), (3, r+s-2), \ldots, (r+1, s), \\ \qquad\qquad (r, 0), \ldots, (r, s-1). \end{cases}$$

In 2015, the authors of [48] presented three different new classes of Baxter objects that do not share many properties known for Baxter objects (for instance, their intrinsic symmetry). Although combinatorial bijections link these structures two by two, the result that they are enumerated by Baxter numbers is exclusively analytical. The succession rule presented in [48] to generate these new Baxter structures is here denoted by Ω_{Bax2}. This succession rule comes out as a particular case of a general result derived in [47], and its generating tree differs from $\mathcal{T}_{Bax}$ starting from the first levels - see Figure 1.21(a),(c).

Proposition 1.4.12 ([48, 47])**.** *The following succession rule, Ω_{Bax2}, generates Baxter* *numbers*

$$\Omega_{Bax2} = \begin{cases} (0,0) & \\ & \\ (i,j) \rightsquigarrow & (i,i),\ (i+1,j), \\ & (i,j),\ (i,j+1),\ \ldots,\ (i,i-1), \qquad \text{if } i>0, \\ & (i-1,j),\ (i-1,j+1),\ \ldots,\ (i-1,i-1), \qquad \text{if } i>0, \\ & (i,j-1),\ (i-1,j-1), \qquad \text{if } i>0 \text{ and } j>0. \end{cases}$$

The last succession rule for Baxter numbers is extremely recent: it was derived in [97] and, as Figure 1.22(d) confirms, it is not equivalent to any of the previous succession rules.

Proposition 1.4.13 ([97], Lemma 4.3)**.** *The following succession rule, Ω_{Bax3}, generates Baxter numbers*

$$\Omega_{Bax3} = \begin{cases} (1,1) & \\ & \\ (p,q) \rightsquigarrow & (1,q+1),\ (2,q+1),\ \ldots,\ (p-1,q+1), \qquad \text{if } p>1, \\ & (1,q+1), \\ & (p+q,1),\ (p+q-1,2),\ \ldots,\ (p+1,q). \end{cases}$$

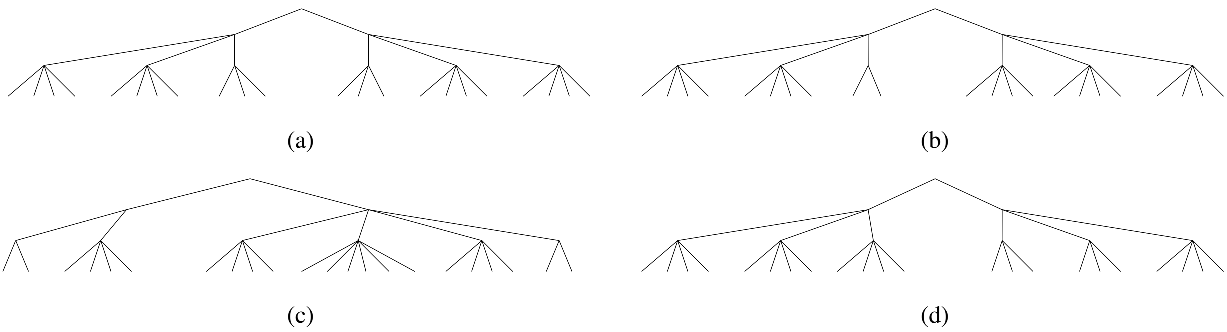


Figure 1.22: The first four levels of the Baxter generating trees: (a) corresponding to Ω_{Bax}; (b) corresponding to Ω_{TBax}; (c) associated with Ω_{Bax2}; (d) associated with Ω_{Bax3}.

1.4.5 Baxter generating function: the obstinate variant of the kernel method

Analogously to Section 1.3.6 for Ω_{Cat}, we can readily translate the succession rule Ω_{Bax} associated with Baxter numbers into a functional equation whose solution is their generating function. The main difference with respect to Section 1.3.6 is that the labels of Ω_{Bax} are arrays of length two, and only one additional variable is not sufficient to keep track of the label production of Ω_{Bax}. In D. Zeilberger's terminology [151], the succession rule Ω_{Bax} yields a linear equation with two catalytic variables y and z. Therefore, we are going to show a functional equation involving a trivariate function $G_{Bax}(x;y,z)$ and a number of its specializations (*i.e.* functions that do not depend on y and z simultaneously). This functional equation has first been derived by M. Bousquet-Mélou in [32], exactly starting from the rule Ω_{Bax}.

For $h, k \geq 1$, let $G_{h,k}(x) \equiv G_{h,k}$ denote the size generating function of non-empty Baxter permutations of label (h, k).

Proposition 1.4.14 (Corollary 3, [32])**.** *The generating function* $G_{Bax}(x; y, z) \equiv G_{Bax}(y, z) = \sum_{h,k\geq 1} G_{h,k}\, y^h z^k$ *satisfies the following functional equation*

$$G_{Bax}(y, z) = xyz + \frac{xyz}{1-y}\left(G_{Bax}(1, z) - G_{Bax}(y, z)\right) + \frac{xyz}{1-z}\left(G_{Bax}(y, 1) - G_{Bax}(y, z)\right) . \tag{1.16}$$

Proof. By using the productions of Ω_{Bax}, the generating function of non-empty Baxter permutations counted by their length (variable x) and labels (variables y and z) can be rewritten as follows,

$$\begin{aligned} G_{Bax}(y, z) &= xyz + x \sum_{h,k\geq 1} G_{h,k} \left((y + y^2 + \cdots + y^h)z^{k+1} + (z + z^2 + \cdots + z^k)y^{h+1}\right) \\ &= xyz + xyz \sum_{h,k\geq 1} G_{h,k} \frac{1-y^h}{1-y} z^k + xyz \sum_{h,k\geq 1} G_{h,k} \frac{1-z^k}{1-z} y^h \\ &= xyz + \frac{xyz}{1-y}\left(G_{Bax}(1, z) - G_{Bax}(y, z)\right) + \frac{xyz}{1-z}\left(G_{Bax}(y, 1) - G_{Bax}(y, z)\right) . \end{aligned}$$ □

It has recently been shown that the generating function solution of similar functional equations with two catalytic variables could be algebraic, or D-finite, or even non D-finite [33, 37, 112], in contrast with the case of only one catalytic variable which yields always an algebraic solution (Section 1.3.6, and 1.3.7).

In order to solve Equation (1.16), we apply the same methodology as [32] that is known with the name of *obstinate kernel method*. According to [32], this method was inspired by Section 2.4 of the book of G. Fayolle, R. Iasnogorodski and V. Malyshev [74] and, among its applications, it was largely used to count some families of walks in the quarter plane [33, 37].

The name is motivated as it is a variant of the usual kernel method: indeed, as previously seen, the kernel method allows to eliminate from the original equation the catalytic variable by annihilating its kernel. The obstinate variant of the kernel method instead, by means of pairs of substitutions that annihilate the kernel, allows to write a system of equations, which relate functions involving only one catalytic variable.

More precisely, the steps to solve Equation (1.16) by means of the obstinate variant of kernel method are listed below:

- Write Equation (1.16) into its kernel form by collecting the terms in $G_{Bax}(y, z)$,

$$\left(1 + \frac{xyz}{1-y} + \frac{xyz}{1-z}\right) G_{Bax}(y, z) = xyz + \frac{xyz}{1-y} G_{Bax}(1, z) + \frac{xyz}{1-z} G_{Bax}(y, 1) .$$

Note that Equation (1.16) is symmetric in y and z, and in particular it holds that $G_{Bax}(y, 1) = G_{Bax}(1, z)$.

- For convenience, set $y = 1+a$ and $z = 1+b$ so that the kernel form of Equation (1.16) becomes

$$\frac{ab - x(1+a)(1+b)(a+b)}{x(1+a)(1+b)} G_{Bax}(1+a, 1+b) = ab - R(b) - R(a), \tag{1.17}$$

where $R(a) = aG_{Bax}(1+a, 1)$. The coefficient of $G_{Bax}(1+a, 1+b)$ (more precisely, only its numerator) is said kernel and denoted with $K_{Bax}(a, b)$.

- As a polynomial in b, $K_{Bax}(a, b) = -x(1+a)b^2 - (x(1+a)^2 - a)b - xa(1+a)$ has two roots β_0 and β_1 such that $\beta_0\beta_1 = a$,

$$\begin{aligned} \beta_0(a) &= (1+a)x + (1+a)^2\left(1+\frac{1}{a}\right)x^2 + O(x^3), \qquad \text{and,} \\ \beta_1(a) &= \frac{a}{1+a}x^{-1} - (1+a) - (1+a)x + O(x^2). \end{aligned}$$

According to the usual kernel method only β_0 is a legal substitution for b in Equation (1.17), since it annihilates the kernel $K_{Bax}(a, b)$ and the term $G_{Bax}(1+a, 1+\beta_0)$ is a well-defined power series in x.

Nevertheless, note that by substituting b for β_0 in Equation (1.17), yields

$$R(a) + R(\beta_0) = a\beta_0,$$

which is unsatisfactory to determine the unknown $R(a)$.

- Apply the obstinate variant of the kernel method by seeking the pairs $(A, B) \neq (0, 0)$ of Laurent series in x such that $K_{Bax}(A, B) = 0$.

In this particular case, the following two involutions

$$\Phi : (a, b) \to \left(\frac{b}{a}\, ,\, b\right) \qquad \text{and} \qquad \Psi : (a, b) \to \left(a\, ,\, \frac{a}{b}\right),$$

by acting on the pair (a, β_0) give rise to a group of order 6 - see Figure 1.23.

$$\begin{array}{ccccccc} & \overset{\Phi}{\nearrow} & (\bar{a}\beta_0, \beta_0) & \xrightarrow{\Psi} & (\bar{a}\beta_0, \bar{a}) & \overset{\Phi}{\searrow} & \\ (a, \beta_0) & & & & & & (\bar{a}\beta_1, \bar{a}) \\ & \underset{\Psi}{\searrow} & (a, \beta_1) & \xrightarrow{\Phi} & (\bar{a}\beta_1, \beta_1) & \underset{\Psi}{\nearrow} & \end{array}$$

Figure 1.23: The orbit of (a, β_0) under the action of Φ and Ψ, with $\bar{a} = 1/a$.

All the 6 pairs of Laurent power series in Figure 1.23 cancel the kernel $K_{Bax}(a,b)$. Yet only those not involving β_1 can be legally substituted for (a,b) in Equation (1.17).

- Substituting (a,β_0), $(\bar{a}\beta_0,\beta_0)$, and $(\bar{a}\beta_0,\bar{a})$ for (a,b) in the main equation (1.17), we obtain a system of three equations for the unknown function $R(a)$,

$$\begin{cases} R(a)+R(\beta_0)=a\beta_0 \\ R(\bar{a}\beta_0)+R(\beta_0)=\bar{a}\beta_0^2 \\ R(\bar{a}\beta_0)+R(\bar{a})=\bar{a}^2\beta_0. \end{cases}$$

By combining these three equations, it holds that

$$R(a)+R(\bar{a})=\bar{a}^2\beta_0(a^3-a\beta_0+1)\,.$$

- Finally, $R(a)=a\,G_{Bax}(1+a,1)$ is a formal power series in x with polynomial coefficients in a whose lowest power of a is 1, whereas $R(\bar{a})$ is a formal power series in x with polynomial coefficients in $\bar{a}$ whose highest power of a is -1.

  Thus, it must hold that

$$a\,G_{Bax}(1+a,1)=\left[\bar{a}^2\beta_0(a^3-a\beta_0+1)\right]^{>}\,, \tag{1.18}$$

  where by $F^{>}$ we denote the *positive part* of F in a. More precisely, if F is a formal power series in x whose coefficients are Laurent polynomials in a, $F=\sum_{n\geq 0,i\in\mathbb{Z}} f(n,i)a^ix^n$, then

$$F^{>}=\sum_{n\geq 0}x^n\sum_{i>0}f(n,i)\,a^i.$$

Equation (1.18) provides effectively an expression for the bivariate generating function of Baxter numbers. The generating function $G_{Bax}(1+a,1)$ is known to be D-finite [32]: Equation (1.18) shows that $G_{Bax}(1+a,1)$ is D-finite being the positive part in a of an algebraic series - see for further details [32, Section 1.4].

In addition, in [32] it is proved the following result.

Corollary 1.4.15 ([32], Corollary 5)**.** *The generating function $G_{Bax}(1+a,1)$ can be expressed as*

$$G_{Bax}(1+a,1)=\sum_{n\geq 1}x^n\sum_{i=0}^{n}\frac{a^i(i+1)}{n(n+1)^2(n+2)}\sum_{k=i}^{n}(2k+ni)\binom{n+2}{k-i}\binom{n+1}{k}\binom{n+1}{k+1}.$$

Extracting the coefficients of a^0 from $G_{Bax}(x;1+a,1)$ yields a proof of the explicit expression of Baxter numbers of Equation (1.13).

Moreover, since the specializations of a D-finite generating function are D-finite, the generating function of Baxter numbers $G_{Bax}(x)\equiv G_{Bax}(1,1)$ results to be D-finite.

1.4.6 Asymptotics

Another remarkable fact pointed out by F. R. K. Chung et al. in [53] is the behaviour of Baxter numbers as n goes to infinity, whose estimate is attributed to A. M. Odlyzko.

Theorem 1.4.16 (Asymptotic form, [53]). *Let B_n be the nth Baxter number. Then, as n goes to infinity,*

$$B_n \sim \frac{32 \cdot 8^n}{\pi\sqrt{3} \cdot n^4}\,.$$

Thus, the growth rate of Baxter numbers is 8, and because of the factor n^{-4} we can conclude that the generating function of Baxter numbers $G_{Bax}(x) = \sum_{n\geq 1} B_n x^n$ is D-finite, but not algebraic - see Theorem 1.2.8.

Chapter 2

Slicings of parallelogram polyominoes

Plan of the chapter

The objective of this chapter is to describe and explain the inclusions "Catalan in Schröder in Baxter". To this purpose, we introduce in Section 2.1 a new family of combinatorial objects, called Baxter slicings of parallelogram polyominoes, whose first appearance was in [G1]. These objects can be generated according to rule Ω_{Bax} and they have a natural subfamily enumerated by Catalan numbers, which is described in Section 2.1.3. In addition, in Section 2.1.4 we introduce the family of Baxter paths, that can be regarded as an alternative way of representing Baxter slicings by means of labelled Dyck paths. To our knowledge, it constitutes a first combinatorial interpretation of Baxter numbers as *single* paths, which we believe was missing so far in the literature on Baxter numbers. In the end of Section 2.1 we point out the problem of reconciling the Catalan and Baxter numbers with the Schröder numbers [132, sequence A006318].

Section 2.2 is intended to collect results about the Schröder number sequence; such as known combinatorial structures and their associated succession rules. Among the Schröder families, we present separately in Section 2.3.1 a completely new interpretation of these numbers, which has been introduced in [G3]. All these known and new combinatorial interpretations of Schröder numbers are to show that the Schröder structures are either generalisations of Catalan structures or restrictions of Baxter structures, without being both at the same time. The only exceptions are given by some families of pattern-avoiding permutations that display a discrete continuity from Catalan to Baxter, yet not at the abstract level of succession rules.

The main goal of this chapter is then accomplished in Section 2.4 by defining the family of Schröder slicings of parallelogram polyominoes. In fact, Section 2.4 shows that a continuum from Catalan to Baxter via Schröder can be visible at the abstract level of generating trees, and consequently succession rules. The result first established in [G1], and then developed in [G2], consists in providing a new succession rule, associated with a growth for Schröder slicings, that interpolates between the two known succession rules for Catalan and Baxter numbers.

By means of this general tool we can exhibit in Section 2.5 Schröder subfamilies of known Baxter structures among those listed in the previous Section 1.4.2: namely a Schröder subset of NILPs, and a Schröder subset of Baxter permutations, and a Schröder subset of mosaic floorplans.

Finally, in the last section we define two subfamilies of Baxter slicings that have been introduced in [G1]: the family of m-skinny slicings and the one of m-row-restricted slicings, $m \in \mathbb{N}$ being a parameter. All these subfamilies are enumerated by intermediate number sequences between Catalan and Baxter numbers motivating thus their study. By using functional equations and the kernel method, we manage in [G2] to compute the generating functions for some special cases, and to prove it is algebraic. On the other hand, for general m, we present an underpinned conjecture about their algebraic nature.

2.1 Baxter slicings of parallelogram polyominoes

In this section we define a new family of Baxter objects that generalise parallelogram polyominoes, whose characterisation is of interest for our purpose. Indeed, with respect to the growth of Section 1.3.4 page 29 for parallelogram polyominoes, we can think of generating parallelogram polyominoes symmetrically, by allowing at the same time insertions of a rightmost column of any possible height, or of a topmost row of any possible width. Of course, this process generates parallelogram polyominoes ambiguously. Yet we can eliminate any ambiguity by recording the "building history" of the polyomino, that is, which columns and rows are added during the growth process.

This observation motivates the definition of new combinatorial objects, that generalise parallelogram polyominoes, and grow unambiguously according to rule Ω_{Bax} - see Section 2.1.1. The objects defined result to be a reinterpretation of the well-known Baxter family of triples of NILPs, as Section 2.1.2 shows, revealing nice combinatorial properties for these objects.

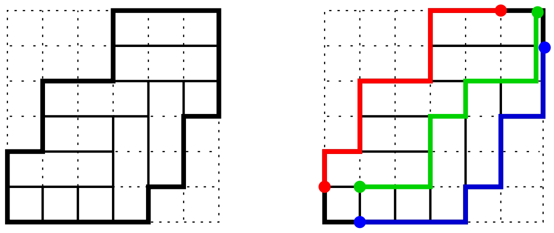
In Section 2.1.3 we formally define the restriction of Baxter slicings to a subfamily enumerated by Catalan numbers. This restriction is obtained by breaking the symmetry of Baxter slicings, and returning to the usual growth for parallelogram polyominoes.

Moreover, by means of the definition of Catalan slicings in Section 2.1.4 will be presented a new combinatorial interpretation of Baxter numbers: Baxter paths. These new objects are of interest because being a generalisation of Dyck paths they form a new occurrence of Baxter numbers in terms of single lattice paths.

2.1.1 Definition and growth of Baxter slicings

The objects we are going to define are parallelogram polyominoes whose interior is divided into blocks, of width or height 1. We call these objects *Baxter slicings of parallelogram polyominoes*, or *Baxter slicings* for short and denote their family by $\mathcal{BS}$ - see an example in Figure 2.1(a).

Definition 2.1.1. A *Baxter slicing* of size n is a parallelogram polyomino S of semi-perimeter $n+1$ whose interior is recursively divided into n blocks as follows: one block is the topmost row (resp. rightmost column) of S – such blocks are called *horizontal* (resp. *vertical*) blocks – and the other $n-1$ blocks form a Baxter slicing of the parallelogram polyomino of semi-perimeter n obtained by deletion of the topmost row (resp. rightmost column) of S.



(a) (b)

Figure 2.1: (a) A Baxter slicing of size 11; (b) the way for determining the triple of NILPs associated with it.

Theorem 2.1.2. *Baxter slicings can be generated by rule Ω_{Bax} and, thus, are enumerated by Baxter numbers.*

Proof. In order to prove that Baxter slicings grow according to rule Ω_{Bax}, we define an ECO operator $\vartheta_{BS} : \mathcal{BS}_n \to \mathcal{BS}_{n+1}$, where $\mathcal{BS}_n$ is the family of Baxter slicings of size n.

Let S be a parallelogram polyomino with topmost row of width h and rightmost column of height k. Then, the operator ϑ_{BS} applied to a Baxter slicing of shape S produces $h+k$ Baxter slicings obtained either by adding a new horizontal block in a new topmost row, of any width from 1 to h, or by adding a new vertical block in a new rightmost column, of any height from 1 to k. The set of Baxter slicings produced through ϑ_{BS} entirely depends on the two parameters: width of the topmost row and height of the rightmost column.

Thus, we label any Baxter slicing with (h, k), where h is the width of its topmost row and k is the height of its rightmost column. Baxter slicings produced by applying ϑ_{BS} have labels $(i, k+1)$, for any $1 \leq i \leq h$, and $(h+1, j)$, for any $1 \leq j \leq k$. As a consequence, Baxter slicings are enumerated by Baxter numbers. □

The set of Baxter slicings produced through the application of ϑ_{BS} to the Baxter slicing in Figure 2.1(a) is depicted in Figure 2.2, where for each Baxter slicing the corresponding label is indicated.

2.1.2 Bijection with triples of NILPs

Among all the Baxter structures presented in Section 1.4.2, one can be seen to be in bijection with Baxter slicings in a very simple way: the triples of NILPs - see Figure 2.1(b).

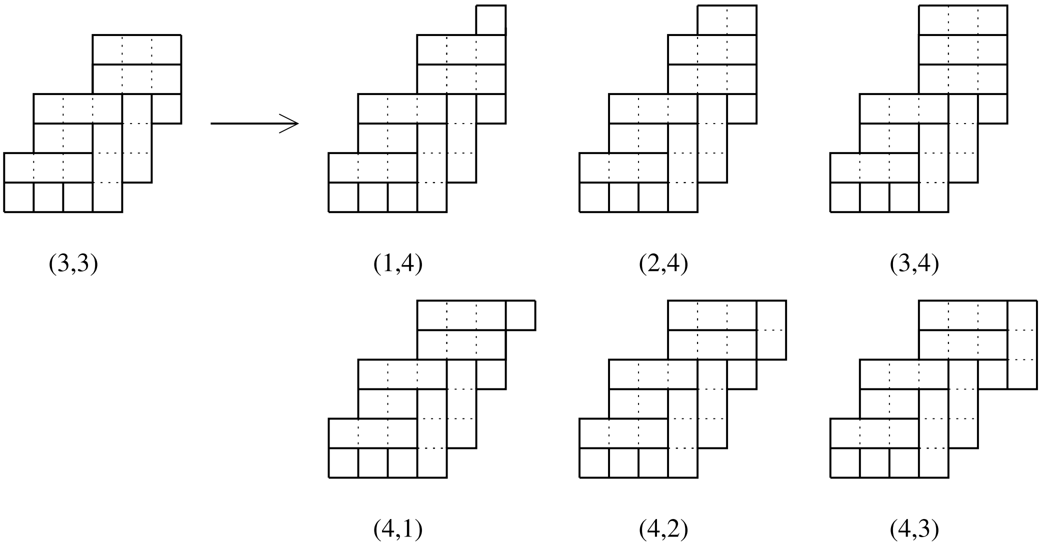


Figure 2.2: The growth of Baxter slicings following rule Ω_{Bax}.

Consider a Baxter slicing of a parallelogram polyomino S, whose bottom-left corner is assumed to be placed at coordinates $(0,0)$. Define the paths

- P, corresponding to the upper border of S, except the first and last steps,
- R, corresponding to the lower border of S, except the first and last steps,
- Q, going from $(1,1)$ to the top-right corner of S, following the lower border of every horizontal block of the slicing, and the left border of every vertical block.

Associate the triple (P,Q,R) to the original Baxter slicing - in Figure 2.1(b) P is drawn in red, and Q in green, and R in blue.

Theorem 2.1.3. *The above construction provides a size-preserving bijection between Baxter slicings and triples of NILPs.*

Proof. Consider a Baxter slicing of a parallelogram polyomino S, and define P,Q and R as above. Shifting by one the path P (resp. R) upwards (resp. rightwards) so that the starting point is at $(0,2)$ (resp. $(2,0)$), we want to prove (P,Q,R) is a triple of NILPs of size n. Note that by construction each step of the path Q is inside or on the border of the polyomino S; this immediately ensures the non-intersecting property. Moreover, by construction all paths P,Q and R have $n-1$ steps, if $n+1$ denotes the semi-perimeter of S. Finally, we easily check that P,Q and R have the same number of east and north steps as follows. Since the path Q separates the horizontal blocks, which remain above it, from the vertical ones, which remain below it, each step of this path is either the right edge of a horizontal block or the upper edge of a vertical block. Then, the paths P (resp. R) and Q have the same number of north (resp. east) steps, as each north (resp. east) step of the path P (resp. R) is the left (resp. lower) edge of a horizontal (resp. vertical) block.

To prove that this construction is a bijection, we describe its inverse. Any triple (P, Q, R) of NILPs corresponds to a unique Baxter slicing of a parallelogram polyomino S, whose contour is defined by P and R and whose block division is obtained by Q. More precisely, we obtain the contour of S by adding an initial and a final step to both the paths P and R and drawing them starting at $(0, 0)$. Let the starting point of the path Q be in $(1, 1)$. Then, the blocks inside S are drawn according to the steps of Q: for every east (resp. north) step Q_i of Q, $1 \leq i \leq n-1$, draw a vertical (resp. horizontal) block whose top (resp. right) edge is Q_i and that extends downwards (resp. leftwards) until the border of S; and finally, add the initial block consisting of one cell extending from $(0, 0)$ to $(1, 1)$. □

Up to the simple bijective correspondence described in Theorem 2.1.3, our Theorem 2.1.2 can also be seen as a description of the growth of triples of NILPs according to succession rule Ω_{Bax}, which was alternatively described in [27]. Moreover, it follows directly from the proof of Theorem 2.1.3 the result below.

Corollary 2.1.4. *Baxter slicings of size n having k horizontal blocks and ℓ vertical blocks are counted by $\theta_{k,\ell}$ of Equation* (1.14).

2.1.3 Definition and growth of Catalan slicings

We start defining Catalan slicings recursively as it has been done for Baxter slicing. It follows from such a definition that every Catalan slicing is indeed a Baxter slicing according to Definition 2.1.1.

Definition 2.1.5. A *Catalan slicing* of size n is a parallelogram polyomino S of semi-perimeter $n+1$ whose interior is recursively divided into n blocks, as follows: if the topmost row of S contains just one cell, then this cell constitutes a horizontal block, and the other $n-1$ blocks form a Catalan slicing of the parallelogram polyomino of semi-perimeter n obtained by deleting this cell in the topmost row of S; otherwise, the rightmost column of S constitutes a vertical block, and the other $n-1$ blocks form a Catalan slicing of the parallelogram polyomino of semi-perimeter n obtained by deleting the rightmost column of S.

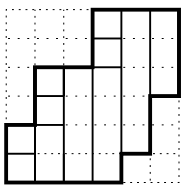


Figure 2.3: A Catalan slicing of size 11.

As expected, because of their deterministic definition, we find exactly one Catalan slicing for every parallelogram polyomino S. More precisely, there exists only one Baxter slicing of shape S whose horizontal blocks all have width 1 and we call it the Catalan slicing of shape S. For instance, the Catalan slicing corresponding to the shape S of the Baxter slicing of Figure 2.1(a) is depicted in Figure 2.3. Therefore, the following proposition trivially holds.

Proposition 2.1.6. *Catalan slicings are enumerated by Catalan numbers.*

In terms of generating trees and succession rules the above proposition can be also proved as follows. The growth described for Baxter slicings in Proposition 2.1.9 and depicted in Figure 2.2 is evident to generalise the one described for parallelogram polyominoes that is depicted in Figure 1.9 (page 29). More precisely, the growth for parallelogram polyominoes of Definition 1.3.10 (page 29) is equivalent to the growth of Catalan slicings, which is obtained by restricting the productions of (h,k) according to Ω_{Bax} to the labels $(h+1,i)$, for $1 \leq i \leq k$, and to label $(1,k+1)$. As pointed out by Observation 1.4.10, on page 46, this reads as: the generating tree $\mathcal{T}_{Bax}$ of Baxter numbers (associated with the growth of Baxter slicings) has a subtree isomorphic to the generating tree $\mathcal{T}_{Cat}$ (associated with the growth of Catalan slicings). Figure 2.16 on page 78 will make visible this embedding.

2.1.4 Definition of Baxter paths, and their bijection with Baxter slicings

Relying on these new notions of Baxter slicing and Catalan slicing, we define here a generalisation of Dyck paths obtained by assigning a label to some up steps, called *free*.

Definition 2.1.7. A *free up step* in a Dyck path P is any up step which does not immediately follow a down step, *i.e.*, any step U which does not appear in a DU factor. If U is a free up step, we usually write it $\bar{U}$.

Definition 2.1.8. A *Baxter path* of semi-length n is a Dyck path P of length $2n$ in which every free up step $\bar{U}$ is labelled. The label assignment of any $\bar{U}$ is defined recursively from left to right, as follows:

- the first free up step is labelled 1;

- any $\bar{U}$, apart from the first one, is labelled with a positive integer value in the range $[1,h]$, where h is the rightmost label assigned to a step $\bar{U}^*$, augmented by the number of DU factors of P occurring between $\bar{U}^*$ and $\bar{U}$.

It follows that all the up steps of the initial sequence are free and have label 1, and that the sequence of labels of any maximal sequence of consecutive up steps is non-increasing. We denote by $\mathcal{B}_n$ the set of Baxter paths of semi-length n. Figure 2.4 depicts on the right a Baxter path of semi-length 9. Note that Dyck paths are retrieved as that subfamily of Baxter paths in which all the free up steps have label 1.

Proposition 2.1.9. *Baxter paths can be generated by the succession rule* Ω_{Bax}.

Figure 2.4: A Dyck path of semi-length 9 whose free up step are labelled in two different ways: the labelling on the left does not satisfy Definition 2.1.8, while the path on the right is a Baxter path of semi-length 9.

Proof. In order to prove the above statement, we define a growth for the family of Baxter paths very similar to the one provided for Dyck paths. Indeed, given a Baxter path B, we insert a peak (*i.e.* a UD factor) with the U step possibly labelled in any point of B's last descent (*i.e.* final sequence of D steps) - see Figure 2.5. First, note that the path obtained removing the last UD factor from a Baxter path of $\mathcal{B}_{n+1}$ belongs to $\mathcal{B}_n$.

Then, assign a label (h, k) to each Baxter path $B \in \mathcal{B}_n$, where h is the label e assigned to the rightmost free up step $\bar{U}$ of B plus the number of DU factors that follow $\bar{U}$ and k is the number of D steps in the last descent of B. The Baxter path UD, in which U is labelled with 1, has label $(1, 1)$. Then, to any Baxter path $B \in \mathcal{B}_n$ of label (h, k) we apply the following operations:

a) We add a peak on top of the last descent of B (*i.e.* just after the rightmost up step of B). Since the up step U of the added peak is a free up step, it must receive a label i in the range $[1, h]$. Then, for each value i in $[1, h]$, we label the added up step with i and the Baxter path obtained is of label $(i, k + 1)$.

b) We add a peak immediately after any down step of the last descent of B. The up step U of the added peak is not free, and hence carries no label. More precisely, denoting $B = wUD^k$ (with this U possibly labelled), all the Baxter paths produced from B are $wUD^{k+1-j}UD^j$, for any $1 \leq j \leq k$, and their rightmost free up step is the same as B. Thus, they have labels $(h + 1, j)$, for $1 \leq j \leq k$, because of the number of DU factors after the rightmost label has been increased by one. □

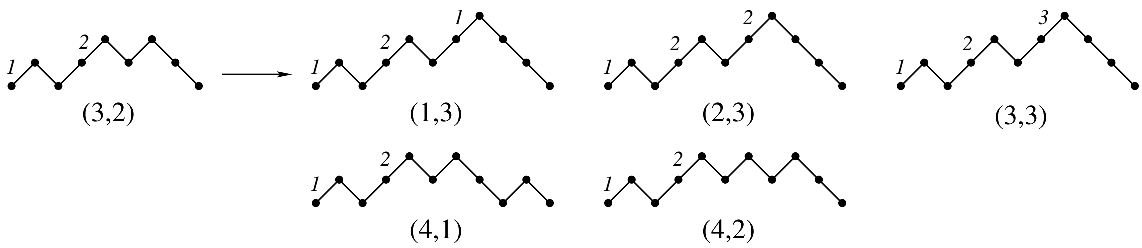


Figure 2.5: The growth of a Baxter path of label $(3, 2)$.

Baxter paths and Baxter slicings are in a size-preserving bijection since they both grow according to Ω_{Bax}. Moreover, this recursive bijection can be directly defined on the objects themselves in a simple way, and its restriction to the family of Baxter paths with all free up steps labelled with 1 yields the family of Catalan slicings as image.

Given a Baxter path B in $\mathcal{B}_n$, we denote by P the Dyck path of semi-length n underlying B. Then,

1) construct the parallelogram polyomino S' corresponding to P via the bijection β between Dyck paths and parallelogram polyominoes described in Section 1.3.3 on page 24;

2) construct the unique Catalan slicing of shape S' and assign a label to each of its horizontal blocks, included the initial unit square block. These labels are precisely the labels of the free up steps of B read left-to-right, which are assigned to horizontal blocks bottom-to-top, starting from the unit square block that indeed takes the label 1 of the first free up step of B;

3) starting from the bottom, replace any labelled horizontal block with a horizontal block of width equal to its label. Thus, the initial unit square block labelled with 1 is replaced by a unit square block. And, after replacing the topmost labelled horizontal block, we obtain a shape S subdivided in horizontal and vertical blocks.

Note that the shape S is likely to be different from S', but they coincide if all the free up step of the Baxter path B are labelled with 1.

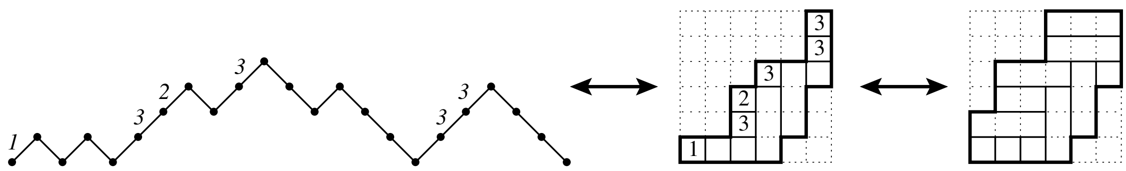


Figure 2.6: The bijection between Baxter paths and Baxter slicings.

Proposition 2.1.10. *The above construction is a bijection between Baxter paths and Baxter slicings, whose restriction to Baxter paths with all free up steps labelled with* 1 *yields a bijection with Catalan slicings.*

Proof. In order to prove this statement, we first show that the labelling process at step **2)** is a proper labelling, *i.e.* the free up steps of P are as many as the horizontal blocks of the Catalan slicing of shape S'. Then, we show that the shape S obtained at the end of step **3)** is a parallelogram polyomino, and thus the object obtained is a Baxter slicing of size n. Finally, we prove that the mapping described above has an inverse, and thus it is a bijection.

First, recall that according to the definition of the mapping $\beta : \mathcal{D} \to \mathcal{PP}$ on page 24, any UD factor of a Dyck path $P \in \mathcal{D}_n$ is associated with a column of the corresponding $S' \in \mathcal{PP}_{n+1}$ determining its height, and any DU factor with a pair of columns determining their edge connection. It follows from it that there is a link between the up steps of P and the steps of the path Q', where (Q', L') is the pair of paths defining S'. More precisely, the path Q' can be obtained directly from P by coding left-to-right any free (resp. non-free) up step of P with a north (resp. east) step of Q', and by adding a further final east step.

Since any north step of Q' is the left border of a horizontal block, this encoding establishes a correspondence between the free up steps of P and the horizontal blocks of the unique Catalan slicing of shape S'. Therefore, the labelling process described at step **2)** makes sense.

Now, according to step **3)**, replacing all the labelled horizontal blocks leads to modify the upper path Q' defining S'. We can argue that such a modified path and L' still define a parallelogram polyomino. In fact, by Definition 2.1.8 the labels of free up steps can increase from left to right if and only if there is between them a sufficient number of DU factors: precisely, the first free up step occurring on the right of any free up step $\bar{U}$ with label e can be labelled at most $e+x$, being x the number of DU factors between them. This property is translated via the bijection β in a condition on the steps of Q', and thus on the labels of the corresponding horizontal blocks. Let $\bar{u}_1$ be a north step of Q' (horizontal block with label ℓ) and $\bar{u}_2$ be the north step (horizontal block with label e) that first follows $\bar{u}_1$ in Q'. Then, the value ℓ is at most $e+x$, being x the number of east steps between $\bar{u}_1$ and $\bar{u}_2$.

Then, replacing the horizontal blocks of the Catalan slicings with horizontal blocks of width according to their labelling corresponds to modify the path Q' by changing the positions of its north steps with respect to the positions of its east steps. So, let Q be the path obtained modifying Q' as above. The number of the north and east steps of Q is the same as Q' as well as its starting and ending point. The pair of paths (Q, L') thus defines uniquely a parallelogram polyomino of semi-perimeter $n+1$, which we denote by S and forms the shape of the Baxter slicing image of $B \in \mathcal{B}_n$.

Finally, we prove that the mapping described above, which sends a Baxter path into a Baxter slicing, is a bijection by showing its inverse mapping.

Given a Baxter slicing of size n and shape S, we replace top-to-bottom any horizontal block u (included the bottommost unit square block) with a unit square block labelled with the width of u. The shape obtained by replacing all the horizontal blocks is denoted by S'. By construction, S' is a parallelogram polyomino of semi-perimeter $n+1$ and (Q', L') is the pair of paths defining it. Via the inverse mapping β^{-1}, S' corresponds to a Dyck path $P \in \mathcal{D}_n$. In particular, the steps of Q' read from bottom to top describe the sequence of free and non-free up steps of P from left to right (as above). Then, we use the block labelling order to label from left to right all the free up steps of P.

We can prove that $B \in \mathcal{B}_n$, namely the above labelling satisfies Definition 2.1.8. The first free up step of B is labelled with 1 as the bottommost unit square block. Then, given a free up step $\bar{U}_2$ of B with label $e \geq 1$, it corresponds to a horizontal block $\bar{u}_2$ labelled e. It holds that the label ℓ of the horizontal block $\bar{u}_1$ lying in the row immediately below $\bar{u}_2$ coincides with the label of the first free up step $\bar{U}_1$ of B on the left of $\bar{U}_2$. Then, the two labels corresponding to the horizontal blocks $\bar{u}_1$ and $\bar{u}_2$ can be either $e \leq \ell$ or $\ell < e$. The relation $\ell < e$ can hold as long as between $\bar{u}_1$ and $\bar{u}_2$ there is a number of columns x at least $e - \ell$. Thus, it follows that $e \in [1, \ell + x]$. This concludes the proof, since x according to β^{-1} is the number of DU factors between $\bar{U}_1$ and $\bar{U}_2$. □

2.1.5 Baxter slicings of a given shape

One of the most basic enumerative questions that one may ask about Baxter slicings is to determine the number of Baxter slicing whose shape is a given parallelogram polyomino S. In the light of the bijection between Baxter slicings and triples of NILPs, this question can be translated in terms of counting triples of NILPs having the same "external" paths (*i.e.* P and R, for any triple (P, Q, R)), which are the two paths defining the shape S of the corresponding Baxter slicing. This question is not the main focus of this chapter, so we just give the extremal cases as observations.

Observation 2.1.11. *Let S be the parallelogram polyomino of rectangular shape, whose bounding rectangle has dimensions $k \times \ell$. The number of Baxter slicings of S is $\binom{k+\ell-2}{\ell-1}$.*

Proof. This follows from Theorem 2.1.3, since the number of Baxter slicings of S coincides with the number of paths from $(1, 1)$ to (k, ℓ) using north and east steps. □

Observation 2.1.12. *Let S be a snake, that is, a parallelogram polyomino not containing four cells placed as ⊞. There is only one Baxter slicing of S.*

Proof. We prove that if S is a snake of size n, then its interior is unambiguously divided in n blocks, each consisting of a single cell. Since S does not contain ⊞, then the topmost cell in the rightmost column is the only cell in its row or the only cell in its column. In the former (resp. latter) case, it forms a horizontal (resp. vertical) block. Removing this block from S, the remaining cells form a snake of size $n - 1$, and the result follows by induction. □

2.1.6 A discrete continuity

The Catalan number sequence is point-wise smaller than the Baxter number sequence; namely, $C_n \leq B_n$, for any $n > 0$. These point-wise comparisons can be extended to include Schröder numbers [132, sequence A006318], which will be treated in the next sections. The Schröder number sequence is point-wise larger than the Catalan number sequence and point-wise smaller than the Baxter number sequence.

The inclusions "Catalan in Schröder in Baxter" are obvious on pattern-avoiding permutations: in fact, as Section 2.2.1 shows, Schröder numbers count separable permutations that are contained in Baxter permutations and contain any permutation class $AV(\tau)$, with $\tau \in \{132, 213, 231, 312\}$. Nevertheless, looking at several other combinatorial objects, it appears that the permutation example is a little miracle, and that the unclarity of these inclusions is rather the rule here. Table 2.1 summarises all the Schröder structures that will be presented in Sections 2.2.1 and 2.3.1 and it compares them both to the Baxter structures of Section 1.4.2 and to the Catalan structures of Section 1.3.2. As Table 2.1 illustrates, these inclusions remain quite obscure on all the other objects, apart from permutations. Many Baxter families can be immediately seen to contain a Catalan subfamily.

Number sequence	Structures				
Baxter (A001181)	Baxter permutations	**?**	Triples of NILPs	Mosaic floorplans	**?**
Schröder (A006318)	Separable permutations (Sec. 2.2.1)	Schröder paths (Sec. 2.2.1)	**?**	Slicing floorplans (Sec. 2.2.1)	Schröder parking functions (Sec. 2.3.1)
Catalan (A000108)	$AV(\tau)$, with $\tau \in \{132, 213, 231, 312\}$	Dyck paths	Pairs of NILPs	**?**	Non-decreasing sequences

Table 2.1: Comparison among families of Catalan, Schröder, and Baxter objects.

For instance, the set of triples of NILPs contains all pairs of NILPs as subfamily enumerated by Catalan numbers, but an intermediate Schröder family of NILPs is missing. On the other hand, consider the family of Schröder parking functions introduced in [G3] and proved in Section 2.3.1 to be counted by Schröder numbers: it contains as subfamily the set of non-decreasing sequences, which is counted by Catalan numbers, leaving Baxter aside.

2.2 Schröder numbers

Schröder numbers are arguably a bit less popular compared to Catalan numbers, yet their history appears to go back to Hipparchus during the second century B.C. according to R. P. Stanley's surveys [135, 136]. Their name is due to the mathematician E. Schröder (1841-1902) for his famous work published in 1870 "Four Combinatorial Problems" (*Vier Kombinatorische Probleme*, see p. 66, 213 [135]), in which these numbers first appear.

2.2.1 Formulas and known structures

The first terms of sequence A006318 [132] of (large) Schröder numbers R_n are

$$1, 2, 6, 22, 90, 394, 1806, 8558, 41586, 206098, 1037718, 5293446, 27297738, \ldots$$

We specify that the Schröder numbers R_n we are interested in are those enumerating separable permutations and Schröder paths, which are known as "large". They are opposed to "little" Schröder numbers L_n (sequence A001003 in [132] counting the ways to insert parentheses in a string of $n+1$ symbols), for which it holds $R_n = 2L_n$, $n \geq 1$. This equality has widely been studied in literature giving insight of Schröder numbers: L. Shapiro and R. Sulanke first provided a bijective proof in [130], and later E. Deutsch established another bijective proof of this remarkable relation in [65].

One of the most concise way to define Schröder numbers R_n, with $n > 0$, is in terms of Catalan numbers C_n (see [28]),

$$R_{n+1} = \sum_{k=0}^{n} \binom{2n-k}{k} C_{n-k}, \text{ for } n \geq 0.$$

Alternatively, we can show their generating function,

$$F_{Sch}(x) = \sum_{n\geq 1} R_n\, x^n = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2}\,, \tag{2.1}$$

which is known to be algebraic and to satisfy the polynomial equation $F^2-(1-x)\,F+x = 0$.

Schröder structures have widely been studied and a long list is exhibited by R. P. Stanley [135, Exercise 6.39]. Nevertheless, these numbers have never stopped attracting attention, since they appear in different contexts: certain polyominoes [6, 13, 140], lattice paths [13, 28, 129], plane trees [13, 80, 89], words [126], as well as pattern-avoiding permutations [87, 103, 129, 144].

We report here a more detailed description of those Schröder structures we will use to explain the aforementioned inclusion “Catalan in Schröder in Baxter”, which resurfaces as

$$C_n \leq R_n \leq B_n, \text{ for all } n > 0\,.$$

Separable permutations

Many permutation classes are enumerated by the Schröder numbers. For instance, up to symmetry, exactly 10 pairs (τ, σ) of patterns of length four are such that the cardinality of $AV_n(\tau, \sigma)$ is the nth Schröder number R_n [103]. Among them, the permutation class $AV(2413, 3142)$ was studied in [29] and identified with the family of *separable* permutations, first defined in D. Avis and M. Newborn’s work on pop-stacks [5].

Definition 2.2.1. A *separable permutation* is any permutation π that can be built from the permutation 1 by repeatedly applying two operations, known as *direct sum* ($\oplus$), and *skew sum* ($\ominus$), which are defined on two smaller permutations τ of length k and σ of length m by

$$(\tau \oplus \sigma)_i = \begin{cases} \tau_i & \text{if } 1 \leq i \leq k, \\ \sigma_{i-k} + k & \text{if } k < i \leq k+m, \end{cases}$$

$$(\tau \ominus \sigma)_i = \begin{cases} \tau_i + m & \text{if } 1 \leq i \leq k, \\ \sigma_{i-k} & \text{if } k < i \leq k+m. \end{cases}$$

All permutations of length 3 are separable, since they can be easily decomposed as direct and skew sum of 12 and 1, or 21 and 1. Figure 2.7 depicts the way to construct the separable permutation $\pi = 312675498$ from the permutation 1 by using operations $\oplus$ and $\ominus$.

Note that there are only two permutations of length four, 2143 and 3142, that cannot be obtained by smaller permutations through the application of $\oplus$ or $\ominus$.

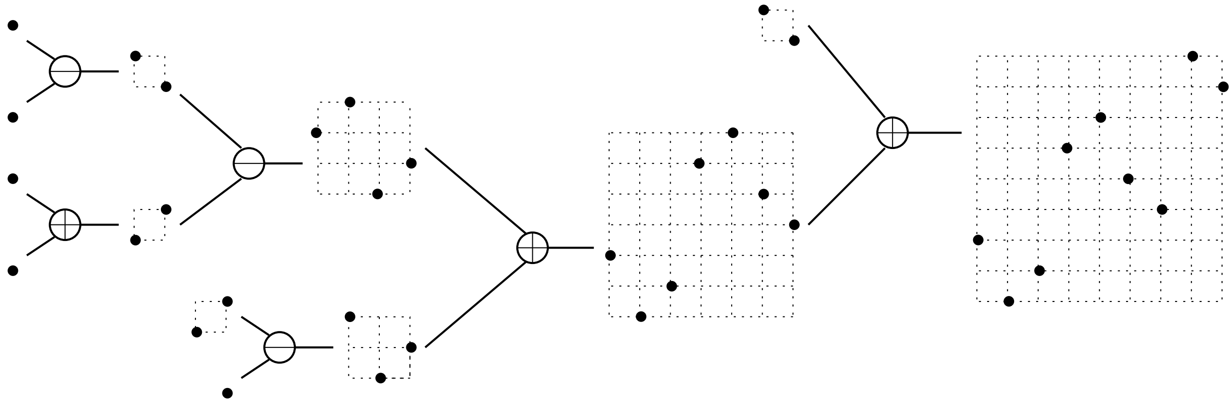


Figure 2.7: The building process of the separable permutation $\pi = 312675498$ by means of $\oplus$ and $\ominus$.

Proposition 2.2.2 ([29])**.** *The family of separable permutations coincides with the permutation class* $AV(2413, 3142)$.

Separable permutations were enumerated by using their characterisation in terms of pattern-avoiding permutations by J. West [144, Lemma 4.1] as well as by L. Shapiro and A. Stephens, considering a family of permutation matrices equivalent to separable permutations [129].

Moreover, note that Proposition 2.2.2 makes clear that the class of separable permutations is a subfamily of Baxter permutations of Definition 1.4.2 (page 39) and contains the permutation class $AV(\tau)$, for τ being in $\{132, 213, 231, 312\}$, as subfamilies.

Schröder paths

In 1993, J. Bonin, and L. Shapiro, and R. Simion [28] defined and enumerated the following new family of paths.

Definition 2.2.3. A *Schröder path* of semi-length n is a path T of length $2n$ in the positive quarter plane that uses *up* steps $U = (1, 1)$, and *down* steps $D = (1, -1)$, and double *horizontal* steps $H = (2, 0)$, starting at the origin and returning to the x-axis.

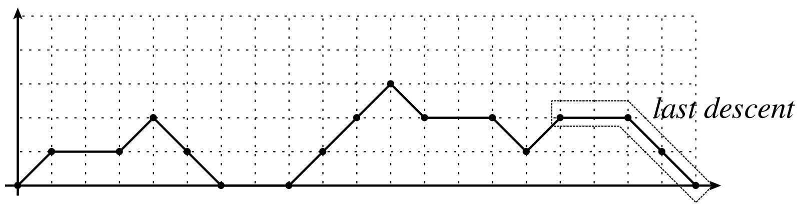


Figure 2.8: A Schröder path of semi-length 10, whose last descent is encircled.

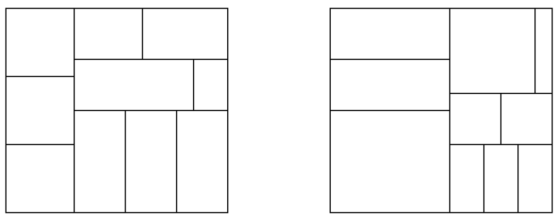
Figure 2.8 shows an example of a Schröder path of semi-length 10. According to [28] as well as [15], Schröder paths of semi-length n are enumerated by the Schröder number

R_{n+1}, for $n \geq 0$. In fact, of semi-length zero there exists only one Schröder path (the empty path), and of semi-length one there are two paths, UD and H. We denote by $\mathcal{SP}$ the family of Schröder paths, and by $\mathcal{SP}_n$ the family of Schröder paths of semi-length n. Then, the family $\mathcal{D}$ of Dyck paths defined in Definition 1.3.2 on page 22 is trivially a subfamily of $\mathcal{SP}$: any Schröder path with no H steps is a Dyck path.

Slicings floorplans

Regarding the family of floorplans introduced in Section 1.4.2 about Baxter structures, a special kind of them are called *slicing floorplans* because of their definition.

Definition 2.2.4. A *slicing floorplan* is a particular rectangular partition of a boundary rectangle, which is obtained by recursively subdividing each rectangle in two smaller rectangles either horizontally or vertically. Slicing floorplans are considered up to equivalence under the action of sliding their internal segments.

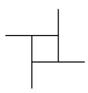
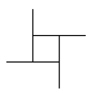


(a) (b)

Figure 2.9: (a) A slicing floorplan with 9 internal segments; (b) a slicing floorplan equivalent to the one in (a).

Figure 2.9 shows two equivalent slicing floorplans. Because of the above definition, slicing floorplans are actually equivalence classes of floorplans, as mosaic floorplans are.

Moreover, any slicing floorplan is a mosaic floorplan. As stated in [1], slicing floorplans can be characterised as those mosaic floorplans whose internal segments avoid a "pin-wheel" structure, namely configurations of type

 and  .

In [148], slicing floorplans are proved to be enumerated by Schröder numbers according to the number of internal segments: the number of slicing floorplans with n internal segments is R_{n+1}, for $n \geq 0$. A bijective proof of this fact appears also in [1]. Indeed, the bijection ϕ described in Section 1.4.3 (page 42) between mosaic floorplans and Baxter permutations can be restricted to the family of slicing floorplans yielding a bijection between slicing floorplans and separable permutations.

2.2.2 Generating trees and succession rules

In order to report generating trees known in literature for enumerating Schröder numbers, we start defining a growth for the family of separable permutations, which comes from the growth described by J. West in [144].

Analogously to Baxter permutations and permutations avoiding 132, the main point in defining such a growth is to recognise, for every separable permutation of length n, the set of active sites in which $n+1$ can be placed.

Therefore, let π be a separable permutation of length n. We divide it in two parts according to the position of n and subdivide each part in blocks of consecutive elements strictly depending on LTR maxima and RTL maxima positions as explained in the following - see Figure 2.10.

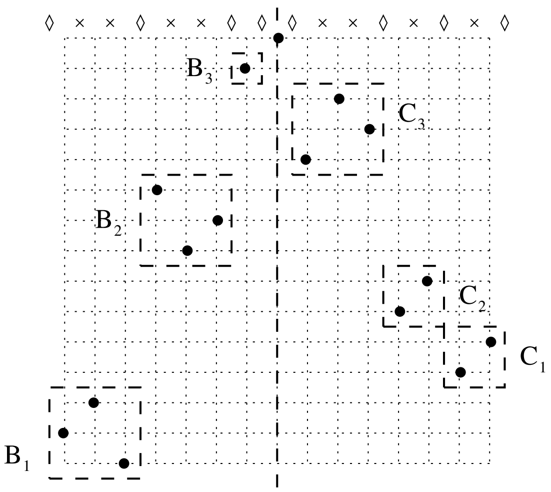


Figure 2.10: The graphical representations of a separable permutation whose points have been subdivided in blocks of consecutive elements, and of its sites classified in active ($\Diamond$) and non-active ($\times$).

Precisely, let $\ell_1, \ldots, \ell_{k+1} = n$ be the LTR maxima of π and suppose that $\ell_1 \neq n$. The first LTR maximum ℓ_1 could have on its right some smaller points that are on the left of n. By definition of separable permutation, the values of these points must be consecutive: if there were a point x smaller than ℓ_1 on the right of n, and a point y smaller than ℓ_1 on the left of n such that $x > y$, then $\ell_1\, y\, n\, x$ would form a 3142 occurrence. Let m_1 be the rightmost of these consecutive points (if there is any). The block B_1 on the left of n is made of all points of π between ℓ_1 (included) and m_1 (included). Note that according to the position of m_1, the block B_1 may contain some other LTR maxima different from ℓ_1, which lie in a smaller position than m_1. All the other blocks $B_2, \ldots, B_h$, with $h \leq k$, are iteratively formed starting from the first LTR maximum not included in any previous block and repeating for each block the same arguments as B_1.

A symmetric reasoning holds on the right of n. The first RTL maximum $r_1 \neq n$ could have on its left some smaller points that are on the right of n and, as above, their values must be consecutive, by definition of separable permutation: if there were a point x on left of n such that $r_1 > x > y$, where y is any point on the right of n smaller than r_1, then $x\,n\,y\,r_1$ would form a 2413 occurrence. Then, the first block C_1 on the right of n is made of r_1 and all the points on the right of n between r_1 and m'_1, where m'_1 is the leftmost among the smaller consecutive element on the left of r_1. And, all the other blocks $C_2, \dots, C_m$ are iteratively formed starting from the first RTL maximum not included in any previous block and repeating for each block the same arguments as C_1.

It should be noticed that all the elements within blocks $B_1, \dots, B_h$ as well as $C_1, \dots, C_m$ are consecutive values of the permutation π: indeed, it follows easily by a reasoning similar to the one done for each single block of π. Whence, blocks form a unimodal sequence that increases on the left of n and decreases on its right.

Lemma 2.2.5 ([144], Lemma 4.1)**.** *Let* $\pi \in AV_n(2413, 3142)$*. Suppose the blocks of consecutive elements of* π *are* $B_1, \dots, B_h$ *on the left of* n *and* $C_1, \dots, C_m$ *on the right of* n*, with* $h, m \geq 0$*. Then, a site of* π *is active if and only if it is:*

- *immediately before any left block* B_i*, with* $1 \leq i \leq h$*;*
- *immediately before, or immediately after* n*;*
- *immediately after any right block* C_j*, with* $1 \leq j \leq m$*.*

Figure 2.11 shows the growth of a separable permutation by the addition of $n+1$ in any of its active sites.

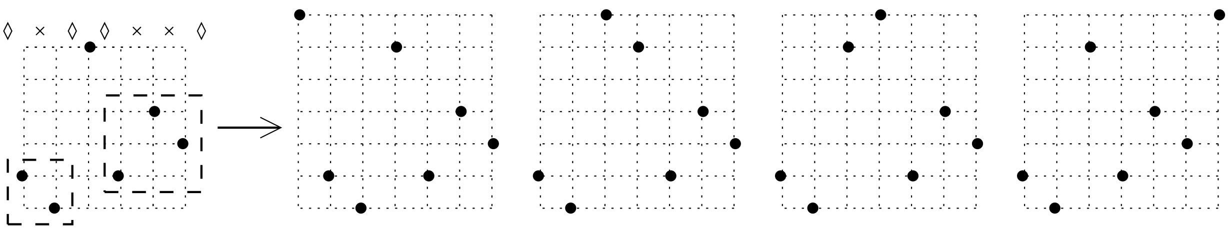


Figure 2.11: The set of separable permutations obtained from $\pi = 216354$ by adding a new maximum point.

Proof. Let π be a non-empty separable permutation of length n. Let $B_1, \dots, B_h$ be its left blocks and $C_1, \dots, C_m$ be its right blocks. Since the forbidden patterns are one reverse of the other, the situation is symmetric with respect to n, and we can consider only insertions of $n+1$ on the left of n - the insertions of $n+1$ on the right of n behave symmetrically.

If $n+1$ is added just before n, there is nothing to prove: if placing $n+1$ here created occurrences of the forbidden patterns, then n would already play the same role in like occurrences. Now, suppose $n+1$ is inserted within a block B_i, for some $i \leq h$, whose cardinality is more than one. Let m_i be the rightmost point of B_i and ℓ_s be the leftmost point of B_i. Specify that the point ℓ_s is a LTR maximum by definition of block. Then

$\ell_s(n+1)m_i n$ is an occurrence of 2413. Conversely, suppose $n+1$ is inserted just before B_i, for any i, and assume for the sake of contradiction that it forms an occurrence of the forbidden patterns. First, suppose that $n+1$ gives rise to a 2413 occurrence. Then, there are three elements $x < y < z$ of π such that $y(n+1)xz$ is a 2413 occurrence. The point y must belong to a block B_j, with $j < i$, and by the definition of block a smaller point such as x cannot be on the left of n without being within B_j. Thus, x must be on the right of n, but $y\,n\,x\,z$ would form a 2413 occurrence. On the other hand, suppose $n+1$ gives rise to a 3142 occurrence. Then, there are three elements of π, $x < y < z$, such that $z\,x\,(n+1)\,y$ is a 3142 occurrence. As above, the point z must belong to a block B_j, with $j < i$, and by definition of block, a smaller point not being in B_j, such as y, cannot be on the left of n. A contradiction is derived, since $z\,x\,n\,y$ forms a 3142 occurrence. □

Figure 2.12 depicts the first level of the generating tree of separable permutations.

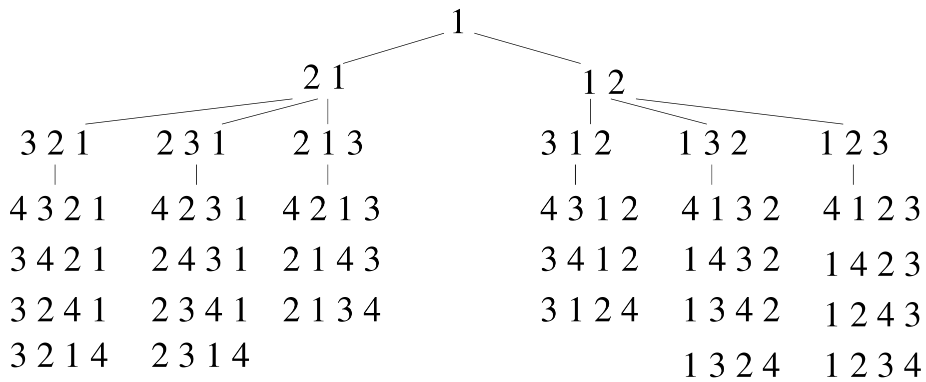


Figure 2.12: The first levels of the generating tree of separable permutations.

Proposition 2.2.6 ([144], Lemma 4.1). *Separable permutations can be generated by*

$$\Omega_{Sch} = \begin{cases} (2) \\ (k) \quad \rightsquigarrow (3),\, (4),\, \ldots,\, (k),\, (k+1)^2. \end{cases}$$

Proof. First, by removing n from a separable permutation of length $n > 0$ we still obtain a separable permutation of length $n-1$, since no occurrences of 2413 and 3142 can be generated. According to Lemma 2.2.5, the growth of each separable permutation can be completely controlled by the number of its active sites. Thus, we can substitute each permutation in the decorated generating tree of Figure 2.12 for a label (k), where k is the number of its active sites - see Figure 2.13.

Then, we have that the permutation 1 has label (2), which is the axiom of Ω_{Sch}. And, let π be a separable permutation with $k \geq 2$ active sites, namely with $k-2$ blocks (see Lemma 2.2.5). It holds that both insertions of $n+1$ next to n produce separable permutations with $k+1$ active sites, since a new block of one single element has been generated by n. Now, let the blocks of π forming an unimodal sequence be numbered increasingly from 1 to $k-2$ from bottom to top. The insertion of $n+1$ just before a left block (or just after a right block) reduces the number of active sites according to their numeration. Precisely, if a left block B_i is numbered q, $1 \leq q \leq k-2$, and $n+1$ is added just before it, then in the permutation produced π' all blocks bigger than B_i are merged together in a unique block of π'. Consequently, their sites become non-active in π', which has only q blocks and thus $q+2$ active sites. The same holds symmetrically if we consider a right block. Hence, a separable permutation with k active sites produces k different separable permutations with $3, 4, \ldots, k, k+1, k+1$ active sites. □

Figure 2.13 shows the first levels of the generating tree associated with Ω_{Sch} decorated with its labels.

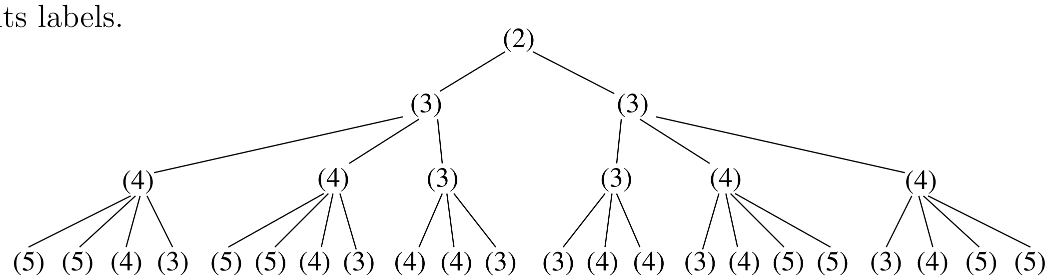


Figure 2.13: The first levels of the generating tree corresponding to Ω_{Sch} decorated with its labels.

Proposition 2.2.7 ([13])**.** *The family of Schröder paths can be generated by* Ω_{Sch}.

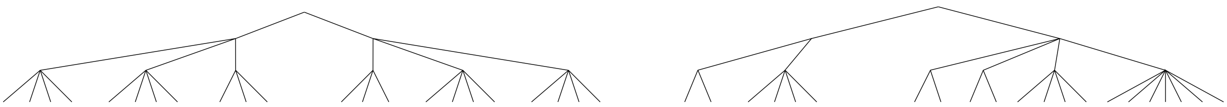
Proof. We can define a growth for the family $\mathcal{SP}$ of Schröder paths slightly modifying the ECO operator $\vartheta_{\mathcal{D}}$ defined for Dyck paths on page 28. More precisely, let $T \in \mathcal{SP}_n$ be a Schröder path of semi-length n. Analogously to Dyck paths, the last descent of T is defined as the set of steps that follows the rightmost U step of T. Thus, the first step of the last descent can be either an H step or a D step - see Figure 2.8. In the first case, if we remove this H step from T, we obtain a path in $\mathcal{SP}_{n-1}$. And, in the other case, if we remove the rightmost UD factor of T, we still obtain a path in $\mathcal{SP}_{n-1}$. Therefore, we can perform local expansions on a Schröder path of $\mathcal{SP}_n$ by inserting either a UD factor in any point of T's last descent, or an H step in the leftmost point of T's last descent.

Now, we label any Schröder path T with (k), where k is obtained by adding one to the number of points of T's last descent. The empty path can be thought as a single point, thus it takes label (2), which is the axiom of Ω_{Sch}. Then, by the above construction, a Schröder path T of $\mathcal{SP}_n$ with label (k) generates k Schröder paths of $\mathcal{SP}_{n+1}$, whose labels range in $(3), (4), \ldots, (k+1), (k+1)$. The double label $(k+1)$ comes from the double possible insertion (either a UD factor or an H step) in the leftmost point of T's last descent. □

Other succession rules are known for enumerating Schröder structures [117]. These succession rules differ from Ω_{Sch} since the first levels of their corresponding generating trees. For instance, one of them is the following rule Ω_{Sch2}. Figure 2.14 depicts the first levels of its corresponding generating tree that are not symmetric as the ones associated with Ω_{Sch}.

Proposition 2.2.8. *The following succession rule, Ω_{Sch2}, generates Schröder numbers*

$$\Omega_{Sch2} = \begin{cases} (2) \\ (2^k) \quad \rightsquigarrow (2)^{2^{k-1}}, \ldots, (2^{k-1})^2, (2^k)(2^{k+1}) . \end{cases}$$



(a) (b)

Figure 2.14: The first four levels of the Schröder generating trees: (a) corresponding to Ω_{Sch}; (b) corresponding to Ω_{Sch2}.

Remark 2.2.9. *Although the inclusions "Catalan in Schröder in Baxter transpire easily on pattern-avoiding permutations, they remain obscure at the more abstract level of generating trees, and succession rules. More precisely, the succession rule Ω_{Sch} that is proved to enumerate separable permutations does not appear to be a restriction of the succession rule Ω_{Bax} for Baxter permutations, or at least not in the same manner as Observation 1.4.10 does for Ω_{Cat} and Ω_{Bax}. On the other hand, as Proposition 2.2.7 shows, Ω_{Sch} generalises Ω_{Cat} by doubling the label $(k+1)$ in the production of (k). Therefore, the two generalisations, Ω_{Bax} and Ω_{Sch}, of the Catalan rule Ω_{Cat} appear to be independent, and not easily reconciled.*

2.3 A new Schröder family of parking functions

The notion of parking functions is recurring in discrete mathematics, and arises naturally in the so-called *parking problem*, which can be stated as follows: there are n cars $\mathsf{C}_1, \ldots, \mathsf{C}_n$ that want to park on a one-way street with ordered parking spaces $0, 1, \ldots, n-1$. Each car C_i has a preferred space a_i, and the cars enter the street one at a time in the order $\mathsf{C}_1, \ldots, \mathsf{C}_n$. Any car tries to park in its preferred space; if that space is occupied, then it parks in the next available space. If there is no space, then the car leaves the street.

Definition 2.3.1. The sequence $a_1 \ldots a_n$ is called a *parking function* of length n if all the cars $\mathsf{C}_1, \ldots, \mathsf{C}_n$ can park, namely no car leaves the street.

It is easy to see that a sequence $a_1 \ldots a_n$ is a parking function if and only if it has at least i terms strictly less than i, for each $1 \leq i \leq n$. In particular, every parking function can be obtained as a rearrangement of a non-decreasing sequence (see Definition 1.3.4) as proved in [135]: a sequence $a_1 \ldots a_n$ is a parking function if and only if there is a permutation σ of length n such that $0 \leq a_{\sigma_i} < i$, for each $1 \leq i \leq n$.

The number of parking functions of length n was analytically proved to be equal to $(n+1)^{n-1}$ in [100], but then several combinatorial explanations of this formula were provided (see for instance [125]). Amongst them, there are also many bijections that show remarkable connections between parking functions and other combinatorial structures, and lead to various generalizations and applications in different fields, notably in algebra, interpolation theory, probability and statistics, representation theory, and geometry. A relation between parking functions and Dyck paths has been established in [84] and it is not reported here, not being needed for our purposes.

In [G3] we have introduced a new family of parking functions counted by Schröder numbers, which directly appears as a generalisation of non-decreasing sequences (Definition 1.3.4 of Section 1.3.2).

2.3.1 Schröder parking functions

Definition 2.3.2. A *Schröder parking function* s of length n is a sequence $s_1\, s_2 \ldots s_n$ such that $0 \leq s_i < i$, and $s_i \geq s_j - 1$, for all $j < i$. A *fall* of s is any element s_i such that either $s_i = 0$ or $s_i < s_j$, for some index $j < i$.

The sequence $s = 0001433676$ is a Schröder parking function of length 10 with falls $s_1 = 0$, $s_2 = 0$, $s_3 = 0$, $s_6 = 3$, $s_7 = 3$, $s_{10} = 6$. Whereas 0020 is not a Schröder parking function being the rightmost element strictly smaller than 1. Obviously, according to Definition 1.3.4, any non-decreasing sequence of length n is a Schröder parking function of length n.

Proposition 2.3.3. *Schröder parking functions are counted by Schröder numbers.*

Proof. To prove that Schröder parking functions are counted by Schröder numbers, we describe a growth for their family, and show that the succession rule associated with it is precisely Ω_{Sch}. Given $s_1 \ldots s_n$ a Schröder parking function of length n, we add a new rightmost element s_{n+1} to s as to form a Schröder parking function of length $n+1$. The possible additions depend on the rightmost value s_n, as follows:

(a) If s_n is not a fall, then the element s_{n+1} added to the sequence $s_1 \ldots s_n$ is any value among $n, n-1, \ldots, s_n, s_n - 1$.

(b) If s_n is a fall, then the element s_{n+1} added to $s_1 \ldots s_n$ is any value among $n, n-1, \ldots, s_n + 1, s_n$.

Note that operation performed in case (a) (resp. (b)) produces $n+2-s_n$ (resp. $n+1-s_n$) sequences of length $n+1$ that satisfy Definition 2.3.2 and among them only one is such that s_{n+1} is a fall: namely, in case $s_{n+1} = s_n - 1$ (resp. $s_{n+1} = s_n$).

Now, we check that the succession rule associated with the above growth is Ω_{Sch}. To a sequence $s = s_1 \ldots s_n$ we assign the label (k), where $k = n - s_n + 2$ if it satisfies the condition at point (a), or $k = n - s_n + 1$ if it satisfies the condition at point (b). The Schröder parking function $s = 0$ has label (2), since it satisfies the condition at point (b).

Then, the sequence $s_1 \dots s_n\, s_{n+1}$, produced in case (a) by setting $s_{n+1} = n$ (resp. $n - 1, \dots, s_n{+}1, s_n, s_n{-}1$), has label (3) (resp. $(4), \dots, (k), (k{+}1), (k{+}1)$). While the sequence $s_1 \dots s_n\, s_{n+1}$, produced in case (b) by setting $s_{n+1} = n$ (resp. $n-1, \dots, s_n+2, s_n+1, s_n$), has label (3) (resp. $(4), \dots, (k), (k{+}1), (k{+}1)$). Thus, in both cases any label (k) produces labels $(3), (4), \dots, (k), (k+1), (k+1)$ concluding the proof. □

A Schröder parking function $s = s_1 \dots s_n$ can be represented uniquely as a word $w = w(s)$ of length $n-1$ in the alphabet $\{a, b, c\}$ as follows.

1. $w(s_1)$ is the empty word ε. (Note that $s_1 = 0$, for every s.)

2. Let $w' = w(s_1 \dots s_{n-1})$, with $n \geq 2$. If s_n is a fall, then $w(s_1 \dots s_n) = w'a$. Otherwise $w(s_1 \dots s_n) = w'c^k b$, where k is determined either by $s_n - s_r$, where s_r is the rightmost non-fall element of $s_1 \dots s_{n-1}$ if there is any, or by $s_n - 1$.

In particular, note that the length n of a Schröder parking function s is given by adding one to the number of occurrences of a and b in the word $w(s)$, namely $|w|_a + |w|_b = n-1$. For example, the Schröder parking function $s = 0001433676$ is encoded by the word $a\,a\,b\,cccb\,a\,a\,ccb\,cb\,a$.

From the definition above, we can provide a combinatorial description for the language $\mathcal{L}_S = \cup_n \mathcal{L}_S(n)$, where

$$\mathcal{L}_S(n) = \{w(s) : s \text{ is a Schröder parking function of length } n+1\}\,.$$

Lemma 2.3.4. *For any n, let $u \in \mathcal{L}_S(n)$ and $s_1 \dots s_{n+1}$ be the Schröder parking function corresponding to u. Then, $|u|_c = s_r - 1$, where s_r is the rightmost non-fall element of $s_1 \dots s_{n+1}$, if there is any, otherwise $|u|_c = 0$.*

Proof. First, note that if all the elements of $s_1 \dots s_{n+1}$ are falls, then by construction $|u|_c = 0$. Else if $s_1 \dots s_{n+1}$ has at least a non-fall element, we denote by s_r the rightmost non-fall element, where $r > 1$. Now, we prove by induction on n that $|u|_c = s_r - 1$.

If $n = 1$, the only Schröder parking function of length 2 having a non-fall element is $s = 01$. Its rightmost non-fall element is $s_2 = 1$. The one-letter word corresponding to s is b, and it holds trivially that $|b|_c = 1 - 1$.

Now, let $n > 1$. We need to distinguish whether or not $s_1 \dots s_n$ has a rightmost non-fall element, and whether or not it coincides with s_r. Let $u' \in \mathcal{L}_S(n-1)$ be the word corresponding to $s_1 \dots s_n$.

Suppose s_r is such that $r < n+1$; namely both s_r is the rightmost non-fall element of $s_1 \dots s_n$ and s_{n+1} is a fall, so that $u = u'a$ holds. Then, the induction hypothesis yields that $|u|_c = |u'|_c = s_r - 1$.

Otherwise, it must be $s_r = s_{n+1}$. We need to consider whether or not $s_1 \dots s_n$ has a rightmost non-fall element. If such an element does not exist, then $|u'|_c = 0$ by induction hypothesis. In addition, $|u|_c = s_r - 1$ holds, since $u = u'c^k b$ with $k = s_{n+1} - 1$ by construction. Else denoting s_t the rightmost non-fall element of $s_1 \dots s_n$, it holds that $|u'|_c = s_t - 1$ by induction hypothesis. Therefore, $|u|_c = |u'|_c + k = s_r - 1$ holds, since $u = u'c^k b$, with $k = s_{n+1} - s_t$ by construction. □

Proposition 2.3.5. *A word w in the alphabet $\Sigma = \{a, b, c\}$ belongs to $\mathcal{L}_S(n)$ if and only if*

a) *for each prefix v of w, $|v|_c \leq |v|_a + |v|_b$,*

b) *w does not contain the factor ca,*

c) *the last letter is not c,*

d) $|w|_a + |w|_b = n$.

Proof. $\Rightarrow$) We prove that each word $w \in \mathcal{L}_S(n)$ satisfies the conditions **a)-b)-c)-d)**, for every n. First, note that properties **b)-c)-d)** hold by construction of $w = w(s)$, for s any Schröder parking function of length $n+1$.

Now, we prove the validity of property **a)** by induction on n. If $n = 0$, the set $\mathcal{L}_S(0)$ comprises only the empty word ε, and property **a)** trivially holds.

Now, suppose $n > 0$. Let $w \in \mathcal{L}_S(n)$ be uniquely subdivided into factors either a or $c^k b$, where k is maximal. By construction, there are n such factors in w. Let $v' \in \mathcal{L}_S(n-1)$ be the prefix of w corresponding to its first $n-1$ factors from left to right. By induction hypothesis, $|v'|_c \leq |v'|_a + |v'|_b$ and $|v'|_a + |v'|_b = n-1$. Now, if the nth factor of w is a or b, *i.e.* $w = v'a$ or $w = v'b$, the property **a)** trivially holds for w. Otherwise, by construction, it must be $w = v'c^k b$, with $0 < k < n$. In this case, we should consider whether or not $|v'|_c = 0$. If $|v'|_c = 0$, the property **a)** obviously holds for w. On the contrary, if $|v'|_c > 0$, the Schröder parking function $s_1 \dots s_n$ corresponding to v' has at least a non-fall element. By Lemma 2.3.4, it follows that $|v'|_c = s_r - 1$, where s_r is the rightmost non-fall element of $s_1 \dots s_n$. The validity of **a)** is equivalent to $|v'|_c + i \leq |v'|_a + |v'|_b$, for all $1 \leq i \leq k$. Therefore, to prove this, it is enough to show that $s_r - 1 + k \leq n - 1$. Noticing that by construction $k = s_{n+1} - s_r$, this inequality is equivalent to $s_{n+1} \leq n$, which trivially holds.

$\Leftarrow$) We prove that each word $w \in \Sigma^*$ satisfying properties **a)-b)-c)-d)** is a word of $\mathcal{L}_S(n)$. By properties **b)-c)-d)** any such word w can be subdivided uniquely in n factors comprising either the only letter a or the sequence $c^k b$, for some $k \geq 0$. We can proceed by induction on the number n of these factors that subdivide w, to show that $w \in \mathcal{L}_S(n)$.

First, let $n = 1$. There are only two words that are composed of only one factor and satisfy **a)-b)-c)-d)**, namely the one-letter words a and b. Both letters are in $\mathcal{L}_S(1)$, since they encode the two Schröder parking functions 00 and 01.

Then, let $n > 1$. By induction hypothesis the first $n-1$ factors from left to right form a word $u \in \mathcal{L}_S(n-1)$. Now, the nth factor can be either a or $c^k b$, for some k such that $|u|_c + k \leq n-1$ (by property **a)**). In both cases $w \in \mathcal{L}_S(n)$: more precisely, there exists a Schröder parking function $s = s_1 \dots s_n s_{n+1}$ corresponding to w, where $s_1 \dots s_n$ is the Schröder parking function corresponding to u. Indeed, in case $w = ua$, it holds that $w = w(s)$ if we set the element s_{n+1} is a fall, and $s_{n+1} = s_n - 1$ if s_n

is a non-fall element, otherwise $s_{n+1} = s_n$. Whereas, if $w = uc^k b$, we set s_{n+1} is a non-fall element of s, and precisely $s_{n+1} = k + |u|_c + 1$. By Lemma 2.3.4, $|u|_c$ can be either $s_r - 1$, where s_r is the rightmost non-fall element of $s_1 \dots s_n$ if there is any, or 0, and thus either $s_{n+1} = k + s_r$ or $s_{n+1} = k + 1$ yields. Hence, s is a Schröder parking function such that $w = w(s)$. □

To our knowledge, the language $\mathcal{L}_S$ of Proposition 2.3.5 provides a new occurrence of Schröder numbers. We describe a bijective proof of this fact in the next section.

2.3.2 Bijection between Schröder parking functions and Schröder paths

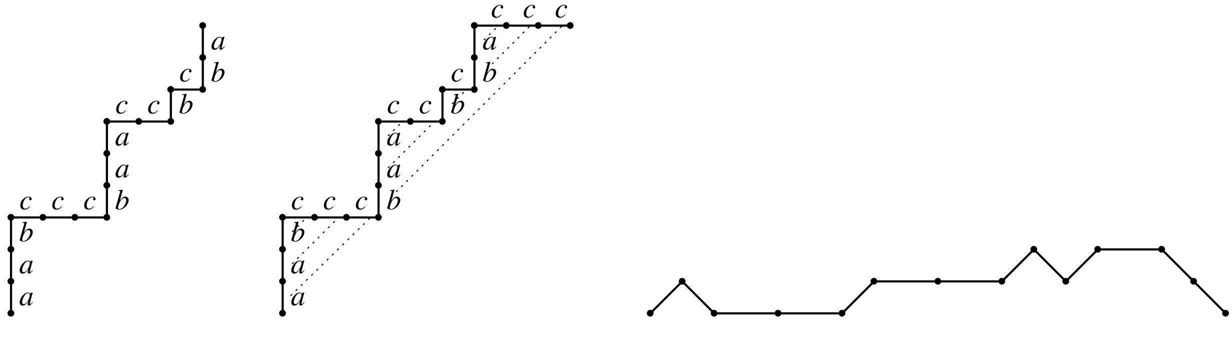
In this section we describe a bijective way to pass from a word w of $\mathcal{L}_S(n)$, encoding a Schröder parking function of length $n+1$, to a Schröder path of semi-length n.

In order to define a mapping from $\mathcal{L}_S(n)$ to the family of Schröder paths $\mathcal{SP}_n$ of semi-length n, we graphically represent each word w of $\mathcal{L}_S(n)$ in the Cartesian plane as a labelled path of length $n + |w|_c$. Such a path starts at the origin and ends at $(|w|_c, n)$, and encodes the word w as follows: each letter a (resp. b) is a north step $(0,1)$ labelled a (resp. b), and each letter c is an east step $(1,0)$ labelled c. From now on, we will use words of $\mathcal{L}_S(n)$ and their graphical representations indifferently.

Another definition we need is the *closure* $\overline{w}$ of a word $w \in \mathcal{L}_S(n)$. Given a path w of $\mathcal{L}_S(n)$ we define its *closure* $\overline{w}$ as the smallest path ending at (n,n) and containing w as prefix, namely $\overline{w} = w\,c^h$, where $h = n - |w|_c \geq 1$. Therefore,

$$\overline{\mathcal{L}_S}(n) = \{\overline{w} : w \in \mathcal{L}_S(n)\}\,.$$

Clearly $\mathcal{L}_S(n)$ and $\overline{\mathcal{L}_S}(n)$ are in bijection. The graphical representations of the word $w = a\,a\,b\,c\,c\,c\,b\,a\,a\,c\,c\,b\,c\,b\,a$ and its closure $\overline{w}$ are shown in Figure 2.15(a),(b).



(a) (b) (c)

Figure 2.15: The graphical representation of: (a) the word $w = a\,a\,b\,c\,c\,c\,b\,a\,a\,c\,c\,b\,c\,b\,a$; (b) its closure $\overline{w} = a\,a\,b\,c\,c\,c\,b\,a\,a\,c\,c\,b\,c\,b\,a\,c\,c\,c$, where dotted lines match each pair (a,c) (resp. (b,c)); (c) the Schröder path of semi-length 9 corresponding to $\overline{w}$ through χ.

Now, by means of the graphical representation of $\overline{w}$, for $w \in \mathcal{L}_S(n)$, we define some matchings among letters of $\overline{w}$ needed to define our mapping.

Given a north step labelled a (resp. b) in a path $\overline{w} \in \overline{\mathcal{L}_S}(n)$, there exists a unique east step labelled c such that it is the first step encountered by drawing from a (resp. b) a line parallel to the main diagonal. We say that the pair (a, c) (resp. (b, c)) forms a *matching*, and a (resp. b) *matches* c. Note that by Proposition 2.3.5, each a (resp. b) is matched to a unique c, and vice versa (see Figure 2.15(b)).

Then, we define the function $\chi : \overline{\mathcal{L}_S}(n) \to \mathcal{SP}_n$, according to the following decompositions for $\overline{w}$,

(1) $\overline{w}$ is the empty path,

(2) $\overline{w} = a\, v'\, c\, v$, where (a, c) is a matching, and $v', v'' \in \overline{\mathcal{L}_S}$,

(3) $\overline{w} = b\, v'\, c\, v$, where (b, c) is a matching, and $v', v'' \in \overline{\mathcal{L}_S}$.

Observe that in cases (2) and (3) owing to the definition of words in $\overline{\mathcal{L}_S}(n)$ we have that v'' is either the empty path or

$$v'' = b\, g_1\, c\, b\, g_2\, c \dots\, b\, g_k\, c,$$

where, for $i \geq 1$, any g_i is in $\overline{\mathcal{L}_S}$ and the pair (b, c), with $b\, g_i\, c$, forms a matching. According to this decomposition, the function χ is defined as follows:

(1) $\chi(\varepsilon) = \emptyset\,;$

(2) $\chi(a\, v'\, c\, v'') = \chi(v')\, H\, \chi(v'')\,;$

(3) $\chi(b\, v'\, c\, v'') = U\, \chi(v')\, D\, \chi(v'')\,.$

where, as usual, ε is the empty word and $\emptyset$ denotes the empty Schröder path, U (resp. D) denotes an up (resp. down) step, while H denotes a double horizontal step.

The Schröder path $\chi(\overline{w})$ obtained from the word $w = a\,a\,b\,c\,c\,c\,b\,a\,a\,c\,c\,b\,c\,b\,a$ of Figure 2.15(a) is $UDHHUHHUDUHDD$ and is depicted in Figure 2.15(c).

Proposition 2.3.6. *The above defined function $\chi : \overline{\mathcal{L}_S}(n) \to \mathcal{SP}_n$ is a bijection. Thus, there exists a bijective correspondence between Schröder parking functions of length $n+1$ and Schröder paths of semi-length n.*

Proof. To prove the main statement it is sufficient to define the function $\psi : \mathcal{SP}_n \to \overline{\mathcal{L}_S}(n)$ and prove that, for all words $\overline{w} \in \overline{\mathcal{L}_S}(n)$, $\psi(\chi(\overline{w})) = \overline{w}$. So, for T a Schröder path of semi-length n, the function ψ is defined differently according to the final step of T by

$$\psi(T) = \begin{cases} \varepsilon & \text{if } T = \emptyset \\ a\, \psi(P)\, c & \text{if } T = P\, H \\ \psi(P)\, b\, \psi(P')c & \text{if } T = P\, U\, P'\, D\,, \end{cases}$$

where P and P' are Schröder paths. Let us now prove that $\psi(\chi(\overline{w})) = \overline{w}$, by induction on the length of $\overline{w}$.

Basis. If $\overline{w} = \varepsilon$, $\psi(\chi(\varepsilon)) = \varepsilon$.

Inductive step. Let $\overline{w} = b\,v'\,c\,v''$ (resp. $\overline{w} = a\,v'\,c\,v''$), where b (resp. a) matches c.

If $v'' = \varepsilon$, then $\psi(\chi(\overline{w})) = \psi(\chi(b\,v'\,c)) = \psi(U\,\chi(v')\,D) = b\,\psi(\chi(v'))\,c = b\,v'\,c$, (resp. $\psi(\chi(\overline{w})) = \psi(\chi(a\,v'\,c)) = \psi(\chi(v')\,H) = a\,\psi(\chi(v'))\,c = a\,v'\,c$).

Else if $v'' = b\,g_1\,c\,\ldots\,b\,g_k\,c$, with $k \geq 1$, then by applying χ recursively it holds

$$\chi(v'') = U\,\chi(g_1)\,D\ldots U\,\chi(g_k)\,D.$$

Therefore, if $\overline{w} = b\,v'\,c\,v''$ (resp. $\overline{w} = a\,v'\,c\,v''$), then

$$\begin{aligned}
\psi(\chi(\overline{w})) &= \psi(U\,\chi(v')\,D\,U\,\chi(g_1)\,D\ldots U\,\chi(g_k)\,D)\\
&= \psi(U\,\chi(v')\,D\,U\,\chi(g_1)\,D\ldots\,U\chi(g_{k-1})\,D)\,b\,\psi(\chi(g_k))\,c = \ldots\\
&= b\,\psi(\chi(v'))\,c\,b\,\psi(\chi(g_1))\,c\ldots b\,\psi(\chi(g_k))\,c = b\,v'\,c\,b\,g_1\,c\,\ldots\,b\,g_k\,c.
\end{aligned}$$

$$\begin{aligned}
\text{(resp. } \psi(\chi(f\overline{w})) &= \psi(\chi(v')\,H\,U\,\chi(g_1)\,D\ldots U\,\chi(g_k)\,D)\\
&= \psi(\chi(v')\,H\,U\,\chi(g_1)\,D\ldots\,U\chi(g_{k-1})\,D)\,b\,\psi(\chi(g_k))\,c = \ldots\\
&= \psi(\chi(v')\,H)\,b\,\psi(\chi(g_1))\,c\ldots b\,\psi(\chi(g_k))\,c = a\,v'\,c\,b\,g_1\,c\,\ldots\,b\,g_k\,c\text{)}. \quad \square
\end{aligned}$$

2.4 Schröder slicings

Our first interest in this section is to define a family of objects enumerated by Schröder numbers, that lies between parallelogram polyominoes (or more precisely, Catalan slicings) and Baxter slicings. Furthermore, this Schröder family of slicings grows according to a succession rule that generalizes Ω_{Cat} while specializing Ω_{Bax}. As Remark 2.2.9 of Section 2.2.2 points out, none among the different succession rules shown for Schröder numbers has this property.

2.4.1 A new Schröder succession rule

Let us consider the following succession rule,

$$\Omega_{NewSch} = \begin{cases} (1,1) \\ \\ (h,k) & \rightsquigarrow (1,k+1),(2,k+1),\ldots,(h,k+1),\\ & (2,1),(2,2),\ldots,(2,k-1),\\ & (h+1,k)\,, \end{cases}$$

whose associated generating tree decorated with labels is denoted by $\mathcal{T}_{Sch}$ and shown in Figure 2.16.

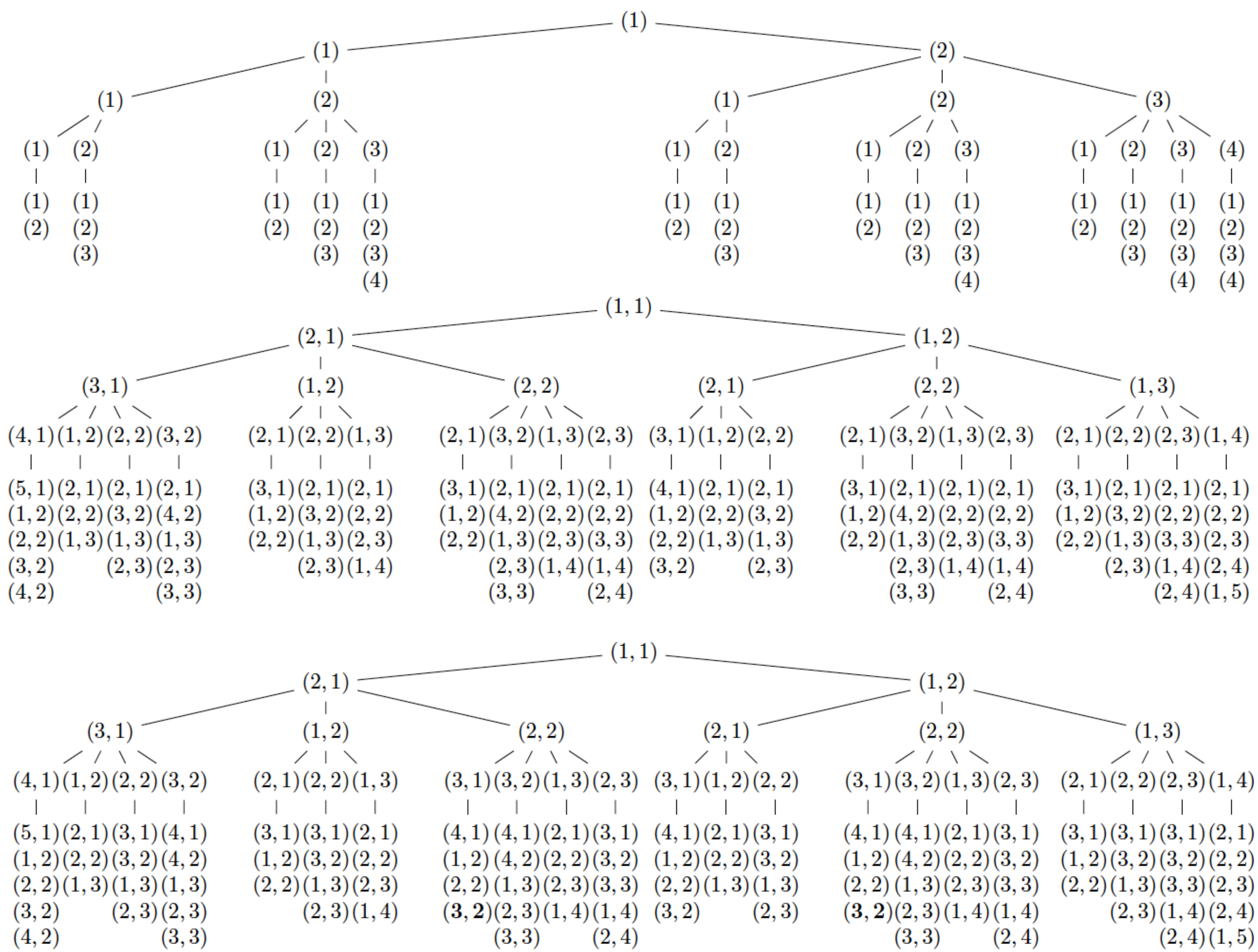


Figure 2.16: The first levels of the generating trees for rules Ω_{Cat}, Ω_{NewSch} and Ω_{Bax}. Bold characters are used to indicate the first vertices of $\mathcal{T}_{Bax}$ that do not appear in $\mathcal{T}_{Sch}$.

Theorem 2.4.1. *The enumeration sequence associated with rule Ω_{NewSch} is that of Schröder numbers.*

Proof. From [144] and Section 2.2.2, we know that the succession rule Ω_{Sch} is associated with Schröder numbers. We claim that rules Ω_{NewSch} and Ω_{Sch} produce the same generating tree. Indeed, replacing each label (h, k) in rule Ω_{NewSch} by the sum $h + k$ of its elements immediately gives rule Ω_{Sch}. □

The rule Ω_{NewSch} can be immediately seen to generalize rule Ω_{Cat}, in the same fashion rule Ω_{Bax} does - see Observation 1.4.10 of the previous chapter. More precisely, in rule Ω_{NewSch}, looking only at the productions $(2, 1), (2, 2), \dots, (2, k-1), (h+1, k)$ and $(1, k+1)$ of a label (h, k), and considering the second component of the labels, we recover rule Ω_{Cat}.

What is further interesting with rule Ω_{NewSch} is that rule Ω_{Bax} for Baxter numbers generalizes it. In other words, it holds that:

Theorem 2.4.2. *$\mathcal{T}_{Sch}$ is (isomorphic to) a subtree of $\mathcal{T}_{Bax}$.*

Our proof of this theorem exhibits one subtree of $\mathcal{T}_{Bax}$ isomorphic to $\mathcal{T}_{Sch}$. We call this subtree "canonical", since it is obtained by mapping the productions in rules Ω_{Bax} and Ω_{NewSch} in the obvious way.

Proof. Note first that the only difference between rules Ω_{Bax} and Ω_{NewSch} is that labels $(h + 1, i)$ for $1 \leq i \leq k - 1$ in the production of rule Ω_{Bax} are replaced by $(2, i)$ in rule Ω_{NewSch}. With this remark, we can prove the following statement by induction on the depth of the vertices in the generating trees: for any h, k, and $h' \geq h$, there exists an injective mapping from the vertices of the generating tree produced from root (h, k) in rule Ω_{NewSch} to the vertices of the generating tree produced from root (h', k) in rule Ω_{Bax}, which preserves the depth, and such that for any vertex labelled (i, j), its image is labelled (i', j) for some $i' \geq i$. Indeed, it is enough to map vertices of the generating trees along the productions of rules Ω_{Bax} and Ω_{NewSch} as follows:

$$\begin{array}{llllllll}
(h,k) \underset{\text{NewSch}}{\rightsquigarrow} & (1,k+1), \dots, & (h,k+1), & & (2,1), & \dots, (2,k-1), & & (h+1,k). \\
& \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow \\
(h',k) \underset{\text{Bax}}{\rightsquigarrow} & (1,k+1), \dots, & (h,k+1), & & (h'+1,1), & \dots, (h'+1,k-1), & & (h'+1,k). \\
& (h+1,k+1), \dots, & (h',k+1), & & & & &
\end{array}$$

The proof is then concluded by applying the statement for $h = h' = k = 1$. □

To our knowledge, this is the first time three succession rules for Catalan, Schröder and Baxter numbers are given, which are each a generalization of the previous one. The first levels of the generating trees for rules Ω_{Cat}, Ω_{NewSch} and Ω_{Bax} are shown in Figure 2.16.

2.4.2 Definition of Schröder slicings, and their growth

We want to define Schröder slicings so that they form a subset of the Baxter slicings, that is enumerated by the Schröder numbers, and whose growth is described by rule Ω_{NewSch}. To do that, recall that a "canonical" subtree of $\mathcal{T}_{Bax}$ isomorphic to $\mathcal{T}_{Sch}$ was built in the proof of Theorem 2.4.2. From there, it is enough to label the vertices of $\mathcal{T}_{Bax}$ by the corresponding Baxter slicings, and to keep only the objects which label a vertex of this "canonical" subtree. With this global approach to the definition of Schröder slicings, the problem is to provide a characterization of these objects that would be local, *i.e.* that could be checked on any given Baxter slicing without reconstructing the whole chain of productions according to rule Ω_{Bax} that resulted in this object.

For the sake of clarity, we have chosen to reverse the order in the presentation of Schröder slicings: we will first give their "local characterization", and then prove that they grow according to rule Ω_{NewSch}. It will be clear in the proof of this statement (see Theorem 2.4.5) that Schröder slicings correspond to the "canonical" subtree of $\mathcal{T}_{Bax}$ on Baxter slicings described earlier.

Definition 2.4.3. Given a Baxter slicing of a parallelogram polyomino S, let u be any of its horizontal blocks. We denote by $\ell(u)$ the width of u. The projection $X(u)$ of u on the lower border of S is the lower-most point of this border whose abscissa is that of the right edge of u. We now define $r(u)$ to be the number of horizontal steps on the lower border of S to the left of $X(u)$ before an up step (or the bottom-left corner of S) is met.

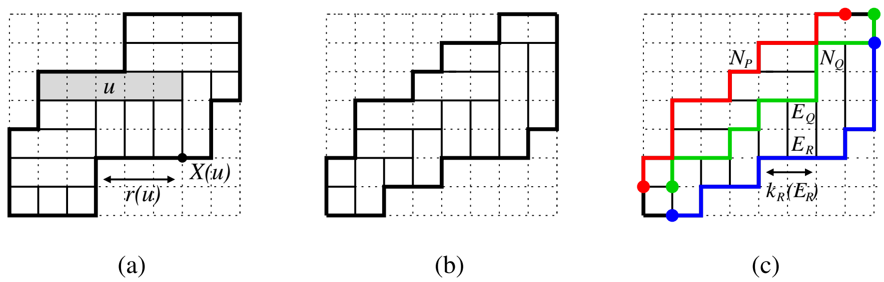


Figure 2.17: (a) Illustration of Definition 2.4.3; (b) an example of a Schröder slicing; (c) illustration of Definition 2.5.1 and Theorem 2.5.3.

Definition 2.4.4. A Schröder slicing is any Baxter slicing such that for any horizontal block u, the following inequality holds:

$$\ell(u) \leq r(u) + 1. \qquad (\ell r_1)$$

Figure 2.17(a),(b) illustrates the definitions of $\ell(u)$ and $r(u)$, and shows an example of Schröder slicing.

Theorem 2.4.5. *Schröder slicings can be generated by* Ω_{NewSch}*.*

Proof. Like Baxter slicings, Schröder slicings grow adding vertical blocks on the right and horizontal blocks on top, but the width of these horizontal blocks is restricted as to satisfy condition (ℓr_1).

To any Schröder slicing S, let us associate the label (h, k) where h (resp. k) denotes the maximal width (resp. height) of a horizontal (resp. vertical) block that may be added to S, without violating condition (ℓr_1). Note that if a horizontal block of width i may be added, then for all $i' \leq i$, the addition of a horizontal block of width i' is also allowed. Consequently, we may add horizontal blocks of width 1 to h to S. Notice also that k denotes the height of the rightmost column of S (since condition (ℓr_1) introduces no restriction on vertical blocks), and that columns of any height from 1 to k may be added to S.

Figure 2.18 illustrates the three cases discussed below in the growth of Schröder slicings according to rule Ω_{NewSch}.

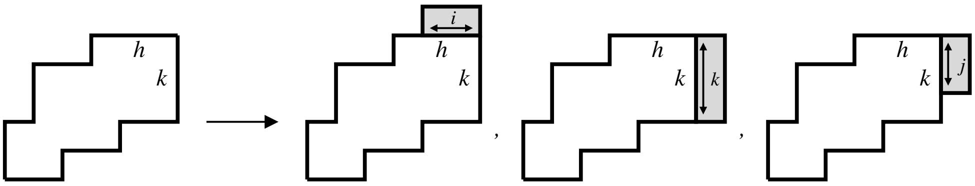


Figure 2.18: The productions of a Schröder slicing of label (h, k) following rule Ω_{NewSch}.

For any $i \leq h$, consider the Schröder slicing P' obtained by adding a horizontal block u of width $\ell(u) = i$. We claim that the label of P' is $(i, k+1)$. Obviously, the height of the last column of P' is $k+1$. Moreover, if we were to add a further horizontal block u' of any width $\ell(u') = i' \leq i$, u' would satisfy condition (ℓr_1), since $X(u) = X(u')$ and $r(u) = r(u')$.

Next, consider the Schröder slicing P' obtained by adding a column of height k to S. We claim that it has label $(h+1, k)$. Of course, the rightmost column of P' has height k. Moreover, the horizontal blocks u' that may be added to P' are of two types: either the block u' is made of one single cell on top of the rightmost column of P', or u' is exactly the same as a horizontal block that could be added to S, except that it is augmented of one cell on the right. In this latter case, condition (ℓr_1) is indeed satisfied since both $\ell(u')$ and $r(u')$ increase by 1, when going from S to P'.

Finally, for any $j < k$, the Schröder slicing P' obtained by adding a column of height j to S has label $(2, j)$. Indeed, the rightmost column of P' has height j, and only horizontal blocks u' of width 1 or 2 may be added to P' without violating condition (ℓr_1), since $r(u') = 1$. □

2.5 Other Schröder restrictions of Baxter objects

For any Baxter class $\mathcal{C}$, whose growth according to rule Ω_{Bax} is understood, it is immediate to define a Schröder subclass of $\mathcal{C}$. Indeed, we can consider the full generating tree of shape $\mathcal{T}_{Bax}$ associated with $\mathcal{C}$, its "canonical" subtree isomorphic to $\mathcal{T}_{Sch}$, and keep only the objects of $\mathcal{C}$ associated with a vertex of $\mathcal{T}_{Sch}$. This method has the advantage of being systematic, but it does not *a priori* provide a characterization of the objects in the Schröder subclass which does not refer to the generating trees.

In this section, we give three examples of Schröder subclasses of Baxter classes, that are not obtained with the above general method, but for which we provide a characterization of the Schröder objects without referring to generating trees.

2.5.1 A Schröder family of NILPs

From Theorem 2.1.3, we have a simple bijection between triples of NILPs and Baxter slicings. In Section 2.4, we have seen a subset of Baxter slicings enumerated by the Schröder numbers. A natural question, which we now solve, is then to give a characterization of the triples of NILPs which correspond to Schröder slicings via the bijection of Theorem 2.1.3.

Definition 2.5.1. Let (P, Q, R) be a triple of NILPs. A pair (N_P, N_Q) of north steps of P and Q is *matched* if there exists i such that N_P (resp. N_Q) is the ith north step of P (resp. Q). Similarly, a pair (E_Q, E_R) of east steps of Q and R is *matched* if there exists i such that E_Q (resp. E_R) is the ith east step of Q (resp. R).

Moreover, for any north step N_P in P (resp. N_Q in Q), we denote by $h_P(N_P)$ (resp. $h_Q(N_Q)$) the number of east steps of P (resp. Q) that occur before N_P (resp. N_Q). And for any east step E_R in R, we denote by $k_R(E_R)$ the largest k such that E^k is a factor of R ending in E_R.

Figure 2.17(c) on page 80 should help understand these definitions.

Definition 2.5.2. A Schröder triple of NILPs is any triple (P, Q, R) of NILPs such that for any north step N_P of the path P, denoting N_Q the north step of Q such that (N_P, N_Q) is matched, E_Q the last east step of Q before N_Q, and E_R the east step of R such that (E_Q, E_R) is matched, the following inequality holds

$$h_P(N_P) - h_Q(N_Q) \leq k_R(E_R). \tag{2.2}$$

Theorem 2.5.3. *Schröder slicings are in one-to-one correspondence with Schröder triples of NILPs by means of the size-preserving bijection described in Theorem 2.1.3.*

Proof. We prove that the image of the class of Schröder slicings under the bijection given in Theorem 2.1.3 coincides with the class of Schröder triples of NILPs of Definition 2.5.2. This will follow since condition (2.2) on triples of NILPs is equivalent to condition (ℓr_1) on Baxter slicings.

Let (P, Q, R) be the image of a Baxter slicing S. By construction (see also Figure 2.17(c)), every horizontal block w of S is associated with a pair (N_P, N_Q) of matched north steps of P and Q, which correspond to the left (for N_P) and right (for N_Q) edges of w. Similarly, every vertical block of S is associated with a pair (E_Q, E_R) of matched east steps of Q and R, corresponding to the upper and lower edges of the block.

Consider a horizontal block w in S, and let (N_P, N_Q) be the associated pair of matched steps. Denote by E_Q the last east step of Q before N_Q, and by E_R the east step of R such that (E_Q, E_R) is matched. This is the situation represented in Figure 2.17(c). We claim that w satisfies condition (ℓr_1) if and only if N_P, N_Q and E_R satisfy condition (2.2). On the one hand, note that the width $\ell(w)$ of w is also expressed as $h_Q(N_Q)+1-h_P(N_P)$. On the other hand, it is not hard to see that $r(w) = k_R(E_R)$. Indeed, the projection $X(w)$ of w on the lower border of S is the ending point of the step E_R in R, so that both $r(w)$ and $k_R(E_R)$ denote the maximal number of east steps seen when reading R (that is to say, the lower border of S) from right to left starting from $X(w)$. It follows that $\ell(w) \leq r(w)+1$ if and only if $h_P(N_P) - h_Q(N_Q) \leq k_R(E_R)$, which concludes the proof. □

Remark 2.5.4. *Note that by restricting the size-preserving bijection described in Theorem 2.1.3 to the family of Catalan slicings yields a bijection between Catalan slicings and pairs of NILPs. This follows from the fact that each triple of NILPs (P, Q, R) image of a Catalan slicing is such that $P = Q$ up to translation. Furthermore, any triple of NILPs (P, Q, R) such that P and Q share the same sequence of north and east steps satisfies Definition 2.5.2.*

2.5.2 Another Schröder subset of Baxter permutations

Although we have not been able to explain the growth of separable permutations according to rule Ω_{NewSch}, by restricting the growth of Baxter permutations according to rule Ω_{Bax}, we are able to describe a new subset of Baxter permutations, enumerated by the Schröder numbers, and whose growth is governed by rule Ω_{NewSch}.

As explained at the beginning of this section, a Schröder subset of Baxter permutations can be obtained by considering the "canonical" embedding of $\mathcal{T}_{Sch}$ in $\mathcal{T}_{Bax}$. Doing so, the two Baxter permutations of length 5 that are not obtained are 13254 and 23154, which correspond to the vertices of $\mathcal{T}_{Bax}$ shown in bold characters in Figure 2.16. Although this subset of Baxter permutations is easy to define from the generating tree perspective, we have not been able to characterize the permutations it contains without referring to the generating trees, which is somewhat unsatisfactory. On the other hand, the subset of Baxter permutations studied below is not as immediate to define from the generating trees themselves, but has a nice characterization in terms of forbidden patterns.

The definition (in a special case) of bivincular patterns is useful to define the subset of Baxter permutations we are considering: a permutation σ avoids the pattern 41323^+ (resp. 42313^+) when no subsequence $\sigma_i\sigma_j\sigma_k\sigma_\ell\sigma_m$ of σ satisfies $\sigma_j < \sigma_\ell < \sigma_k$ (resp. $\sigma_\ell < \sigma_j < \sigma_k$), $\sigma_m = \sigma_k + 1$, and $\sigma_m < \sigma_i$.

Theorem 2.5.5. *Let $\mathcal{A}$ be the subset of Baxter permutations defined by avoidance of the (bi)vincular patterns $2\underline{41}3$, $3\underline{14}2$, 41323^+ and 42313^+. The family $\mathcal{A}$ can be generated by rule Ω_{NewSch}, and consequently $\mathcal{A}$ is enumerated by the Schröder numbers.*

Note that the two Baxter permutations of length 5 that are not in $\mathcal{A}$ are 51324 and 52314.

Proof. First, note that if $\sigma \in \mathcal{A}$, then the permutation obtained by removing the maximal element of σ also belongs to $\mathcal{A}$. So we can make permutations of $\mathcal{A}$ grow by insertion of the maximum point.

Second, observe that $\mathcal{A}$ is a subset of Baxter permutations. So the active sites (*i.e.* positions where the new maximum can be inserted while remaining in the class) is a subset of the active sites in the growth of Baxter permutations according to rule Ω_{Bax}, that are the sites immediately to the right of RTL maxima, and the sites immediately to the left of LTR maxima. In particular, the two sites surrounding the current maximum are always active.

We claim that the active sites of $\sigma \in \mathcal{A}$ are the following, where n denotes the length of σ:

- the sites immediately to the right of RTL maxima, and
- for any LTR maximum σ_i, the site immediately to the left of σ_i, provided that the sequence $\sigma_{i+1} \ldots \sigma_n$ contains no pattern 212^+ where 2 is mapped to a value larger than σ_i.

More formally, the condition above on $\sigma_{i+1} \ldots \sigma_n$ is expressed as follows: there is no subsequence $\sigma_a \sigma_b \sigma_c$ of $\sigma_{i+1} \ldots \sigma_n$ such that $\sigma_a > \sigma_i$, $\sigma_b < \sigma_a$ and $\sigma_c = \sigma_a + 1$.

For the first item, it is enough to notice that the insertion of $n+1$ to the right of n cannot create a 41323^+ or 42313^+ pattern (if it would, then n instead of $n+1$ would give a forbidden pattern in σ).

For the second item, consider a LTR maximum σ_i. The insertion of $n+1$ immediately to the left of σ_i creates a 41323^+ or 42313^+ pattern if and only if it creates such a pattern where $n+1$ is used as the 4.

Assume first that the sequence $\sigma_{i+1} \ldots \sigma_n$ contains a pattern 212^+ where 2 is mapped to a value larger than σ_i. Then together with $n+1$ and σ_i, we get a 41323^+ or 42313^+ pattern: such insertions do not produce a permutation in $\mathcal{A}$.

On the other hand, assume that the sequence $\sigma_{i+1} \ldots \sigma_n$ contains no pattern 212^+ where 2 is mapped to a value larger than σ_i. If the insertion of $n+1$ immediately to the left of σ_i creates a 41323^+ or 42313^+ pattern, say $(n+1)\sigma_a\sigma_b\sigma_c\sigma_d$, then $\sigma_b\sigma_c\sigma_d$ is a 212^+ pattern in $\sigma_{i+1} \ldots \sigma_n$, and by assumption $\sigma_b < \sigma_i$. This implies that σ_i is larger than all of σ_a, σ_b, σ_c and σ_d, so that $\sigma_i\sigma_a\sigma_b\sigma_c\sigma_d$ is a 41323^+ or 42313^+ pattern in σ, contradicting the fact that $\sigma \in \mathcal{A}$. In conclusion, under the hypothesis that the sequence $\sigma_{i+1} \ldots \sigma_n$ contains no pattern 212^+ where 2 is mapped to a value larger than σ_i, then the insertion of $n+1$ immediately to the left of σ_i produces a permutation in $\mathcal{A}$.

To any permutation σ of $\mathcal{A}$, associate the label (h, k) where h (resp. k) denotes the number of active sites to the left (resp. right) of its maximum. Of course, the permutation 1 has label $(1, 1)$. We shall now see that the permutations produced inserting a new maximum in σ have the labels indicated by rule Ω_{NewSch}, concluding our proof of Theorem 2.5.5.

Denote by n the length of σ. When inserting $n+1$ in the ith active site (from the left) on the left of n, this increases by 1 the number of RTL maxima. Moreover, no pattern 212^+ is created, so that all sites to the left of n that were active remain so, provided they remain

LTR maxima. The permutations so produced therefore have labels $(i, k+1)$ for $1 \leq i \leq h$. Similarly, when inserting $n+1$ immediately to the right of n, no 212^+ is created, and the subsequent permutation has label $(h+1, k)$. On the contrary, when inserting $n+1$ to the right of a RTL maximum $\sigma_j \neq n$, a pattern 212^+ is created (as $n\sigma_j(n+1)$). Consequently, there are only two LTR maxima such that there is no pattern 212^+ after them with a 2 of a larger value: namely, those are n and $n+1$. If σ_j was the ith RTL maximum of σ, starting their numbering from the right, then the resulting permutation has label $(2, i)$. □

Remark 2.5.6. *Note that any permutation class* $AV(\tau)$*, for* τ *being in* $\{132, 213, 231, 312\}$*, is a subfamily of the family of permutations* $\mathcal{A}$*.*

2.5.3 A Schröder family of mosaic floorplans

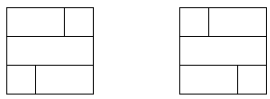
In this section, we explain the growth of mosaic floorplans according to rule Ω_{Bax}, *i.e.* along the generating tree $\mathcal{T}_{Bax}$. Then, we define a subfamily of mosaic floorplans enumerated by Schröder numbers, which we call *Schröder floorplans*, and prove that they grow according to Ω_{NewSch}.

In order not to deal with equivalent mosaic floorplans, to our purposes we prefer to use the notion of packed floorplan introduced in Section 1.4.2 and here formally recalled.

Definition 2.5.7. A packed floorplan (PFP, for short) of dimension (d, ℓ) is a partition of a rectangle of width ℓ and height d into $d + \ell - 1$ rectangular blocks whose sides have integer lengths such that the pattern $\lrcorner\ulcorner$ is avoided, *i.e.* for every pair of blocks (b_1, b_2), denoting (x_1, y_1) the coordinates of the bottom rightmost corner of b_1 and (x_2, y_2) those of the top leftmost corner of b_2, it is not possible to have both $x_1 \leq x_2$ and $y_1 \geq y_2$.

The size of a PFP of dimension (d, ℓ) is $n = d + \ell - 1$.

Figure 2.19(a) shows an example of packed floorplan, while Figure 2.19(b) shows another (non-packed) representative of the same mosaic floorplan.



(a) (b)

Figure 2.19: (a) An example of packed floorplan of dimension $(3, 3)$, (b) a non-packed representative of the same mosaic floorplan.

It results from [4] that the cardinality of the set of PFPs of size n is the Baxter number B_n. Moreover, we can prove the following result.

Theorem 2.5.8. *The family of PFPs can be generated by rule* Ω_{Bax}*.*

Observe that a generating tree for PFPs is presented in [4] (via a procedure called *InsertTile* for adding a new block in PFPs). Considering only the first few levels of this generating tree, it appears immediately that it is not isomorphic to $\mathcal{T}_{Bax}$. Therefore, to prove Theorem 2.5.8, we need to define a new way of adding a block to a PFP.

Proof. Consider a PFP F of dimension (d, ℓ) and size n. Let h (resp. k) be one greater than the number of internal segments of F (*i.e.* segments that are not part of the bounding rectangle of F) that meet the right (resp. upper) border of the bounding rectangle of F. We define a growth for PFPs that builds $h + k$ children of size $n + 1$ for any PFP F, as follows.

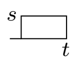
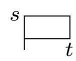
The first h, of dimension $(d, \ell + 1)$, are obtained by adding a new block b on the right of the north-east corner of F: the left side of b then forms a new internal segment that can reach the bottom border of the floorplan or stop when meeting any segment s incident with the right border of F (note that there are $h - 1$ such segments). The segments reaching the right border of F which are below s (and the corresponding blocks) are then extented to reach the right border of the wider rectangle $(d, \ell + 1)$.

The other k, of dimension $(d+1, \ell)$, are obtained by adding a new block b on top of the north-east corner of F: similarly, the bottom side of b then forms a new internal segment that can reach the left border of the floorplan or stop when meeting any segment s incident with the upper border of F (note that there are $k-1$ such segments). Again, the segments reaching the upper border of F which are to the left of s (and the corresponding blocks) are extented to reach the upper border of the higher rectangle $(d + 1, \ell)$.

With h and k defined as above, and giving label (h, k) to PFPs, it is clear that the children of a PFP with label (h, k) have labels $(i, k + 1)$ for $1 \leq i \leq h$ (insertion of a new block on the right of F) and $(h+1, j)$ for $1 \leq j \leq k$ (insertion of a new block on top of F). Moreover, the unique packed floorplan of size 1 (having dimension $(1, 1)$) has no internal segment, so its label is $(1, 1)$.

To prove that PFPs grow according to rule Ω_{Bax}, it is then enough to show that the above construction generates exactly once each PFP.

First, we prove by induction that this construction generates only PFPs. The relation between the number of blocks and the dimensions of the bounding rectangle is clearly satisfied. So we only need to check that, if F is a PFP, then all of its children avoid the pattern ⌟⌜. Consider a child F' of F obtained by adding a new block b on the right (resp. on top) of the north-east corner of F. The bottom right (resp. top left) corners of the existing blocks may only be modified by being moved to the right (resp. in the upper direction). So those cannot create any pattern ⌟⌜. And the new block b cannot create any such pattern either, since it has no block strictly above it nor strictly to its right.

Next, we prove by induction that all PFPs are generated. Consider a PFP F of size $n \geq 2$. Let b be the block in the north-east corner of F and s (resp. t) be the left (resp. bottom) side of b. Their graphical configurations can be either  or  .

In the first (resp. second) case, define F' by deleting the part of F on the right of the line on which s lies (resp. the part of F above the line on which t lies). Since by Definition 2.5.7 F does not contain ⌟⌜, it follows that in both cases the only block removed is b. So F' is indeed a PFP of size one less than F, and F is by construction one of the children of F'.

Finally, it remains to prove that no PFP is generated several times. Obviously, the children of a given PFP are all different. So we only need to make sure that the parent of a PFP F is uniquely determined. Looking again at the block b in the north-east corner of

F, and at the type of the T-junction at the bottom-left corner of b, we determine whether b was added on top or on the right of the north-east corner of its parent. By construction, the parent is then uniquely determined: it is necessarily obtained from F by deleting the parts of F described above. □

Figure 2.20 shows the growth of a packed floorplan of dimension $(3,3)$ having label $(3,2)$.

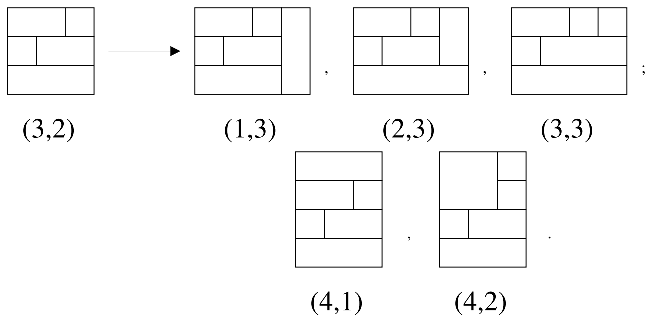


Figure 2.20: The growth of packed floorplans following rule Ω_{Bax}.

Definition 2.5.9. A Schröder PFP is a PFP as in Definition 2.5.7, whose internal segments avoid the following configuration:

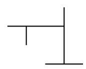
 .

Figure 2.21 shows some packed floorplans that contain the forbidden configuration of Definition 2.5.9 and so, they are not Schröder PFPs.

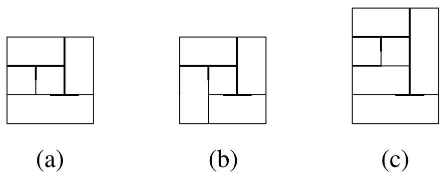
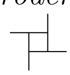
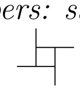
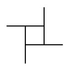
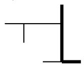


Figure 2.21: (a)-(b) The two packed floorplan of size 5 which are not Schröder PFPs; (c) a non-Schröder packed floorplan of size 6.

Remark 2.5.10. *In Section 2.2.1, we described the subfamily of mosaic floorplans enumerated by Schröder numbers: slicing floorplans. They are defined by the avoidance of the configurations*  *and*  *. Our Schröder floorplans are also defined by a forbidden configuration of segments – see Definition 2.5.9. However, slicing floorplans do not coincide with our Schröder floorplans. Nevertheless, both slicing floorplans and Schröder floorplans avoid the configuration*  *, and the similarity of the forbidden configurations is striking. We leave open the problem of explaining this similarity combinatorially, for* *instance by describing an explicit bijection between slicing floorplans and Schröder floorplans. Note that we were not able to describe a growth of slicing floorplans that follows rule* Ω_{NewSch}.

Theorem 2.5.11. *The generating tree obtained by letting Schröder PFPs grow by insertion of a new block as in the proof of Theorem 2.5.8 is* $\mathcal{T}_{Sch}$. *More precisely, they grow following rule* Ω_{NewSch}.

Proof. Let F be a PFP, and b be the block in the north-east corner of F. Recall that the parent F' of F was described in the proof of Theorem 2.5.8. It follows immediately that if F is a Schröder PFP, then F' is also a Schröder PFP. Consequently, we can make Schröder PFPs grow by addition of a new block either on the right of the north-east corner or above it, as in the proof of Theorem 2.5.8.

Let F be a Schröder PFP. We consider all its children following the growth of PFPs described in the proof of Theorem 2.5.8, and we determine which of them are Schröder PFPs. Let b be a new block added to F. Note first that the addition of b may only create forbidden configurations involving the sides of b. Moreover, if such a forbidden configuration is created, the sides of b are necessarily the segments shown in bold line on the following picture:  . In particular, the T-junction at the bottom left corner of b is of type $\perp$.

If b is added above the north-east corner of F, then by construction the bottom side of b reaches the left border of F or forms a T-junction of type $\vdash$ with a segment meeting the upper border of F. So the forbidden configurations cannot be created, and all PFPs obtained by adding blocks above the north-east corner of F are Schröder PFPs.

On the contrary, if b is added on the right of the north-east corner of F, then the T-junction at the bottom left corner of b is of type $\perp$, so a forbidden configuration may be created. More precisely, the forbidden configuration is generated if and only if the following situation occurs: the segment corresponding to the left side of b reaches an internal segment meeting the right border of F, which in turn is below another internal segment that is incident with the right border of F and that forms a T-junction of type $\top$ with some internal segment. So, to determine which children of F are Schröder PFPs, among those obtained by adding b on the right of the north-east corner of F, it is essential to identify the topmost internal segment, denoted p_F, which meets the right border of F and which forms a T-junction of type $\top$ with some internal segment of F. Then, adding b to F, a Schröder PFP is obtained exactly when the bottom side of b is either the bottom border of F or an internal segment meeting the right border of F which is above p_F (p_F included).

With the above considerations, it is not hard to prove that Schröder PFPs grow according to rule Ω_{NewSch}. To any Schröder PFP F, we assign the label (h, k) where h is one greater than the number of internal segments meeting the right border of F above p_F (included) and k is one greater than the number of internal segments meeting the upper border of F. Of course, the only (Schröder) PFP of size 1 has label $(1, 1)$. Following the growth previously described, a Schröder PFP F of label (h, k) produces:

- h Schröder PFPs obtained by adding a block b on the right of the north-east corner of F. The left side of b may reach the bottom border of F, and then a Schröder PFP of label $(1, k+1)$ is obtained. It may also reach any internal segment s incident with the right border of F that is above p_F (included), and Schröder PFPs of labels $(2, k+1), \dots, (h, k+1)$ are obtained in this way.

- k Schröder PFPs obtained by adding a block b above the north-east corner of F. The bottom side of b may reach the rightmost segment incident with the upper border of F, and then a Schröder PFP of label $(h+1, k)$ is obtained. But if it reaches any other segment incident with the upper border of F (left border of F included), then a T-junction of type $\top$ is formed with at least one internal segment meeting the upper border of F. By definition, for the Schröder PFP F' produced, we therefore have that $p_{F'}$ is the segment that supports the bottom edge of b. Consequently, the labels of the Schröder PFPs produced are $(2, k-1), \dots, (2, 1)$.

This concludes the proof that Schröder PFPs grow with rule Ω_{NewSch}, and so along the generating tree $\mathcal{T}_{Sch}$. □

To illustrate the growth of Schröder PFPs with rule Ω_{NewSch}, note that, seen as a Schröder PFP, the object whose growth is depicted in Figure 2.20 has label $(2, 2)$ and it has only four children (the middle object of the first line is not produced, and indeed it is not a Schröder PFP).

Figure 2.22 shows an example of the growth of a Schröder PFP F of dimension $(4, 2)$ having label $(3, 1)$. The segment p_F (the topmost internal segment of F which meets the right border and forms a T-junction of type $\top$ with an internal segment of F) is highlighted in bold line.

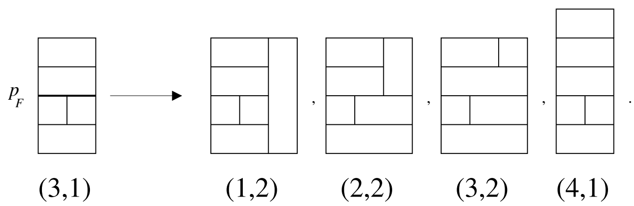


Figure 2.22: The growth of Schröder PFPs following rule Ω_{NewSch}.

Remark 2.5.12. *In the same fashion, we can define a subfamily of PFP enumerated by the Catalan numbers, and prove that they grow according to rule Ω_{Cat}. A Catalan PFP would be a PFP as in Definition 2.5.7, whose internal segments avoid the configuration $\perp$. The proof that they grow according to rule Ω_{Cat} is omitted, but the one of Theorem 2.5.11 is alike.*

2.6 Generalisation of Schröder and Catalan slicings

With the Schröder slicings, we have seen one way of specializing the succession rule Ω_{Bax}. In this section, we are interested in other specializations of rule Ω_{Bax}, which allow to define m-skinny slicings and m-row-restricted slicings, for any integer $m \geq 0$. Section 2.6.5 will explore the properties of their generating functions.

2.6.1 Skinny slicings

We have seen in Definition 2.4.4 that Schröder slicings are defined by condition (ℓr_1), that is to say, $\ell(u) \leq r(u) + 1$, for any horizontal block u. Figure 2.17(a) on page 80 shows which quantities are to be checked for satisfying the above condition.

A rough idea to characterize a Schröder slicings of a parallelogram polyomino S is: every corner of the lower path defining S must have above only horizontal blocks that do not exceed more than *one cell* leftwards its x-coordinate. Therefore, this condition (ℓr_1) can be naturally generalized for number $m \geq 0$ of cells: for any horizontal block u,

$$\ell(u) \leq r(u) + m. \qquad (\ell r_m)$$

Definition 2.6.1. An *m-skinny slicing* is a Baxter slicing such that for any horizontal block u, the inequality (ℓr_m) holds.

Note that an m'-skinny slicing, with $m' \leq m$, is an m-skinny slicing as well; Figure 2.17(a) depicts an m-skinny slicing, for any $m \geq 3$.

Theorem 2.6.2. *The family of m-skinny slicings can be generated by the following succession rule*

$$\Omega_{m\text{-}Sk} = \begin{cases} (1,1) & \\ (h,k) & \rightsquigarrow (1,k+1),(2,k+1),\ldots,(h,k+1), \\ & \quad (h+1,1),\ldots,(h+1,k-1), \qquad \text{if } h < m, \\ & \quad (m+1,1),\ldots,(m+1,k-1), \qquad \text{if } h \geq m, \\ & \quad (h+1,k). \end{cases}$$

Proof. The proof follows the exact same steps as the proof of Theorem 2.4.5, which corresponds to $m = 1$. The only difference is that the maximal width of the horizontal block that may be added in the third case is $\min(h+1, m+1)$ instead of 2. □

Considering the case $m = 0$, we obtain a family of Baxter slicings which is intermediate between Catalan slicings (for which $\ell(u) = 1$, for all horizontal blocks u) and Schröder slicings (*i.e.* 1-skinny slicings). The first few terms of the enumeration sequence of 0-skinny slicings are

$$1, 2, 6, 21, 80, 322, 1347, 5798, 25512, 114236, 518848, 2384538, 11068567, \ldots.$$

These terms appear to match the sequence A106228 [132]. This sequence, and a curious enumerative result relating to it, are further explored in Section 2.6.5.

2.6.2 Row-restricted slicings

Conditions (ℓr_m) naturally generalize the condition that defines Schröder slicings, but it is not the most natural restriction on horizontal blocks of Baxter slicings one may think of. Indeed, for some parameter $m \geq 1$, we could simply impose that horizontal blocks have width no larger than m. In what follows, we study these objects under the name of *m-row-restricted slicings*.

Note that, taking $m = 1$, we recover Catalan slicings, and that the case $m = 0$ is degenerate, since there is only one 0-row-restricted slicing of any given size: the horizontal bar of height 1 and width n divided in (vertical) blocks made of one cell only.

Theorem 2.6.3. *The family of m-row-restricted slicings can be generated by the succession rule*

$$\Omega_{m\text{-}RR} = \begin{cases} (1,1) \\ \\ (h,k) & \rightsquigarrow (1,k+1),(2,k+1),\dots,(h,k+1), \\ & \qquad (h+1,1),(h+1,2),\dots,(h+1,k), \quad \text{if } h < m \\ & \qquad (m,1),(m,2),\dots,(m,k). \qquad\qquad\quad \text{if } h = m \end{cases}$$

Proof. Again, the proof is similar to those of Theorem 2.1.2 and 2.6.2, and when a slicing has label (h,k), h (resp. k) indicates the maximal width of a horizontal block that may be added (resp. the maximal height of a vertical block that may be added). In the case of m-row-restricted slicings, when a vertical block is added to the right, the maximal width of a horizontal block that may be added afterward increases by 1, except if it was m already, in which case it stays at m. □

2.6.3 Functional equations for skinny and row-restricted slicings

In this subsection we will set out the functional equations satisfied by the generating functions of m-skinny slicings and m-row-restricted slicings. The solutions of these functional equations will then be discussed in the following two subsections.

We begin by treating separately the set of 0-skinny slicings. From Theorem 2.6.2, 0-skinny slicings grow according to rule $\Omega_{\text{0-Sk}}$,

$$\Omega_{\text{0-Sk}} = \begin{cases} (1,1) \\ \\ (h,k) & \rightsquigarrow (1,k+1),(2,k+1)\ \dots,(h,k+1), \\ & \qquad (1,1),(1,2),\dots,(1,k-1), \\ & \qquad (h+1,k). \end{cases}$$

Now let $F_{0\text{-}Sk}(u,v)$ be the generating function of 0-skinny slicings,

$$F_{0\text{-}Sk}(x;u,v) \equiv F_{0\text{-}Sk}(u,v) = \sum_{\alpha} x^{n(\alpha)} u^{h(\alpha)} v^{k(\alpha)}$$

where α ranges over all 0-skinny slicings, and the variable x takes into account the size $n(\cdot)$ of the slicing, while u and v mark the labels h and k of the object. The rule $\Omega_{\text{0-Sk}}$ can be translated into the following functional equation

$$\begin{aligned} F_{0\text{-}Sk}(u,v) = xuv &+ \frac{xuv}{1-u}\left[F_{0\text{-}Sk}(1,v) - F_{0\text{-}Sk}(u,v)\right] \\ &+ \frac{xu}{1-v}\left[vF_{0\text{-}Sk}(1,1) - F_{0\text{-}Sk}(1,v)\right] + xuF_{0\text{-}Sk}(u,v). \end{aligned} \tag{2.3}$$

Next, recall that 1-skinny slicings are exactly Schröder slicings, whose generating function is given by $F_{\text{Sch}}(x)$ in Equation (2.1).

Therefore, fix some $m \geq 2$. For any $i < m$ (resp. $i = m$), let $F_i(x;u,v)$ be the trivariate generating function of m-skinny slicings whose label according to rule $\Omega_{m\text{-Sk}}$ is of the form $(i,\cdot)$ (resp. $(j,\cdot)$ for any $j \geq m$),

$$F_i(x;u,v) \equiv F_i(u,v) = \sum_{\alpha} x^{n(\alpha)} u^{h(\alpha)} v^{k(\alpha)},$$

where α ranges over all m-skinny slicings such that $h(\alpha) = i$ (resp. $h(\alpha) = j$, with $j \geq m$).

Then, for any $m \geq 2$, the trivariate generating function of m-skinny slicings is given by

$$F_{m\text{-}Sk}(x;u,v) \equiv F_{m\text{-}Sk}(u,v) = \sum_{i} F_i(u,v),$$

and the rule $\Omega_{m\text{-Sk}}$ translates into the following system of functional equations,

$$\begin{cases} F_1(u,v) = xuv + xuv\left[F_1(1,v) + F_2(1,v) + \ldots + F_m(1,v)\right] & (2.4) \\ \qquad\vdots & \\ F_i(u,v) = \dfrac{xu^i v}{1-v}\left[F_{i-1}(1,1) - F_{i-1}(1,v)\right] + xu^i v\left[F_i(1,v) + \ldots + F_m(1,v)\right] & (2.5) \\ \qquad\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 1 < i < m, & \\ F_m(u,v) = \dfrac{xu^m v}{1-v}\left[F_{m-1}(1,1) - F_{m-1}(1,v)\right] + \dfrac{xu^{m+1}}{1-v}\left[vF_m(1,1) - F_m(1,v)\right] & (2.6) \\ \qquad\qquad + xuF_m(u,v) + \dfrac{xuv}{1-u}\left[u^{m-1}F_m(1,v) - F_m(u,v)\right]. & \end{cases}$$

Note that by definition $F_i(u,v) = u^i F_i(1,v)$ for all $i < m$, but this does not hold for $i = m$.

Lastly, we consider m-row-restricted slicings. As previously mentioned, $m = 0$ leads to a trivial combinatorial class, while $m = 1$ yields the family of Catalan slicings, whose size generating function is reported in Section 1.3.6 on page 33 in Equation (1.7).

We thus fix some $m \geq 2$. The succession rule $\Omega_{m\text{-RR}}$ yields a system of functional equations satisfied by the generating function of m-row-restricted slicings. More precisely, for any $i \leq m$, let $G_i(x; u, v)$ be the trivariate generating function of m-row-restricted slicings whose label according to rule $\Omega_{m\text{-RR}}$ is of the form $(i, \cdot)$,

$$G_i(x; u, v) \equiv G_i(u, v) = \sum_{\alpha} x^{n(\alpha)} u^{h(\alpha)} v^{k(\alpha)},$$

where α ranges over all m-skinny slicings such that $h(\alpha) = i$. Also in this case, for any $m \geq 2$, the trivariate generating function of m-row-restricted slicings is given by

$$G_{m\text{-}RR}(x; u, v) \equiv G_{m\text{-}RR}(u, v) = \sum_{i} G_i(u, v).$$

Note that $G_i(u, v) = u^i G_i(1, v)$ for all $i \leq m$, which makes the variable u unnecessary. Rule $\Omega_{m\text{-RR}}$ translates into the following system,

$$\begin{cases} G_1(u,v) &= xuv + xuv\,[G_1(1,v) + G_2(1,v) + \ldots + G_m(1,v)] \\ &\vdots \\ G_i(u,v) &= \dfrac{xu^i v}{1-v}\,[G_{i-1}(1,1) - G_{i-1}(1,v)] + xu^i v\,[G_i(1,v) + \ldots + G_m(1,v)] \\ &\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 1 < i < m, \\ G_m(u,v) &= \dfrac{xu^m v}{1-v}\,[G_m(1,1) - G_m(1,v) + G_{m-1}(1,1) - G_{m-1}(1,v)] + xu^m v G_m(1,v), \end{cases}$$

or equivalently, written without u in $H_i(v) \equiv G_i(1, v)$,

$$\begin{cases} H_1(v) = xv + xv\,[H_1(v) + H_2(v) + \ldots + H_m(v)] & (2.7) \\ \qquad\vdots \\ H_i(v) = \dfrac{xv}{1-v}\,[H_{i-1}(1) - H_{i-1}(v)] + xv\,[H_i(v) + \ldots + H_m(v)] \quad \text{for } 1 < i < m, & (2.8) \\ \qquad\vdots \\ H_m(v) = \dfrac{xv}{1-v}\,[H_m(1) - H_m(v) + H_{m-1}(1) - H_{m-1}(v)] + xvH_m(v). & (2.9) \end{cases}$$

2.6.4 The special case of 0-skinny and 2-row-restricted slicings

In this subsection we prove the following surprising result, for which we presently have no bijective explanation.

Theorem 2.6.4. *The number of 2-row-restricted slicings is equal to the number of 0-skinny slicings, for any fixed size.*

We first solve the case of 2-row-restricted slicings, and obtain the following.

Theorem 2.6.5. *The generating function $H(x)$ of 2-row-restricted slicings satisfies the functional equation*

$$H(x) = \frac{x(H(x)+1)}{1-x(H(x)+1)^2}. \tag{2.10}$$

Proof. The succession rule associated with the growth of 2-row-restricted slicings is

$$\Omega_{\text{2-RR}} = \begin{cases} (1,1) \\ (h,k) \rightsquigarrow (1,k+1),\ldots,(h,k+1), \\ \qquad\quad (2,1),(2,2),\ldots,(2,k) \end{cases}$$

and the corresponding system of functional equations is

$$\begin{cases} H_1(v) &=& xv + xv(H_1(v)+H_2(v)) \\ H_2(v) &=& \dfrac{xv}{1-v}(H_2(1)-H_2(v)+H_1(1)-H_1(v)) + xvH_2(v). \end{cases} \tag{2.11}$$

The quantity we wish to solve is the generating function of 2-row-restricted slicings, given by $H(x) \equiv G_{2\text{-}RR}(x;1,1) = H_1(1)+H_2(1)$. Cancelling $H_1(v)$ between (2.11), we arrive at

$$K(v)H_2(v) = \frac{xv}{1-v}\left(\frac{-xv}{1-xv} + H_1(1)+H_2(1)\right)$$

where

$$K(v) = 1 - xv + \frac{xv}{1-v} + \frac{x^2v^2}{(1-v)(1-xv)}.$$

This equation is susceptible to the kernel method (see Section 1.3.6 on page 33). The equation $K(v)=0$ is cubic in v, and one of the three roots has a power series expansion in x (the other two are not analytic at $x=0$). Letting $\lambda(x) \equiv \lambda$ denote this root, we then have

$$H(x) = H_1(1)+H_2(1) = \frac{x\lambda}{1-x\lambda}.$$

It follows that

$$\lambda = \frac{H}{x(H+1)},$$

and the condition $K(\lambda)=0$ rewrites as

$$xH^3 + 2xH^2 + (2x-1)H + x = 0\ , \tag{2.12}$$

or equivalently, Equation (2.10). □

Remark 2.6.6. *It follows that the sequence for 2-row-restricted slicings is (up to the first term) the same as sequence A106228 in [132]. Indeed, the generating function S of sequence A106228 is characterized by $xS^3 - xS^2 + (x-1)S + 1 = 0$ [3], and with* (2.12) *it is immediate to check that $H+1$ satisfies this equation.*

Proof of Theorem 2.6.4. The generating function $F_{0\text{-}Sk}(u,v)$ of 0-skinny slicings satisfies Equation (2.3), and this equation can also be solved via the kernel method. However, things are somewhat more complicated here, due to the presence of two catalytic variables. First, we rearrange the equation into the kernel form

$$L(u,v)F_{0\text{-}Sk}(u,v) = xuv + xu\left(\frac{v}{1-u} - \frac{1}{1-v}\right)F_{0\text{-}Sk}(1,v) + \frac{xuv}{1-v}F_{0\text{-}Sk}(1,1)$$

where

$$L(u,v) = 1 - xu + \frac{xuv}{1-u}.$$

The equation $L(u,v) = 0$ is quadratic in u, and one of the two roots is a power series in x with coefficients in $\mathbb{Z}[v]$ (the other is not analytic at $x = 0$). We denote this root by

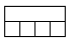
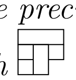
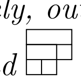
$$\mu(x,v) \equiv \mu(v) = \frac{1 + x - xv - \sqrt{1 - 2x - 2xv + x^2 - 2x^2v + x^2v^2}}{2x}.$$

It follows that

$$M(v)F_{0\text{-}Sk}(1,v) = v + \frac{v}{1-v}F_{0\text{-}Sk}(1,1) \quad \text{where} \quad M(v) = \frac{1}{1-v} - \frac{v}{1-\mu(v)}.$$

Now the kernel method can be applied *again* – the equation $M(v) = 0$ is (after rearrangement) quartic in v, namely, it is $4xv(1 - v + xv - xv^2 + xv^3) = 0$. One of the three non-zero roots of this equation has a power series expansion in x. Denoting by $\kappa(x) \equiv \kappa$ this root, we finally have $F_{0\text{-}Sk}(1,1) = \kappa - 1$. Some elementary manipulations in MAPLE show that $F_{0\text{-}Sk}(1,1)$ also satisfies (2.10). □

We point out that D. Callan indicates in [132] that $F_{0\text{-}Sk} \equiv F_{0\text{-}Sk}(1,1)$ is also the generating function of Schröder paths with no triple descents, *i.e.* having no occurrences of the factor DDD. It would be interesting to provide a bijection between Schröder slicings and Schröder paths whose restriction to 0-skinny slicings yields a bijection with Schröder paths having no triple descents.

Remark 2.6.7. *It does not hold in general that there are as many m-skinny slicings as $(m+2)$-row-restricted slicings: already for $m = 1$, there are* 91 *3-row-restricted slicings but* 90 *Schröder (*i.e. *1-skinny) slicings of size* 5*. More precisely, out of the* 92 *Baxter slicings of size* 5*, only*  *is not* 3*-row-restricted, but both*  *and*  *are not Schröder slicings.*

2.6.5 Generating functions of m-skinny and m-row-restricted slicings for general m

In this final subsection, we outline an approach for solving the generating functions of m-skinny and m-row-restricted slicings, for arbitrary m. While this method is *provably* correct for small m, we do not know how to prove that all of the steps always work, and the following thus remains a conjecture.

m	0	1	2	3	4	5
m-row -restricted	$1/(1-x)$ §2.6.2	$G_{\text{Cat}}(x;1)$ §1.3.6	Eq. (2.10) Thm 2.6.5	Eq. (2.17) p.99	Eq. (2.18) p.99	Eq. (2.19) p.99
m-skinny	Eq. (2.10) Thm 2.6.4	$F_{\text{Sch}}(x)$ Thm 2.5.11	Eq. (2.20) p.100	Eq. (2.21) p.100		

Table 2.2: For small values of m, the statement of Conjecture 2.6.8 holds. Each cell of the table gives the corresponding generating function and/or an equation characterizing it.

Conjecture 2.6.8. *For all finite $m \geq 0$, the generating functions of m-skinny and m-row-restricted slicings are algebraic.*

Table 2.2 summarizes the cases for which we know that the above statement holds, either from previous results in this paper, or from the method described below.

We will mostly focus on m-row-restricted slicings, and briefly explain at the end how to modify the method to solve m-skinny slicings. In the following it is assumed that $m \geq 3$.

Generating functions of m-row-restricted slicings

The method used to treat the case of m-row-restricted slicings can be summarised as follows:
Step 1. Note that the system (2.7)–(2.9) can be rewritten in the form of a matrix equation

$$\mathbf{K}_m(v)\mathbf{H}_m(v) = \mathbf{B}_m(v)\mathbf{H}_m(1) + \mathbf{C}_m(v), \tag{2.13}$$

where

$$\mathbf{H}_m(v) = \begin{pmatrix} H_1(v) \\ \vdots \\ H_m(v) \end{pmatrix}, \quad \mathbf{K}_m(v) = \begin{pmatrix} 1-xv & -xv & -xv & -xv & \cdots & -xv \\ \frac{xv}{1-v} & 1-xv & -xv & -xv & \cdots & -xv \\ 0 & \frac{xv}{1-v} & 1-xv & -xv & \cdots & -xv \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{xv}{1-v} & 1-xv & -xv \\ 0 & 0 & 0 & \cdots & \frac{xv}{1-v} & 1-xv+\frac{xv}{1-v} \end{pmatrix},$$

$$\mathbf{B}_m(v) = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{xv}{1-v} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{xv}{1-v} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{xv}{1-v} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{xv}{1-v} & \frac{xv}{1-v} \end{pmatrix} \quad \text{and} \quad \mathbf{C}_m(v) = \begin{pmatrix} xv \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Step 2. The determinant $|\mathbf{K}_m(v)|$ is a rational function of x and v which can be shown to be not identically zero for any m. It follows that, in general, $\mathbf{K}_m(v)$ has an inverse.

Write $\mathbf{K}_m^*(v) = |\mathbf{K}_m(v)|\mathbf{K}_m^{-1}(v)$ (the transpose of the matrix of cofactors of $\mathbf{K}_m(v)$). It can further be shown that none of the elements of the last row of $\mathbf{K}_m^*(v)$ are identically zero.

Step 3. Multiply (2.13) on the left by $\mathbf{K}_m^*(v)$ to give

$$|\mathbf{K}_m(v)|\mathbf{H}_m(v) = \mathbf{K}_m^*(v)\left[\mathbf{B}_m(v)\mathbf{H}_m(1) + \mathbf{C}_m(v)\right]. \tag{2.14}$$

This can be viewed as a system of m kernel equations, where the kernel (namely $|\mathbf{K}_m(v)|$) is the same for each. The left-hand side of the m-th equation of (2.14) is $|\mathbf{K}_m(v)|H_m(v)$, while the right-hand side is a linear combination of all the m unknowns $H_1(1), \ldots, H_m(1)$. Furthermore, note that in (2.7)–(2.9), the unknowns $H_{m-1}(1)$ and $H_m(1)$ only appear together as $H_{m-1}(1) + H_m(1)$. Writing this latter quantity as $H_{(m-1)+m}(1)$, we now see that there are really only $m-1$ unknowns on the right-hand side of (2.14).

Step 4. The equation $|\mathbf{K}_m(v)| = 0$ can be shown to have $m-2$ roots (in the variable v) which are Puiseux series in x. Denote these roots by $\nu_1(x), \ldots, \nu_{m-2}(x)$.

Step 5. Substitute $v = \nu_i(x)$ into the first of the m equations comprising the system (2.14), for $i = 1, \ldots, m-2$. This yields a system of $m-2$ linear equations in $m-1$ unknowns.

Step 6. To obtain one more equation, set $v = 1$ in (2.7) (again combining $H_{m-1}(1)+H_m(1)$ as $H_{(m-1)+m}(1)$).

Step 7. Solve this entire linear system of $m-1$ equations with $m-1$ unknowns, and add all solutions together to obtain $H(x)$.

It is the validity of Step 7 which we are unable to verify in general, as further discussed in Remark 2.6.12. The validity of Steps 1 to 6 is explained below. Theorem 2.6.9 and 2.6.10 are used to justify Step 2, while Corollary 2.6.11 is needed for Step 4.

Theorem 2.6.9. *For $m \geq 3$, the determinant $|\mathbf{K}_m(v)|$ of $\mathbf{K}_m(v)$ satisfies the recurrence*

$$|\mathbf{K}_m(v)| = (1-xv)|\mathbf{K}_{m-1}(v)| + xv\sum_{j=2}^{m-2}(-1)^j\left(\frac{xv}{1-v}\right)^{j-1}|\mathbf{K}_{m-j}(v)| + (-1)^{m+1}xv(1-xv)\left(\frac{xv}{1-v}\right)^{m-2} \tag{2.15}$$

with

$$|\mathbf{K}_2(v)| = 1 - 2xv + x^2v^2 + \frac{xv}{1-v}.$$

Moreover, for $m \geq 3$, $|\mathbf{K}_m(v)|$ is of the form

$$|\mathbf{K}_m(v)| = \frac{(1-v)^{m-2} + P_m(x,v)}{(1-v)^{m-2}}, \tag{2.16}$$

where $P_m(x,v)$ is a non-trivial polynomial in x and v satisfying $P_m(0,v) = 0$, and with $(1-v)$ not being a factor of $P_m(x,v)$.

Proof. These results about the determinant of $\mathbf{K}_m(v)$ come from its almost-triangular structure.

First, to prove Equation (2.15), expand the determinant of $\mathbf{K}_m(v)$ through its first row. The $(1,1)$ minor is $|\mathbf{K}_{m-1}(v)|$. For $j = 2, \ldots, m-2$, the $(1,j)$ minor can be expanded a further $j-1$ times to eventually yield $|\mathbf{K}_{m-j}(v)|$. The $(1, m-1)$ and $(1,m)$ minors are determinants of upper triangular matrices, and they can be combined to give the final term in (2.15).

Note that, comparing the expressions of $|\mathbf{K}_m(v)|$ and $|\mathbf{K}_{m-1}(v)|$ obtained from (2.15), the same equation can be rewritten as a three-term recurrence, namely

$$|\mathbf{K}_m(v)| = \frac{1}{1-v}\left(\left(1-v-2xv+xv^2\right)|\mathbf{K}_{m-1}(v)| + xv|\mathbf{K}_{m-2}(v)|\right),$$

with initial conditions

$$|\mathbf{K}_1(v)| = \frac{1-v+xv^2}{1-v}, \qquad \text{and} \qquad \mathbf{K}_2(v)| = 1-2xv+x^2v^2+\frac{xv}{1-v}.$$

From that, Equation (2.16) can be proved by induction on m. □

Theorem 2.6.10. *None of the elements of the last row of* $\mathbf{K}^*_m(v)$ *are identically zero.*

Proof. From Theorem 2.6.9, $|\mathbf{K}_m(v)|$ is not identically zero. So, proving this theorem amounts to showing that none of the elements of the last row of $\mathbf{K}^{-1}_m(v)$ are identically zero, which in turn means that none of the cofactors $C_{i,m}$ of $\mathbf{K}_m(v)$ are zero, for $1 \le i \le m$.

Let $\hat{\mathbf{K}}_{m-1}(v)$ be the matrix obtained by deleting the last row and column of $\mathbf{K}_m(v)$. Similarly to the proof of Theorem 2.6.9, we can see that $|\hat{\mathbf{K}}_m(v)|$ is of the form $Q/(1-v)^p$ with Q a polynomial and p some non-negative integer. Moreover $|\hat{\mathbf{K}}_m(0)| = 1$. We can conclude that $|\hat{\mathbf{K}}_m(v)|$ is not identically zero for any m.

It is straightforward to recursively expand the cofactors of $\mathbf{K}_m(v)$; one obtains

$$\begin{aligned} C_{1,m} &= (-1)^{m+1}\left(\frac{xv}{1-v}\right)^{m-1} \\ C_{2,m} &= (-1)^m(1-xv)\left(\frac{xv}{1-v}\right)^{m-2} \\ C_{i,m} &= (-1)^{m+i}\left(\frac{xv}{1-v}\right)^{m-i} |\hat{\mathbf{K}}_{i-1}(v)| \qquad \text{for } 3 \le i \le m. \end{aligned}$$

It follows that none of these terms are identically zero. □

Corollary 2.6.11. *For* $m \ge 3$ *the equation* $|\mathbf{K}_m(v)| = 0$ *has* $m-2$ *solutions in* v *which can be expressed as Puiseux series in* x *with no negative powers; that is, there are* $m-2$ *roots in* v *which are finite at* $x = 0$.

Proof. The fact that $(1-v)$ does not divide $P_m(x, v)$ means that nothing cancels between the numerator and denominator of Equation (2.16), so finding roots of $|\mathbf{K}_m(v)|$ devolves to finding roots of the numerator

$$N_m(x, v) = (1-v)^{m-2} + P_m(x, v).$$

The fact that $P_m(0, v) = 0$ means that $N_m(x, v) \to (1-v)^{m-2}$ as $x \to 0$. If $P_m(x, v)$ has degree p in v, then $N_m(x, v)$ is a polynomial of degree $\max(m-2, p)$. Since the roots of any polynomial are continuous functions of the coefficients of the polynomial, it follows that as $x \to 0$, we must have $m-2$ roots which approach 1. If $p > m-2$ there will be $p-m+2$ remaining roots; these cannot approach any finite complex number, and since they are algebraic functions of x, they must each diverge like $cx^{-\alpha}$ for some complex c and real $\alpha > 0$. □

Remark 2.6.12. *It is important to note that the $m-2$ solutions to $|\mathbf{K}_m(v)| = 0$ of Corollary 2.6.11 and denoted by $\nu_1(x), \ldots \nu_{m-2}(x)$ in Step 4 have not been shown to be* distinct. *In order to prove the correctness of our method for arbitrary m, we would need to show that those $m-2$ roots, viewed as functions (or Puiseux series) of x, are* linearly independent. *Of course linear independence would automatically imply distinctness.*

Moreover, as to prove the validity of Step 7, it would be also necessary to show that the $(m-1)$-th equation obtained in Step 6 is independent of those obtained in Step 5.

Although the above method is not proved for general m, it has been verified manually for $m \leq 5$.

The series expansion of the generating function of 3-row-restricted slicings is

$$x+2x^2+6x^3+22x^4+91x^5+405x^6+1893x^7+9163x^8+45531x^9+230902x^{10}+O(x^{11}). \quad (2.17)$$

With some help from Mathematica, and here specifically from M. Kauers' "Guess" package, one finds that this generating function is a root of the cubic polynomial

$$x + 2x^2 + x^3 + (-1 - 2x + 2x^2 + 3x^3)H + (2 - 2x^2 + 3x^3)H^2 + (-1 + 3x - 2x^2 + x^3)H^3.$$

The generating functions for $m = 4$ and $m = 5$ have the respective series expansions

$$x+2x^2+6x^3+22x^4+92x^5+421x^6+2051x^7+10449x^8+55023x^9+297139x^{10}+O(x^{11}), \quad (2.18)$$

$$x+2x^2+6x^3+22x^4+92x^5+422x^6+2073x^7+10724x^8+57716x^9+320312x^{10}+O(x^{11}). \quad (2.19)$$

By construction these functions must be algebraic, but as the order of the kernel equation $|\mathbf{K}_m(v)| = 0$ increases with m, we have been unable to determine precisely the polynomials satisfied by these generating functions.

Generating functions of m-skinny slicings

We now briefly turn to m-skinny slicings. The method is largely the same, with some minor differences. Firstly, an additional step is required at the start.

Step 0*. Substitute $u = \mu(v)$ into (2.6), where $\mu(v)$ is the power series root of $L(u, v)$ as defined in the proof of Theorem 2.6.4, eliminating the term $F_m(u, v)$, leaving an equation relating $F_{m-1}(1, 1), F_{m-1}(1, v), F_m(1, 1)$ and $F_m(1, v)$. Meanwhile, the variable u is unnecessary in Equations (2.4) and (2.5) for $1 \leq i < m$, so set it to 1.

The remaining steps (Steps 1-7) can then be adapted to this system of equations, with $F_i(1, v)$ taking the place of $H_i(v)$. One key difference is that $F_{m-1}(1, 1)$ and $F_m(1, 1)$ cannot be combined, so there are m unknowns that need to be solved instead of $m - 1$. However, this time the kernel (again the determinant of a matrix) has $m - 1$ Puiseux series roots instead of $m - 2$, which exactly compensates for this problem.

When $m = 2$ the desired solution $F_1(1, 1) + F_2(1, 1)$ has the form

$$x+2x^2+6x^3+22x^4+92x^5+419x^6+2022x^7+10168x^8+52718x^9+279820x^{10}+O(x^{11}). \quad (2.20)$$

This generating function is a root of the quintic polynomial

$$x^3 - x^2(1 - 6x)F - 3x^2(2 - 5x)F^2 + x(2 - 13x + 19x^2)F^3 + x(5 - 12x + 12x^2)F^4 - (1 - 3x + 4x^2 - 3x^3)F^5.$$

When $m = 3$ the desired solution $F_1(1, 1) + F_2(1, 1) + F_3(1, 1)$ has the form

$$x+2x^2+6x^3+22x^4+92x^5+422x^6+2070x^7+10668x^8+57061x^9+314061x^{10}+O(x^{11}). \quad (2.21)$$

By construction it is certainly algebraic, but we make no attempt to write down the polynomial of which it is a root.

Chapter 3

Semi-Baxter permutations

Plan of the chapter

In this chapter we aim to enumerate a family of pattern-avoiding permutations, introduced in [G4, G5] as *semi-Baxter permutations* owing to their close relation with the two Baxter families of Baxter and twisted Baxter permutations. Indeed, in Section 3.1 we define semi-Baxter permutations the permutations avoiding the vincular pattern $2\,\underline{41}\,3$, and we call semi-Baxter numbers the numbers enumerating them. Then, we provide a tool to establish if a combinatorial class is equinumerous to the family of semi-Baxter permutations: it is the semi-Baxter succession rule of Section 3.1.2, which first appeared in [G4].

In Section 3.2, by using this tool other combinatorial structures, apart from the semi-Baxter permutations, are proved to be enumerated by semi-Baxter numbers. In particular, in Section 3.2.1 we tackle the problem of enumerating *plane permutations* (permutations avoiding $2\,\underline{14}\,3$), which was set as an open problem by M. Bousquet-Mélou and S. Butler [35]. The number sequence enumerating plane permutations, registered as sequence A117106 on [132], coincides with the semi-Baxter number sequence. This result was first established in [96], yet in [G4] we provide another alternative proof of this fact. Moreover, there were several conjectures related to sequence A117106 [22, 23, 110] that we will be able to prove along this chapter. One of them is shown in Section 3.2.2, and involves another family of combinatorial objects called *inversion sequences*. Whereas, in Section 3.2.3 we introduce a completely new occurrence of semi-Baxter numbers in terms of lattice paths [G5], more precisely in terms of labelled Dyck paths extending the definition of Baxter paths of Chapter 2.

By means of standard tools, in Section 3.3, we translate the succession rule of Section 3.1.2 into a functional equation whose solution is the generating function of semi-Baxter numbers. Then, the functional equation is solved using an obstinate variant of the kernel method, which ensures that the generating function solution is D-finite. Moreover, in Section 3.3.2 we give an expression for that generating function which will allow us to provide a closed formula for its coefficients by using the Lagrange inversion (Section 3.4.1).

In fact, Section 3.4 collects different expressions for the semi-Baxter numbers. By means of the first explicit formula provided we find a recurrence relation applying the method of *creative telescoping*, which is described in Section 3.4.2. In addition, some other closed formulas that had been conjectured in [23] are proved to be the semi-Baxter numbers in Section 3.4.4. Eventually, careful estimates of binomial coefficients occurring in the closed formulas for the semi-Baxter coefficients give their asymptotic behavior, as discussed in Section 3.5.

3.1 Semi-Baxter numbers

We start defining the sequence of semi-Baxter numbers as the enumerative sequence of a particular family of pattern-avoiding permutations. Although the first terms of this number sequence have been known in literature as sequence A117106 on [132], no closed formulas or recurrence were proved for these numbers. Nonetheless, the succession rule of Section 3.1.2 generating semi-Baxter numbers enables us to solve completely this enumerative problem in the next few sections.

3.1.1 Definition of semi-Baxter permutations, and context

According to the definition of vincular pattern (Definition 1.4.3), we define semi-Baxter permutations as follows.

Definition 3.1.1. A *semi-Baxter permutation* is a permutation that avoids the pattern $2\,\underline{41}\,3$.

Definition 3.1.2. The sequence of *semi-Baxter numbers*, $\{SB_n\}_{n\geq 1}$, is defined by taking SB_n to be the number of semi-Baxter permutations of length n.

The name "semi-Baxter" has been chosen because $2\,\underline{41}\,3$ is one of the two patterns whose avoidance defines the family of the Baxter permutations - namely, the patterns $2\,\underline{41}\,3$ and $3\,\underline{14}\,2$ as defined in Section 1.4.2. It is well worth noticing that up to symmetry, we could have defined semi-Baxter permutations by the avoidance of $3\,\underline{14}\,2$, and still obtained the same number sequence enumerating semi-Baxter permutations. Nevertheless, the pattern $2\,\underline{41}\,3$ is also one of the two patterns whose avoidance defines the family of twisted Baxter permutations - namely, $2\,\underline{41}\,3$ and $3\,\underline{41}\,2$ as described in Section 1.4.2 - which also are enumerated by the Baxter number sequence. Therefore, the family of permutations avoiding $2\,\underline{41}\,3$ contains both families of Baxter and twisted Baxter permutations as subfamilies, motivating the name of semi-Baxter permutations.

The family of semi-Baxter permutations has already appeared in the literature, at least on a few occasions. Indeed, it is an easy exercise to see that the avoidance of $2\,\underline{41}\,3$ is equivalent to that of the barred pattern $25\bar{3}14$, which has been studied with a quite experimental perspective as one case among others by L. Pudwell in [121]. As already specified the definition of barred patterns is not essential to our work, so we address to [121] for a more precise definition. In addition, in that paper L. Pudwell suggests that the enumerative sequence of semi-Baxter permutations and the one of permutations avoiding $2\,\underline{14}\,3$

coincide, which was not expected. Her conjecture first formulated by means of enumeration schemes, has later been proved in [96, Corollary 1.9(b)] as a special case of a more general statement. Then, the sequence enumerating permutations avoiding $2\underline{14}3$, which is registered on the OEIS [132] as sequence A117106, results to be our number sequence $\{SB_n\}_{n\geq 1}$. We will provide later, in Section 3.2.1, an alternative and self-contained proof that semi-Baxter permutations and permutations avoiding $2\underline{14}3$ are indeed equinumerous.

The first terms of the sequence A117106 [132] of semi-Baxter numbers SB_n are

$$1, 2, 6, 23, 104, 530, 2958, 17734, 112657, 750726, 5207910, 37387881, 276467208, \dots$$

D. Bevan in [22, Theorem 13.1] first provided a functional equation whose solution is the generating function of permutations avoiding $2\underline{14}3$. Although this solution is not known, by iterating the functional equation D. Bevan computed up to 37 terms (determined in twelve hours) of the number sequence A117106 [132]. Moreover, no formula (closed or recursive) are proved in [22] for the semi-Baxter numbers SB_n. There is however a conjectured explicit formula related to Apéry numbers, which, in addition, conjectures information about their asymptotic behavior (see Remark 3.5.2 in Section 3.5).

Another recursive formula for SB_n has been conjectured by M. Martinez and C. Savage in [110], in relation with *inversion sequences* avoiding some patterns (definition and precise statement are provided in Subsection 3.2.2). Finally, some other closed formulas for SB_n have been conjectured by D. Bevan in [23].

All the above partial and conjectured results on the semi-Baxter number sequence will be resolved along this chapter thanks to the succession rule for semi-Baxter permutations provided in the next subsection.

3.1.2 Semi-Baxter succession rule

Throughout this and the following chapters we define growths for family of permutations by performing "local expansions" on the right of any permutation π. More precisely, when inserting $a \in \{1, \dots, n+1\}$ on the right of any π of length n, we obtain the permutation $\pi' = \pi'_1 \dots \pi'_n \pi'_{n+1}$ where $\pi'_{n+1} = a$, $\pi'_i = \pi_i$ if $\pi_i < a$ and $\pi'_i = \pi_i + 1$ if $\pi_i \geq a$. We use the notation $\pi \cdot a$ to denote π'. For instance, $1\,4\,2\,3 \cdot 3 = 1\,5\,2\,4\,3$. This is easily understood on the graphical representation of permutations: a local expansion corresponds to adding a new point on the right of the grid, which lies vertically between two existing points (or below the lowest, or above the highest), and finally normalizing the picture obtained - see Figure 3.1.

Proposition 3.1.3. *Semi-Baxter permutations can be generated by the following succession rule*

$$\Omega_{semi} = \begin{cases} (1,1) \\ \\ (h,k) \rightsquigarrow (1,k+1), \dots, (h,k+1) \\ \qquad\qquad (h+k,1), \dots, (h+1,k). \end{cases}$$

Proof. First, observe that removing the last point of a permutation avoiding $2\,\underline{41}\,3$, we obtain a permutation that still avoids $2\,\underline{41}\,3$. So, a growth for semi-Baxter permutations can be obtained with local expansions on the right. For π being a semi-Baxter permutation of length n, the active sites are by definition the points a (or equivalently the values a) such that $\pi \cdot a$ is also semi-Baxter, *i.e.*, avoids $2\,\underline{41}\,3$. Necessarily, all the other points $a \in \{1, \dots, n+1\}$ are called non-active sites.

An occurrence of $2\,\underline{31}$ in π is a subsequence $\pi_j \pi_i \pi_{i+1}$ (with $j < i$) such that $\pi_{i+1} < \pi_j < \pi_i$. Obviously, the non-active sites a of π are characterized by the fact that $a \in (\pi_j, \pi_i]$ for some occurrence $\pi_j \pi_i \pi_{i+1}$ of $2\,\underline{31}$. We call a *non-empty descent* of π a pair $\pi_i \pi_{i+1}$ such that there exists π_j that makes $\pi_j \pi_i \pi_{i+1}$ an occurrence of $2\,\underline{31}$. Note that in the case where $\pi_{n-1}\pi_n$ is a non-empty descent, choosing $\pi_j = \pi_n + 1$ always gives an occurrence of $2\,\underline{31}$, and it is the smallest possible value of π_j for which $\pi_j \pi_{n-1} \pi_n$ is an occurrence of $2\,\underline{31}$.

To each semi-Baxter permutation π of length n, we assign a label (h, k), where h (resp. k) is the number of the active sites of π smaller than or equal to (resp. greater than) π_n. Remark that $h, k \geq 1$, since 1 and $n+1$ are always active sites. Moreover, the label of the permutation $\pi = 1$ is $(1, 1)$, which is the axiom in Ω_{semi}.

Consider a semi-Baxter permutation π of length n and label (h, k). Proving Proposition 3.1.3 amounts to showing that permutations $\pi \cdot a$ have labels $(1, k+1), \dots, (h, k+1), (h+k, 1), \dots, (h+1, k)$ when a runs over all active sites of π. Figure 3.1, which shows an example of semi-Baxter permutation π with label $(2, 2)$ and all the corresponding $\pi \cdot a$ with their labels, should help understanding the case analysis that follows.

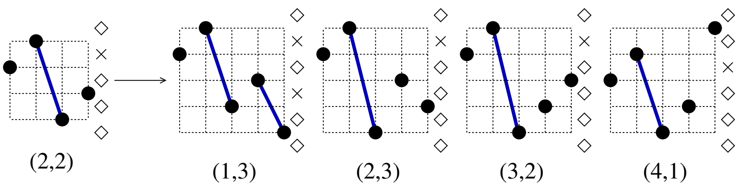


Figure 3.1: The growth of a semi-Baxter permutation. Active sites are marked with $\diamond$, non-active sites by $\times$, and non-empty descents are represented with bold blue lines.

Let a be an active site of π. Assume first that $a > \pi_n$ (this happens exactly k times), so that $\pi \cdot a$ ends with an ascent. The occurrences of $2\,\underline{31}$ in $\pi \cdot a$ are the same as in π. Consequently, the active sites are not modified, except that the active site a of π is now split into two actives sites of $\pi \cdot a$: one immediately below a and one immediately above. It follows that $\pi \cdot a$ has label $(h + k + 1 - i, i)$, if a is the ith active site from the top. Since i ranges from 1 to k, this gives the second row of the production of Ω_{semi}.

Assume next that $a = \pi_n$. Then, $\pi \cdot a$ ends with a descent, but an empty one. Similar to the above case, we therefore get one more active site in $\pi \cdot a$ than in π, and $\pi \cdot a$ has label $(h, k+1)$, the last label in the first row of the production of Ω_{semi}.

Finally, assume that $a < \pi_n$ (this happens exactly $h - 1$ times). Now, $\pi \cdot a$ ends with a non-empty descent, which is $(\pi_n + 1)a$. It follows from the discussion at the beginning of this proof that all sites of $\pi \cdot a$ in $(a + 1, \pi_n + 1]$ become non-active, while all others remain active if they were so in π (again, with a replaced by two active sites surrounding it, one below it and one above). If a is the ith active site from the bottom, it follows that $\pi \cdot a$ has label $(i, k+1)$, hence giving all missing labels in the first row of the production of Ω_{semi}. □

3.2 Other semi-Baxter structures

This section shows that semi-Baxter numbers are not solely the enumerative sequence of semi-Baxter permutations. As already mentioned in Section 3.1, they form the enumerative sequence of another family of pattern-avoiding permutations (Section 3.2.1), as well as of a special family of inversion sequences (Section 3.2.2). A last occurrence of these numbers is presented in Section 3.2.3 in terms of lattice paths: the semi-Baxter paths defined in [G5] provide a completely new combinatorial interpretation of the semi-Baxter number sequence and naturally extend the definition of Baxter paths of Section 2.1.4.

3.2.1 Plane permutations

Definition 3.2.1. A *plane permutation* is a permutation that avoids the vincular pattern $2\,\underline{14}\,3$ (or equivalently, the barred pattern $21\bar{3}54$).

The enumeration of plane permutations has received a fair amount of attention in the literature. The problem first arose as an open problem in [35], where permutations avoiding $21\bar{3}54$ were identified as a superset of forest-like permutations investigated in [35]. A forest-like permutation is any permutation whose Hasse graph is a forest - the Hasse graph of $\pi \in \mathcal{S}_n$ is constructed on the vertex set $\{1, \ldots, n\}$ by joining i and j if and only if $\pi(i) < \pi(j)$, with $i < j$, and there is no k, $i < k < j$, such that $\pi(i) < \pi(k) < \pi(j)$. In [35] a characterisation of forest-like permutations is provided in terms of pattern-avoiding permutations, from which it follows that the Hasse graph of a permutation avoiding $21\bar{3}54$ is plane (*i.e.* non-crossing). Hence, the authors of [35] named plane permutations those avoiding $21\bar{3}54$ and called for their enumeration.

Now, we go further in enumerating plane permutations proving that there exists a growth for plane permutations, which yields Ω_{semi} as succession rule.

Proposition 3.2.2. *Plane permutations can be generated by* Ω_{semi}.

Proof. The proof of this statement follows applying the same steps as in the proof of Proposition 3.1.3. First, observe that removing the last point of a permutation avoiding $2\,\underline{14}\,3$, we obtain a permutation that still avoids $2\,\underline{14}\,3$. So, a generating tree for plane permutations can be obtained with local expansions on the right.

For π a plane permutation of length n, the active sites are by definition the values a such that $\pi \cdot a$ avoids $2\,\underline{14}\,3$. An occurrence of $2\,\underline{13}$ in π is a subsequence $\pi_j \pi_i \pi_{i+1}$ (with $j < i$) such that $\pi_i < \pi_j < \pi_{i+1}$. Note that the non-active sites a of π are characterized by the fact that $a \in (\pi_j, \pi_{i+1}]$ for some occurrence $\pi_j \pi_i \pi_{i+1}$ of $2\,\underline{13}$. We call a *non-empty* *ascent* of π a pair $\pi_i\pi_{i+1}$ such that there exists π_j that makes $\pi_j\pi_i\pi_{i+1}$ an occurrence of $2\underline{13}$. As in the proof of Proposition 3.1.3, if $\pi_{n-1}\pi_n$ is a non-empty ascent, $\pi_j = \pi_{n-1}+1$ is the smallest value of π_j such that $\pi_j\pi_{n-1}\pi_n$ is an occurrence of $2\underline{13}$.

Now, to each plane permutation π of length n, we assign a label (h,k), where h (resp. k) is the number of the active sites of π greater than (resp. smaller than or equal to) π_n. Remark that $h, k \geq 1$, since 1 and $n+1$ are always active sites. Moreover, the label of the permutation $\pi = 1$ is $(1,1)$, which is the axiom in Ω_{semi}. The proof is concluded by showing that the permutations $\pi \cdot a$ have labels $(1,k+1),\dots,(h,k+1),(h+k,1),\dots,(h+1,k)$, when a runs over all active sites of π.

If $a \leq \pi_n$, $\pi \cdot a$ ends with a descent, and it follows as in the proof of Proposition 3.1.3 that the active sites of $\pi \cdot a$ are the same as those of π (with a split into two sites). This gives the second row of the production of Ω_{semi} (the label $(h+k+1-i,i)$ for $1 \leq i \leq k$ corresponding to a being the ith active site from the bottom).

If $a = \pi_n + 1$, $\pi \cdot a$ ends with an empty ascent, and hence has label $(h,k+1)$ again as in the proof of Proposition 3.1.3.

Finally, if $a > \pi_n + 1$ (which happens $h-1$ times), $\pi \cdot a$ ends with a non-empty ascent. The discussion at the beginning of the proof implies that all sites of $\pi \cdot a$ in $(\pi_n+1, a]$ are deactivated while all others remain active. If a is the ith active site from the top, it follows that $\pi \cdot a$ has label $(i,k+1)$, hence giving all missing labels in the first row of the production of Ω_{semi}. □

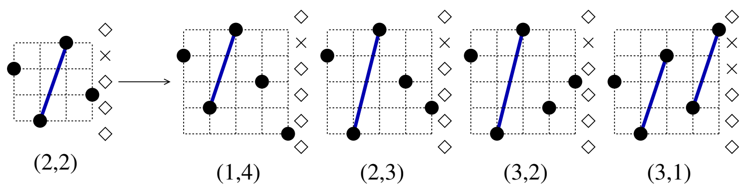


Figure 3.2: The growth of a plane permutation. Active sites are marked with ◇, non-active sites by ×, and non-empty ascents are represented with bold blue lines.

Because the two families of semi-Baxter and plane permutations grow according to the same succession rule, we obtain the following.

Corollary 3.2.3. *Semi-Baxter permutations and plane permutations are in bijection. Thus, SB_n is also the number of plane permutations of length n.*

Indeed, the two generating trees for semi-Baxter and for plane permutations which both are encoded by Ω_{semi} are of course isomorphic. This provides a size-preserving bijection between these two families. However, it is not defined directly on the objects themselves, but only referring to the generating tree structure. So we leave open the problem of describing a direct bijection between the family of semi-Baxter permutations and the one of plane permutations.

3.2.2 Inversion sequences $\mathbf{I}_n(>, \geq, -)$

In line with the study of patterns in permutations, the authors of [58] started studying a similar notion of pattern in inversion sequences.

Definition 3.2.4. An *inversion sequence* of size n is an integer sequence $(e_1, e_2, \dots, e_n)$ satisfying $0 \leq e_i < i$, for all $i \in \{1, 2, \dots, n\}$.

Given a word $q = q_1 \dots q_k$, an inversion sequence *contains* the pattern q, if there exist k indices $i_1 < \dots < i_k$ such that the word $e_{i_1} \dots e_{i_k}$ is order-isomorphic to q.

The notion of pattern in inversion sequences was further analysed in [110], where the authors considered the following generalisation of pattern avoidance.

Definition 3.2.5. Let (ρ_1, ρ_2, ρ_3) be a triple of binary relations. The set $\mathbf{I}_n(\rho_1, \rho_2, \rho_3)$ is defined as the set of all inversion sequences $(e_1, e_2, \dots, e_n)$ of size n such that there are no three indices $i < j < k$, for which it holds $e_i \, \rho_1 \, e_j \, \rho_2 \, e_k$ and $e_i \, \rho_3 \, e_k$.

For instance, the inversion sequence $(0, 1, 1, 2)$ does not belong to $\mathbf{I}_4(=, <, <)$, because $e_2 = e_3 < e_4$. The authors of [110] attempted to provide a comprehensive classification of all the possible number sequences $\{i_n\}_{n \geq 0}$, where $i_n = |\mathbf{I}_n(\rho_1, \rho_2, \rho_3)|$, for any triple (ρ_1, ρ_2, ρ_3) of relations in the set $\{<, >, \leq, \geq, =, \neq, -\}$.[1]

In this framework many conjectures arise [110, Table 2], and one of them involves the family of inversion sequences avoiding the triple $(>, \geq, -)$, which is thought to be equinumerous to the family of plane permutations [110, Section 2.27]. Thanks to the succession rule of Section 3.1.2 this conjecture can be easily proved by showing that this family of inversion sequence can be generated by Ω_{semi}.

In addition, in [110, Section 2.27] it is proved that the set $\mathbf{I}_n(>, \geq, -)$ is equinumerous to the sets $\mathbf{I}_n(-, <, \geq)$, and $\mathbf{I}_n(\geq, >, -)$, and $\mathbf{I}_n(-, \leq, >)$. Thus, proving that semi-Baxter numbers enumerate inversion sequences avoiding the triple of relations $(>, \geq, -)$ implicitly solves the enumeration problem of exactly four cases of [110, Table 2].

The set of inversion sequences $\mathbf{I}_n(>, \geq, -)$ is proved in [110] to coincide with the set of inversion sequences of size n avoiding both patterns 210 and 100. Based on the results previously established in [58] for the family of inversion sequences avoiding 210, the following properties and formulas are proved in [110].

Let a *weak left-to-right maximum* of an inversion sequence $e = (e_1, e_2, \dots, e_n)$ be any entry e_i satisfying $e_i \geq e_j$, for all $j \leq i$. Every inversion sequence e can be decomposed in e^{top}, which is the (weakly increasing) sequence of weak left-to-right maxima of e, and e^{bottom}, which is the (possibly empty) sequence of the remaining entries of e.

Proposition 3.2.6 ([110], Observation 10)**.** *An inversion sequence e avoids* 210 *and* 100 *if and only if e^{top} is weakly increasing and e^{bottom} is strictly increasing.*

[1] The relation $-$ on a set S coincides with $S \times S$.

The enumeration of inversion sequences avoiding 210 and 100 is solved in [110, Theorem 32], with a summation formula reported in Proposition 3.2.7 below. The first terms produced by iterating the following equation (3.1) had been seen to match those enumerating plane permutations produced by D. Bevan in [22].

Let $\mathrm{top}(e) = \max(e^{top})$ and $\mathrm{bottom}(e) = \max(e^{bottom})$. If e^{bottom} is empty, the convention is to take $\mathrm{bottom}(e) = -1$.

Proposition 3.2.7 ([110], Theorem 32)**.** *Let $Q_{n,a,b}$ be the number of $e \in \mathbf{I}_n(>, \geq, -)$ with top$(e) = a$ and bottom$(e) = b$. Then,*

$$Q_{n,a,b} = \sum_{i=-1}^{b-1} Q_{n-1,a,i} + \sum_{j=b+1}^{a} Q_{n-1,j,b}\,,$$

with initial conditions $Q_{n,a,b} = 0$, if $n \leq a$, and $Q_{n,a,-1} = \frac{n-a}{n}\binom{n-1+a}{a}$. Hence,

$$|\mathbf{I}_n(>, \geq, -)| = \sum_{a=0}^{n-1} \sum_{b=-1}^{a-1} Q_{n,a,b} = \frac{1}{n+1}\binom{2n}{n} + \sum_{a=0}^{n-1} \sum_{b=0}^{a-1} Q_{n,a,b}. \tag{3.1}$$

We prove the following conjecture stated in [110], thus showing that Equation (3.1) generates semi-Baxter numbers.

Theorem 3.2.8. *There are as many inversion sequences of size n avoiding 210 and 100 as plane permutations of length n. In other words, $|\mathbf{I}_n(>, \geq, -)| = SB_n$.*

Proof. We prove the statement by showing a growth for the family $\cup_n \mathbf{I}_n(>, \geq, -)$ according to Ω_{semi}. Given an inversion sequence $e \in \mathbf{I}_n(>, \geq, -)$, we define this growth by adding a new rightmost entry.

Let $a = \mathrm{top}(e)$ and $b = \mathrm{bottom}(e)$. From Proposition 3.2.6, it follows that $f = (e_1, \ldots, e_n, p)$ is an inversion sequence of size $n+1$ avoiding 210 and 100 if and only if $n \geq p > b$.

Moreover, if $p \geq a$, then f^{top} comprises p in addition to the elements of e^{top}, and $f^{bottom} = e^{bottom}$; else if $b < p < a$, then $f^{top} = e^{top}$ and f^{bottom} comprises p in addition to the elements of e^{bottom}. Now, we assign to any $e \in \mathbf{I}_n(>, \geq, -)$ the label (h, k), where $h = a - b$ and $k = n - a$.

The sequence $e = (0)$ has label $(1, 1)$, which is the axiom of Ω_{semi}, since $a = \mathrm{top}(e) = 0$ and $b = \mathrm{bottom}(e) = -1$. Let e be an inversion sequence of $\mathbf{I}_n(>, \geq, -)$ with label (h, k). The labels of the inversion sequences of $\mathbf{I}_{n+1}(>, \geq, -)$ produced adding a rightmost entry p to e are

- $(h+k, 1), (h+k-1, 2), \ldots, (h+1, k)$ when $p = n, n-1, \ldots, a+1$,
- $(h, k+1)$ when $p = a$,
- $(1, k+1), \ldots, (h-1, k+1)$ when $p = a-1, \ldots, b+1$,

which concludes the proof that $\cup_n \mathbf{I}_n(>, \geq, -)$ grows according to Ω_{semi}. □

3.2.3 Semi-Baxter paths

To our knowledge, only families of restricted permutations and inversion sequences are known to be counted by the semi-Baxter numbers. Here, we provide a new occurrence of these numbers in terms of labelled Dyck paths, that generalises the family of Baxter paths of Section 2.1.4.

To this purpose, we recall that according to Definition 2.1.7 a free up step of a Dyck path P is any up step of P not forming a DU factor.

Definition 3.2.9. A *semi-Baxter path* of semi-length n is a Dyck path of length $2n$ having all its free up steps labelled according to the following constraint: the leftmost free up step is labelled 1 and for every pair of free up steps (U', U''), with U' occurring before U'' and no free up step between them, the label of U'' is in the range $[1, h]$, where $h \geq 1$ is the sum of the label of U' with the number of D steps between U' and U''.

A semi-Baxter path can be obtained from a Dyck path by properly labelling its free up steps as shown by the path on the left in Figure 3.3. Whereas the labelled path on the right in Figure 3.3 is not a semi-Baxter path, because the last free up step is labelled by 6, which is here a value outside of the range of Definition 3.2.9. One can notice that semi-Baxter paths generalise Baxter paths comparing the following picture to Figure 2.4 (page 59).

Observe that as in any Baxter paths, the sequence of labels corresponding to consecutive free up steps of a semi-Baxter paths has to be non-increasing.

Figure 3.3: A Dyck path of semi-length 9 whose free up step are labelled in two different ways: the path on the left is a semi-Baxter path of semi-length 9, while the labelling on the right does not satisfy Definition 3.2.9.

Let $\mathcal{SB}_n$ denote the set of semi-Baxter paths of semi-length n. With Proposition 3.2.10 below, we prove that $\mathcal{SB} = \cup_n \mathcal{SB}_n$ is enumerated by the sequence of semi-Baxter numbers.

Proposition 3.2.10. *Semi-Baxter paths can be generated by rule* Ω_{semi}.

Proof. Similarly to the growths of Dyck paths and Baxter paths, we make semi-Baxter paths grow by insertion of a peak in any point of the last descent, as shown in Figure 3.4. To any $S \in \mathcal{SB}_n$, denoting e the label of its rightmost free up step $\bar{U}$ (which always exists, since the first step of the path is always a free up step), we assign the label (h, k), where h is equal to e plus the number of down steps between $\bar{U}$ and the rightmost up step of S (they might coincide) and k is the number of steps of the last descent of S. With this labelling, we shall see that the growth of semi-Baxter paths can be encoded by Ω_{semi}.

The unique semi-Baxter path in $\mathcal{SB}_1$ receives the label $(1,1)$, which is the axiom of Ω_{semi}. From any $S \in \mathcal{SB}_n$ of label (h,k), we perform two kinds of insertions, which we shall see correspond to all productions of (h,k) in Ω_{semi}.

a) We add a peak at the beginning of the last descent of S. This means that the added U step follows another up step and hence is free, while the number of down steps in the last descent increases by one. Moreover, U receives a label, which can be any value in the range $[1,h]$. If i is the label assigned to U, for $1 \leq i \leq h$, then the path produced has label $(i, k+1)$.

b) We add a peak immediately after any down step of the last descent of S. In this case the added step U is not free, and hence carries no label. Denoting $S = w \cdot UD^k$ (with this U possibly labelled), the children of S so produced are $w \cdot UD^j\boldsymbol{UD}D^{k-j}$ for $1 \leq j \leq k$, so they have labels $(h+j, k+1-j)$. □

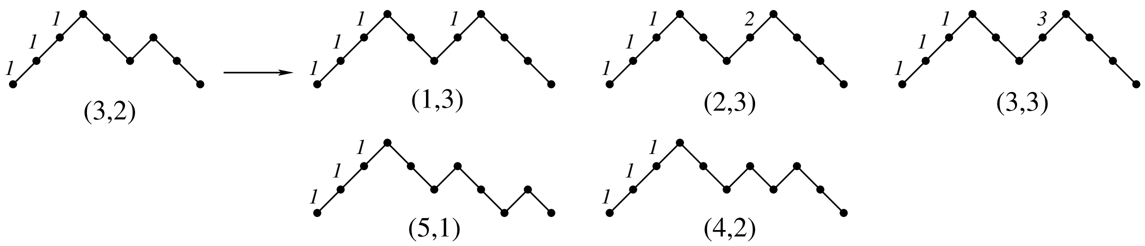


Figure 3.4: The growth of a semi-Baxter path of label $(3,2)$.

Observation 3.2.11. *The set $\mathcal{B}_n$ of Baxter paths of semi-length n forms a subset of $\mathcal{SB}_n$, for every n.*

Proof. It holds that the label range for Baxter paths of Definition 2.1.8 on page 58 is always contained into the label range of Definition 2.1.8, because any valley consists of exactly one D step. □

Moreover, note that by restricting the growth of Proposition 3.2.10 to the family of Baxter paths we retrieve the growth provided in the proof of Proposition 2.1.9.

3.3 Generating function

In this section, we first translate the succession rule Ω_{semi} provided for enumerating semi-Baxter permutations into a functional equation whose solution is the (multivariate) generating function of semi-Baxter numbers. Then, we approach the problem of solving the resulting functional equation by using some variant of the kernel method, which has already been used in Section 1.4.5 to solve the Baxter case. From it, a lot of information can be derived about the generating function of semi-Baxter numbers, and about these numbers themselves. The results we obtain are shown in the next sections, and a Maple worksheet recording the computations of the following proofs has been reported in Appendix A.

3.3.1 Functional equation

For $h, k \geq 1$, let $S_{h,k}(x) \equiv S_{h,k}$ denote the generating function of semi-Baxter permutations having label (h, k). The rule Ω_{semi} translates into a functional equation for the generating function $S(x; y, z) \equiv S(y, z) = \sum_{h,k\geq 1} S_{h,k}\, y^h z^k$.

Proposition 3.3.1. *The generating function $S(y, z)$ satisfies the following functional equation*

$$S(y,z) = xyz + \frac{xyz}{1-y}\left(S(1,z) - S(y,z)\right) + \frac{xyz}{z-y}\left(S(y,z) - S(y,y)\right). \tag{3.2}$$

Proof. Starting from the growth of semi-Baxter permutations according to Ω_{semi} we write

$$\begin{aligned} S(y,z) &= xyz + x\sum_{h,k\geq 1} S_{h,k}\left((y+y^2+\cdots+y^h)z^{k+1} + (y^{h+k}z + y^{h+k-1}z^2 + \cdots + y^{h+1}z^k)\right) \\ &= xyz + x\sum_{h,k\geq 1} S_{h,k}\left(\frac{1-y^h}{1-y}y\, z^{k+1} + \frac{1-\left(\frac{y}{z}\right)^k}{1-\frac{y}{z}}y^{h+1}z^k\right) \\ &= xyz + \frac{xyz}{1-y}\left(S(1,z) - S(y,z)\right) + \frac{xyz}{z-y}\left(S(y,z) - S(y,y)\right). \end{aligned}$$

□

3.3.2 Semi-Baxter generating function

The aim of this subsection is to establish the nature of the function $S(y, z)$, which denotes the multivariate generating function of semi-Baxter permutations, where y (resp. z) takes into account the first (resp. second) entry of each label of a semi-Baxter permutation.

From the previous section, $S(y, z)$ satisfies Equation (3.2) that is a linear functional equation with two catalytic variables, y and z, in the sense of Zeilberger [151]. A similar functional equation has been given for Baxter permutations in Section 1.4.5 on page 48. Nevertheless, Equation (3.2), and thus its solution $S(y, z)$, is not symmetric in y and z; hence, it differs substantially from the Baxter functional equation, although its shape resembles it. This similarity however allows us to apply the obstinate variant of the kernel method presented in Section 1.4.5 in order to solve Equation (3.2).

First, it is convenient to set $y = 1 + a$ and collect all the terms having $S(1 + a, z)$ in them, obtaining the so-called kernel form of Equation (3.2):

$$K(a,z)S(1+a,z) = xz(1+a) - \frac{xz(1+a)}{a}S(1,z) - \frac{xz(1+a)}{z-1-a}S(1+a,1+a), \tag{3.3}$$

where the kernel is

$$K(a,z) = 1 - \frac{xz(1+a)}{a} - \frac{xz(1+a)}{z-1-a}.$$

For brevity, we refer to the right-hand side of Equation (3.3) by using the expression $\mathcal{R}(x, a, z, S(1, z), S(1 + a, 1 + a))$, where

$$\mathcal{R}(x_0, x_1, x_2, w_0, w_1) = x_0x_2(1 + x_1) - \frac{x_0x_2(1 + x_1)}{x_1}w_0 - \frac{x_0x_2(1 + x_1)}{x_2 - 1 - x_1}w_1.$$

The kernel function $K(a, z)$ is quadratic in z, thus equation $K(a, z) = 0$ has two solutions in z. Denoting $Z_+(a)$ and $Z_-(a)$ the solutions of $K(a, z) = 0$ with respect to z, we have

$$Z_+(a) = \frac{1}{2}\frac{a + x + ax - Q}{x(1 + a)} = (1 + a) + (1 + a)^2x + \frac{(1 + a)^3(1 + 2a)}{a}x^2 + O(x^3),$$

$$Z_-(a) = \frac{1}{2}\frac{a + x + ax + Q}{x(1 + a)} = \frac{a}{(1 + a)x} - a - (1 + a)^2x - \frac{(1 + a)^3(1 + 2a)}{a}x^2 + O(x^3),$$

where $Q = \sqrt{a^2 - 2ax - 6a^2x + x^2 + 2ax^2 + a^2x^2 - 4a^3x}$.

The kernel root Z_- is not a well-defined power series in x, whereas the other kernel root Z_+ is a power series in x whose coefficients are Laurent polynomials in a. So, setting $z = Z_+$, the function $S(1 + a, z)$ is a convergent power series in x and the right-hand side of Equation (3.3) is equal to zero,

$$\mathcal{R}(x, a, Z_+, S(1, Z_+), S(1 + a, 1 + a)) = 0.$$

At this point following the usual kernel method approach (see Section 1.3.6 or Section 1.3.7), we are stuck and cannot find a suitable expression for the solution $S(1+a, 1+a)$, because of the unknown term $S(1, Z_+)$.

Therefore, we follow the steps of the obstinate variant of the kernel method (Section 1.4.5) and attempt to eliminate the term $S(1, Z_+)$ by exploiting transformations that leave the kernel, $K(a, z)$, unchanged. Examining the kernel shows that the transformations

$$\Phi : (a, z) \to \left(\frac{z - 1 - a}{1 + a}, z\right) \quad \text{and} \quad \Psi : (a, z) \to \left(a, \frac{z + za - 1 - a}{z - 1 - a}\right)$$

leave the kernel unchanged and generate a group of order 10 - see Figure 3.5.

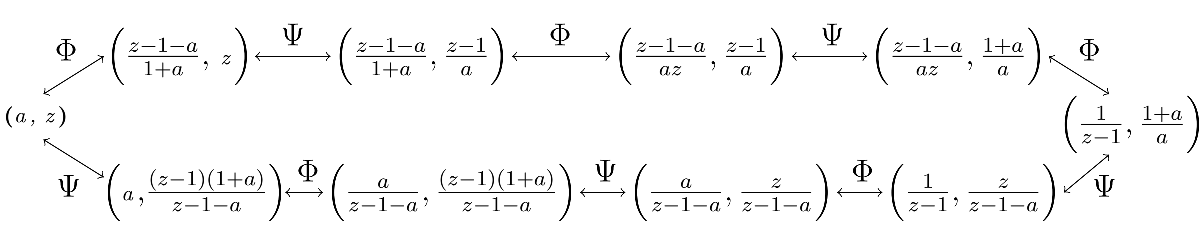


Figure 3.5: The orbit of (a, z) under the action of Φ and Ψ.

Among all the elements of this group we consider the pairs $(f_1(a,z), f_2(a,z))$ such that $f_1(a, Z_+)$ and $f_2(a, Z_+)$ are well-defined power series in x with Laurent polynomial coefficients in a. More precisely, they are exactly the four pairs $(f_1(a,z), f_2(a,z))$ of the first line of Figure 3.5, in addition to the pair (a, z). Consequently, these pairs share the property that $S(1 + f_1(a, Z_+), f_2(a, Z_+))$ are convergent power series in x.

Hence, it follows that substituting each of these pairs for (a, z) in Equation (3.3) and $z = Z_+$, we obtain a system of five equations and with six overlapping unknowns, whose left-hand sides are all equal to 0,

$$\begin{cases} 0 = \mathcal{R}\left(x, a, Z_+, S(1, Z_+), S(1+a, 1+a)\right) \\ 0 = \mathcal{R}\left(x, \dfrac{Z_+ - 1 - a}{1+a}, Z_+, S(1, Z_+), S\left(1 + \dfrac{Z_+ - 1 - a}{1+a}, 1 + \dfrac{Z_+ - 1 - a}{1+a}\right)\right) \\ 0 = \mathcal{R}\left(x, \dfrac{Z_+ - 1 - a}{1+a}, \dfrac{Z_+ - 1}{a}, S\left(1, \dfrac{Z_+ - 1}{a}\right), S\left(1 + \dfrac{Z_+ - 1 - a}{1+a}, 1 + \dfrac{Z_+ - 1 - a}{1+a}\right)\right) \\ 0 = \mathcal{R}\left(x, \dfrac{Z_+ - 1 - a}{aZ_+}, \dfrac{Z_+ - 1}{a}, S\left(1, \dfrac{Z_+ - 1}{a}\right), S\left(1 + \dfrac{Z_+ - 1 - a}{aZ_+}, 1 + \dfrac{Z_+ - 1 - a}{aZ_+}\right)\right) \\ 0 = \mathcal{R}\left(x, \dfrac{Z_+ - 1 - a}{aZ_+}, \dfrac{1+a}{a}, S\left(1, \dfrac{1+a}{a}\right), S\left(1 + \dfrac{Z_+ - 1 - a}{aZ_+}, 1 + \dfrac{Z_+ - 1 - a}{aZ_+}\right)\right). \end{cases} \tag{3.4}$$

Now, by eliminating all unknowns except $S(1+a, 1+a)$ and $S(1, 1+\bar{a})$, where as usual $\bar{a}$ denotes $1/a$, System (3.4) reduces (after some work) to the following equation,

$$S(1+a, 1+a) + \frac{(1+a)^2 x}{a^4} S\left(1, 1+\bar{a}\right) + P(a, Z_+) = 0, \tag{3.5}$$

where

$$P(a,z) = \frac{(-z+1+a)}{za^4(z-1)}\Big(- za^4 + z^2a^4 - za^3 + z^2a^3 - z^3a^2 - 2a^2 + z^2a^2 + za^2 - 4a \\ + 5az - 3az^2 + z^3a + 3z - z^2 - 2\Big).$$

Note that the coefficient of $S(1, 1+\bar{a})$ in (3.5) results to be equal to $(1+a)^2 x\bar{a}^4$ only after setting $z = Z_+$ and simplifying the expression obtained. This coefficient is remarkable since the function $S(1, 1+\bar{a})$ is multiplied by a polynomial in which the highest power of a is -2.

Then, the form of Equation (3.5) allows us to separate its terms according to the power of a:

- $S(1+a, 1+a)$ is a power series in x with polynomial coefficients in a whose lowest power of a is 0,

- $S(1, 1+\bar{a})$ is a power series in x with polynomial coefficients in $\bar{a}$ whose highest power of a is 0; consequently, since $(1+a)^2 x\bar{a}^4 = x(a^{-4}+2a^{-3}+a^{-2})$, we obtain that $(1+a)^2 x\,\bar{a}^4 S(1, 1+\bar{a})$ is a power series in x with polynomial coefficients in $\bar{a}$ whose highest power of a is -2.

Then, when we expand the series $-P(a, Z_+)$ as a power series in x, the non-negative powers of a in the coefficients must be equal to those of $S(1+a, 1+a)$, while the negative powers of a come from $(1+a)^2 x\,\bar{a}^4 S(1, 1+\bar{a})$.

In order to have a better expression for the series $P(a, z)$, we perform a further substitution setting $z = w+1+a$. More precisely, let $W \equiv W(x; a)$ be the power series in x defined by $W = Z_+ - (1+a)$. We have the following expression for $F(a, W) := -P(a, Z_+)$,

$$\begin{aligned} F(a,W) = -P(a, W+1+a) = {} & (1+a)^2\, x + \left(\frac{1}{a^5} + \frac{1}{a^4} + 2 + 2a\right) x\, W \\ & + \left(-\frac{1}{a^5} - \frac{1}{a^4} + \frac{1}{a^3} - \frac{1}{a^2} - \frac{1}{a} + 1\right) x\, W^2 \\ & + \left(\frac{1}{a^4} - \frac{1}{a^2}\right) x\, W^3. \end{aligned} \tag{3.6}$$

Since the kernel function annihilates if $z = W+1+a$, namely $K(a, W+1+a) = 0$, the function W is recursively defined by

$$W = x\bar{a}(1+a)(W+1+a)(W+a), \tag{3.7}$$

Therefore, by using Equation (3.6) and Equation (3.7), we can express the generating function of semi-Baxter permutations as follows.

Theorem 3.3.2. *Let $W(x; a) \equiv W$ be the unique formal power series in x such that*

$$W = x\bar{a}(1+a)(W+1+a)(W+a).$$

The series solution $S(y, z)$ of Equation (3.2) satisfies $S(1+a, 1+a) = [F(a, W)]^{\geq}$, where $[F(a, W)]^{\geq}$ stands for the formal power series in x obtained by considering only those terms in the series expansion that have non-negative powers of a, and the function $F(a, W)$ is defined by

$$\begin{aligned} F(a, W) = {} & (1+a)^2\, x + (\bar{a}^5 + \bar{a}^4 + 2 + 2a)\, x\, W \\ & + (-\bar{a}^5 - \bar{a}^4 + \bar{a}^3 - \bar{a}^2 - \bar{a} + 1)\, x\, W^2 + (\bar{a}^4 - \bar{a}^2)\, x\, W^3. \end{aligned}$$

Note that in Theorem 3.3.2, W and $F(a, W)$ are algebraic series in x whose coefficients are Laurent polynomials in a. It follows, as in [32, page 6], that $S(1+a, 1+a) = [F(a, W)]^{\geq}$ is D-finite, and hence also its specialisation $S(1, 1)$.

3.4 Semi-Baxter formulas

In this section we provide some explicit closed formulas for semi-Baxter numbers, as well as a recurrence relation. The first explicit formula is shown in Section 3.4.1 and is obtained directly from Theorem 3.3.2 by applying the Lagrange inversion formula. This expression is rather complicated, yet hides a simple and nice recurrence presented in Section 3.4.3. Other expressions are presented in Section 3.4.4: they were initially thought for enumerating plane permutations though without any proof, and now, thanks to the recurrence of Section 3.4.3, we are able to prove them. Appendix A.2 shows the calculations performed in this section.

3.4.1 Explicit closed formula

Using the Lagrange inversion formula of Theorem 1.2.6 on page 18, we can obtain from the expression of $S(1+a,1+a)$ of Theorem 3.3.2 an explicit, though complicated, expression for the coefficients of the semi-Baxter generating function $S(1,1)$.

Corollary 3.4.1. *The number SB_n of semi-Baxter permutations of length n is, for all $n>1$,*

$$SB_n = \tfrac{1}{n-1}\sum_{j=0}^{n}\tbinom{n-1}{j}\Bigg[\tbinom{n-1}{j+1}\Big[\tbinom{n+j+1}{j+5}+2\tbinom{n+j+1}{j}\Big]+2\tbinom{n-1}{j+2}\Big[-\tbinom{n+j+2}{j+5}+\tbinom{n+j+1}{j+3} \\ -\tbinom{n+j+2}{j+2}+\tbinom{n+j+1}{j}\Big]+3\tbinom{n-1}{j+3}\Big[\tbinom{n+j+2}{j+4}-\tbinom{n+j+2}{j+2}\Big]\Bigg].$$

Proof. The nth semi-Baxter number, SB_n, is the coefficient of x^n in $S(1,1)$, which we denote as usual $[x^n]S(1,1)$. Note that this number is also the coefficient $[a^0x^n]S(1+a,1+a)$, and so by Theorem 3.3.2 it is the coefficient of a^0x^n in $F(a,W)$, namely

$$SB_n = [a^0x^{n-1}]\Big((1+a)^2+\left(\bar{a}^5+\bar{a}^4+2+2a\right)W+\left(-\bar{a}^5-\bar{a}^4+\bar{a}^3-\bar{a}^2-\bar{a}+1\right)W^2 \\ +\left(\bar{a}^4-\bar{a}^2\right)W^3\Big).$$

This expression can be evaluated from $[a^sx^k]W^i$, for $i=1,2,3$. Precisely,

$$SB_n = [a^5x^{n-1}]W+[a^4x^{n-1}]W+2[a^0x^{n-1}]W+2[a^{-1}x^{n-1}]W-[a^5x^{n-1}]W^2-[a^4x^{n-1}]W^2 \\ +[a^3x^{n-1}]W^2-[a^2x^{n-1}]W^2-[a^1x^{n-1}]W^2+[a^0x^{n-1}]W^2+[a^4x^{n-1}]W^3-[a^2x^{n-1}]W^3.$$

The Lagrange inversion (Theorem 1.2.6) together with Equation (3.7) then prove that

$$[a^sx^k]W^i = \frac{i}{k}\sum_{j=0}^{k-i}\binom{k}{j}\binom{k}{j+i}\binom{k+j+i}{j+s}, \qquad \text{for } i=1,2,3.$$

We can then substitute this into the above expression for SB_n and so, for $n \geq 2$, express SB_n as

$$SB_n = \sum_{j=0}^{n-1} F_{SB}(n,j), \qquad \text{where}$$

$$\begin{aligned} F_{SB}(n,j) = \frac{1}{n-1}\binom{n-1}{j}\Bigg[\binom{n-1}{j+1}\left[\binom{n+j+1}{j+5} + 2\binom{n+j+1}{j}\right] \\ +2\binom{n-1}{j+2}\left[-\binom{n+j+2}{j+5} + \binom{n+j+1}{j+3} - \binom{n+j+2}{j+2} + \binom{n+j+1}{j}\right] \\ +3\binom{n-1}{j+3}\left[\binom{n+j+2}{j+4} - \binom{n+j+2}{j+2}\right]\Bigg]. \end{aligned} \tag{3.8}$$

□

3.4.2 Creative telescoping

In this section we report a strategy that allows us to prove several expressions for semi-Baxter numbers: a recurrence relation in Section 3.4.3 and different explicit formulas in Section 3.4.4.

The method we are using was developed by D. Zeilberger [150], and it is known as the method of *creative telescoping*. We report here the main guidelines of this method and a rather simple example that explains how the method works. For a detailed description, we refer to the book [118, Chapter 6].

The object of interest is the sum

$$f_n = \sum_k F(n,k),$$

where $F(n,k)$ is a hypergeometric term in both arguments, namely $F(n+1,k)/F(n,k)$ and $F(n,k+1)/F(n,k)$ are both rational functions of n and k, and k runs over all integers. The method of creative telescoping provides a recurrence relation for the term f_n, starting from a recurrence relation for the summand $F(n,k)$. The name telescoping associated with it is motivated by the way in which the recurrence for $F(n,k)$ is determined.

More precisely, let N (resp. K) denote the forward shift operator in n (resp. k), *i.e.* $Ng(n,k) = g(n+1,k)$ and $Kg(n,k) = g(n,k+1)$. In [118, Theorem 6.2.1] it is proved that if $F(n,k)$ is a hypergeometric term, then F satisfies a non-trivial recurrence

$$p(n,N)F(n,k) = (K-1)G(n,k),$$

where $p(n,N) = a_0(n) + a_1(n)N + a_2(n)N^2 + \ldots + a_J(n)N^J$ with $a_i(n)$ polynomials for every $1 \leq i \leq J$, and $G(n,k)/F(n,k)$ a rational function.

In other words, Theorem 6.2.1 in [118] proves the existence of a "telescoped" recurrence of type

$$\sum_{j=0}^{J} a_j(n)F(n+j,k) = G(n,k+1) - G(n,k), \tag{3.9}$$

where $G(n,k) = R(n,k)F(n,k)$ for a rational function $R(n,k)$. From it, we can obtain a recurrence relation for the sum f_n. Indeed, since the coefficients on the left-hand side of Equation (3.9) are independent of k, we can sum (3.9) over all interger values of k and obtain

$$\sum_{j=0}^{J} a_j(n)f_{n+j} = 0, \tag{3.10}$$

provided that $G(n,k)$ has compact support in k for each n, namely the summand $G(n,k)$ vanishes automatically if $k < 0$ or $k > n$.

In [118, Section 6.3] it is shown how Zeilberger's algorithm of creating telescoping works, namely how to build the recurrence for $F(n,k)$ and the function $G(n,k)$, which is the certificate that (3.10) holds.

We report here only a few lines of a simple example of the application of this method: in fact, according to [118], very few examples can be worked out by hand. Along this dissertation to do all the hard computations we use the Maple package `SumTools` where the method of creative telescoping has been implemented: the command `Zeilberger` on input $F(n,k)$ gives as output the recurrence relation in (3.10) and the certificate function $G(n,k)$.

Example 3.1. *Given the summand $F(n,k) = \binom{n}{k}$, Zeilberger's algorithm in [118, Section 6.3] proves that the sum $f_n = \sum_k F(n,k)$ satisfies a recurrence of order $J = 1$.*

Indeed, the algorithm first proves that $F(n,k)$ satisfies a recurrence in telescoped form of order $J = 1$, which is

$$-2F(n,k) + F(n+1,k) = G(n,k+1) - G(n,k), \tag{3.11}$$

where the function $G(n,k)$ is expressed by

$$G(n,k) = \frac{k}{k-n-1}\binom{n}{k}.$$

Then, we sum the recurrence (3.11) *over all integers k. The right-hand side collapses to* 0 *and we find that*

$$-2f_n + f_{n+1} = 0, \qquad \textit{for } n \geq 1, \textit{ and } f_0 = 1\,.$$

Note that this recurrence is exactly the well-known relation $\sum_k \binom{n+1}{k} = 2\sum_k \binom{n}{k}$.

3.4.3 Recursive formula

The explicit closed formula for SB_n in Corollary 3.4.1 is extremely complicated, yet by means of the method described in Section 3.4.2 we can obtain from it a very simple recurrence.

Proposition 3.4.2. *The numbers SB_n are recursively defined by $SB_0 = 0$, $SB_1 = 1$ and for $n \geq 2$*

$$SB_n = \frac{11n^2 + 11n - 6}{(n+4)(n+3)} SB_{n-1} + \frac{(n-3)(n-2)}{(n+4)(n+3)} SB_{n-2}. \tag{3.12}$$

Proof. From Corollary 3.4.1, we can write $SB_n = \sum_j F_{SB}(n, j)$, where the summand $F_{SB}(n, j)$ given by Equation (3.8) on page 116 is hypergeometric, and prove the announced recurrence using the method of creative telescoping. More precisely, in the following we show the results of the calculation performed by MAPLE, where this approach has been implemented. By using $F_{SB}(n, j)$ as input, Zeilberger's method proves that

$$(n+5)(n+6)\, F_{SB}(n+2, j) - (11n^2 + 55n + 60)\, F_{SB}(n+1, j) - n(n-1)\, F_{SB}(n, j)$$
$$= G_{SB}(n, j+1) - G_{SB}(n, j), \tag{3.13}$$

where the expression of the certificate function $G_{SB}(n, j)$ is quite cumbersome and we do not report it here - it can be read from the MAPLE worksheet associated[2].

To complete the proof of the recurrence it is sufficient to sum both sides of Equation (3.13) over j, j ranging from 0 to $n+1$. Since the coefficients on the left-hand side of Equation (3.13) are independent of j, summing it over j gives

$$(n+5)(n+6)\, SB_{n+2} - (11n^2 + 55n + 60)\, SB_{n+1} - n(n-1)\, SB_n$$
$$- (11n^2 + 55n + 60)\, F_{SB}(n+1, n+1) - n(n-1)\, (F_{SB}(n, n) + F_{SB}(n, n+1)) \tag{3.14}$$

Summing the right-hand side over j gives a telescoping series, and simplifies as $G_{SB}(n, n+2) - G_{SB}(n, 0)$. From the explicit expression of $F_{SB}(n, j)$ and $G_{SB}(n, j)$, it is elementary to check that

$$F_{SB}(n+1, n+1) = F_{SB}(n, n) = F_{SB}(n, n+1) = G_{SB}(n, n+2) = G_{SB}(n, 0) = 0.$$

Summing Equation (3.13) therefore gives

$$(n+5)(n+6)\, SB_{n+2} - (11n^2 + 55n + 60)\, SB_{n+1} - n(n-1)\, SB_n = 0.$$

Shifting $n \mapsto n-2$ and rearranging finally gives the recurrence of Proposition 3.4.2. □

[2]See Appendix A.2

3.4.4 Alternative formulas

From the recurrence of Proposition 3.4.2, we can in turn prove closed formulas for semi-Baxter numbers, which have been conjectured in [23]. These are much simpler than the one given in Corollary 3.4.1 by the Lagrange inversion, and also very much alike the summation formula for Baxter numbers of Section 1.4.1 on page 38.

Theorem 3.4.3. *For any $n \geq 2$, the number SB_n of semi-Baxter permutations of length n satisfies*

$$
\begin{aligned}
SB_n &= \frac{24}{(n-1)n^2(n+1)(n+2)} \sum_{j=0}^{n} \binom{n}{j+2}\binom{n+2}{j}\binom{n+j+2}{j+1} \\
&= \frac{24}{(n-1)n^2(n+1)(n+2)} \sum_{j=0}^{n} \binom{n}{j+2}\binom{n+1}{j}\binom{n+j+2}{j+3} \\
&= \frac{24}{(n-1)n^2(n+1)(n+2)} \sum_{j=0}^{n} \binom{n+1}{j+3}\binom{n+2}{j+1}\binom{n+j+3}{j}.
\end{aligned}
$$

Proof. For each of the summation formulas given in Theorem 3.4.3, we apply the method of creative telescoping, as in the proof of Proposition 3.4.2. In all three cases, this produces a recurrence satisfied by these numbers, and every time we find exactly the recurrence given in Proposition 3.4.2.[3] Checking that the initial terms of the sequences coincide completes the proof. □

There is actually a fourth formula that has been conjectured in [23], namely

$$SB_n = \frac{24}{(n-1)n(n+1)^2(n+2)} \sum_{j=0}^{n} \binom{n+1}{j}\binom{n+1}{j+3}\binom{n+j+2}{j+2}.$$

Taking the multiplicative factors inside the sums, it is easy to see (for instance going back to the definition of binomial coefficients as quotients of factorials) that it is term by term equal to the second formula of Theorem 3.4.3.

As indicated in Section 3.1.1, in addition to the formulas reported in Theorem 3.4.3 above, other conjectural formulas for SB_n have been proposed in the literature, in different contexts.

The first one has already been shown in Proposition 3.2.7 and its validity results as a consequence of Theorem 3.2.8.

The second one is attributed to M. Van Hoeij and reported by D. Bevan in [22]. The conjecture is an explicit formula for semi-Baxter numbers that involves Apéry numbers $a_n = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{j}$ (sequence A005258 on [132]). We will prove the validity of this

[3] For further details see Appendix A.2

conjecture using the recursive formula for semi-Baxter numbers (Proposition 3.4.2) and the following recurrence satisfied by the Apéry numbers, for $n \geq 1$,

$$a_{n+1} = \frac{11n^2 + 11n + 3}{(n+1)^2}\, a_n + \frac{n^2}{(n+1)^2}\, a_{n-1}, \text{ with } a_0 = 1 \text{ and } a_1 = 3. \tag{3.15}$$

Proposition 3.4.4 ([22], Conjecture 13.2)**.** *For $n \geq 2$,*

$$SB_n = 24\, \frac{(5n^3 - 5n + 6)a_{n+1} - (5n^2 + 15n + 18)a_n}{5(n-1)n^2(n+2)^2(n+3)^2(n+4)}$$

Proof. For the sake of brevity we write $A(n) = 5n^3 - 5n + 6$ and $B(n) = 5n^2 + 15n + 18$ so that the statement becomes

$$SB_n = \frac{24(A(n)\, a_{n+1} - B(n)\, a_n)}{5(n-1)n^2(n+2)^2(n+3)^2(n+4)}. \tag{3.16}$$

The validity of Equation (3.16) is proved by induction on n: for $n = 2, 3$, it holds that $SB_2 = (A(2)a_3 - B(2)a_2)/(2000) = (36 \cdot 147 - 68 \cdot 19)/(2000) = 2$ and $SB_3 = (A(3)a_4 - B(3)a_3)/(23625) = (126 \cdot 1251 - 108 \cdot 147)/(23625) = 6$.

Then, suppose that Equation (3.16) is valid for $n-1$ and $n-2$. In order to prove it for n, consider the recursive formula of Equation (3.12) and substitute in it SB_{n-1} and SB_{n-2} by using Equation (3.16). Now, after some work of manipulation and by using Equation (3.15) we can write SB_n as in Equation (3.16). □

Remark 3.4.5. *With Corollary 3.4.1, Theorem 3.4.3 and Proposition 3.4.4, we get five expressions for the nth semi-Baxter number as a sum over j. Note that although the sums are equal, the corresponding summands in each sum are not (this is readily checked for $n = 8$ and $j = 5$ for instance). Therefore, Corollary 3.4.1, and Theorem 3.4.3, and Proposition 3.4.4 give five essentially different ways of expressing the semi-Baxter numbers.*

3.5 Asymptotics of the semi-Baxter numbers

From the first (or any) of the formulas provided in Theorem 3.4.3, we can derive the dominant asymptotics of SB_n, revealing that their generating function cannot be algebraic.

Corollary 3.5.1. *Let $\varphi = \frac{1}{2}(\sqrt{5} - 1)$. It holds that*

$$SB_n \sim A\, \frac{\mu^n}{n^6},$$

where $A = \frac{12}{\pi}\, 5^{-1/4} \varphi^{-15/2} \approx 94.34$ and $\mu = \varphi^{-5} = (11 + 5\sqrt{5})/2$.

Remark 3.5.2. *We point out that the result stated in Corollary 3.5.1 can altenatively be derived by considering Equation* (3.16) *combined with the asymptotic expansion of the Apéry number a_n, which has been studied by McIntosh in [111].*

Proof of Corollary 3.5.1. From Theorem 3.4.3, and letting

$$A(n;j) \equiv A(j) = \frac{24}{(n-1)n^2(n+1)(n+2)} \binom{n}{j+2} \binom{n+2}{j} \binom{n+j+2}{j+1},$$

we have that for all $n \geq 2$, $SB_n = \sum_{j=0}^{n} A(j)$. From this expression, the proof of Corollary 3.5.1 follows the same strategy as in [40, Section 2.6]. We first show that the summands $A(j)$ form a unimodal sequence, and we identify the value j_0 where $A(j)$ is maximal. Second, we find an estimate of $A(j)$ when j is close to j_0 (in an interval of width $\mathcal{O}(n^{1/2+\varepsilon})$). We next split the sum $\sum_{j=0}^{n} A(j)$ into two parts: the terms for j outside of this interval, and those for j inside. The third step is to prove that the first part is negligible with respect to the second part. And the fourth step is to estimate the second part of the sum, using the estimate of $A(j)$ when j is close to j_0.

This is achieved in a series of four lemmas below. Combining Lemmas 3.5.6 and 3.5.7, it then follows immediately that, for any $\varepsilon \in (0, 1/6)$, we have

$$SB_n = A\mu^n n^{-6} \left(1 + O\left(n^{3\varepsilon - 1/2}\right)\right),$$

where $\varphi = \frac{1}{2}(\sqrt{5}-1)$, $A = \frac{12}{\pi} 5^{-1/4} \varphi^{-15/2} \approx 94.34$ and $\mu = \varphi^{-5} = (11 + 5\sqrt{5})/2$. And this completes the proof of Corollary 3.5.1. □

Remark 3.5.3. *The computations presented in the lemmas below are actually simpler than those in [40, Section 2.6], because we are interested in the dominant asymptotics only. But following [40] more closely and keeping higher order terms in our expansions, one could establish further subdominant terms in the asymptotic expansion of SB_n.*

Lemma 3.5.4. *For large enough n, the numbers $A(j)$ form a unimodal sequence. Moreover, the maximum of this sequence occurs at $j = \varphi n + O(1)$, for $\varphi = \frac{1}{2}(\sqrt{5}-1)$.*

Proof. Let R be the ratio $R \equiv R(j) = A(j)/A(j+1) = (j+1)(j+2)(j+3)/((n+2-j)(n-j-2)(n+j+3))$. Rewrite it as $R = F \cdot G \cdot H$, where $F = (j+1)/(n-j-2)$, $G = (j+2)/(n-j+2)$ and $H = (j+3)/(n+j+3)$. Computing the derivatives in j of F, G and H, we check that each of them is an increasing function of j, thus R is increasing with j. Moreover, for n large enough, it holds that $R(0) \leq 1$ and $R(n-3) \geq 1$. Therefore, $A(j)$ is a unimodal (and actually log-concave) sequence and there exists $j_0 \in [0, n-3]$ such that $R(j_0) = 1$.

To find this value j_0 where $A(j)$ reaches its maximum, we simply set $R = 1$ and solve it for j. The equation $R = 1$ is quadratic in j and we choose the only solution lying in the range $[0, n-3]$. Expanding it in n gives $j_0 = \varphi n - \frac{3}{2} + \frac{3}{10}\sqrt{5} + O(n^{-1})$, with $\varphi = \frac{1}{2}(\sqrt{5}-1)$. □

Lemma 3.5.5. *Let $\varepsilon \in (0, 1/6)$ and $j = \varphi n + r$, with $r = s\sqrt{n}$ and $|s| \leq n^{\varepsilon}$. Then:*

$$A(j) = \frac{24}{\sqrt{8\pi^3}} \cdot \varphi^{-5n-9} \cdot n^{-13/2} \cdot e^{-(1/\varphi^3 + 1/2)\cdot s^2} \left(1 + O\left(n^{3\varepsilon - 1/2}\right)\right).$$

Moreover, this estimate is uniform in j.

Proof. We start with an estimate of each of the binomial coefficients occurring in $A(j)$, obtained using expansions of the Gamma function.

First, recall the Stirling expansion $\Gamma(n+1) = n^n \cdot \sqrt{2\pi n} \cdot e^{-n} \cdot \left(1 + O\left(\frac{1}{n}\right)\right)$ - see Section 1.3.8. Hence, with $j = \varphi n + r$ and $r = s\sqrt{n}$, we also have

$$\begin{aligned}\Gamma(j+3) &= (j+2)^{j+2} \cdot \sqrt{2\pi} \cdot \sqrt{\varphi n}\left(1 + O(n^{\varepsilon-1/2})\right) \cdot e^{-(j+2)} \cdot \left(1 + O\left(\frac{1}{j+2}\right)\right) \\ &= (j+2)^{j+2} \cdot \sqrt{2\pi\varphi n} \cdot e^{-(j+2)} \cdot \left(1 + O(n^{\varepsilon-1/2})\right),\end{aligned}$$

and similarly $\Gamma(n-j-1) = (n-j-2)^{n-j-2} \cdot \sqrt{2\pi(1-\varphi)n} \cdot e^{-(n-j-2)} \cdot \left(1 + O(n^{\varepsilon-1/2})\right)$.

Note that the above expansions for $\Gamma(j+3)$ and $\Gamma(n-j-1)$ are both uniform in j. This will also be the case for the other expansions obtained later in this proof, but we will not remark on it every time.

We further expand

$$\begin{aligned}\log \tfrac{n^n}{(j+2)^{j+2}(n-j-2)^{n-j-2}} &= n \log \tfrac{n}{n-j-2} - (j+2)\log \tfrac{j+2}{n-j-2} \\ &= n \log\left(\tfrac{1}{1-\varphi} \cdot \tfrac{1}{1 - \frac{s}{(1-\varphi)\sqrt{n}} - \frac{2}{(1-\varphi)n}}\right) \\ &\quad -(j+2)\log\left(\tfrac{\varphi}{1-\varphi} \cdot \tfrac{1 + \frac{s}{\varphi\sqrt{n}} + \frac{2}{\varphi n}}{1 - \frac{s}{(1-\varphi)\sqrt{n}} - \frac{2}{(1-\varphi)n}}\right) \\ &= n\left(\log \tfrac{1}{1-\varphi} + \tfrac{s}{(1-\varphi)\sqrt{n}} + \tfrac{s^2}{2(1-\varphi)^2 n} + \tfrac{2}{(1-\varphi)n} + O(n^{3\varepsilon-3/2})\right) \\ &\quad -(j+2)\left(\log \tfrac{\varphi}{1-\varphi} + \tfrac{s}{\varphi(1-\varphi)\sqrt{n}} + \tfrac{s^2(2\varphi-1)}{2\varphi^2(1-\varphi)^2 n} + \tfrac{2}{(1-\varphi)n} + O(n^{3\varepsilon-3/2})\right) \\ &= n \log \tfrac{1}{1-\varphi} + \tfrac{s\sqrt{n}}{(1-\varphi)} + \tfrac{s^2}{2(1-\varphi)^2} + \tfrac{2}{(1-\varphi)} + O(n^{3\varepsilon-1/2}) \\ &\quad -(j+2)\log \tfrac{\varphi}{1-\varphi} - \tfrac{s\sqrt{n}}{(1-\varphi)} - \tfrac{s^2}{2\varphi(1-\varphi)^2} - \tfrac{2}{(1-\varphi)} + O(n^{3\varepsilon-1/2}) \\ &= n\log\tfrac{1}{1-\varphi} - (j+2)\log\tfrac{\varphi}{1-\varphi} + \tfrac{s^2}{2(1-\varphi)^2}\left(1 - \tfrac{1}{\varphi}\right) + O(n^{3\varepsilon-1/2}).\end{aligned}$$

So, exponentiating, we obtain

$$\frac{n^n}{(j+2)^{j+2}(n-j-2)^{n-j-2}} = \left(\frac{1}{1-\varphi}\right)^n \left(\frac{1-\varphi}{\varphi}\right)^{j+2} \cdot e^{-s^2/2\varphi(1-\varphi)}\left(1 + O\left(n^{3\varepsilon-1/2}\right)\right).$$

This gives our estimate for the binomial coefficient $\binom{n}{j+2} = \frac{\Gamma(n+1)}{\Gamma(j+3)\Gamma(n-j-1)}$ occurring in $A(j)$:

$$\binom{n}{j+2} = \frac{1}{\sqrt{2\pi\varphi(1-\varphi)n}} \cdot \left(\frac{1}{1-\varphi}\right)^n \left(\frac{1-\varphi}{\varphi}\right)^{j+2} \cdot e^{-s^2/2\varphi(1-\varphi)}\left(1 + O\left(n^{3\varepsilon-1/2}\right)\right).$$

We need to compute similar estimates for the other two binomial coefficients occurring in $A(j)$, namely $\binom{n+2}{j}$ and $\binom{n+j+2}{j+1}$. Skipping the details, we obtain

$$\binom{n+2}{j} = \frac{1}{\sqrt{2\pi\varphi(1-\varphi)n}} \cdot \left(\frac{1}{1-\varphi}\right)^{n+2} \left(\frac{1-\varphi}{\varphi}\right)^{j} \cdot e^{-s^2/2\varphi(1-\varphi)} \left(1 + O\left(n^{3\varepsilon-1/2}\right)\right)$$

and $$\binom{n+j+2}{j+1} = \sqrt{\frac{1+\varphi}{2\pi\varphi n}} \cdot (1+\varphi)^{n+j+2}\varphi^{-(j+1)} \cdot e^{-s^2/2} \left(1 + O\left(n^{3\varepsilon-1/2}\right)\right).$$

Consequently, since $\varphi^2 = 1 - \varphi$ and $1 + \varphi = 1/\varphi$, we obtain

$$\binom{n}{j+2}\binom{n+2}{j}\binom{n+j+2}{j+1} = \frac{1}{\sqrt{8\pi^3}} \cdot \varphi^{-9} \cdot \varphi^{-5n} \cdot n^{-3/2} \cdot e^{-(1/\varphi^3+1/2)\cdot s^2} \left(1 + O\left(n^{3\varepsilon-1/2}\right)\right).$$

This finally gives us our claimed estimate for $A(j) = \frac{24}{(n-1)n^2(n+1)(n+2)}\binom{n}{j+2}\binom{n+2}{j}\binom{n+j+2}{j+1}$:

$$A(j) = \frac{24}{\sqrt{8\pi^3}} \cdot \varphi^{-5n-9} \cdot n^{-13/2} \cdot e^{-(1/\varphi^3+1/2)\cdot s^2} \cdot \left(1 + O\left(n^{3\varepsilon-1/2}\right)\right).$$

□

Lemma 3.5.6. *Let $\varepsilon \in (0, 1/6)$. Then for all $m \geq 0$,*

$$\sum_{|j-\varphi n|>n^{1/2+\varepsilon}} A(j) = o(\varphi^{-5n} n^{-m}).$$

Proof. Let $j_+ = \lfloor \varphi n + n^{1/2+\varepsilon} \rfloor$ and $j_- = \lceil \varphi n - n^{1/2+\varepsilon} \rceil$. By Lemma 3.5.4 (unimodality), we have

$$\sum_{|j-\varphi n|>n^{1/2+\varepsilon}} A(j) \leq (\varphi n - n^{1/2+\varepsilon}) \cdot A(j_-) + (n - \varphi n - n^{1/2+\varepsilon}) \cdot A(j_+).$$

Moreover, by Lemma 3.5.5, it holds that

$$A(j_\pm) = \frac{24}{\sqrt{8\pi^3}} \cdot \varphi^{-5n-9} \cdot n^{-13/2} \cdot e^{-(1/\varphi^3+1/2)\cdot n^{2\varepsilon}} \cdot \left(1 + O\left(n^{3\varepsilon-1/2}\right)\right).$$

It follows that for every $m \geq 0$,

$$(\varphi n - n^{1/2+\varepsilon})A(j_-) = o(\varphi^{-5n} n^{-m}) \text{ and } (n - \varphi n - n^{1/2+\varepsilon})A(j_+) = o(\varphi^{-5n} n^{-m}).$$

□

Lemma 3.5.7. *Let $\varepsilon \in (0, 1/6)$. Then,*

$$\sum_{|j-\varphi n|\leq n^{1/2+\varepsilon}} A(j) = A\mu^n n^{-6} \left(1 + O\left(n^{3\varepsilon-1/2}\right)\right),$$

where $A = \frac{12}{\pi} 5^{-1/4} \varphi^{-15/2} \approx 94.34$ and $\mu = \varphi^{-5} = (11 + 5\sqrt{5})/2$.

Proof. The estimate of Lemma 3.5.5 being uniform in j, we can write

$$\sum_{|j-\varphi n|\le n^{1/2+\varepsilon}} A(j) = \sum_{|j-\varphi n|\le n^{1/2+\varepsilon}} \frac{24}{\sqrt{8\pi^3}} \cdot \varphi^{-5n-9} \cdot n^{-13/2} \cdot e^{-(1/\varphi^3+1/2)\cdot(j-\varphi n)^2/n} \cdot \left(1+O\left(n^{3\varepsilon-1/2}\right)\right)$$
$$= \frac{24}{\sqrt{8\pi^3}} \cdot \varphi^{-5n-9} \cdot n^{-13/2} \cdot \left(1+O\left(n^{3\varepsilon-1/2}\right)\right) \sum_{|j-\varphi n|\le n^{1/2+\varepsilon}} e^{-(1/\varphi^3+1/2)\cdot(j-\varphi n)^2/n}.$$

Using the Euler-Maclaurin formula, we rewrite this sum as an integral as follows:

$$\sum_{|j-\varphi n|\le n^{1/2+\varepsilon}} e^{-(1/\varphi^3+1/2)\cdot(j-\varphi n)^2/n} = \sqrt{n} \cdot \int_{-\infty}^{\infty} e^{-(1/\varphi^3+1/2)s^2} \mathrm{d}s + o(n^{-m})$$
$$= \sqrt{n} \cdot \sqrt{\frac{2\pi\varphi^3}{\sqrt{5}}} + o(n^{-m}).$$

In this formula, m is any positive integer (since the error term in the Euler-Maclaurin formula is smaller than any polynomial in n). Note also that the leading $\sqrt{n}$ comes from changing integration variables from j to s (with $j = \varphi n + s\sqrt{n}$). The estimate given in Lemma 3.5.7 finally follows by elementary computations. □

Chapter 4

Strong-Baxter permutations

Plan of the chapter

In this chapter we deal with the enumeration of another family of pattern-avoiding permutations introduced in [G4, G5] under the name of *strong-Baxter permutations*. This family is closely related to Baxter and twisted Baxter permutations, like the family of semi-Baxter permutations of Chapter 3. More precisely, in Section 4.1 the family of strong-Baxter permutations is defined as the intersection between the family of Baxter permutations and the family of twisted Baxter permutations, and their enumerative sequence is called the sequence of strong-Baxter numbers. Thus, we now turn to a sequence point-wise smaller than the Baxter one [132, A001181], whereas Chapter 3 was about a sequence point-wise larger than the Baxter one (semi-Baxter permutations contain both Baxter and twisted Baxter permutations). The main result of Section 4.1 is a succession rule for generating strong-Baxter numbers. To our knowledge, these numbers has not appeared first in the literature, and they had not been recorded on [132].

Furthermore, strong-Baxter permutations do not form the only occurrence of strong-Baxter numbers: Section 4.2 illustrates that also a family of paths is enumerated by it. More precisely, we can provide an interpretation of these numbers in terms of labelled Dyck paths which form a subfamily of Baxter paths introduced in Chapter 2.

Finally, in Section 4.3 we study the functional equation obtained from the succession rule of Section 4.1.1. The solution of this functional equation is the generating function of strong-Baxter permutations, and a very surprising fact is that it results not to be D-finite. More precisely, in Section 4.3.2 we relate the solution of the strong-Baxter functional equation to the solution of an enumerative problem involving walks in the quarter plane [33, 37, 30]. This relation, and in particular the results of [30], allow us to conclude that the generating function of strong-Baxter permutations is not D-finite. Examples of non D-finite generating function are quite rare in the literature of pattern-avoiding permutations, see the analysis in [3, 83], thus providing a major reason for their study.

Lastly, we precise that the sequence of strong-Baxter numbers is now registered on [132] as sequence A281784.

4.1 Strong-Baxter numbers

Strong-Baxter numbers are defined in Section 4.1.1 in association with a family of pattern-avoiding permutations, which we call strong-Baxter permutations. In Section 4.1.1 we provide a succession rule for enumerating strong-Baxter permutations, and thus for generating the sequence of strong-Baxter numbers.

Eventually, in Section 4.1.2 we highlight a remarkable combinatorial property that this number sequence displays, and establish a close link among the sequences of semi-Baxter, Baxter and strong-Baxter numbers.

4.1.1 Definition and growth of strong-Baxter permutations

Definition 4.1.1. A *strong-Baxter permutation* is a permutation that avoids all three vincular patterns $2\,\underline{41}\,3$, $3\,\underline{14}\,2$ and $3\,\underline{41}\,2$.

Definition 4.1.2. The sequence of *strong-Baxter numbers* is the sequence that enumerates strong-Baxter permutations.

The pattern-avoidance definition makes it clear that the family of strong-Baxter permutations is the intersection of the two families of Baxter and twisted Baxter permutations. In that sense, such permutations "satisfy two Baxter conditions", hence the name strong-Baxter.

A succession rule for strong-Baxter permutations is given by the following proposition.

Proposition 4.1.3. *Strong-Baxter permutations can be generated by the following succession rule*

$$\Omega_{strong} = \begin{cases} (1,1) & \\ & \\ (h,k) & \rightsquigarrow (1,k),\dots,(h-1,k), \\ & (h,k+1), \\ & (h+1,1),\dots,(h+1,k). \end{cases}$$

Proof. As in the proofs of the previous chapter, we define a growth for strong-Baxter permutations performing local expansions on the right, as illustrated in Figure 4.1. Note that this is possible since removing the rightmost point from any strong-Baxter permutation, the permutation obtained still avoids all the three forbidden patterns.

Let π be a strong-Baxter permutation of length n. Recall that when inserting $a \in \{1,\dots,n+1\}$ on the right of any π of length n, we obtain the permutation $\pi' = \pi'_1 \dots \pi'_n \pi'_{n+1}$ where $\pi'_{n+1} = a$, $\pi'_i = \pi_i$ if $\pi_i < a$ and $\pi'_i = \pi_i + 1$ if $\pi_i \geq a$, and we use the notation $\pi \cdot a$ to denote π'. By definition, the active sites of π are the points a such that $\pi \cdot a$ is a strong-Baxter permutation.

By definition, any non-empty descent (resp. ascent) of π is a pair $\pi_i\pi_{i+1}$ such that there exists π_j that makes $\pi_j\pi_i\pi_{i+1}$ an occurrence of $2\,\underline{31}$ (resp. $2\,\underline{13}$). Then, the non-active sites a of π are characterized by the fact that $a \in (\pi_{i+1}, \pi_i]$ (resp. $a \in (\pi_i, \pi_j]$), for some occurrence $\pi_j\pi_i\pi_{i+1}$ of $2\,\underline{31}$ (resp. $2\,\underline{13}$). Note that in the case where $\pi_{n-1}\pi_n$ is a non-empty descent (resp. ascent), choosing $\pi_j = \pi_n + 1$ (resp. $\pi_j = \pi_n - 1$) always gives an occurrence of $2\,\underline{31}$ (resp. $2\,\underline{13}$), and it is the smallest (resp. largest) possible value of π_j for which $\pi_j \pi_{n-1} \pi_n$ is an occurrence of $2\,\underline{31}$ (resp. $2\,\underline{13}$).

To the strong-Baxter permutation π we assign the label (h, k) , where h (resp. k) is the number of active sites that are smaller than or equal to (resp. greater than) π_n. The permutation 1 has label $(1, 1)$, which is the axiom of Ω_{strong}. Now, we need to show that the labels of the permutations $\pi \cdot a$, when a runs over all active sites of π, are $(1, k), \ldots, (h-1, k), (h, k+1), (h+1, 1), \ldots, (h+1, k)$. So, let a be such an active site.

If $a < \pi_n$, then $\pi \cdot a$ ends with a non-empty descent, which is $(\pi_n + 1)a$. Recall that the site a of π is split into two sites of $\pi \cdot a$, one immediately above the point a and one immediately below. Then, all sites of $\pi \cdot a$ in $(a, \pi_n + 1]$ become non-active. All other active sites of π remain active in $\pi \cdot a$. Hence, a running over all active sites, $\pi \cdot a$ has label (i, k), with $1 \leq i < h$ being such that a is the ith active site from the bottom. Thus, the first line of the production of Ω_{strong} is obtained.

If $a = \pi_n$, no site of π becomes non-active, giving the label $(h, k + 1)$ in the production of Ω_{strong}.

Finally, if $a > \pi_n$, $\pi \cdot a$ ends with an ascent, which is $\pi_n a$. Because of the avoidance of $3\,\underline{14}\,2$, all sites of $\pi \cdot a$ in $(\pi_n, a - 1]$ become non-active, while others remain active if they were so in π (with a split into two active sites, one above and one below). Hence, when a runs over all the active sites of π, the permutations $\pi \cdot a$ give the missing labels in the production of Ω_{strong}: $(h+1, i)$ for $1 \leq i \leq k$, where i indicates that a is the ith active site from the top. □

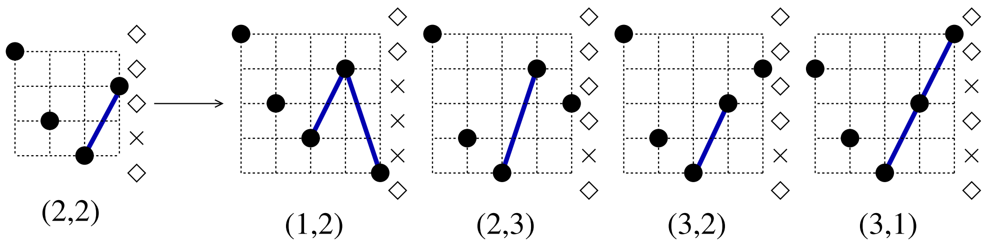


Figure 4.1: The growth of a strong-Baxter permutation. Active sites are marked with $\diamondsuit$, non-active sites by $\times$, and non-empty descents/ascents with bold blue lines.

The first few terms of the sequence of strong-Baxter numbers are obtained by iterating the rule Ω_{strong} and they are

$$1, 2, 6, 21, 82, 346, 1547, 7236, 35090, 175268, 897273, 4690392, 24961300, \ldots$$

As we pointed out this number sequence had not been registered in the On-line Encyclopedia of Integer Sequences before, so we have recorded it and now its reference is [132, sequence A281784].

4.1.2 A restriction of two Baxter succession rules

In this section we investigate further the connection between strong-Baxter numbers and Baxter numbers.

It is clear from their definition in terms of pattern avoidance that the families of Baxter and twisted Baxter permutations are supersets of the family of strong-Baxter permutations. Nevertheless, more evidence of these inclusions can be visualised by comparing their corresponding succession rules. Indeed, the strong-Baxter succession rule Ω_{strong} can be defined by restricting the succession rules for both Baxter permutations and twisted Baxter permutations described below.

Proposition 4.1.4. *Baxter permutations can be generated by inserting a new rightmost point, and this growth yields Ω_{Bax} as succession rule.*

Proof. To prove the above statement we recall that Baxter permutations are invariant under the 8 symmetries of the square [86]. Therefore, the growth provided in Proposition 1.4.8 on page 45 by means of a 90 degree clockwise rotation inserts a new rightmost point in any Baxter permutation. Then, Baxter permutations grow by insertion of a new rightmost point according to Ω_{Bax}, where h counts the number of RTL maxima, and k counts the number of RTL minima. □

Turning to twisted Baxter permutations, we obtain the following.

Proposition 4.1.5. *Twisted Baxter permutations can be generated by*

$$\Omega_{TwBax} = \begin{cases} (1,1) & \\ & \\ (h,k) & \rightsquigarrow (1,k), \dots, (h-1,k), \\ & \quad (h,k+1), \\ & \quad (h+k,1), \dots, (h+1,k). \end{cases}$$

Proof. As in the proof of Proposition 4.1.3, we let twisted Baxter permutations grow by performing local expansions on the right, as illustrated in Figure 4.2. This is possible since removing the rightmost element in a twisted Baxter permutation produces a twisted Baxter permutation.

Let π be a twisted Baxter permutation of length n. By definition an active site of π is an element a such that $\pi \cdot a$ avoids both $2\underline{41}3$ and $3\underline{41}2$. Analogously to the proof of Proposition 4.1.3, we consider any non-empty descent of π, namely a pair $\pi_i\pi_{i+1}$ such that there exists π_j that makes $\pi_j\pi_i\pi_{i+1}$ an occurrence of $2\underline{31}$. Then, the non-active sites a of π are characterized by the fact that $a \in (\pi_{i+1}, \pi_i]$, for some occurrence $\pi_j\pi_i\pi_{i+1}$ of $2\underline{31}$.

Then, as usual, we assign to π a label (h,k), where h (resp. k) is the number of active sites smaller than or equal to (resp. greater than) π_n. As in the proof of Proposition 4.1.3, the permutation 1 has label $(1,1)$ and now we describe the labels of the permutations $\pi \cdot a$ when a runs over all the active sites of π.

If $a < \pi_n$, then $\pi \cdot a$ ends with a non-empty descent, which is $(\pi_n + 1)a$ and, all sites of π in the range $(a, \pi_n + 1]$ become non-active. More precisely, due to the avoidance of $2\underline{41}3$

(resp. $3\,\underline{41}\,2$), all sites of π in the range $(a+1, \pi_n+1]$ (resp. $(a, \pi_n]$) become non-active. All other active sites of π remain active in $\pi \cdot a$. Hence, the permutation $\pi \cdot a$ gives the label (i,k), with $1 \leq i < h$, when a is the ith active site from the bottom. Thus, the first line of the production of Ω_{TwBax} is obtained.

If $a = \pi_n$, no sites of π become non-active, giving the label $(h, k+1)$.

If $a > \pi_n$, then $\pi \cdot a$ ends with an ascent and no site of π become non-active. Hence, we obtain the missing labels in the production of Ω_{TwBax}, which are $(h+k+1-i, i)$, for $1 \leq i \leq k$. Indeed, the permutation $\pi \cdot a$ has label $(h+k+1-i, i)$ if a is the ith active site from the top. □

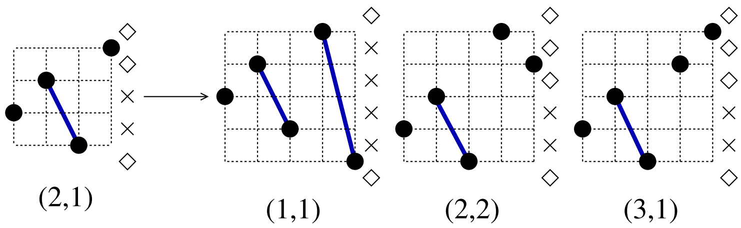


Figure 4.2: The growth of a twisted Baxter permutation (same notation as Figure 4.1).

We note that Ω_{TwBax} is not precisely the succession rule presented in [41] for twisted Baxter permutations, and reported in Proposition 1.4.11 of Section 1.4.4. Nonetheless, it is an obvious variant of it: starting from the rule Ω_{TBax} of [41], it is enough to replace every label (r,s) by $(s+1, r-1)$ to recover Ω_{TwBax}.

Now, we show how the succession rule Ω_{strong} can be obtained by combining the two Baxter rules Ω_{Bax} and Ω_{TwBax}. In fact, not only is the rule Ω_{strong} associated with the intersection of the two families of pattern-avoiding permutations, but it can be obtained simply by taking in the production of Ω_{Bax} and Ω_{TwBax} the label with minimum entries, as shown in the following representation:

$$\begin{array}{llllllllll}
\Omega_{Bax}: & (h,k) & \rightarrow & (1,k+1) & \ldots & (h-1,k+1) & (h,k+1) & (h+1,1) & \ldots & (h+1,k) \\
\Omega_{TwBax}: & (h,k) & \rightarrow & (1,k) & \ldots & (h-1,k) & (h,k+1) & (h+k,1) & \ldots & (h+1,k) \\
\Omega_{strong}: & (h,k) & \rightarrow & (1,k) & \ldots & (h-1,k) & (h,k+1) & (h+1,1) & \ldots & (h+1,k).
\end{array}$$

This is easily explained. Note first that in all three cases h (resp. k) records the number of active sites below (resp. above) the rightmost element of a permutation. Then, it is enough to remark that among the active sites of a permutation avoiding $2\,\underline{41}\,3$, the avoidance of $3\,\underline{41}\,2$ deactivates only sites above the rightmost element of the permutation, while the avoidance of $3\,\underline{14}\,2$ deactivates only sites below it.

This remark then allows us to enlarge the above chart as to include also the semi-Baxter succession rule of Chapter 3:

$$
\begin{array}{lllllllll}
\Omega_{semi}: & (h,k) & \rightarrow & (1,k+1) & \dots & (h-1,k+1) & (h,k+1) & (h+k,1) & \dots & (h+1,k) \\
\Omega_{Bax}: & (h,k) & \rightarrow & (1,k+1) & \dots & (h-1,k+1) & (h,k+1) & (h+1,1) & \dots & (h+1,k) \\
\Omega_{TwBax}: & (h,k) & \rightarrow & (1,k) & \dots & (h-1,k) & (h,k+1) & (h+k,1) & \dots & (h+1,k) \\
\Omega_{strong}: & (h,k) & \rightarrow & (1,k) & \dots & (h-1,k) & (h,k+1) & (h+1,1) & \dots & (h+1,k).
\end{array}
$$

Indeed, the growth of semi-Baxter permutations according to Ω_{semi} provided in Section 3.1.2 can be restricted to each of the above growths for Baxter and twisted-Baxter permutations, deactivating sites either below (for Baxter permutations) or above (for twisted Baxter permutations) the rightmost element of a permutation. Note that, similarly to Ω_{strong}, the rule Ω_{semi} can be obtained simply by taking in the production of Ω_{Bax} and Ω_{TwBax} the label with maximum entries.

On the one hand, this property reinforces the idea of Chapter 3 that the generalisation of Baxter numbers to semi-Baxter numbers is natural; on the other hand, it shows that the two Baxter specialisations are somehow "independent".

4.2 Another occurrence: strong-Baxter paths

Like we did for the Baxter and semi-Baxter numbers, we can provide a family of labelled Dyck paths which is enumerated by the strong-Baxter numbers. To do this, we recall that free up steps of a Dyck path are those not forming a DU factor.

Definition 4.2.1. A *strong-Baxter path* of semi-length n is a Dyck path of length $2n$ having all its free up steps labelled according to the following constraint: the leftmost free up step is labelled 1 and for every pair of free up steps (U', U''), with U' occurring before U'' and no free up step between them, the label of U'' is in the range $[1, k]$, where $k \geq 1$ is the sum of the label of U' with the number of UDU factors between U' (included) and U''.

It follows immediately from their definition that the family of Baxter paths (and, hence in turn that of semi-Baxter paths) contains strong-Baxter paths as subfamily (see also examples in Figure 4.3).

Our goal is to prove that they are enumerated by strong-Baxter numbers, which is immediate from the following proposition.

Proposition 4.2.2. *Strong-Baxter paths can be generated by* Ω_{strong}.

As usual, we provided a growth for strong-Baxter paths by insertion of a peak in the last descent. There is however a subtlety in the way this growth is encoded in the labels (h, k) with respect to the growth provided for Baxter paths in Chapter 2 and for semi-Baxter paths in Chapter 3.

First, remark that Ω_{Bax} is completely symmetric in h and k. Therefore, interchanging the interpretation of the two entries in each label, the same growth for Baxter structures

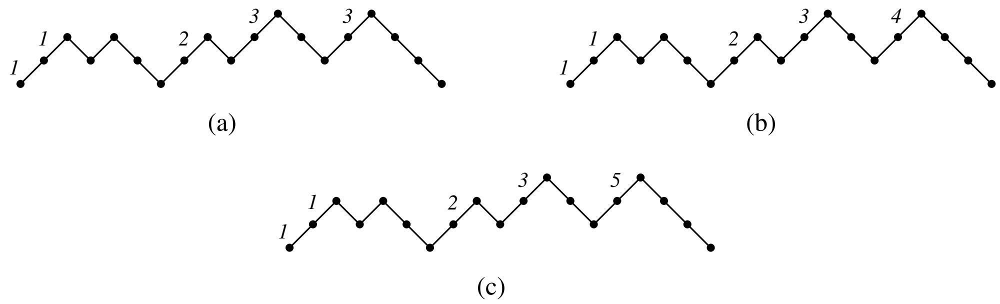


Figure 4.3: Labelling of the same Dyck path: (a) a strong-Baxter path; (b) a Baxter path, which is not a strong-Baxter path; (c) a semi-Baxter path, which is neither a Baxter path, nor a strong-Baxter path.

would also be encoded by Ω_{Bax}. More precisely, with respect to Proposition 2.1.9, interchanging the interpretation of each label (h,k) by taking h to be the number of steps of the last descent and k to be the label of the rightmost free up step plus the number of DU factors after it, the same growth for Baxter paths would also be encoded by Ω_{Bax} (Proposition 2.1.9). This "interchanged" interpretation of labels is the appropriate one to prove that the growth in Proposition 4.2.2 is a restriction of the one in Proposition 2.1.9.

Proof of Proposition 4.2.2. Similarly to Proposition 3.2.10 on page 109, we make strong-Baxter paths grow by insertion of a peak in the last descent, as shown in Figure 4.4. To any strong-Baxter path S, denoting e the label of its rightmost free up step $\overline{U}$, we assign the label (h,k), where h is the number of steps of the last descent of S and k is equal to e plus the number of UDU factors occurring after $\overline{U}$ (included).

The unique strong-Baxter path of semi-length 1, UD with U labelled 1, has label $(1,1)$, which is the axiom of Ω_{strong}.

Let S be a strong-Baxter path of label (h,k). The insertions of a peak in the last descent of S produce the following strong-Baxter paths, whose labels correspond to the production of Ω_{strong}.

a) We add a peak at the beginning of the last descent of S. The added U step is free, and receives a label which is any value in the range $[1,k]$. Denoting by i the label assigned to U, the produced path has label $(h+1,i)$.

b) We add a peak immediately after any down step of the last descent of S. The added U step following a down step, it carries no label. Therefore, if $S = w \cdot UD^h$ (with this U possibly labelled), the children of S are $w \cdot UD^j \boldsymbol{UD} D^{h-j}$ for $1 \leq j \leq h$. When $j=1$, one UDU factor is created after the rightmost free up step of S, and the obtained path has label $(h,k+1)$. Otherwise, no such factor is created, and the obtained paths have labels $(h-j+1,k)$ for $1 < j \leq h$. □

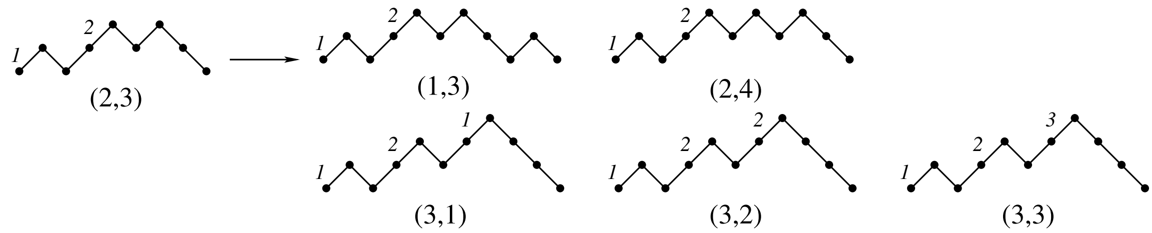


Figure 4.4: The growth of a strong-Baxter path of label $(2,3)$.

4.3 Strong-Baxter generating function

In this section we tackle the problem of studying the generating function of strong-Baxter permutations. First, in Section 4.3.1 we derive a functional equation whose solution is their generating function, by means of the succession rule given in Section 4.1.1. Then, we establish a link between this functional equation and a particular equation studied by A. Bostan et al. in [30] regarding some families of walks in the quarter plane. This link enables us to conclude the remarkable fact that the generating function of strong-Baxter permutations is not D-finite. Moreover, in Section 4.3.3 it allows us to provide information about the growth rate of strong-Baxter numbers.

4.3.1 Functional equation

Let $I_{h,k}(x) \equiv I_{h,k}$ denote the generating function of strong-Baxter permutations having label (h,k), with $h,k \geq 1$, and let $I(x;y,z) \equiv I(y,z) = \sum_{h,k\geq 1} I_{h,k}\, y^h z^k$.

Proposition 4.3.1. *The generating function $I(y,z)$ satisfies the following functional equation*

$$I(y,z) = xyz + \frac{x}{1-y}(y\,I(1,z) - I(y,z)) + xz\,I(y,z) + \frac{xyz}{1-z}(I(y,1) - I(y,z)). \quad (4.1)$$

Proof. From the growth of strong-Baxter permutations according to Ω_{strong} we write

$$\begin{aligned} I(y,z) &= xyz + x\sum_{h,k\geq 1} I_{h,k}\left((y+y^2+\ldots+y^{h-1})z^k + y^h z^{k+1} + y^{h+1}(z+z^2+\ldots+z^k)\right) \\ &= xyz + x\sum_{h,k\geq 1} I_{h,k}\left(\frac{1-y^{h-1}}{1-y}y\,z^k + y^h z^{k+1} + \frac{1-z^k}{1-z}y^{h+1}\,z\right) \\ &= xyz + \frac{x}{1-y}\left(y\,I(1,z) - I(y,z)\right) + xz\,I(y,z) + \frac{xyz}{1-z}\left(I(y,1) - I(y,z)\right). \end{aligned}$$

□

In order to study Equation (4.1), we write it in the kernel form

$$K(y,z)\,I(y,z) = xyz + \frac{xy}{1-y}\,I(1,z) + \frac{xyz}{1-z}\,I(y,1)\,,$$

where the kernel is

$$K(y,z) = 1 + x\left(\frac{1}{1-y} - z + \frac{yz}{1-z}\right). \tag{4.2}$$

We perform the substitutions $y = 1+a$ and $z = 1+b$ so that Equation (4.2) is rewritten as

$$K(1+a, 1+b) = 1 - x\,Q(a,b), \quad \text{where} \quad Q(a,b) = \frac{1}{a} + \frac{1}{b} + \frac{a}{b} + a + 2 + b. \tag{4.3}$$

We could look for the birational transformations Φ and Ψ in a and b that leave the kernel unchanged, trying to apply the obstinate variant of the kernel method of Section 3.3.2. These transformations are

$$\Phi : (a,b) \to \left(a, \frac{1+a}{b}\right), \quad \text{and} \quad \Psi : (a,b) \to \left(-\frac{b}{a(1+b)}, b\right).$$

One observes, using Maple for example, that the group generated by these two transformations is not of small order and it is likely to be of infinite order preventing us from using the obstinate kernel method to solve Equation (4.1).

Nevertheless, after the substitution $y = 1+a$ and $z = 1+b$, the kernel we obtain in Equation (4.3) resembles kernels of functional equations associated with the enumeration of families of walks in the quarter plane [37]. In order to make this link precise, in the next section we turn to the problem of counting walks in the quarter plane.

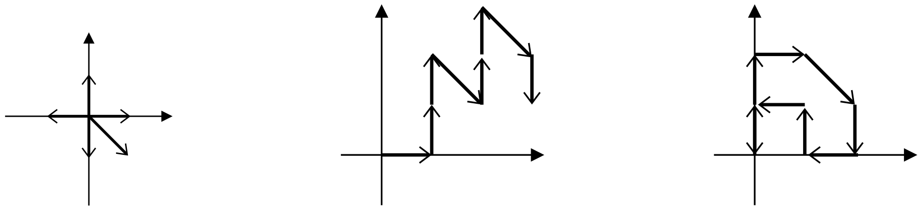
4.3.2 The case of walks confined in the quarter plane

In recent years a fair amount of attention has been paid to the problem of enumerating walks constrained in the quarter plane $\mathbb{N}^2$, see for instance [30, 33, 37, 112, 114] and their references. In particular, some of the cases analysed raised interest since their generating function appear not to be D-finite [30, 112, 114].

In order to study the nature of the generating function $I(1,1)$ of strong-Baxter numbers defined in Equation (4.1), we first consider one of the aforementioned problems that involves a particular family of walks in the quarter plane, whose set of steps is in $\{0, \pm 1\}^2$, and not contained in a half-plane. More precisely, we are interested in walks confined in the quarter plane $\mathbb{N}^2$ that use $\mathfrak{S}_1 = \{(-1,0), (0,-1), (1,-1), (1,0), (0,1)\}$ as step set - see Figure 4.5.

Proposition 4.3.2. *Let $W(t;a,b) \equiv W(a,b)$ be the generating function of walks confined in the quarter plane and using $\mathfrak{S}_1$ as step set, where t counts the number of steps and a (resp. b) records the x-coordinate (resp. y-coordinate) of the ending point. The function $W(a,b)$ satisfies the following functional equation*

$$W(a,b) = 1 + t\left(\frac{1}{a} + \frac{1}{b} + \frac{a}{b} + a + b\right)W(a,b) - \frac{t}{a}W(0,b) - t\frac{(1+a)}{b}W(a,0). \tag{4.4}$$



(a) (b) (c)

Figure 4.5: (a) The step set $\mathfrak{S}_1$; (b) a walk in the quarter plane using $\mathfrak{S}_1$ as step set; (c) an excursion in the quarter plane using $\mathfrak{S}_1$ as step set.

Proof. To prove Equation (4.4) we repeat a reasoning similar to the one used for Dyck prefixes in Chapter 1 (Section 1.3.7 on page 35). Indeed, let $\mathcal{W}_{\mathfrak{S}_1}$ be the family of walks in the quarter plane using $\mathfrak{S}_1$ as step set. We can add a final step to any walk of $\mathcal{W}_{\mathfrak{S}_1}$ of length n and obtain a walk of $\mathcal{W}_{\mathfrak{S}_1}$ of length $n+1$, provided that the added step is in $\mathfrak{S}_1$ and the constraint to remain in the quarter plane is not violated.

Thus, any walk of $\mathcal{W}_{\mathfrak{S}_1}$ is either the empty walk, which gives the contribute 1 in Equation (4.4), or obtained from a walk of $\mathcal{W}_{\mathfrak{S}_1}$ of smaller length by adding a final step. Owing to the steps of $\mathfrak{S}_1$, all the possible additions of a final step to a walk of $\mathcal{W}_{\mathfrak{S}_1}$ give the contribution $t(\bar{a}+\bar{b}+a\bar{b}+a+b)W(a,b)$, where as usual $\bar{a}$ denotes $1/a$ (resp. $\bar{b}=1/b$).

Yet we have to eliminate some walks from those obtained: precisely, the walks whose final step infringes the constraint to remain in $\mathbb{N}^2$. In particular, we should not consider the additions of the step $(-1,0)$ to the walks ending on the y-axis, giving the correction term $-t\bar{a}W(0,b)$, and neither the additions of the steps $(0,-1),(1,-1)$ to the walks ending on the x-axis, giving the correction term $-t(\bar{b}+a\bar{b})W(a,0)$. This concludes the proof of Equation (4.4). □

In the right-hand side of Equation (4.4), the coefficient of $W(a,b)$ is, up to a t factor, a Laurent polynomial in a and b. Such a polynomial is defined for any step set $\mathfrak{S}$, as $\chi_{\mathfrak{S}}(a,b) := \sum_{(i,j)\in\mathfrak{S}} a^i\, b^j$ and it is called *characteristic polynomial* of the step set $\mathfrak{S}$. It appears that

$$\chi_{\mathfrak{S}_1}(a,b) = \frac{1}{a}+\frac{1}{b}+\frac{a}{b}+a+b,$$

which resembles the polynomial $Q(a,b)$ of Equation (4.3). And, moreover, the kernel of Equation (4.4) strictly depends on the polynomial $\chi_{\mathfrak{S}_1}$, as it reads simply as $1-t\chi_{\mathfrak{S}_1}(a,b)$, that if compared to Equation (4.3) motivates our choice of the step set $\mathfrak{S}_1$.

In addition, in [37] all the characteristic polynomials $\chi_{\mathfrak{S}}(a,b)$, for every step set $\mathfrak{S} \subseteq \{0,\pm 1\}^2$, have been classified according to the cardinality of the group generated by the rational transformations that leave invariant $\chi_{\mathfrak{S}}(a,b)$.

It results that our step set $\mathfrak{S}_1$ forms one case among the 51 classified in [37] - and reported in [30, Table 1] - such that the group of rational transformations that leave invariant the characteristic polynomial has infinite order. Thus, the application of the obstinate variant of the kernel method, as done in Section 3.3.2, appears to fail to solve Equation (4.4).

Nevertheless, we can say which is the nature of the generating function $W(a, b)$ of Equation (4.4) thanks to the following remarkable result by A. Bostan et al. in [30].

Theorem 4.3.3 (Theorem 1, [30]). *Let $\mathfrak{S} \subseteq \{0, \pm 1\}^2$ be any of the 51 step sets in $\mathbb{N}^2$ such that the group of rational transformations that leave invariant its characteristic polynomial $\chi_{\mathfrak{S}}(a, b)$ has infinite order, and let $F_{\mathfrak{S}}(t; a, b)$ denote the generating function of walks in the quarter plane using $\mathfrak{S}$ as step set, where t marks the number of steps, and a (resp. b) the x-coordinate (resp. y-coordinate) of the final step. Then, the generating function $F_{\mathfrak{S}}(t; 0, 0)$ of $\mathfrak{S}$-excursions is not D-finite. In particular, the full generating function $F_{\mathfrak{S}}(t; a, b)$ is not D-finite.*

From Theorem 4.3.3 it follows that the full generating function solution of Equation (4.4), $W(a, b)$, is not D-finite: indeed, our step set $\mathfrak{S}_1$ forms the 23^{rd} entry of Table 1 in [30, Appendix] listing all the 51 cases.

Moreover, in Table 1 of [30, Appendix] it has been calculated also the asymptotic behaviour of the coefficients of the generating function $W(0, 0)$ of $\mathfrak{S}$-excursions (*i.e.* walks having the origin as ending point - see Figure 4.5(c)). In particular, such an estimate is obtained starting from a result of D. Denisov and V. Wachtel [62] that is stated in [30] as follows.

Theorem 4.3.4 (Theorem 4, [30]). *Let $\mathfrak{S} \subseteq \{0, \pm 1\}^2$ be the step set of a walk in the quarter plane $\mathbb{N}^2$, which is not contained in a half-plane. Let e_n denote the number of $\mathfrak{S}$-excursions of length n using only steps in $\mathfrak{S}$. Then, there exist constants K, ρ and α which depend only on $\mathfrak{S}$, such that:*

- *if the walk is aperiodic,* $e_n \sim K\, \rho^n\, n^\alpha$,
- *if the walk is periodic (then of period 2),* $e_{2n} \sim K\, \rho^{2n}\, (2n)^\alpha$, $\;e_{2n+1} = 0$.

Corollary 4.3.5 (Table 1-2, [30]). *The growth constant associated with the coefficients of $W(0, 0)$ is an algebraic number ρ_W, whose minimal polynomial is*

$$x^3 + x^2 - 18x - 43.$$

The numerical value for ρ_W is 4.729031538.

Now, we take inspiration from the above literature on walks confined in the quarter plane for solving Equation (4.1).

4.3.3 Strong-Baxter generating function, and the growth rate of its coefficients

In this section we exploit the link between Equation (4.1) having the generating function of strong-Baxter permutations as solution and Equation (4.4) provided for $\mathfrak{S}_1$-walks confined in the quarter plane. Indeed, modifying the step set considered in Equation (4.4), we can arrange that the kernel $K(1+a,1+b)$ of Equation (4.3) is **exactly** the kernel arising in the functional equation for enumerating a particular family of walks.

Let $\mathfrak{S}_2$ be the step (multi-)set $\{(-1,0),(0,-1),(1,-1),(1,0),(0,1),(0,0),(0,0)\}$. The difference with the step set of Proposition 4.3.2 is that we have added to the step set two copies of the trivial step $(0,0)$, which are distinguished (they can be considered as counterclockwise and clockwise loops for instance). An example of walk confined in the quarter plane using $\mathfrak{S}_2$ as step set is depicted in Figure 4.6, where we highlight the distinction between the two possibilities for the step $(0,0)$ by using different colours.

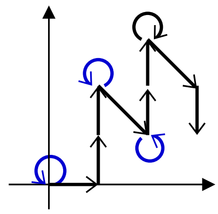


Figure 4.6: A walk in the quarter plane using $\mathfrak{S}_2$ as step set.

Lemma 4.3.6. *Let $W_2(t;a,b)$ be the generating function of walks confined in the quarter plane and using $\mathfrak{S}_2$ as step (multi-)set, where t counts the number of steps and a (resp. b) records the x-coordinate (resp. y-coordinate) of the ending point. The generating functions $W(t;a,b)$ of Proposition 4.3.2 and $W_2(t;a,b)$ are related by*

$$W_2(x;a,b) = W\left(\frac{x}{1-2x};a,b\right)\frac{1}{1-2x}. \tag{4.5}$$

Moreover, writing $J(x;a,b) := I(x;1+a,1+b)$ the generating function of strong-Baxter permutations, it holds that

$$J(x;a,b) = (1+a)(1+b)\,x\,W_2(x;a,b). \tag{4.6}$$

Proof. First, $\mathfrak{S}_2$-walks can be described starting from $\mathfrak{S}_1$-walks as follows: a $\mathfrak{S}_2$-walk is a (possibly empty) sequence of trivial steps, followed by a $\mathfrak{S}_1$-walk where, after each step, we insert a (possibly empty) sequence of trivial steps. This simple combinatorial argument proves Equation (4.5).

Next, consider the kernel form of the original equation (4.1) for the strong-Baxter generating function, after substituting $y = 1+a$ and $z = 1+b$, which is

$$\begin{aligned}(1-&xQ(a,b))J(x;a,b) = \\ &x(1+a)(1+b) - x\,\frac{1+a}{a}\,J(x;0,b) - x\,\frac{(1+a)(1+b)}{b}\,J(x;a,0).\end{aligned} \tag{4.7}$$

Compare it to the kernel form of Equation (4.4)

$$(1-t(Q(a,b)-2))W(t;a,b) = 1 - \frac{t}{a}\,W(t;0,b) - t\,\frac{(1+a)}{b}\,W(t;a,0). \tag{4.8}$$

Substituting t with $\frac{x}{1-2x}$ in (4.8), and multiplying this equation by $(1+a)(1+b)x$, we see that $(1+a)(1+b)\,x\,W_2(x;a,b)$ satisfies (4.7), proving our claim. □

Now, by using Theorem 4.3.3, this easily gives the following result.

Theorem 4.3.7. *The generating function $I(1,1)$ of strong-Baxter permutations is not D-finite. The same holds for the refined generating function $I(a+1,b+1)$.*

Proof. With the notation of Lemma 4.3.6, our goal is to prove that $J(x;a,b)$ and $J(x;0,0)$ are not D-finite. Recall from Equation (4.6) that $J(x;a,b) = (1+a)(1+b)\,x\,W_2(x;a,b)$, so $J(x;0,0)$ and $W_2(x;0,0)$ (resp. $J(x;a,b)$ and $W_2(x;a,b)$) coincide up to a factor x (resp. $(1+a)(1+b)\,x$). Therefore, proving that $W_2(x;0,0)$ and $W_2(x;a,b)$ are non D-finite is enough.

Since by Theorem 4.3.3 it is proved that neither $W(t;a,b)$ nor $W(t;0,0)$ are D-finite, and being $\frac{1}{1-2x}$ and $\frac{x}{1-2x}$ rational series, it follows from Equation (4.5) that both $W_2(x;a,b)$ and $W_2(x;0,0)$ are not D-finite. □

Moreover, some information on the asymptotic behaviour of the number of strong-Baxter permutations can be derived starting from Corollary 4.3.5. Indeed, by Corollary 4.3.5 the growth constant ρ_W associated with the generating function $W(t;0,0)$ is an algebraic number. Then, we show below that the growth constant of strong-Baxter numbers is closely related to the number ρ_W.

Corollary 4.3.8. *The growth constant for the strong-Baxter numbers is*

$$\rho_W + 2 \approx 6.729031538.$$

Proof. From Lemma 4.3.6, it follows that

$$I(x;1,1) = x\,W_2(x;0,0) = x\,W\left(\frac{x}{1-2x};0,0\right)\frac{1}{1-2x}.$$

By Corollary 4.3.5, $1/\rho_W$ is the radius of convergence of $W(t;0,0)$. Whereas the radius of convergence of $g(x) = x/(1-2x)$ is $\frac{1}{2}$, and $\lim_{\substack{x\to 1/2\\ x<1/2}} g(x) = +\infty > \frac{1}{\rho_W}$.

So, the composition $W(g(x);0,0)$ is supercritical (see [79, p. 411]), and the radius of convergence of $W(\frac{x}{1-2x};0,0)$ is

$$g^{-1}\left(\frac{1}{\rho_W}\right) = \frac{1}{\rho_W + 2}\,. \tag{4.9}$$

Since $1/(\rho_W+2)$ is smaller than the radius of convergence of $1/(1-2x)$, which is $1/2$, the radius of convergence of

$$x\,W\left(\frac{x}{1-2x};0,0\right)\frac{1}{1-2x} = I(x;1,1),$$

is equal to the one reported in Equation (4.9) proving our claim. □

Chapter 5

Inversion sequences and steady paths

Plan of the chapter

This chapter can be divided in two parts according to the combinatorial objects treated. The first part (Section 5.1) focuses on inversion sequences, in particular on some of the families introduced in the literature in [58, 110]. The second part (Sections 5.2 and 5.3) is dedicated to the combinatorial structures enumerated by the sequence A113227 on [132]. The link between the two parts is a particular family of inversion sequences among those listed in the first part that has been proved in [110] to be enumerated by this sequence.

In Section 5.1 we show some families of inversion sequences that have been introduced in [110], and whose enumeration was set as an open problem. In particular, two of them have been conjectured to be enumerated by the recurring Catalan and Baxter numbers, motivating our choice to study them. We succeed in resolving different cases conjectured in [110], and specifically those whose associated families form a hierarchy if ordered by inclusion. Moreover, we discover that not only is this hierarchy visible on these combinatorial objects, but also it is mirrored by a chain of succession rules that are provided for enumerating each of these families. Nonetheless we have not been able to generate the family at the top of this hierarchy by means of a succession rule of the chain (Section 5.1.5). This family has, however, a major role in this chapter being its enumeration sequence A113227.

Section 5.2 provides general results on sequence A113227 [132]. The numbers of this sequence have been called in this dissertation *powered Catalan numbers*, because of the succession rule provided in Section 5.1.5. Section 5.2.1 collects combinatorial structures enumerated by A113227 and results about this sequence already known in the literature [19, 49]. On the other hand, Section 5.2.2 shows a new result about the enumeration of permutations avoiding the vincular pattern $1\,\underline{23}\,4$, which provides a second succession rule generating powered Catalan numbers.

In Section 5.3 we provide a new occurrence of the sequence of powered Catalan numbers in terms of lattice paths, and we relate it to the combinatorial structures presented in Section 5.2. More precisely, we introduce in Section 5.3.1 the family of steady paths, proving that they are enumerated by the powered Catalan numbers. Then, Sections 5.3.2

and 5.3.3 are to establish a link between the new combinatorial interpretation of the powered Catalan numbers and the already known objects of Section 5.2.1. For some of them we prove a one-to-one correspondence with steady paths, and for some others we only succeed in conjecturing it. It may appear as if there are two different kinds of powered Catalan structures, as we point out in Section 5.3.4. Finally, in Section 5.3.5 we generalise the definition of steady paths and find out two super families of lattice paths of non-trivial enumeration.

5.1 A hierarchy of inversion sequences

In this section we introduce a hierarchy of families of inversion sequences that are related to well-known number sequences occurring along this work. Figure 5.1 depicts a chain that involves inversion sequences ordered by inclusion and their corresponding enumeration sequences. It is well worth recalling that the families of inversion sequences of Figure 5.1 were first introduced in [58, 110], yet the majority of them were only conjectured in [110] to be counted by these numbers.

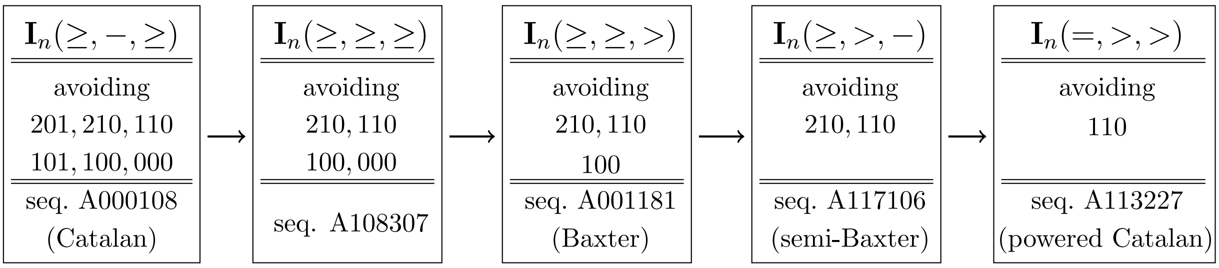


Figure 5.1: A chain of inversion sequences ordered by inclusion, with their characterisation in terms of pattern avoidance, and their enumerative sequence.

Moreover, we recall that inversion sequences have already been introduced in Chapter 3 as a particular occurrence of the semi-Baxter number sequence following the conjectures in [110]. In this section we investigate further the families of inversion sequences conjectured in [110] to be counted by the recurring Catalan, Baxter, and semi-Baxter numbers. To this purpose, we recall here the formal definition of inversion sequence and the notion of pattern containment, as defined in Section 3.2.2 on page 107.

By Definition 3.2.4 and 3.2.5 of Section 3.2.2, an inversion sequence of size n is an integer sequence $(e_1, e_2, \ldots, e_n)$ satisfying $0 \leq e_i < i$, for all $i \in \{1, 2, \ldots, n\}$. Given a word $q = q_1 \ldots q_k \in \{0, \ldots, k-1\}^k$, an inversion sequence contains the pattern q, if there exist k indices $i_1 < \ldots < i_k$ such that the word $e_{i_1} \ldots e_{i_k}$ is order-isomorphic to q. Given a triple of relations (ρ_1, ρ_2, ρ_3), the set $\mathbf{I}_n(\rho_1, \rho_2, \rho_3)$ comprises all inversion sequences $(e_1, e_2, \ldots, e_n)$ of size n such that there are no three indices $i < j < k$, for which it holds $e_i \, \rho_1 \, e_j \, \rho_2 \, e_k$ and $e_i \, \rho_3 \, e_k$.

The objective of the section is twofold: on the one hand we provide (and/or collect) enumerative results about the families of inversion sequences of Figure 5.1, on the other hand we aim to treat all these families in a unified way. More precisely, in each subsection we first provide a simple combinatorial characterisation for the corresponding family of inversion sequences, and then we show a recursive growth that yields a succession rule.

The main noticeable property of the succession rules provided in Sections 5.1.1, 5.1.2, 5.1.3, and 5.1.4 is that they reveal the hierarchy of Figure 5.1 at the abstract level of succession rules.

More precisely, the recursive construction (and consequently the succession rule) provided for each family of these sections is obtained by extending the construction (and thus the succession rule) of the immediately smaller family. We start from $\mathbf{I}_n(\geq, -, \geq)$ in Section 5.1.1, and by defining a growth for this family of inversion sequences we provide a new succession rule for Catalan numbers. Then, by progressive generalisations we achieve a succession rule for the family of inversion sequences $\cup_n \mathbf{I}_n(\geq, >, -)$.

Nevertheless, an important remark involves the set $\mathbf{I}_n(=, >, >)$, the rightmost element of the chain of Figure 5.1. Indeed, although its combinatorial characterisation appears to generalise naturally the family $\mathbf{I}_n(\geq, >, -)$, we do not have a growth for $\mathbf{I}_n(=, >, >)$ that generalises the one of $\mathbf{I}_n(\geq, >, -)$, motivating the study of its enumerative sequence separately in Section 5.2.

5.1.1 Inversion sequences $\mathbf{I}_n(\geq, -, \geq)$

The first family of inversion sequences considered is $\mathbf{I}(\geq, -, \geq) = \cup_n \mathbf{I}_n(\geq, -, \geq)$. This family of inversion sequences is conjectured in [110] to be counted by the well-known sequence of Catalan numbers [132, A000108]. It is well worth specifying at the very beginning of this section that this enumerative result has recently been proved independently from us by D. Kim and Z. Lin in [97]. Nevertheless, we will provide another proof of this fact in Proposition 5.1.3 by showing that there exists a growth for $\mathbf{I}(\geq, -, \geq)$ according to the well-known Catalan succession rule Ω_{Cat}. Consequently, there exists a (recursive) bijection between the family $\mathbf{I}(\geq, -, \geq)$ and any other Catalan family that can be generated according to Ω_{Cat}.

Then, we show a direct bijection between $\mathbf{I}(\geq, -, \geq)$ and a family of pattern-avoiding permutations, which results thus enumerated by Catalan numbers. Finally, we provide a growth for the family $\mathbf{I}(\geq, -, \geq)$ that yields a succession rule different from Ω_{Cat}, yet appropriated to be generalised in the following sections.

The family $\mathbf{I}(\geq, -, \geq)$ has a simple characterisation in terms of inversion sequences avoiding patterns of length three.

Proposition 5.1.1. *An inversion sequence is in* $\mathbf{I}(\geq, -, \geq)$ *if and only if it avoids* 000*,* 100*,* 101*,* 110*,* 201 *and* 210*.*

Proof. The proof is rather straightforward, since containing e_i, e_j, e_k such that $e_i \geq e_j, e_k$, with $i < j < k$, is equivalent to containing the listed patterns. □

In addition to the above characterisation, we remark the following combinatorial description of $\mathbf{I}_n(\geq, -, \geq)$, as it will be useful to define a growth according to the Catalan succession rule Ω_{Cat}.

Proposition 5.1.2. *Any inversion sequence* $e = (e_1, \ldots, e_n)$ *is in* $\mathbf{I}_n(\geq, -, \geq)$ *if and only if for any* i*, with* $1 \leq i < n$*,*

$$\text{if } e_i \text{ forms a weak descent, i.e. } e_i \geq e_{i+1}, \text{ then } e_i < e_j, \text{ for all } j > i + 1.$$

Proof. One direction is clear. The other direction can be proved by contrapositive. More precisely, suppose there are three indices $i < j < k$, such that $e_i \leq e_j, e_k$. Then, if $e_j = e_{i+1}$, then e_i forms a weak descent and the fact that $e_i \geq e_k$ concludes the proof. Otherwise, since $e_i \geq e_j$, there must be an index i', with $i \leq i' < j$, such that $e_{i'}$ forms a weak descent and $e_{i'} \geq e_k$. This concludes the proof. □

Proposition 5.1.3. *The family* $\mathbf{I}(\geq, -, \geq)$ *can be generated by* Ω_{Cat}*,*

$$\Omega_{Cat} = \begin{cases} (1) \\ (k) \rightsquigarrow (1), (2), \ldots, (k), (k+1). \end{cases}$$

Proof. We prove the statement by showing a growth for the family $\mathbf{I}(\geq, -, \geq)$ according to Ω_{Cat}. Given an inversion sequence $e = (e_1, \ldots, e_n)$, we define the inversion sequence $e \odot i$ as the sequence $(e_1, \ldots, e_{i-1}, i-1, e_i, \ldots, e_n)$, where the entry $i-1$ is inserted in position i, for some $1 \leq i \leq n+1$, and the entries $e_i, \ldots, e_n$ are shifted rightwards by one. By definition of inversion sequence, $i-1$ is the largest possible value that the ith entry can assume. And moreover, letting $e' := e \odot i$, it holds that $e'_j = e_{j-1} < j-1$, for all $j > i$; namely the index i is the rightmost index such that $e'_k = k-1$. For example, if $i = 4$ and $e = (0, 0, 1, 3, 4, 5)$, then $e \odot i = (0, 0, 1, 3, 3, 4, 5)$.

Then, given an inversion sequence $e \in \mathbf{I}_n(\geq, -, \geq)$ by removing from e the rightmost entry whose value is equal to its position minus one, we obtain an inversion sequence that is in $\mathbf{I}_{n-1}(\geq, -, \geq)$. Note that $e_1 = 0$ for every $e \in \mathbf{I}_n(\geq, -, \geq)$, thus such an entry always exists.

Now, we can describe a growth for the family $\mathbf{I}(\geq, -, \geq)$ by inserting an entry $i-1$ in position i. By Proposition 5.1.2, since the entry $i-1$ forms a weak descent in $e \odot i$, the inversion sequence $e \odot i$ is in the set $\mathbf{I}_{n+1}(\geq, -, \geq)$ if and only if $e_{i+1}, \ldots, e_n > i-1$. Then, we call *active positions* all the indices i, with $1 \leq i \leq n+1$, such that $e \odot i$ is in $\mathbf{I}_{n+1}(\geq, -, \geq)$. According to this definition, $n+1$ and n are always active positions: indeed, both $e \odot (n+1) = (e_1, \ldots, e_n, n)$ and $e \odot n = (e_1, \ldots, n-1, e_n)$ are in $\mathbf{I}_{n+1}(\geq, -, \geq)$.

We label an inversion sequence $e \in \mathbf{I}_n(\geq, -, \geq)$ with (k), where k is the number of its active positions decreased by one. The smallest inversion sequence has label (1), which is the axiom of rule Ω_{Cat}.

Now, we show that given an inversion sequence $e \in \mathbf{I}_n(\geq, -, \geq)$ with label (k), the labels of $e \odot i$, where i ranges over all the active positions, are precisely the label productions of (k) in Ω_{Cat}.

More precisely, let $i_1, \ldots, i_{k+1}$ be the active positions of e from left to right. Note that $i_k = n$ and $i_{k+1} = n+1$. We argue that, for any $1 \leq j \leq k+1$, the active positions of the inversion sequence $e \odot i_j = (e_1, \ldots, i_j - 1, e_{i_j}, \ldots, e_n)$ are $i_1, \ldots, i_{j-1}$, $n+1$ and $n+2$. Indeed, on the one hand any position which is non-active in e is still non-active in $e \odot i_j$. On the other hand, by Proposition 5.1.2, the index i_j becomes non-active in $e \odot i_j$, since the entry $e_{i_j} < i_j$ by definition. Similarly, any position i_h, with $i_j < i_h < n+1$, which is active in e becomes non-active in $e \odot i_j$. Indeed, by Proposition 5.1.2, the index i_h is active in $e \odot i_j$ if and only if $e_{i_h}, \ldots, e_n > i_h$. Since $e_{i_h} < i_h$ by definition, the active positions of $e \odot i_j$ are $i_1, \ldots, i_{j-1}$, $n+1$ and $n+2$. Hence, $e \odot i_j$ has label (j), for any $1 \leq j \leq k+1$. □

An interesting result is the characterisation of the set $\mathbf{I}_n(\geq, -, \geq)$ as the set of left inversion tables of some pattern-avoiding permutations of length n. In particular, it results that the family of pattern-avoiding permutations $AV(1\underline{23}, 2\underline{14}3)$ forms a new occurrence of the Catalan numbers. In order to prove it, the following lemma is useful.

Lemma 5.1.4. *Let* $\mathtt{T}$ *be the mapping of Definition 1.3.6. Let* $\pi \in \mathcal{S}_n$*, and* $\mathtt{T}(\pi) = (t_1, \ldots, t_n)$*. For every* $i < j$*, if* $\pi_i > \pi_j$*, then* $t_i > t_j$*.*

Proof. It follows straightforward by the definitions of inversion and left inversion table of π (Definition 1.3.6 on page 27). □

Proposition 5.1.5. *For any* n*, the set* $\mathbf{I}_n(\geq, -, \geq)$ *and the set* $AV_n(1\underline{23}, 2\underline{14}3)$ *are in bijection.*

Proof. We prove our statement by using the mapping $\mathtt{R} \circ \mathtt{T}$, which is a bijection between the family of permutations and integer sequences $(e_1, \ldots, e_n)$ such that $0 \leq e_i < i$ - see Proposition 1.3.7 in Section 1.3.3 on page 27. Then, we will show that the restriction of the bijection $\mathtt{R} \circ \mathtt{T}$ to the family $AV(1\underline{23}, 2\underline{14}3)$ yields a bijection with the family $\mathbf{I}(\geq, -, \geq)$. Precisely, we want to prove that for every n, an inversion sequence is in the set $\{(\mathtt{R} \circ \mathtt{T})(\pi) : \pi \in AV_n(1\underline{23}, 2\underline{14}3)\}$ if and only if it is in $\mathbf{I}_n(\geq, -, \geq)$.

⇒) We prove the contrapositive: if $e \notin \mathbf{I}_n(\geq, -, \geq)$, then $\pi = (\mathtt{R} \circ \mathtt{T})^{-1}(e)$ contains $1\underline{23}$ or $2\underline{14}3$. Let $t = (t_1, \ldots, t_n) = (e_n, \ldots, e_1)$. Then, t is the left inversion table of a permutation $\pi \in \mathcal{S}_n$, *i.e* $\mathtt{T}(\pi) = t$. Since $e \notin \mathbf{I}_n(\geq, -, \geq)$, there exist three indices, $i < j < k$, such that $t_i \leq t_k$ and $t_j \leq t_k$.

Without any loss of generality, we can suppose that there is no index h, such that $j < h < k$ and $t_i \leq t_h$ and $t_j \leq t_h$. Namely t_k is the leftmost entry of t that is greater than both t_i and t_j. Then, we have two possibilities:

1. either $j + 1 = k$,
2. or $j + 1 \neq k$, and in this case it holds that $t_j > t_{k-1}$ or $t_i > t_{k-1}$.

First, by using Lemma 5.1.4 it follows from $t_i \leq t_j$ and $t_i \leq t_k$ that $\pi_i < \pi_k$ and $\pi_j < \pi_k$.

Now, we prove that both in case 1. and in case 2. above we have $\pi \notin AV_n(1\underline{23}, 2\underline{14}3)$.

1. Let us consider the subsequence $\pi_i\pi_j\pi_{j+1}$. We have $\pi_i < \pi_{j+1}$ and $\pi_j < \pi_{j+1}$. If also $\pi_i < \pi_j$, then it forms a $1\underline{23}$.

   Otherwise, it must hold that $\pi_i > \pi_j$, and thus $t_j < t_i \leq t_{j+1}$. Since the pair (π_i, π_j) is an inversion of π and $t_i \leq t_{j+1}$, there must be a point π_s on the right of π_{j+1} such that (π_{j+1}, π_s) is an inversion and (π_i, π_s) is not. Thus, $\pi_i\pi_j\pi_{j+1}\pi_s$ forms a $2\underline{14}3$.

2. First, if $t_j > t_{k-1}$, consider the subsequence $\pi_i\pi_j\pi_{k-1}\pi_k$. It follows that $t_{k-1} < t_k$, since $t_j \leq t_k$, and by Lemma 5.1.4 $\pi_{k-1} < \pi_k$. In addition, we know that $\pi_j < \pi_k$. Then, $\pi_j\pi_{k-1}\pi_k$ forms an occurrence of $1\underline{23}$ if $\pi_j < \pi_{k-1}$. Otherwise, it must hold that $\pi_j > \pi_{k-1}$. As in case 1. the pair (π_j, π_{k-1}) is an inversion, and $t_j \leq t_k$. Therefore, there must be an element π_s on the right of π_k such that (π_k, π_s) is an inversion and (π_j, π_s) is not. Hence $\pi_j\pi_{k-1}\pi_k\pi_s$ forms a $2\underline{14}3$.

   Now, suppose $t_j \leq t_{k-1}$, and consider the subsequence $\pi_i\pi_j\pi_{k-1}\pi_k$. According to case 2. it must be that $t_i > t_{k-1}$, and since $t_i \leq t_k$, it holds that $t_{k-1} < t_k$. By Lemma 5.1.4 both $\pi_j < \pi_{k-1}$ and $\pi_{k-1} < \pi_k$ hold. Thus, $\pi_j\pi_{k-1}\pi_k$ forms an occurrence of $1\underline{23}$.

$\Leftarrow$) By contrapositive, if a permutation π contains $1\underline{23}$ or $2\underline{14}3$, then $e = (\mathtt{R} \circ \mathtt{T})(\pi)$ is not in $\mathbf{I}_n(\geq, -, \geq)$.

- If $1\underline{23} \preccurlyeq \pi$, there must be two indices i and j, with $i < j$, such that $\pi_i\pi_j\pi_{j+1}$ forms an occurrence of $1\underline{23}$. We can assume that no points $\pi_{i'}$ between π_i and π_j are such that $\pi_{i'} < \pi_i$. Otherwise we consider $\pi_{i'}\pi_j\pi_{j+1}$ as our occurrence of $1\underline{23}$.

  Then, two relations hold: $t_i \leq t_{j+1}$ and $t_j \leq t_{j+1}$, and thus $e \notin \mathbf{I}_n(\geq, -, \geq)$.

- If $2\underline{14}3 \preccurlyeq \pi$, and $1\underline{23} \not\preccurlyeq \pi$, there must be three indices i, j and k, with $i < j < j+1 < k$, such that $\pi_i\pi_j\pi_{j+1}\pi_k$ forms an occurrence of $2\underline{14}3$. We can assume that no points $\pi_{i'}$ between π_i and π_j are such that $\pi_{i'} < \pi_i$. Indeed, in case $\pi_{i'} < \pi_j$ held, $\pi_{i'}\pi_j\pi_{j+1}$ would be an occurrence of $1\underline{23}$; whereas, if $\pi_j < \pi_{i'} < \pi_i$, we could consider $\pi_{i'}\pi_j\pi_{j+1}\pi_k$ as our occurrence of $2\underline{14}3$.

  Then, as above $t_j \leq t_{j+1}$, and $t_i + 1 \leq t_{j+1}$ because (π_i, π_j) is an inversion of π. Nevertheless, (π_{j+1}, π_k) is an inversion of π as well, and $\pi_i < \pi_k$. Thus, $t_i \leq t_{j+1}$ and $e \notin \mathbf{I}_n(\geq, -, \geq)$. □

We specify that although inversion sequences are actually a coding for permutations, it is not easy to characterise the families $\mathbf{I}(\rho_1, \rho_2, \rho_3) = \cup_n \mathbf{I}_n(\rho_1, \rho_2, \rho_3)$ in terms of families of pattern-avoiding permutations. In fact, the above example is the unique one provided in this section, and to our knowledge in the literature, about the families $\mathbf{I}(\rho_1, \rho_2, \rho_3)$.

Corollary 5.1.6. *The family $AV(1\underline{23}, 2\underline{14}3)$ is enumerated by Catalan numbers.*

Furthermore, we can provide a new succession rule for generating the family $\mathbf{I}(\geq, -, \geq)$: the growth we provide in the following is remarkable as we will show to allow generalisations.

Proposition 5.1.7. *The family* $\mathbf{I}(\geq,-,\geq)$ *can be generated by*

$$\Omega_{\mathbf{I}(\geq,-,\geq)} = \begin{cases} (1,1) \\ \\ (h,k) & \rightsquigarrow (0,k+1)^h, \\ & \quad (h+k,1),(h+k-1,2),\ldots,(h+1,k). \end{cases}$$

Proof. We prove the statement by showing a growth for the family $\mathbf{I}(\geq,-,\geq)$ that defines the succession rule $\Omega_{\mathbf{I}(\geq,-,\geq)}$. Given an inversion sequence $e=(e_1,\ldots,e_n)$, we define this growth by adding a new rightmost entry. Obviously, it is different from the one provided in the proof of Proposition 5.1.3.

Let $\max(e)$ be the maximum value among the entries of e. And let $\mathrm{mwd}(e)$ be the maximum value of the set of all entries e_i that form a weak descent of e; if e has no weak descents, then $\mathrm{mwd}(e) := -1$. By Proposition 5.1.1, since e avoids 100, 201 and 210, the value $\max(e)$ is e_{n-1} or e_n. In particular, if $\max(e) = e_{n-1} \geq e_n$, then $\max(e) = \mathrm{mwd}(e)$.

By proposition 5.1.2, it follows that $f = (e_1,\ldots,e_n,p)$ is an inversion sequence of $\mathbf{I}_{n+1}(\geq,-,\geq)$ if and only if $\mathrm{mwd}(e) < p \leq n$. Moreover, if $\mathrm{mwd}(e) < p \leq \max(e)$, then e_n forms a new weak descent of f, and $\mathrm{mwd}(f)$ becomes the value e_n; whereas, if $\max(e) < p \leq n$, then $\mathrm{mwd}(f) = \mathrm{mwd}(e)$ since the weak descents of f and e coincide.

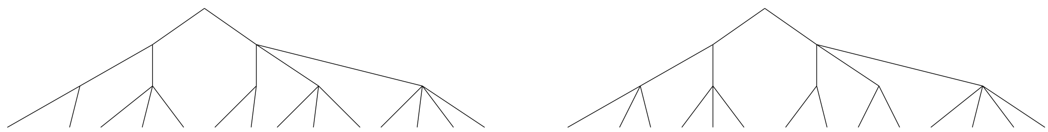
Now, we assign to any $e \in \mathbf{I}_n(\geq,-,\geq)$ the label (h,k), where $h = \max(e) - \mathrm{mwd}(e)$ and $k = n - \max(e)$. In other words, h (resp. k) marks the number of possible additions smaller than or equal to (resp. greater than) the maximum entry of e.

The sequence $e=(0)$ has no weak descents, thus it has label $(1,1)$, which is the axiom of $\Omega_{\mathbf{I}(\geq,-,\geq)}$. Let e be an inversion sequence of $\mathbf{I}_n(\geq,-,\geq)$ with label (h,k). The labels of the inversion sequences of $\mathbf{I}_{n+1}(\geq,-,\geq)$ produced by adding a rightmost entry p to e are

- $(0,k+1)$, for any $p \in \{\mathrm{mwd}(e)+1,\ldots,\max(e)\}$,
- $(h+k,1),(h+k-1,2),\ldots,(h+1,k)$, when $p = n, n-1, \ldots, \max(e)+1$,

which concludes the proof that $\mathbf{I}(\geq,-,\geq)$ grows according to $\Omega_{\mathbf{I}(\geq,-,\geq)}$. □

It is well worth noticing that although the above succession rule $\Omega_{\mathbf{I}(\geq,-,\geq)}$ generates the well-known Catalan numbers, we do not have knowledge of this succession rule in the literature. Figure 5.2 compares the first levels of the two generating trees respectively associated with rules Ω_{Cat} (on the left) and $\Omega_{\mathbf{I}(\geq,-,\geq)}$ (on the right).



(a) (b)

Figure 5.2: The first four levels of the Catalan generating trees: (a) corresponding to Ω_{Cat}; (b) corresponding to $\Omega_{\mathbf{I}(\geq,-,\geq)}$.

5.1.2 Inversion sequences $\mathbf{I}_n(\geq,\geq,\geq)$

Following the hierarchy of Figure 5.1, the next family of inversion sequences we turn to is $\mathbf{I}(\geq,\geq,\geq) = \cup_n \mathbf{I}_n(\geq,\geq,\geq)$. This family was conjectured in [110] to be counted by sequence A108307 on [132], which is defined as the enumerative sequence of set partitions of $\{1,\dots,n\}$ that avoid enhanced 3-crossings [40]. In [40, Proposition 2] it is proved that the number $E_3(n)$ of these set partitions is given by $E_3(0) = E_3(1) = 1$, and by the recursive relation

$$8(n+3)(n+1)E_3(n) + (7n^2+53n+88)E_3(n+1) - (n+8)(n+7)E_3(n+2) = 0\,, \quad (5.1)$$

which holds for all $n \geq 0$.

Thus, the first terms of sequence A108307 according to recurrence (5.1) are

$$1, 1, 2, 5, 15, 51, 191, 772, 3320, 15032, 71084, 348889, 1768483, 9220655, 49286863, \dots$$

This section is aimed at proving that the enumerative sequence of the family $\mathbf{I}(\geq,\geq,\geq)$ is indeed the sequence A108307 [132]. To our knowledge, we provide the first proof of this result conjectured in [110], and we succeed in this goal by means of a succession rule that generalises the one in Proposition 5.1.7.

To start, we provide a combinatorial description of the family $\mathbf{I}(\geq,\geq,\geq)$, which will be used later to define a growth in the proof of Proposition 5.1.10. Then, with the aim of enumerating completely the family $\mathbf{I}(\geq,\geq,\geq)$, we solve the functional equation that the succession rule of Proposition 5.1.10 yields. The method we use to treat this functional equation is the so-called obstinate kernel method (Section 1.4.5). And, in Proposition 5.1.15 we successfully prove that the enumerative sequence of family $\mathbf{I}(\geq,\geq,\geq)$ is A108307 on [132].

As Figure 5.1 shows, the family $\mathbf{I}(\geq,\geq,\geq)$ includes $\mathbf{I}(\geq,-,\geq)$ as subfamily. For instance, the inversion sequence $(0,0,1,1,4,2,6,5)$ is both in $\mathbf{I}_8(\geq,-,\geq)$ and in $\mathbf{I}_8(\geq,\geq,\geq)$, while $(0,1,0,1,4,2,3,5)$ is not in $\mathbf{I}_8(\geq,-,\geq)$ despite being in $\mathbf{I}_8(\geq,\geq,\geq)$. The following characterisation makes explicit this fact.

Proposition 5.1.8. *An inversion sequence belongs to* $\mathbf{I}(\geq,\geq,\geq)$ *if and only if it avoids* 000*,* 100*,* 110 *and* 210*.*

Proof. The proof is a quick check that containing e_i, e_j, e_k such that $e_i \geq e_j \geq e_k$, with $i<j<k$, is equivalent to containing the above patterns. □

Recalling that $\mathbf{I}(\geq,-,\geq)$ coincides with the family of inversions sequences that avoid 000, 100, 101, 110, 201 and 210, the inclusion of $\mathbf{I}(\geq,-,\geq)$ in $\mathbf{I}(\geq,\geq,\geq)$ is now clear. In addition, Proposition 5.1.8 proves the following property stated in [110, Observation 7].

Remark 5.1.9. *Let any inversion sequence* $e=(e_1,\dots,e_n)$ *be decomposed in two subsequences* e^{LTR}*, which is the increasing sequence of left-to-right maxima of* e *(*i.e. *entries*

e_i such that $e_i > e_j$, for all $j < i$), and e^{bottom}, which is the (possibly empty) sequence comprised of all the remaining entries of e.

Then, an inversion sequence e is in the set $\mathbf{I}(\geq,\geq,\geq)$ if and only if it e^{LTR} and e^{bottom} are both strictly increasing sequences.

Proposition 5.1.10. *The family $\mathbf{I}(\geq,\geq,\geq)$ can be generated by*

$$\Omega_{\mathbf{I}(\geq,\geq,\geq)} = \begin{cases} (1,1) \\ \\ (h,k) & \rightsquigarrow (0,k+1),(1,k+1),\ldots,(h-1,k+1), \\ & \quad (h+k,1),(h+k-1,2),\ldots,(h+1,k). \end{cases}$$

Proof. We prove the statement by showing a growth for the family $\mathbf{I}(\geq,\geq,\geq)$ according to $\Omega_{\mathbf{I}(\geq,\geq,\geq)}$, that extends the growth of Proposition 5.1.7. Then, as in Proposition 5.1.7, we define a growth by adding a new rightmost entry.

Let $\mathrm{last}(e)$ be the rightmost entry of e^{bottom}, if there is any, otherwise $\mathrm{last}(e) := -1$. By Remark 5.1.9, it follows that $f = (e_1,\ldots,e_n,p)$ is an inversion sequence of $\mathbf{I}_{n+1}(\geq,\geq,\geq)$ if and only if $\mathrm{last}(e) < p \leq n$. Moreover, if $\mathrm{last}(e) < p \leq \max(e)$, where $\max(e)$ is the maximum value of e, then p cannot be a left-to-right maximum and $\mathrm{last}(f)$ becomes p; whereas, if $\max(e) < p \leq n$, then $\mathrm{last}(f) = \mathrm{last}(e)$ since p is a left-to-right maximum of f.

Now, we assign to any $e \in \mathbf{I}_n(\geq,\geq,\geq)$ the label (h,k), where $h = \max(e) - \mathrm{last}(e)$ and $k = n - \max(e)$. Note that the label interpretation extends the one in the proof of Proposition 5.1.7.

The sequence $e = (0)$ of size one has label $(1,1)$, which is the axiom of $\Omega_{\mathbf{I}(\geq,\geq,\geq)}$. Let e be an inversion sequence of $\mathbf{I}_n(\geq,\geq,\geq)$ with label (h,k). The labels of the inversion sequences of $\mathbf{I}_{n+1}(\geq,\geq,\geq)$ produced by adding a rightmost entry p to e are

- $(0,k+1),\ldots,(h-1,k+1)$, when $p = \max(e),\ldots,\mathrm{last}(e)+1$,
- $(h+k,1),(h+k-1,2),\ldots,(h+1,k)$, when $p = n, n-1,\ldots,\max(e)+1$,

which concludes the proof that $\mathbf{I}(\geq,\geq,\geq)$ grows according to $\Omega_{\mathbf{I}(\geq,\geq,\geq)}$. □

Now, we turn to a functional equation whose solution is the generating function for the family $\mathbf{I}(\geq,\geq,\geq)$.

By using the usual standard technique, we can translate the succession rule $\Omega_{\mathbf{I}(\geq,\geq,\geq)}$ into a functional equation.

For $h,k \geq 0$, let $A_{h,k}(x) \equiv A_{h,k}$ denote the size generating function for inversion sequences of the family $\mathbf{I}(\geq,\geq,\geq)$ having label (h,k). The rule $\Omega_{\mathbf{I}(\geq,\geq,\geq)}$ translates into a functional equation for the generating function $A(x;y,z) \equiv A(y,z) = \sum_{h,k\geq 0} A_{h,k}\, y^h z^k$.

Proposition 5.1.11. *The generating function $A(y,z)$ satisfies the following functional equation*

$$A(y,z) = xyz + \frac{xz}{1-y}\left(A(1,z) - A(y,z)\right) + \frac{xyz}{z-y}\left(A(y,z) - A(y,y)\right) . \tag{5.2}$$

Proof. Starting from the succession rule $\Omega_{\mathbf{I}(\geq,\geq,\geq)}$ we write

$$\begin{aligned}
A(y,z) &= xyz + x\sum_{h,k\geq 0} A_{h,k}\left((1+y+\cdots+y^{h-1})z^{k+1} + (y^{h+k}z + y^{h+k-1}z^2 + \cdots + y^{h+1}z^k)\right)\\
&= xyz + x\sum_{h,k\geq 0} A_{h,k}\left(\frac{1-y^h}{1-y}\,z^{k+1} + \frac{1-\left(\frac{y}{z}\right)^k}{1-\frac{y}{z}}y^{h+1}z^k\right)\\
&= xyz + \frac{xz}{1-y}\left(A(1,z)-A(y,z)\right) + \frac{xyz}{z-y}\left(A(y,z)-A(y,y)\right)\,. \qquad \square
\end{aligned}$$

Equation (5.2) is a linear functional equation with two catalytic variables, y and z, in the sense of Zeilberger [151]. An extremely similar functional equation has been solved in Section 3.3 on page 111 about semi-Baxter permutations.

Such a similarity thus allows us to repeat the same steps of the procedure of Section 3.3 for finding the generating function of semi-Baxter permutations. In other words, we are going to use the obstinate variant of the kernel method to provide an expression of the generating function for $\mathbf{I}(\geq,\geq,\geq)$.

First, set $y = 1+a$ and collect all the terms with $A(y,z)$ of Equation (5.2) to obtain the kernel form

$$K(a,z)A(1+a,z) = xz(1+a) - \frac{xz}{a}A(1,z) - \frac{xz(1+a)}{z-1-a}A(1+a,1+a), \tag{5.3}$$

where the kernel is

$$K(a,z) = 1 - \frac{xz}{a} - \frac{xz(1+a)}{z-1-a}.$$

For brevity, we refer to the right-hand side of Equation (5.3) by using the expression $\mathcal{H}(x,a,z,A(1,z),A(1+a,1+a))$, where

$$\mathcal{H}(x_0,x_1,x_2,w_0,w_1) = x_0x_2(1+x_1) - \frac{x_0x_2}{x_1}w_0 - \frac{x_0x_2(1+x_1)}{x_2-1-x_1}w_1.$$

The kernel equation $K(a,z)=0$ is quadratic in z, and thus it has two roots. As usual, we denote $Z_+(a)$ and $Z_-(a)$ the two solutions of $K(a,z)=0$ with respect to z,

$$\begin{aligned}
Z_+(a) &= \frac{1}{2}\frac{a+x-a^2x-\sqrt{a^2-2ax-2a^3x+x^2-2a^2x^2+a^4x^2-4a^2x}}{x}\\
&= (1+a) + (1+a)^2x + \frac{(1+a)^4}{a}\,x^2 + \frac{(a^2+3a+1)(1+a)^4}{a^2}\,x^3 + O(x^4),
\end{aligned}$$

$$\begin{aligned}
Z_-(a) &= \frac{1}{2}\frac{a+x-a^2x+\sqrt{a^2-2ax-2a^3x+x^2-2a^2x^2+a^4x^2-4a^2x}}{x}\\
&= \frac{a}{x} - (1+a)a - (1+a)^2x - \frac{(1+a)^4}{a}\,x^2 - \frac{(a^2+3a+1)(1+a)^4}{a^2}\,x^3 + O(x^4)\,.
\end{aligned}$$

Note that the kernel root Z_- is not a well-defined power series in x. Whereas, the other kernel root, Z_+, is a well-defined power series in x, whose coefficients are Laurent polynomials in a. Then, the function $A(1+a, Z_+)$ is a well-defined power series in x and the right-hand side of Equation (5.3) is equal to zero by setting $z = Z_+$, *i.e.*

$$\mathcal{H}(x, a, Z_+, A(1, Z_+), A(1+a, 1+a)) = 0.$$

As in Section 3.3, we follow the steps of the obstinate variant of the kernel method and exploit the birational transformations that leave $K(a, z)$ unchanged.

Examining the kernel shows that the transformations

$$\Phi : (a, z) \to \left(\frac{z-1-a}{1+a}, z\right) \quad \text{and} \quad \Psi : (a, z) \to \left(a, \frac{z + za - 1 - a + a^2 + a^3}{z-1-a}\right)$$

leave the kernel unchanged and generate a group of order 12 - see Figure 5.3.[1]

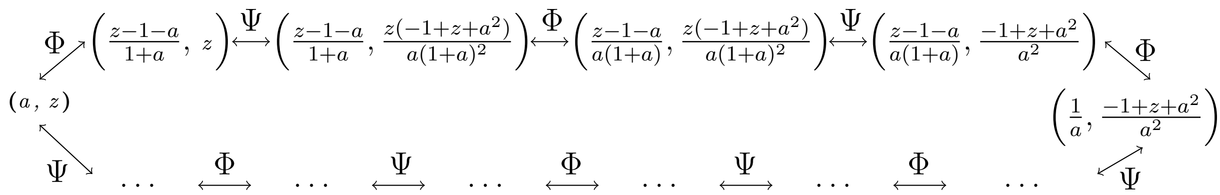


Figure 5.3: The action of Φ and Ψ on the pair (a, z).

Among all the elements of this group we consider those pairs $(f_1(a, z), f_2(a, z))$ such that $f_1(a, Z_+)$ and $f_2(a, Z_+)$ are well-defined power series in x with Laurent polynomial coefficients in a. More precisely, we consider the pairs $(f_1(a, z), f_2(a, z))$ depicted in Figure 5.3.

Consequently, these pairs share the property that $A(1 + f_1(a, Z_+), f_2(a, Z_+))$ is a convergent power series in x. Whence, substituting each of these pairs for (a, z) in Equation (5.3), we obtain a system of six equations whose left-hand sides are all equal to 0 by setting $z = Z_+$,

$$\begin{cases} 0 = \mathcal{H}(x, a, Z_+, A(1, Z_+), A(1+a, 1+a)) \\ 0 = \mathcal{H}\left(x, \frac{Z_+-1-a}{1+a}, Z_+, A(1, Z_+), A\left(1 + \frac{Z_+-1-a}{1+a}, 1 + \frac{Z_+-1-a}{1+a}\right)\right) \\ 0 = \mathcal{H}\left(x, \frac{Z_+-1-a}{1+a}, \frac{Z_+(-1+Z_++a^2)}{a(1+a)^2}, A\left(1, \frac{Z_+(-1+Z_++a^2)}{a(1+a)^2}\right), A\left(1 + \frac{Z_+-1-a}{1+a}, 1 + \frac{Z_+-1-a}{1+a}\right)\right) \\ 0 = \mathcal{H}\left(x, \frac{Z_+-1-a}{a(1+a)}, \frac{Z_+(-1+Z_++a^2)}{a(1+a)^2}, A\left(1, \frac{Z_+(-1+Z_++a^2)}{a(1+a)^2}\right), A\left(1 + \frac{Z_+-1-a}{a(1+a)}, 1 + \frac{Z_+-1-a}{a(1+a)}\right)\right) \\ 0 = \mathcal{H}\left(x, \frac{Z_+-1-a}{a(1+a)}, \frac{-1+Z_++a^2}{a^2}, A\left(1, \frac{-1+Z_++a^2}{a^2}\right), A\left(1 + \frac{Z_+-1-a}{a(1+a)}, 1 + \frac{Z_+-1-a}{a(1+a)}\right)\right) \\ 0 = \mathcal{H}\left(x, \frac{1}{a}, \frac{-1+Z_++a^2}{a^2}, A\left(1, \frac{-1+Z_++a^2}{a^2}\right), A\left(1 + \frac{1}{a}, 1 + \frac{1}{a}\right)\right). \end{cases} \tag{5.4}$$

[1] Appendix B.1 shows all the elements of this group.

Now, by eliminating all unknowns except for $A(1+a,1+a)$ and $A(1,1+\bar{a})$, where as usual $\bar{a}$ denotes $1/a$, System (5.4) reduces to the following equation,

$$(1+a)^3A(1+a,1+a)-\frac{(1+a)^3}{a^4}A\left(1,1+\bar{a}\right)+P(a,Z_+)=0, \qquad \text{where} \tag{5.5}$$

$$P(a,z)=\frac{(-z+1+a)(a-1)}{a^6}\Big(a^8+4a^7+7a^6+a^5z+8a^5-z^2a^4+2za^4+8a^4+3za^3 \\ +7a^3-a^3z^2+6a^2z+4a^2-3z^2a^2+a+az^3-4z^2a+6az-z^2+2z\Big).$$

Appendix B.1 shows in detail how to obtain the above reduction.

Now, as in Section 3.3, the form of Equation (5.5) allows us to separate its terms according to the power of a:

- $(1+a)^3A(1+a,1+a)$ is a power series in x with polynomial coefficients in a whose lowest power of a is 0,
- $A(1,1+\bar{a})$ is a power series in x with polynomial coefficients in $\bar{a}$ whose highest power of a is 0; consequently, since $(1+a)^3\bar{a}^4=a^{-4}+3a^{-3}+3a^{-2}+a^{-1}$, we obtain that $(1+a)^3\,\bar{a}^4A(1,1+\bar{a})$ is a power series in x with polynomial coefficients in $\bar{a}$ whose highest power of a is -1.

Then, when we expand the series $-P(a,Z_+)$ as a power series in x, the non-negative powers of a in the coefficients must be equal to those of $(1+a)^3A(1+a,1+a)$, while the negative powers of a come from $-(1+a)^3\,\bar{a}^4A(1,1+\bar{a})$.

In order to have a better expression for the series $P(a,z)$, we perform a further substitution setting $z=w+1+a$. More precisely, let $W\equiv W(x;a)$ be the power series in x defined by $W=Z_+-(1+a)$. We have the following expression for $Q(a,W):=-P(a,Z_+)$,

$$\begin{aligned}Q(a,W)=-P(a,W+1+a)=&\left(-\frac{1}{a^6}-\frac{3}{a^5}-\frac{3}{a^4}-\frac{1}{a^3}+1+3a+3a^2+a^3\right)W\\&+\left(\frac{1}{a^5}+\frac{1}{a^4}-\frac{1}{a}-1\right)W^2\\&+\left(\frac{1}{a^6}-\frac{1}{a^4}+\frac{1}{a^3}-\frac{1}{a}\right)W^3\\&+\left(-\frac{1}{a^5}+\frac{1}{a^4}\right)W^4.\end{aligned} \tag{5.6}$$

Since the kernel annihilates if $z=W+1+a$, namely $K(a,W+1+a)=0$, the function W is recursively defined by

$$W=x\bar{a}(W+1+a)(W+a+a^2)\,. \tag{5.7}$$

Therefore, by using Equation (5.6) and Equation (5.7), we can express the generating function for $\mathbf{I}(\geq,\geq,\geq)$ as follows.

Theorem 5.1.12. *Let $W(x;a) \equiv W$ be the unique formal power series in x such that*

$$W = x\bar{a}(W + 1 + a)(W + a + a^2).$$

The series solution $A(y,z)$ of Equation (5.2) satisfies

$$A(1+a, 1+a) = \left[\frac{Q(a,W)}{(1+a)^3}\right]^{\geq},$$

where the function $Q(a,W)$ is defined by Equation (5.6), and the notation $[Q(a,W)/(1+a)^3]^{\geq}$ stands for the formal power series in x obtained by considering only those terms in the series expansion of $Q(a,W)/(1+a)^3$ that have non-negative powers of a.

Note that in Theorem 5.1.12, W and $Q(a,W)$ are algebraic series in x whose coefficients are Laurent polynomials in a. It follows that $A(1+a,1+a)$ is D-finite, as stated in Section 3.3 for the semi-Baxter generating function. Hence, the specialisation $A(1,1)$, which is the generating function for $\mathbf{I}(\geq,\geq,\geq)$, is D-finite as well.

Then, we can obtain from the expression of $A(1+a,1+a)$ of Theorem 5.1.12 an explicit, yet very complicated, expression for the coefficients of the generating function $A(1,1)$, *i.e.* $[x^n]A(1,1) = [x^n a^0]A(1+a,1+a)$. In order to do this, we need to calculate first the coefficients $[x^n a^i]W^j$, for $j = 1,2,3,4$.

Lemma 5.1.13. *Let $W(x;a) \equiv W$ be the unique formal power series in x such that Equation (5.7) holds. Then, for $r \geq 1$*

$$[x^n a^s]W^r = \frac{r}{n}\sum_{k=0}^{n-r}\binom{n}{k}\binom{n}{k+r}\binom{n+r}{k-s+2r}. \tag{5.8}$$

Proof. It follows straightforward from Equation (5.7) by applying the Lagrange inversion formula (Theorem 1.2.6 on page 18). □

Proposition 5.1.14. *The number of inversion sequences of the set $\mathbf{I}_n(\geq,\geq,\geq)$, for all $n \geq 1$, is given by $\sum_{k=0}^{n} I(n,k)$, where*

$$\begin{aligned} I(n,k) = \tfrac{1}{n}\tbinom{n}{k}\Big[&\tbinom{n}{k+1}\left[-\tbinom{n+1}{k-4} - 3\tbinom{n+2}{k-2} - \tbinom{n+1}{k-1} + \tbinom{n+1}{k+2} + 3\tbinom{n+2}{k+4} + \tbinom{n+1}{k+5}\right] \\ &+2\tbinom{n}{k+2}\left[\tbinom{n+3}{k} - \tbinom{n+3}{k+4}\right] + 3\tbinom{n}{k+3}\left[\tbinom{n+3}{k} - \tbinom{n+3}{k+2} + \tbinom{n+3}{k+3} - \tbinom{n+3}{k+5}\right] \\ &+4\tbinom{n}{k+4}\left[-\tbinom{n+4}{k+3} + \tbinom{n+4}{k+4}\right]\Big]. \end{aligned}$$

Proof. The number of inversion sequences of $\mathbf{I}_n(\geq,\geq,\geq)$ is the coefficient of x^n in $A(1,1)$, which in turn is the coefficient of $x^n a^0$ in $A(1+a,1+a)$. Note that this number is also the coefficient $[x^n a^0](1+a)^3 A(1+a,1+a)$, and so by Theorem 5.1.12 it is the coefficient of $x^n a^0$ in $Q(a,W)$ as expressed in Equation (5.6), namely

$$\begin{aligned}[x^n a^0]Q(a,W) = &-[x^n a^6]W - 3[x^n a^5]W - 3[x^n a^4]W - [x^n a^3]W + [x^n a^0]W + 3[x^n a^{-1}]W \\ &+ 3[x^n a^{-2}]W + [x^n a^{-3}]W + [x^n a^5]W^2 + [x^n a^4]W^2 - [x^n a]W^2 - [x^n a^0]W^2 \\ &+ [x^n a^6]W^3 - [x^n a^4]W^3 + [x^n a^3]W^3 - [x^n a]W^3 - [x^n a^5]W^4 + [x^n a^4]W^4.\end{aligned}$$

Then, by Lemma 5.1.13 we can substitute into the above expression the coefficients of $[x^n a^s]W^i$, for each s and $i = 1,2,3,4$. Therefore, we obtain the following expression for the number of inversion sequences of $\mathbf{I}_n(\geq,\geq,\geq)$, proving Proposition 5.1.14,

$$\begin{aligned}[x^n a^0]Q(a,W) = \textstyle\sum_{k=0}^{n} \ & \tfrac{1}{n}\tbinom{n}{k}\Big[\tbinom{n}{k+1}\left[-\tbinom{n+1}{k-4} - 3\tbinom{n+2}{k-2} - \tbinom{n+1}{k-1} + \tbinom{n+1}{k+2} + 3\tbinom{n+2}{k+4} + \tbinom{n+1}{k+5}\right] \\ &+2\tbinom{n}{k+2}\left[\tbinom{n+3}{k} - \tbinom{n+3}{k+4}\right] + 3\tbinom{n}{k+3}\left[\tbinom{n+3}{k} - \tbinom{n+3}{k+2} + \tbinom{n+3}{k+3} - \tbinom{n+3}{k+5}\right] \\ &+4\tbinom{n}{k+4}\left[-\tbinom{n+4}{k+3} + \tbinom{n+4}{k+4}\right]\Big]. \qquad \square\end{aligned}$$

Although Proposition 5.1.14 shows a rather complicated expression for the number of inversion sequences of $\mathbf{I}_n(\geq,\geq,\geq)$, Zeilberger's method of creative telescoping [118, 150] illustrated in Section 3.4.2 allows us to provide a recursive formula satisfied by these numbers and to prove that they are indeed the sequence A108307 on [132].

Proposition 5.1.15. *Let* $a_n = |\mathbf{I}_n(\geq,\geq,\geq)|$. *The numbers* a_n *are recursively defined by* $a_0 = a_1 = 1$ *and for* $n \geq 0$,

$$8(n+3)(n+1)a_n + (7n^2+53n+88)a_{n+1} - (n+8)(n+7)a_{n+2} = 0\,.$$

Thus, $\{a_n\}_{n\geq 0}$ *is sequence A108307 on [132].*

Proof. From Proposition 5.1.15, we can write $a_n = \sum_{k=0}^{n} I(n,k)$, where the summand $I(n,k)$ is hypergeometric. Then, we prove the announced recurrence using the method of creative telescoping - see Appendix B.2. The Maple package `SumTools` includes the command `Zeilberger`, which implements this approach. On input $I(n,k)$ it shows that

$$\begin{aligned}-(n+9)(n+8)(n+6)I(n+3,k) + (464n + 6n^3 + 776 + 92n^2)I(n+2,k)& \\ +(n+2)(15n^2+133n+280)I(n+1,k) + 8(n+3)(n+2)(n+1)I(n,k)& \\ = G(n,k+1) - G(n,k),& \qquad (5.9)\end{aligned}$$

where the certificate function $G(n,k)$ has an expression extremely complicated and is not reported here. Nevertheless, it can be checked that $G(n,0) = G(n,n+9) = 0$.

Thus, to complete the proof it is sufficient to sum both sides of Equation (5.9) over k, k ranging from 0 to $n+9$. Since the coefficients on the left-hand side of Equation (5.9) are independent of k, summing Equation (5.9) over k gives

$$\begin{aligned}&-(n+9)(n+8)(n+6)a_{n+3} + (464n + 6n^3 + 776 + 92n^2)a_{n+2} \\ &+(n+2)(15n^2+133n+280)a_{n+1} + 8(n+3)(n+2)(n+1)a_n \;=\; 0\,. \end{aligned} \tag{5.10}$$

Now, it is straightforward to check that the sequence defined in Proposition 5.1.15 also satisfies the above P-recursion.[2] Indeed, Equation (5.10) can be obtained by applying the operator $(n+2)+(n+6)N$ to the recursion of Proposition 5.1.15, where as in Section 3.4.2 N denotes the shift operator. The proof is completed by checking that the recursion of Proposition 5.1.15 coincides with Equation (5.1) on page 146. □

5.1.3 Inversion sequences $\mathbf{I}_n(\geq, \geq, >)$

The next family of inversion sequences according to the hierarchy of Figure 5.1 on page 140 is $\mathbf{I}(\geq, \geq, >) = \cup_n \mathbf{I}_n(\geq, \geq, >)$. Indeed, the family $\mathbf{I}(\geq, \geq, >)$ includes $\mathbf{I}(\geq, \geq, \geq)$, as the following characterisation makes evident.

Proposition 5.1.16. *An inversion sequence is in* $\mathbf{I}(\geq, \geq, >)$ *if and only if it avoids* 100*,* 110 *and* 210*.*

Proof. The proof is a check, as usual. By definition an inversion sequence $e = (e_1, \ldots, e_n)$ is in $\mathbf{I}_n(\geq, \geq, >)$ if and only if there are no three indices $i < j < k$ such that $e_i \geq e_j \geq e_k$, and $e_i > e_k$. Thus, any inversion sequence e is in $\mathbf{I}_n(\geq, \geq, >)$ if and only if it avoids 100, 110 and 210. □

For example, the inversion sequence $(0, 1, 0, 1, 4, 1, 2, 4)$ is in $\mathbf{I}_8(\geq, \geq, >)$, yet it contains an occurrence of 000 and thus it is not in $\mathbf{I}_8(\geq, \geq, \geq)$. Another characterisation can be provided for the family $\mathbf{I}(\geq, \geq, >)$, as follows.

Proposition 5.1.17. *Let* $e = (e_1, \ldots, e_n)$ *be an inversion sequence. As in Proposition 5.1.9, we call an entry* e_i *a LTR maximum (resp. RTL minimum), if* $e_i > e_j$*, for all* $j < i$ *(resp.* $e_i < e_j$*, for all* $j > i$*).*

An inversion sequence e *is in* $\mathbf{I}(\geq, \geq, >)$ *if and only if for every* i *and* j*, with* $i < j$ *and* $e_i > e_j$*, both* e_i *is a LTR maximum and* e_j *is a RTL minimum.*

Proof. The proof in both directions is straightforward by considering the characterisation of Proposition 5.1.16. □

This family of inversion sequences was conjectured in [110] to be counted by the sequence A001181 [132] of Baxter numbers. This conjecture has recently been proved in [97,

[2] This calculation can be found in Appendix B.2, as well.

Theorem 4.1], by means of analytical tools: precisely, in [97, Lemma 4.3] it has been provided a succession rule for enumerating $\mathbf{I}(\geq,\geq,>)$ that results analytically to generate Baxter numbers. Despite this proof, no bijections involving any of the Baxter families of Section 1.4.2 are known, or equivalently no Baxter families are proved to grow according to this succession rule.

Moreover, we precise that the succession rule of [97, Lemma 4.3] is the same as Ω_{Bax3} reported in Section 1.4.4 on page 43 among the known Baxter succession rules. Despite this known result, we choose to report here a proof of [97, Lemma 4.3], since it displays that there exists a growth for the family $\mathbf{I}(\geq,\geq,>)$ that generalises the growth for the family $\mathbf{I}(\geq,\geq,\geq)$ provided in Proposition 5.1.10.

Proposition 5.1.18. *The family* $\mathbf{I}(\geq,\geq,>)$ *can be generated by*

$$\Omega_{Bax3} = \begin{cases} (1,1) \\ \\ (h,k) & \rightsquigarrow (1,k+1),\ldots,(h-1,k+1), \\ & \quad (1,k+1), \\ & \quad (h+k,1),\ldots,(h+1,k). \end{cases}$$

Proof. We prove the statement by showing a growth for the family $\mathbf{I}(\geq,\geq,>)$ according to Ω_{Bax3}, which extends the growth of Proposition 5.1.10, as well as Proposition 5.1.7. As previously, we define this growth by adding a new rightmost entry.

As in the proof of Proposition 5.1.10, let $\mathrm{last}(e)$ be the value of the rightmost entry of e, which is not a LTR maximum, if there is any. Note that $\mathrm{last}(e)$ is also the largest value not being a LTR maximum, since e avoids 210 by Proposition 5.1.16. Otherwise, if such an entry does not exists, we set $\mathrm{last}(e)$ equal to the smallest value of e, *i.e.* $\mathrm{last}(e) := 0$.

Moreover, if there exists this rightmost entry of e, which is not a LTR maximum, it can either form an inversion (*i.e.* there exists an entry e_i on its left such that $e_i > \mathrm{last}(e)$) or not. We need to distinguish two cases in order to define the addition of a new rightmost entry to e:

(a) Suppose either all the entries of e are LTR maxima, or the rightmost entry of e, which is not a LTR maximum, does not form an inversion.

(b) Suppose the rightmost entry of e, which is not a LTR maximum, does form an inversion.

Then, according to Proposition 5.1.18 we have that

(a) The sequence $f = (e_1,\ldots,e_n,p)$ is in $\mathbf{I}_{n+1}(\geq,\geq,>)$ if and only if $\mathrm{last}(e) \leq p \leq n$. Moreover, if $\mathrm{last}(e) \leq p < \max(e)$, where as usual $\max(e)$ is the maximum value of e, then $\mathrm{last}(f) = p$ and f falls in case **(b)**. Else if $p = \max(e)$, then again $\mathrm{last}(f) = p$, yet f falls in case **(a)**. While, if $\max(e) < p \leq n$, p is a LTR maximum of f, which thus falls in the same case **(a)** of e, and $\mathrm{last}(f) = \mathrm{last}(e)$.

(b) The sequence $f = (e_1, \ldots, e_n, p)$ is in $\mathbf{I}_{n+1}(\geq, \geq, >)$ if and only if $\text{last}(e) < p \leq n$. In particular, if $\text{last}(e) < p < \max(e)$, then $\text{last}(f) = p$ and f falls in case **(b)**. Else if $p = \max(e)$, then again $\text{last}(f) = p$ and f falls in case **(a)**. While, if $\max(e) < p \leq n$, as above p is a LTR maximum of f, which thus falls in the same case **(b)** of e, and $\text{last}(f) = \text{last}(e)$.

Now, we assign to any $e \in \mathbf{I}_n(\geq, \geq, >)$ a label according to the above distinction: in case **(a)** (resp. **(b)**) we assign the label (h, k), where $h = \max(e) - \text{last}(e) + 1$ (resp. $h = \max(e) - \text{last}(e)$) and $k = n - \max(e)$.

The sequence $e = (0)$ of size one falls in case **(a)**, thus it has label $(1, 1)$, which is the axiom of Ω_{Bax3}. Now, let e be an inversion sequence of $\mathbf{I}_n(\geq, \geq, >)$ with label (h, k). Following the above distinction, the inversion sequences of $\mathbf{I}_{n+1}(\geq, \geq, >)$ produced by adding a rightmost entry p to e have labels:

(a)
- $(1, k+1), \ldots, (h-1, k+1)$, when $p = \max(e) - 1, \ldots, \text{last}(e)$,
- $(1, k+1)$, for $p = \max(e)$,
- $(h+k, 1), (h+k-1, 2), \ldots, (h+1, k)$, when $p = n, n-1, \ldots, \max(e) + 1$,

(b)
- $(1, k+1), \ldots, (h-1, k+1)$, when $p = \max(e) - 1, \ldots, \text{last}(e) + 1$,
- $(1, k+1)$, for $p = \max(e)$,
- $(h+k, 1), (h+k-1, 2), \ldots, (h+1, k)$, when $p = n, n-1, \ldots, \max(e) + 1$,

which concludes the proof that $\mathbf{I}(\geq, \geq, >)$ grows according to Ω_{Bax3}. □

5.1.4 Inversion sequences $\mathbf{I}_n(\geq, >, -)$

The next family of inversion sequences according to the hierarchy of Figure 5.1 is the family $\mathbf{I}(\geq, >, -) = \cup_n \mathbf{I}(\geq, >, -)$. The following proposition clarifies the inclusion of the family $\mathbf{I}(\geq, \geq, >)$, considered in the previous section, into the family $\mathbf{I}(\geq, >, -)$.

Proposition 5.1.19. *An inversion sequence is in* $\mathbf{I}(\geq, >, -)$ *if and only if it avoids* 110 *and* 210*.*

The proof of the above statement is elementary, and we omit it. Yet it shows clearly that the inversion sequences of family $\mathbf{I}(\geq, >, -)$ avoid only two of the three patterns avoided by the inversion sequences of $\mathbf{I}(\geq, \geq, >)$ (see Proposition 5.1.16 for a comparison). For example, the inversion sequence $(0, 1, 0, 0, 1, 4, 2, 2)$ is in $\mathbf{I}_8(\geq, >, -)$, but not in $\mathbf{I}_8(\geq, \geq, >)$.

Moreover, the following characterisation is an extension of that provided in Proposition 5.1.17 for the family $\mathbf{I}(\geq, \geq, >)$.

Proposition 5.1.20. *Let* $e = (e_1, \ldots, e_n)$ *be an inversion sequence. As in proposition 5.1.17, we call an entry* e_i *a LTR maximum, if* $e_i > e_j$*, for all* $j < i$*, and we say that* e_i *and* e_j *form an inversion, if* $i < j$ *and* $e_i > e_j$*.*

An inversion sequence e *is in* $\mathbf{I}(\geq, >, -)$ *if and only if for every* e_i *and* e_j *that form an inversion,* e_i *is a LTR maximum.*

Proof. Equivalently to the proof of Proposition 5.1.17, the above statement follows immediately by considering that e is an inversion sequence of $\mathbf{I}(\geq, >, -)$ if and only if it avoids 110 and 210. □

This family of inversion sequences was conjectured in [110] to be counted by the sequence A117106 [132] of the semi-Baxter numbers SB_n. It is well worth specifying that we have already proved this conjecture in [G4], and reported its proof in Section 3.2.2 on page 107. Indeed, according to [110] for every n, inversion sequences of $\mathbf{I}_n(>, \geq, -)$ are as many as those of $\mathbf{I}_n(\geq, >, -)$, and in Section 3.2.2 we proved that $|\mathbf{I}_n(>, \geq, -)| = SB_n$.

For the sake of completeness, we choose to report here a proof of the fact that the family $\mathbf{I}(\geq, >, -)$ can be generated by the rule Ω_{semi}. Indeed, the growth of the family $\mathbf{I}(\geq, \geq, >)$ in the proof of Proposition 5.1.18 can be easily generalised to a growth for the family $\mathbf{I}(\geq, >, -)$.

Proposition 5.1.21. *The family* $\mathbf{I}(\geq, >, -)$ *can be generated by*

$$\Omega_{semi} = \begin{cases} (1,1) \\ \\ (h,k) \rightsquigarrow (1,k+1), \ldots, (h,k+1), \\ \qquad\qquad (h+k,1), \ldots, (h+1,k). \end{cases}$$

Proof. As previously, we define a growth for the family $\mathbf{I}(\geq, >, -)$ according to Ω_{semi}, by adding a new rightmost entry.

As in the proof of Proposition 5.1.18, let last(e) be the value of the rightmost entry of e which is not a LTR maximum, if there is any. Otherwise, last$(e) := 0$. Note that differently from Proposition 5.1.18, here we do not need to distinguish cases depending on whether or not the rightmost entry of e not being a LTR maximum forms an inversion.

According to Proposition 5.1.20, it follows that $f = (e_1, \ldots, e_n, p)$ is an inversion sequence of $\mathbf{I}_{n+1}(\geq, >, -)$ if and only if last$(e) \leq p \leq n$. Moreover, if last$(e) \leq p \leq$ max(e), where as usual max(e) is the maximum value of e, then last$(f) = p$; if max$(e) < p \leq n$, then last$(f) =$ last(e), since p is a LTR maximum.

Now, we assign to any $e \in \mathbf{I}_n(\geq, >, -)$ the label (h, k), where $h = \max(e) - \text{last}(e) + 1$ and $k = n - \max(e)$.

The sequence $e = (0)$ of size one has label $(1, 1)$, which is the axiom of Ω_{semi}, since last$(e) = 0$. Let e be an inversion sequence of $\mathbf{I}_n(\geq, >, -)$ with label (h, k). The labels of the inversion sequences of $\mathbf{I}_{n+1}(\geq, >, -)$ produced adding a rightmost entry p to e are

- $(1, k+1), \ldots, (h, k+1)$, when $p = \max(e), \ldots, \text{last}(e)$,
- $(h+k, 1), (h+k-1, 2), \ldots, (h+1, k)$, when $p = n, n-1, \ldots, \max(e)+1$,

which concludes the proof that $\mathbf{I}(\geq, >, -)$ grows according to Ω_{semi}. □

5.1.5 Inversion sequences $\mathbf{I}_n(=,>,>)$

The family of inversion sequences $\mathbf{I}(=,>,>) = \cup_n \mathbf{I}_n(=,>,>)$ is depicted as the ultimate element of the chain in Figure 5.1. Indeed, among the inversion sequences of $\mathbf{I}(=,>,>)$ we can find the inversion sequences of $\mathbf{I}(\geq,>,-)$ (and consequently of $\mathbf{I}(\geq,\geq,>)$, $\mathbf{I}(\geq,\geq,\geq)$, and $\mathbf{I}(\geq,-,\geq)$). The following characterisation will express distinctly this inclusion.

Proposition 5.1.22. *An inversion sequence is the set* $\mathbf{I}(=,>,>)$ *if and only if it avoids* 110.

Proof. It follows from the definition of the set $\mathbf{I}_n(=,>,>)$. Indeed, any inversion sequence $e = (e_1,\ldots,e_n)$ in $\mathbf{I}_n(=,>,>)$ is such that if there are e_i and e_j, with $i<j$ and $e_i = e_j$, then e_k cannot be smaller than e_j, for every $k>j$. Thus, e is in $\mathbf{I}_n(=,>,>)$ if and only if it avoids 110. □

An example of inversion sequence of the set $\mathbf{I}_8(=,>,>)$ that does not belong to the set $\mathbf{I}_8(\geq,>,-)$ is $(0,1,0,0,1,4,3,2)$.

Thanks to the characterisation of Proposition 5.1.22, the family $\mathbf{I}(=,>,>)$ was completely enumerated in [58, Theorem 13]. Its enumerative number sequence is registered on [132] as sequence A113227, whose first terms are

$$1, 1, 2, 6, 23, 105, 549, 3207, 20577, 143239, 1071704, 8555388, 72442465, 647479819, \ldots$$

In [49] it is proved that the nth number p_n of the sequence A113227 can be obtained as $\sum_{k=0}^{n} c_{n,k}$, where the term $c_{n,k}$ is recursively defined by

$$\begin{cases} c_{0,0} = 1, & \\ c_{n,0} = 0, & \text{for } n \geq 1 \\ c_{n,k} = c_{n-1,k-1} + k\sum_{j=k}^{n-1} c_{n-1,j}, & \text{for } n \geq 1, \text{ and } 1 \leq k \leq n\,. \end{cases} \tag{5.11}$$

Proposition 5.1.23 (Theorem 13, [58])**.** *For $n \geq 1$ and $0 \leq k \leq n$, the number of inversion sequences of* $\mathbf{I}_n(=,>,>)$ *having k zeros is given by the term $c_{n,k}$ of Equation* (5.11). *Thus,* $|\mathbf{I}_n(=,>,>)| = p_n$, *for every $n \geq 1$.*

The recursive formula (5.11) and Proposition 5.1.23 can be translated into the following result.

Proposition 5.1.24. *The family* $\mathbf{I}(=,>,>)$ *can be generated by*

$$\Omega_{pCat} = \begin{cases} (1) & \\ (k) & \rightsquigarrow (1), (2)^2, (3)^3, \ldots, (k)^k, (k+1). \end{cases}$$

Remark 5.1.25. *The succession rule Ω_{pCat} is encoded by the recursive formula* (5.11). *Indeed, for $n \geq 1$ and $k \geq 1$, the number of nodes at level n that carry the label (k) in the generating tree associated with Ω_{pCat} is precisely the quantity $c_{n,k}$ given by Equation* (5.11).

Proof of Proposition 5.1.24. We prove the above statement by showing a growth for the family $\mathbf{I}(=,>,>)$. Let $e = (e_1, \ldots, e_n) \in \mathbf{I}_n(=,>,>)$. Suppose e has k entries equal to 0, and let $i_1, \ldots, i_k$ be their indices.

Then, we define a growth that changes the number of the 0 entries of e, as follows:

a) first, increase by one all the entries of e that are greater than 0; namely $e' = (e'_1, \ldots, e'_n)$, where $e'_i = e_i$, if $i = i_1, \ldots, i_k$, otherwise $e'_i = e_i + 1$. Note that $e'_1 = e_1 = 0$.

b) insert a new leftmost 0 entry; namely $e'' = (0, e'_i, \ldots, e'_n)$. Note that e'' is an inversion sequence of size $n+1$, and moreover it has $k+1$ zeros at positions $1, i_1+1, \ldots, i_k+1$.

c) build the following inversion sequences, starting from e''.

(1) Replace all the zeros at positions $i_1+1, i_2+1, \ldots, i_k+1$ by 1; namely $e^{(1)} = (0, e^*_1, \ldots, e^*_n)$, where $e^*_i = e_i + 1$, for all i. Note that $e^{(1)}$ has only one zero.

(j) For all $1 < j < k+1$, replace all the zeros at positions $i_{j+1}+1, \ldots, i_k+1$ by 1, and furthermore replace by 1 only one entry among $i_1+1, \ldots, i_j+1$. There are thus j different inversion sequences $e^{(m)} = (0, e^*_1, \ldots, e^*_n)$, with $1 \leq m \leq j$, such that $e^*_i = e_i + 1$ except for the indices $i_1+1, \ldots, i_{m-1}+1, i_{m+1}+1, \ldots, i_j+1$. Note that $e^{(m)}$ has exactly j zeros, for any $1 \leq m \leq j$.

(k+1) Set $e^{(k+1)} = e''$.

Note that all the inversion sequences of size $n+1$ produced at step c) avoid 110, since the initial inversion sequence e avoids 110. Thus, in each of the above cases we build an inversion sequence of $\mathbf{I}_{n+1}(=,>,>)$.

Moreover, given any inversion sequence $f \in \mathbf{I}_{n+1}(=,>,>)$, it is easy to retrieve the unique inversion sequence $e \in \mathbf{I}_n(=,>,>)$ that produces f according to the operations of c): it is sufficient to replace all the entries equal to 1 by 0, remove the leftmost 0 entry, and finally decrease by one all the entries greater than 0. This procedure is in fact a) - b) - c) backwards.

Finally, if we label an inversion sequence e of $\mathbf{I}_n(=,>,>)$ with (k), where k is its number of 0 entries. It is straightforward, and the above itemised list suggests it, that the inversion sequences of $\mathbf{I}_{n+1}(=,>,>)$ produced by e following the construction at step c) have labels $(1), (2)^2, (3)^3, \ldots, (k)^k, (k+1)$. □

The family $\mathbf{I}(=,>,>) = \cup_n \mathbf{I}_n(=,>,>)$ deserves attention mainly for two reasons. On the one hand, we have not been able to write a succession rule for the family $\mathbf{I}(=,>,>)$ that generalises the one of Proposition 5.1.21, thus provoking a rift in the chain of succession rules according to Figure 5.1. On the other hand, the enumerative sequence of the family $\mathbf{I}(=,>,>)$, (sequence A113227 [132]), is extremely rich of combinatorial interpretations, as the following sections will illustrate.

Of particular interest is indeed the similarity between the Catalan succession rule Ω_{Cat} of Section 1.3.4 and the succession rule shown in Proposition 5.1.24 to generate the family

I$(=,>,>)$. In other words, the rule Ω_{Cat} has the production

$$(k) \rightsquigarrow (1), (2), \ldots, (k), (k+1)\,,$$

while the succession rule Ω_{pCat} has

$$(k) \rightsquigarrow (1), (2)^2, \ldots, (k)^k, (k+1)\,.$$

It looks as if the succession rule Ω_{pCat} is the weighted version of the Catalan succession rule Ω_{Cat}. In the next section, the above observation will allow us to call the numbers p_n of sequence A113227 [132] *powered Catalan* numbers.

Remark 5.1.26. *We point out that apparently there is no information about the ordinary generating function* $F_{pCat}(x) = \sum_{n\geq 0} p_n x^n$. *On the contrary, the exponential generating function* $E_{pCat}(x) = \sum_{n\geq 0} p_n x^n/n!$ *has been studied in [72], as well as in [49], where by means of the recurrence* (5.11) *a refined version of this exponential generating function is provided.*

5.2 Powered Catalan numbers

This section is to collect combinatorial results about the number sequence A113227 on [132], which we call the sequence of powered Catalan numbers.

Definition 5.2.1. Let $p_n = \sum_{k=0}^{n} c_{n,k}$, where $c_{n,k}$ is defined by Equation (5.11), *i.e.*

$$\begin{cases} c_{0,0} = 1, & \\ c_{n,0} = 0, & \text{for } n \geq 1 \\ c_{n,k} = c_{n-1,k-1} + k\sum_{j=k}^{n-1} c_{n-1,j}, & \text{for } n \geq 1, \text{ and } 1 \leq k \leq n\,. \end{cases} \tag{5.11}$$

We call p_n the nth *powered Catalan number*, and the number sequence A113227 on [132] the sequence of powered Catalan numbers.

Table 5.1 shows the terms $c_{n,k}$, for $n \geq 0$ and $0 \leq k \leq n$.

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$n=0$	1						
$n=1$	0	1					
$n=2$	0	1	1				
$n=3$	0	2	3	1			
$n=4$	0	6	10	6	1		
$n=5$	0	23	40	31	10	1	
$n=6$	0	105	187	166	75	15	1

Table 5.1: The first terms generated by the recursive formula (5.11).

The name is clearly motivated by the similarity noticed earlier between the well-known Catalan succession rule Ω_{Cat} and the rule Ω_{pCat} of Section 5.1.5.

Now, in Section 5.2.1 we collect some combinatorial structures known to be counted by the powered Catalan numbers. Next in Section 5.2.2 we provide another succession rule that generates powered Catalan numbers.

5.2.1 Combinatorial structures enumerated by the powered Catalan number sequence

Valley-marked Dyck paths

This family of paths first defined in [49, Section 7] generalises the well-known family of Dyck paths - see Figure 5.4.

Definition 5.2.2. A *valley-marked Dyck path* of semi-length n is a Dyck path P of length $2n$ in which, for each valley (*i.e.* DU factor), one of the lattice points between the valley vertex and the x-axis is marked. In other words, if (i, k) pinpoints any valley of P, then a valley-marked Dyck path associated with P must take a mark in a point (i, j), where $0 \leq j \leq k$.

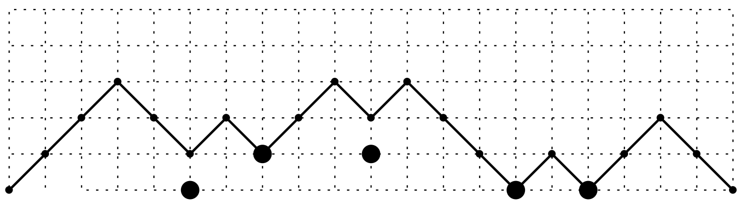


Figure 5.4: A valley-marked Dyck path.

Valley-marked Dyck paths are enumerated by powered Catalan number according to their semi-length and, moreover, the parameter k of Table 5.1 can be interpreted on them as follows.

Proposition 5.2.3. *The number of valley-marked Dyck paths of semi-length n having k down steps in the last descent is given by the term $c_{n,k}$ of Equation* (5.11)*, for every $n \geq 0$ and $0 \leq k \leq n$.*

Proof. We prove the above statement by showing a growth for valley-marked Dyck paths according to Ω_{pCat}. Indeed, to provide a growth according to Ω_{pCat} is equivalent to provide an interpretation of the term $c_{n,k}$ of Equation (5.11) by Remark 5.1.25.

The growth we provide is a slight modification of the one known for Dyck paths (see Section 1.3.4). Let P be a Dyck path of semi-length n with k down steps in its last descent. By adding a rightmost UD factor in any point of its last descent, we obtain the Dyck paths $P^{(1)}, \ldots, P^{(k+1)}$ with $1, \ldots, k+1$ down steps in the last descent, respectively. Note that $P^{(k+1)}$ has as many valleys as P, whereas $P^{(1)}, \ldots, P^{(k)}$ have a new rightmost valley.

Then, let V be a valley-marked Dyck path of semi-length n with k down steps in the last descent. The path V is by definition a Dyck path P of semi-length n with some marks. Then, V produces the following paths:

- for any $1 \leq j \leq k$, the path $P^{(j)}$ with the rightmost valley marked at $(2n+1-j, i)$, for any $0 \leq i \leq j$, and all the other valley-marks as V,
- $P^{(k+1)}$ with all the valleys marked as V.

Now, we label each valley-marked Dyck path V with (k), where k is the number of steps in its last descent. The path UD of semi-length 1 has label (1), which is the axiom of Ω_{pCat}. Then, given a valley-marked Dyck path V of label (k), the labels of the paths produced by V are $(1), (2)^2, (3)^3, \ldots, (k)^k, (k+1)$, concluding the proof. □

Increasing ordered trees

Another family of objects described in [49] to be counted by sequence A113227 is formed by labelled ordered trees.

Definition 5.2.4. An *increasing ordered tree* of size n is a plane tree with $n+1$ labelled vertices, the standard label set being $\{0, 1, 2, \ldots, n\}$, such that each child exceeds its parent. An increasing ordered tree *has increasing leaves* if its leaves, taken in pre-order, are increasing.

Figure 5.5 shows two increasing ordered trees, the first has increasing leaves, while the second does not.

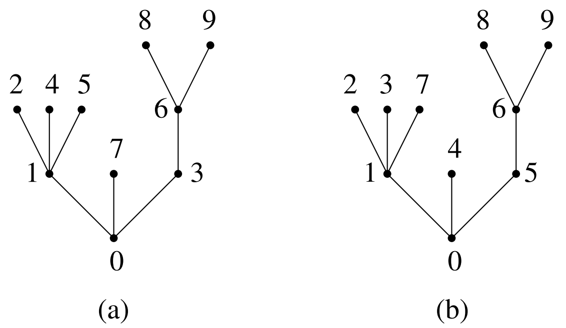


Figure 5.5: Two increasing ordered trees: (a) with increasing leaves; (b) with non-increasing leaves.

The number of increasing ordered trees of size n is given by the odd double factorial $(2n-1)!!$ [50]. If we require the additional constraint of having increasing leaves, then the number of these increasing ordered trees of size n results to be the nth powered Catalan number.

Proposition 5.2.5 (Section 2, [49])**.** *The number of increasing ordered trees with increasing leaves of size n and root degree k is given by the number $c_{n,k}$ of Equation* (5.11).

A proof of the above statement can be also provided similarly to Proposition 5.2.3, by showing a growth according to Ω_{pCat} for increasing ordered trees with increasing leaves.

Pattern-avoiding permutations

Some families of pattern-avoiding permutations are known to be counted by the sequence of powered Catalan numbers. Indeed, sequence A113227 is actually registered on [132] as the enumerative sequence of permutations avoiding the generalised pattern $1\,\underline{23}\,4$.

The family $AV(1\,\underline{23}\,4)$ has been completely enumerated by D. Callan in [49]. Indeed, he shows in [49] a bijection between $AV(1\,\underline{23}\,4)$ and labelled ordered trees with increasing leaves. Therefore, the number $|AV_n(1\,\underline{23}\,4)|$ satisfies Definition 5.2.1, although the interpretation of the parameter k is rather complicated on $1\,\underline{23}\,4$-avoiding permutations.

Furthermore, in [18], some other families of pattern-avoiding permutations are presented as related to the sequence A113227 [132]. In particular, in [18] and subsequent papers, the families $AV(1\,\underline{32}\,4)$, and $AV(1\,\underline{34}\,2)$, and $AV(1\,\underline{43}\,2)$ are proved to be equinumerous to $1\,\underline{23}\,4$-avoiding permutations. It has been conjectured in [19] that also the family $AV(\underline{23}\,14)$ is equinumerous to $AV(1\,\underline{23}\,4)$. We attempted to prove this conjecture by defining a growth for the family $AV(\underline{23}\,14)$ according to Ω_{pCat}. This fact leads us to refine this conjecture as follows.

Conjecture 5.2.6. *The number of permutations of $AV_n(\underline{23}\,14)$ with k RTL minima is given by $c_{n,k}$ as defined in Equation* (5.11)*.*

Although we have a little evidence of the above fact (only up to $n \leq 9$), we suspect that a growth for 213-avoiding permutations according to Ω_{Cat}, where the parameter k marks the number of RTL minima, could be generalised as to obtain one for $\underline{23}\,14$-avoiding permutations according to Ω_{pCat}. Nevertheless, we have not been able to find such a growth up to now.

5.2.2 A second succession rule for powered Catalan numbers

In this section we provide a second succession rule different from Ω_{pCat} for the powered Catalan number sequence A113227 [132]. We are able to provide it by showing a growth for the family of permutations $AV(1\,\underline{23}\,4)$.

According to D. Callan [49], permutations avoiding $1\,\underline{23}\,4$ have a simple characterisation in terms of LTR minima and RTL maxima, as follows.

Proposition 5.2.7. *A permutation π of length n belongs to $AV(1\,\underline{23}\,4)$ if and only if for every index $1 \leq i < n$,*

if $\pi_i\pi_{i+1}$ is an ascent ($\pi_i < \pi_{i+1}$), then π_i is a LTR minimum **or** *π_{i+1} is a RTL maximum.*

Proof. The proof is straightforward. In fact, suppose there exists an index i, $1 \leq i < n$, such that $\pi_i < \pi_{i+1}$, and neither π_i is a LTR minimum nor π_{i+1} is a RTL maximum. Then, there exists an index $j < i$ such that $\pi_j < \pi_i$, and an index $k > i + 1$ such that $\pi_k > \pi_{i+1}$. Thus, $\pi_j\pi_i\pi_{i+1}\pi_k$ forms an occurrence of $1\,\underline{23}\,4$. Conversely, if π contains an occurrence of $1\,\underline{23}\,4$, by definition of pattern containment there exists an index i, $1 \leq i < n$, such that $\pi_i < \pi_{i+1}$, and neither π_i is a LTR minimum nor π_{i+1} is a RTL maximum. □

We show now a recursive growth for the family $AV(1\,\underline{23}\,4)$ that yields a succession rule whose labels are arrays of length two.

Proposition 5.2.8. *Permutations avoiding* $1\,\underline{23}\,4$ *can be generated by the following succession rule*

$$\Omega_{1\underline{23}4} = \begin{cases} (1,1) \\ \\ (1,k) \quad \rightsquigarrow (1,k+1),(2,k),\ldots,(1+k,1), \\ (h,k) \quad \rightsquigarrow (1,h+k),(2,h+k-1),\ldots,(h,k+1), \\ \qquad\qquad (h+1,0),\ldots,(h+k,0), \qquad\qquad\qquad\qquad \text{if } h \neq 1. \end{cases}$$

Proof. First, observe that removing the rightmost point of a permutation avoiding $1\,\underline{23}\,4$, we obtain a permutation that still avoids $1\,\underline{23}\,4$. So, a growth for the permutations avoiding $1\,\underline{23}\,4$ can be obtained with local expansions on the right. By using the notation introduced in Chapter 3, we denote by $\pi\cdot a$, where $a \in \{1,\ldots,n{+}1\}$, the permutation $\pi' = \pi'_1 \ldots \pi'_n \pi'_{n+1}$ where $\pi'_{n+1} = a$, and $\pi'_i = \pi_i$, if $\pi_i < a$, $\pi'_i = \pi_i + 1$ otherwise.

For π a permutation in $AV_n(1\,\underline{23}\,4)$, the active sites of π are by definition the points a (or equivalently the values a) such that $\pi \cdot a$ avoids $1\,\underline{23}\,4$. The other points a are called non-active sites.

An occurrence of $1\,\underline{23}$ in π is a subsequence $\pi_j\pi_i\pi_{i+1}$ (with $j < i$) such that $\pi_j < \pi_i < \pi_{i+1}$. Note that the non-active sites a of π are the values larger than π_{i+1}, for some occurrence $\pi_j\pi_i\pi_{i+1}$ of $1\,\underline{23}$. Then, given $\pi \in AV_n(1\,\underline{23}\,4)$, we denote by $\pi_s\pi_{t-1}\pi_t$ the occurrence of $1\,\underline{23}$ (if there is any), in which the point π_t is minimal. Then the active sites of π form a consecutive sequence from the bottommost site to π_t, *i.e.* they are $[1,\pi_t]$. Figure 5.6 should help understanding which sites are active (represented by diamonds, as usual). If $\pi \in AV_n(1\,\underline{23}\,4)$ has no occurrence of $1\,\underline{23}$, then the active sites of π are $[1,n+1]$.

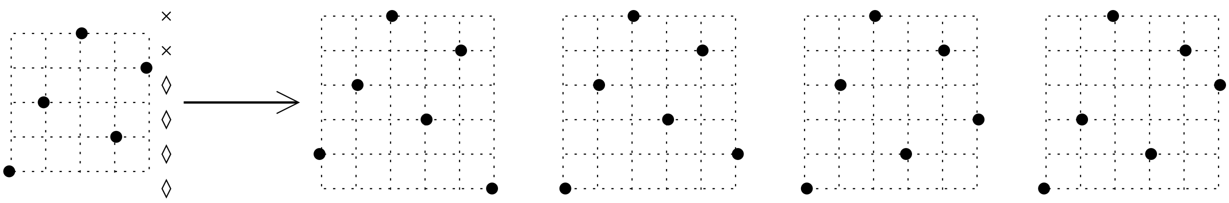
Now, we assign a label (h,k) to each permutation $\pi \in AV_n(1\,\underline{23}\,4)$, where h (resp. k) is the number of its active sites smaller than or equal to (resp. greater than) π_n. Remark that $h \geq 1$, since 1 is always an active site. Moreover, $h = \pi_n$: indeed, let $\pi_s\pi_{t-1}\pi_t$ be the occurrence of $1\,\underline{23}$ with π_t minimal. There must hold that $\pi_t > \pi_n$, otherwise $\pi_s\pi_{t-1}\pi_t\pi_n$ would form an occurrence of $1\,\underline{23}\,4$.

The label of the permutation $\pi = 1$ is $(1,1)$, which is the axiom in $\Omega_{1\underline{23}4}$. The proof then is concluded by showing that for any $\pi \in AV_n(1\,\underline{23}\,4)$ of label (h,k), the permutations $\pi \cdot a$ have labels according to the productions of $\Omega_{1\underline{23}4}$ when a runs over all active sites of π. To prove this we need to distinguish whether $\pi_n = 1$ or not.

If $\pi_n = 1$, no new occurrence of $1\,\underline{23}$ can be generated in the permutation $\pi \cdot a$, for any a active site of π. Thus, the active sites of $\pi \cdot a$ are as many as those of π plus one (since the active site a of π splits into two actives sites of $\pi \cdot a$). Then, since $\pi_n = 1$, permutation π has label $(1,k)$, for some $k > 0$ (at least one site above 1 in active), and permutations $\pi \cdot a$, for a ranging over all the active sites of π from bottom to top, have labels $(1,k+1),(2,k),\ldots,(1+k,1)$, which is the first production of $\Omega_{1\underline{23}4}$.

Otherwise, we have that π has label (h,k), with $h > 1$, and $\pi_n = h$. In this case a new occurrence of $1\,\underline{23}$ is generated in the permutation $\pi \cdot a$, for every $a > \pi_n$: indeed,

$1\,\pi_n a$ forms an occurrence of $1\,\underline{23}$, and moreover is such that a is minimal. Else if $a \leq \pi_n$, no new occurrence of $1\,\underline{23}$ can be generated in the permutation $\pi \cdot a$. Thus, permutations $\pi \cdot a$ have labels $(1, h+k), (2, h+k-1), \ldots, (h, k+1)$, for any active site $a \leq \pi_n$, and labels $(h+1, 0), (h+2, 0), \ldots, (h+k, 0)$, for any active site $a > \pi_n$. Note that this label production coincides with the two lines of the second production of $\Omega_{1\,\underline{23}\,4}$ concluding the proof. Figure 5.6 shows an example of the above construction. □



(4,0) (1,4) (2,3) (3,2) (4,1)

Figure 5.6: The growth of a permutation of label $(4, 0)$.

5.3 The family of steady paths

In the last section we provide a further combinatorial interpretation of powered Catalan numbers in terms of lattice paths. This new occurrence reveals an interesting pecularity of sequence A113227: indeed, it appears that the combinatorial structures enumerated by powered Catalan numbers could be subdivided into two groups according to the succession rule that generates them, as Section 5.3.4 describes.

In Section 5.3.1 we start by defining this new family of paths, called steady paths, and provide a growth for them. The succession rule defined by this growth proves that steady path are equinumerous to $1\,\underline{23}\,4$-avoiding permutations, since both families grow according to the same succession rule. Thus, there is a recursive bijection between these two families.

In Section 5.3.2, we show another bijection, which is direct and extremely simple, between the family of steady paths and the family $AV(1\,\underline{34}\,2)$. Whereas, in Section 5.3.3, we investigate what could be a bijection between the family of steady paths and the family of valley-marked Dyck paths, showing a possible equidistribution of some statistics.

Last, in Section 5.3.5, we show two different generalisations of steady paths, for which we are able to provide a succession rule, yet not to study their ordinary generating functions.

5.3.1 Definition, and growth of steady paths

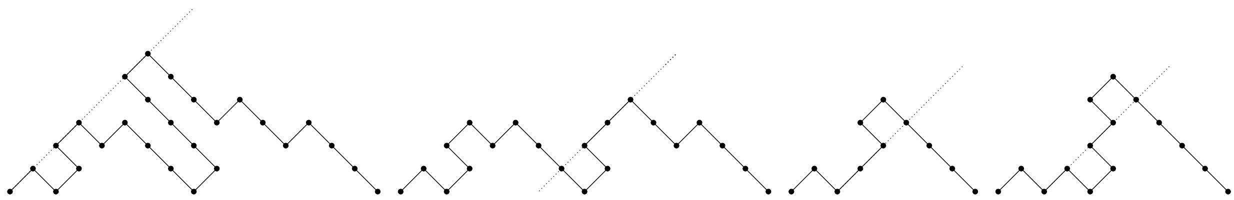
Definition 5.3.1. We call *steady path* of size n a lattice path T confined to the cone $\mathfrak{C} = \{(x, y) \in \mathbb{N}^2 : y \leq x\}$, which uses $U = (1, 1)$, $D = (1, -1)$ and $W = (-1, 1)$ as steps, but without any factors WD nor DW, starting at $(0, 0)$ and ending at $(2n, 0)$, such that:

(S1) for any factor UU, the suffix of T following this UU factor lies weakly below the line parallel to $y = x$ passing through the UU factor;

(S2) for any factor WU, the suffix of T following this WU factor lies weakly below the line parallel to $y = x$ passing through the up step of the WU factor.

We call *edge line* of T the line $y = x - t$, with $t \geq 0$ even integer, passing through the up step of the rightmost occurrence of either UU or WU.

The name "steady" is motivated by the two restrictions (S1) and (S2), which force these paths to remain weakly below a line that moves rightwards and conveys more stability to the mountain range the path would represent. Figure 5.7 (a) shows an example of a steady path whose edge line coincides with $y = x$, whereas the edge line of the steady path depicted in Figure 5.7 (b) is $y = x - 6$. Figures 5.7 (c),(d) show two different examples of paths confined to $\mathfrak{C}$ that are not steady paths.



(a) (b) (c) (d)

Figure 5.7: (a) An example of a steady path T of size 8 with edge line $y = x$; (b) An example of a steady path T of size 8 with edge line $y = x - 6$; (c) a path in $\mathfrak{C}$ that violates (S1); (d) a path in $\mathfrak{C}$ that violates (S2).

Remark 5.3.2. *By Definition 5.3.1, the size of a steady path T is equal to the number of its U steps. Moreover, any steady path T of size n is uniquely determined by the set of positions of its up steps $U^{(1)}, \dots, U^{(n)}$ recorded from left to right: precisely, by the set of starting points (i_k, j_k) for any $U^{(k)}$. Indeed, since neither WD nor DW can occur, there is only one way to draw a steady path given the set of positions $\{(0,0) = (i_1, j_1), \dots, (i_n, j_n)\}$ of its up steps from left to right.*

Furthermore, let a set of points $\{(i_1, j_1), \dots, (i_n, j_n)\}$ in $\mathfrak{C}$ be such that for every index $1 \leq k \leq n$, $j_k = k - 1 - i_k$. This set uniquely defines a steady path of size n provided that for every $1 < k \leq n$, if $i_k \leq i_{k-1} + 1$, then all the points (i_ℓ, j_ℓ), with $\ell > k$, lies weakly below the line parallel to $y = x$ passing through the point (i_k, j_k).

We are able to provide a succession rule for the family of steady paths that results in the following proposition.

Proposition 5.3.3. *The family of steady paths can be generated by*

$$\Omega_{steady} = \begin{cases} (0,2) \\ \\ (h,k) \rightsquigarrow (h+k-1,2), \dots, (h+1,k), \\ \qquad\qquad (0,k+1), \dots, (0,h+k+1). \end{cases}$$

Proof. The proof of the above statement is provided by showing a growth for the family of steady paths that defines the succession rule Ω_{steady}. Analogously to other families of paths occurring along this dissertation, we provide a growth for steady paths by adding a new rightmost occurrence of an up step that makes increase the size by one.

By Remark 5.3.2, given a steady path T of size n, we still obtain a steady path of size $n-1$ by removing its rightmost point (i_n, j_n), namely the rightmost up step of T.

Now, let T be a steady path of size n, and $(0,0) = (i_1, j_1), \dots, (i_n, j_n)$ be the positions of its up steps. We describe in which position (i_{n+1}, j_{n+1}) a new rightmost up step can be inserted so that the path obtained is still a steady path. In particular, according to Definition 5.3.1 if the edge line of T is $y = x - 2t$, with t a non-negative integer, then the point (i_{n+1}, j_{n+1}) must remain weakly below this line, which is $j_{n+1} \leq i_{n+1} - 2t$. Then, we add a new rightmost up step at any position $(2n, 0), (2n-1, 1), (2n-2, 2), \dots, (2n-s, s)$, where $s = n - t$. By Remark 5.3.2, there exists a unique path of size $n+1$ corresponding to $(0,0) = (i_1, j_1), \dots, (i_n, j_n), (i_{n+1}, j_{n+1})$, where (i_{n+1}, j_{n+1}) is any point among $(2n, 0), (2n-1, 1), (2n-2, 2), \dots, (2n-s, s)$, and it is steady by construction.

Moreover, the positions $(2n, 0), (2n-1, 1), (2n-2, 2), \dots, (2n-s, s)$ can be divided into two groups: the positions that are ending points of D steps of the last descent of T, and those which are not. This distinction is crucial. Indeed, when we insert a U step in an ending point of a D step of T's last descent no factors WU or UU are generated. On the contrary, denoting $(2n-r, r)$ the topmost point of the last descent of T, when we insert the new rightmost U step at position $(2n-r, r)$, a UU factor is formed, and when we insert it in any point $(2n-i, i)$, with $r < i \leq s$, a WU factor is formed. In both cases, the edge line of the obtained steady path must pass through the point $(2n-r, r)$ (resp. $(2n-i, i)$, for $r < i \leq s$). Then, the edge line may move rightwards as to include this point.

Now, we assign the label $(h, k) \equiv (h, r+1)$ to any steady path T of size n and edge line $y = x - 2t$, where $r \geq 1$ is the number of steps in the last descent of T and $h = (n-t) - r$. In order words, the label interpretation is such that h counts the positions that do not belong to the last descent of T, in which we insert a new rightmost U step.

The steady path UD of size 1 has edge line $y = x$. Thus its label is $(0, 2)$, which is the axiom of Ω_{steady}. Given a steady path T of size n, edge line $y = x - 2t$, and label $(h, k) \equiv (h, r+1)$, we now prove that the labels of the steady paths obtained by inserting a U step at positions $(2n, 0), \dots, (2n-s, s)$, with $s = n-t$, are precisely the label productions of Ω_{steady}. Indeed, by inserting the U step at positions $(2n, 0), \dots, (2n-(r-1), r-1)$ the edge line does not change and the paths obtained have labels $(h+k-1, 2), \dots, (h+1, k)$, respectively. Whereas, by inserting the U step at position $(2n-i, i)$, for every $r \leq i \leq s$, the edge line becomes (or remains) $y = x - 2(n-i)$ and the path has label $(0, i+2)$. Thus, we obtain the labels $(0, k+1), \dots, (0, h+k+1)$, which are the second line of the production of Ω_{steady}, completing the proof. □

Figure 5.8 depicts the growth of a steady path of size n with edge line $y = x - 2$; for any path, the corresponding edge line is drawn.

Although at a first sight the succession rule Ω_{steady} does not resemble the rule $\Omega_{1\underline{23}4}$ of Proposition 5.2.8, the following result follows by the fact that Ω_{steady} and $\Omega_{1\underline{23}4}$ actually

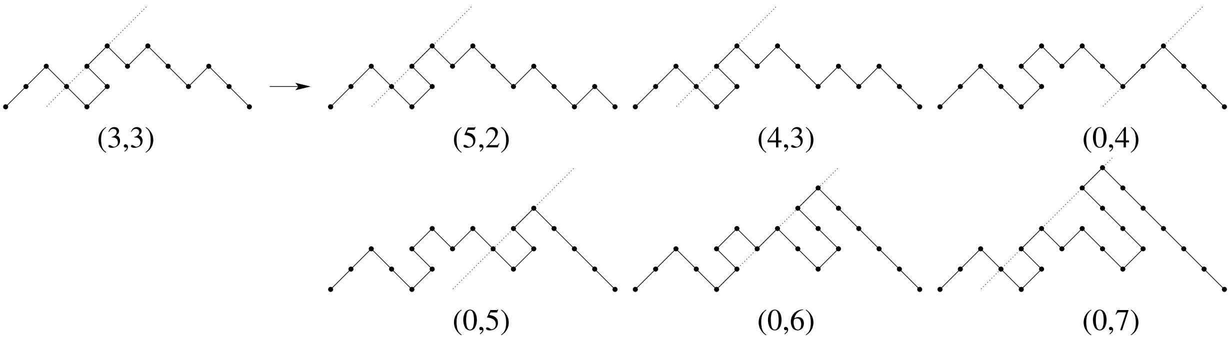


Figure 5.8: The growth of a steady path according to rule Ω_{steady}.

define the same generating tree.

Proposition 5.3.4. *The number of steady paths of size n is equal to the number of permutations in* $AV_n(1\,\underline{23}\,4)$*, thus is the nth powered Catalan number.*

Proof. We prove the above proposition by showing that the succession rule $\Omega_{1\underline{23}4}$ provided for the family $AV(1\,\underline{23}\,4)$ is isomorphic to the rule Ω_{steady}.

First, recall the production of the label $(1,k)$ according to $\Omega_{1\underline{23}4}$, which appears as

$$(1,k) \rightsquigarrow (1,k+1),(2,k),\dots,(1+k,1)\,. \tag{5.12}$$

The same succession rule $\Omega_{1\underline{23}4}$ yields that the label $(h,0)$ produces according to

$$(h,0) \rightsquigarrow (1,h),(2,h-1),\dots,(h,1)\,. \tag{5.13}$$

Now, consider the generating tree defined by $\Omega_{1\underline{23}4}$ and replace all the labels $(1,k)$ by $(k+1,0)$. According to the production (5.12) the children of the node with a replaced label are

$$(k+1,0) \rightsquigarrow (k+2,0),(2,k),\dots,(1+k,1)\,,$$

namely

$$(h,0) \rightsquigarrow (h+1,0),(2,h-1),\dots,(h,1)\,,$$

which is exactly the production (5.13) after substituting $(1,h)$ for $(h+1,0)$ in it. This substitution allows us to rewrite the succession rule $\Omega_{1\underline{23}4}$ as follows

$$\begin{cases} (2,0) & \\ & \\ (h,k) & \rightsquigarrow \quad (h+k+1,0), \\ & \qquad (2,h+k-1),\dots,(h,k+1), \\ & \qquad (h+1,0),\dots,(h+k,0). \end{cases}$$

It is straightforward to check that the growth provided for steady paths in Proposition 5.3.3 defines the above succession rule by exchanging the interpretations of the two parameters h and k with respect to Ω_{steady}. □

5.3.2 Bijection with some pattern-avoiding permutations

Proposition 5.3.4 establishes a recursive bijection between steady paths and permutations of the family $AV(1\underline{23}4)$. Here we provide a more natural and simple bijection between steady paths and another family of pattern-avoiding permutations counted by the powered Catalan number sequence.

Theorem 5.3.5. *The family of steady paths and $AV(1\underline{34}2)$ are in bijection.*

Proof. By Remark 5.3.2, any steady path T of size n is uniquely determined by the positions of its up steps, namely by the points $(0,0) = (i_1, j_1), \dots, (i_n, j_n)$. These points that encode a unique steady path can in turn be encoded from right to left by a sequence $(t_1, \dots, t_n)$ of integers that records the Euclidian distance between these points and the main diagonal $y = x$. More precisely, the entry t_k is the distance between the point (i_{n+1-k}, j_{n+1-k}) and the line $y = x$, for any $1 \leq k \leq n$. Note that $t_n = 0$, because the point $(0,0)$ belongs to the main diagonal. Moreover, for any $1 \leq k \leq n$, the entry t_k is in the range $[0, n-k]$, since steady paths are constrained into the cone $\mathfrak{C} = \{(x,y) \in \mathbb{N}^2 : y \leq x\}$. For instance, the steady path depicted in Figure 5.7(a) is encoded by the sequence $(5,3,0,4,1,0,1,0)$.

Then, we have that any steady path of size n is defined by a particular sequence $(t_1, \dots, t_n)$, for which $0 \leq t_k \leq n-k$, for every k. Certainly, the set of all these particular sequences of size n forms a subset of the set $\{\mathtt{T}(\pi) : \pi \in \mathcal{S}_n\}$ of the left inversion tables of permutations of length n. Our aim is to prove that a left inversion table $t = (t_1, \dots, t_n)$, with $0 \leq t_k \leq n-k$, defines a steady path of size n if and only if $t \in \{\mathtt{T}(\pi) : \pi \in AV_n(1\underline{34}2)\}$.

⇒) We prove the contrapositive. Suppose $t = (t_1, \dots, t_n)$ is the left inversion table of a permutation $\pi \notin AV_n(1\underline{34}2)$. Then, since $1\underline{34}2 \preccurlyeq \pi$, there must be three indices i, j, ℓ, with $i < j < j+1 < \ell$, such that $\pi_i < \pi_\ell < \pi_j < \pi_{j+1}$. Moreover, we can suppose without loss of generality that there are no points π_s between π_i and π_j such that $\pi_s < \pi_i$. Otherwise, we could take $\pi_s \pi_j \pi_{j+1} \pi_\ell$ as our occurrence of $1\underline{34}2$.

Then, by definition of the left inversion table $t = \mathtt{T}(\pi)$, since $\pi_j < \pi_{j+1}$ and $\pi_j > \pi_\ell$, we have that $0 < t_j \leq t_{j+1}$. In addition, since there are no points π_s between π_i and π_j such that $\pi_s < \pi_i$, and $\pi_j > \pi_\ell > \pi_i$, it holds that $t_i < t_j$. From this it follows that t cannot encode a steady path T. Indeed, assuming such a path T would exists, t_i (resp. t_j, resp. t_{j+1}) must be the distance between the line $y = x$ and an up step $U^{(i)}$ (resp. $U^{(j)}$, resp. $U^{(j+1)}$), where $U^{(j+1)}$, $U^{(j)}$, and $U^{(i)}$ appear in this order from left to right. Since $t_{j+1} \geq t_j$, the up step $U^{(j)}$ must form either a UU factor or WU factor. Note that the line parallel to the main diagonal passing through $U^{(j)}$ cannot be $y = x$, since $t_j > 0$. Let this line be $y = x - g$, with g even positive number. Then, from $0 \leq t_i < t_j$ it follows that the suffix of T containing the up step $U^{(i)}$ exceeds the line $y = x - g$ passing through $U^{(j)}$.

⇐) Conversely, suppose for the sake of contradiction that there exists a left inversion table $t = (t_1, \dots, t_n)$ which encodes a non-steady path T of size n.

By definition of steady path, there must be in T an up step $U^{(j)}$ not lying on the main diagonal such that it forms a factor UU or WU, and an up step $U^{(i)}$, which is on the right of $U^{(j)}$, lying above the line parallel to $y = x$ and passing through $U^{(j)}$. This means that $0 < t_j \leq t_{j+1}$, where $U^{(j+1)}$ is the up step which $U^{(j)}$ immediately follows, and $0 \leq t_i < t_j$, with $i < j$. Thus, let $\pi = \mathtt{T}^{-1}(t)$. By Lemma 5.1.4 on page 143, we have that $\pi_i < \pi_j < \pi_{j+1}$. In addition, from $0 \leq t_i < t_j$ it follows that there exists at least a point π_ℓ, with $j < \ell$, such that (π_j, π_ℓ) is an inversion of π and (π_i, π_ℓ) is not. Consequently, $\pi_i \pi_j \pi_{j+1} \pi_\ell$ forms an occurrence of $1\,\underline{34}\,2$. □

5.3.3 Relation with valley-marked Dyck paths

In this section we seek for a bijection between steady paths and valley-marked Dyck paths. Although both families are comprised of lattice paths confined to the region $\mathfrak{C}$, it is not obvious to establish a bijection between the two families, and we still do not have such a correspondence. We believe that a direct bijection between these two families of lattice paths may be of help as to show the relation between the succession rule Ω_{steady}, according to which steady paths grow, and the rule Ω_{pCat}, according to which valley-marked Dyck paths grow.

Moreover, one should stress that every Dyck path is a steady path, thus Dyck paths of semi-length n form a subset of the set of steady paths of size n. In addition, given a Dyck path P of semi-length n, if we fix a "default" marking for each valley (for instance, each valley is marked at height 0), then P with standard marks is a valley-marked Dyck paths of semi-length n. Therefore, both families come out as generalisations of the Catalan family of Dyck paths.

We could think of establishing a bijection that extends the trivial bijection between steady paths that are Dyck paths and Dyck paths whose valleys are all marked at height 0. To this purpose, we conjecture the following equidistribution of parameters.

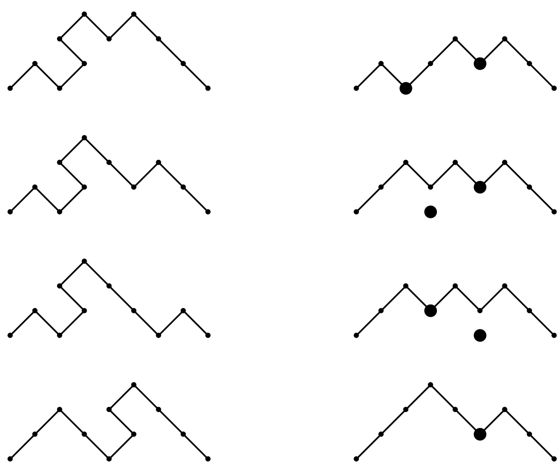


Figure 5.9: Steady paths and valley-marked Dyck paths with $n = 4$, $m = 2$ and $t = 1$.

Conjecture 5.3.6. *Steady paths of size n having m U steps lying on the main diagonal and t W steps are as many as valley-marked Dyck paths of semi-length n having m D steps in the last descent and t being the total sum of the valley-mark heights.*

For instance, Figure 5.9 depicts on the left all the steady paths of size $n = 4$ with $m = 2$ U steps on the main diagonal and $t = 1$ W step, and on the right valley-marked Dyck paths of semi-length $n = 4$ with $m = 2$ D steps in the last descent and total height $t = 1$ of the valley-marks.

Conjecture 5.3.6 is supported by numerical evidence calculated by means of the two generating functions

$$S(x; y, z) = \sum_{n,m,t \geq 0} s_{n,m,t}\, x^n y^m z^t; \qquad \text{and} \qquad V(x; y, z) = \sum_{n,m,t \geq 0} v_{n,m,t}\, x^n y^m z^t\,,$$

where $s_{n,m,t}$ (resp. $v_{n,m,t}$) is the number of steady paths of size n having m U steps lying on the main diagonal and t W steps (resp. valley-marked Dyck paths of semi-length n with m D steps in the last descent and total height t of the valley-marks). Indeed, by specialising the growth for both families, provided in Proposition 5.3.3 for steady paths and Proposition 5.2.3 for valley-marked Dyck paths, we can generate the first terms of $S(x; y, z)$ and $V(x; y, z)$ and check (by using MAPLE, for instance) that $s_{n,m,t} = v_{n,m,t}$, for any m, t, and $n \leq 22$. Table 5.2 displays the terms $s_{n,m,t}$, for $n = 5, 6$ and any possible value of m, t.

$\mathbf{n = 5}$	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$
$m = 1$	14	7	2		
$m = 2$	14	16	8	2	
$m = 3$	9	10	8	3	1
$m = 4$	4	3	2	1	
$m = 5$	1				

$\mathbf{n = 6}$	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$	$t = 6$
$m = 1$	42	36	20	6	1		
$m = 2$	42	64	49	24	7	1	
$m = 3$	28	43	45	29	15	5	1
$m = 4$	14	18	18	14	7	3	1
$m = 5$	5	4	3	2	1		
$m = 6$	1						

Table 5.2: The number $s_{n,m,t}$ of steady paths of size n having m U steps lying on the main diagonal and t W steps, for $n = 5, 6$ and any possible value of m, t.

The first specialisation which allows us to state Conjecture 5.3.6 involves valley-marked Dyck paths, and comes from a refinement of the succession rule Ω_{pCat}.

Proposition 5.3.7. *The number of valley-marked Dyck paths of semi-length n with m D steps in the last descent, and t being the total sum of the valley-mark heights is given by the number of labels (m,t) at level n in the generating tree defined by*

$$\begin{cases} (1,0) \\ \\ (m,t) & \rightsquigarrow (1,t), \quad (2,t), \quad \ldots, \quad (m,t), \qquad (m+1,t), \\ & \qquad\qquad (2,t+1), \quad \ldots, \quad (m,t+1), \\ & \qquad\qquad\qquad\qquad \ddots \quad \vdots \\ & \qquad\qquad\qquad\qquad\quad (m,t+m-1). \end{cases}$$

Proof. The proof of the above statement follows straightforward by considering the growth of Proposition 5.2.3. □

The second specialisation we show involves steady paths and a refinement of the succession rule Ω_{steady}. It needs more parameters than those entailed by the refinement of Proposition 5.3.7. For this reason, we admit the possibility of considering all the possible values for some parameters, and we denote it by $*$.

Proposition 5.3.8. *The number of steady paths of size n with m U steps on the main diagonal, and t W steps is given by the number of labels $(*,*,m,t,*)$ at level n in the generating tree defined by*

$$\begin{cases} (0,2,1,0,\top) \\ \\ (h,k,m,t,\top) & \rightsquigarrow (h+k-1,2,m,t,\top),\ldots,(h+1,k,m,t,\top), \\ & \quad (0,k+1,m,t,\bot),\ldots,(0,h+k,m,t+h-1,\bot), \\ & \quad (0,h+k+1,m+1,t+h,\top), \\ \\ (h,k,m,t,\bot) & \rightsquigarrow (h+k-1,2,m,t,\bot),\ldots,(h+1,k,m,t,\bot), \\ & \quad (0,k+1,m,t,\bot),\ldots,(0,h+k,m,t+h-1,\bot), \\ & \quad (0,h+k+1,m,t+h,\bot). \end{cases}$$

Proof. The proof of the above statement is obtained by considering the growth provided in Proposition 5.3.3 and labelling each steady path with (h,k,m,t,q), where h,k have the same interpretation as Ω_{steady}, m marks the number of U steps on the main diagonal, t the number of W steps, and $q=\top$ (resp. $\bot$) marks steady path whose edge line is (resp. is not) $y=x$. □

5.3.4 Two different families of powered Catalan structures

The above section and the general difficulty in finding a simple bijective relation between valley-marked Dyck paths and steady paths, or between valley-marked Dyck paths and $1\underline{23}4$-avoiding permutations, leads us to classify powered Catalan structures into two

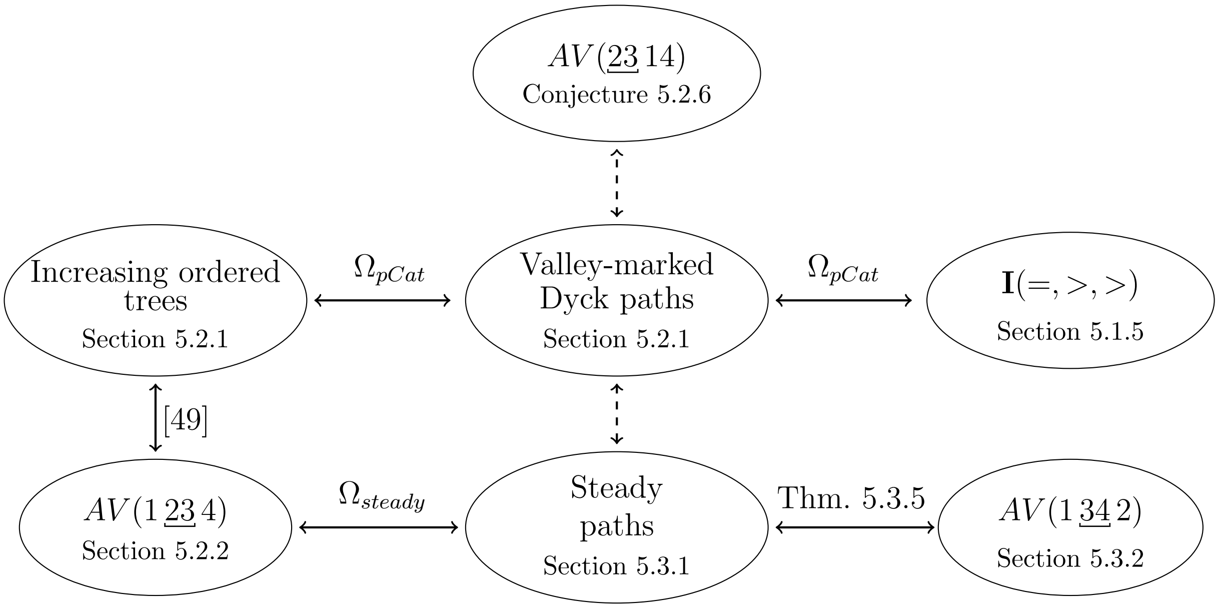


Figure 5.10: All the structures known or conjectured to be enumerated by the powered Catalan numbers and their relations: a solid-line arrow indicates a bijection (either recursive, or direct), while a dashed-line arrow indicates a missing bijection.

groups. In fact, among all the structures known or conjectured to be enumerated by the powered Catalan number sequence [132, A113227], we distinguish two types:

- those that appear as a rather simple generalisation of Catalan structures, for which a growth according to the rule Ω_{pCat} can be found easily;
- those that generalise Catalan structures, but for which a growth according to Ω_{pCat} is not immediate, and the parameter k of Equation (5.11) is not clearly understood.

Valley-marked Dyck paths are the emblem of the first group; while, steady paths as well as $1\,\underline{23}\,4$-avoiding permutations rather belong to the second group of structures. Figure 5.10 shows a map with all the powered Catalan structures of this chapter.

5.3.5 Generalisations of steady paths

In this section we describe two families of lattice paths that come out as natural generalisations of steady paths, when only one of the two conditions (S1) and (S2) is kept.

Definition 5.3.9. A *UU-constrained* path of size n is a lattice path P confined to the cone $\mathfrak{C} = \{(x, y) \in \mathbb{N}^2 : y \leq x\}$, which uses $U = (1, 1)$, $D = (1, -1)$ and $W = (-1, 1)$ as steps, but without any factors WD or DW, starting at $(0, 0)$ and ending at $(2n, 0)$, such that:

(S1) for any factor UU, the suffix of T starting at this UU factor lies weakly below the line parallel to $y = x$ passing through this UU.

Definition 5.3.10. A WU*-constrained* path of size n is a lattice path P confined to the cone $\mathfrak{C} = \{(x, y) \in \mathbb{N}^2 : y \le x\}$, which uses $U = (1, 1)$, $D = (1, -1)$ and $W = (-1, 1)$ as steps, but without any factors WD or DW, starting at $(0, 0)$ and ending at $(2n, 0)$, such that:

(S2) for any factor WU, the suffix of T starting at this WU factor lies weakly below the line parallel to $y = x$ passing through the up step of this WU.

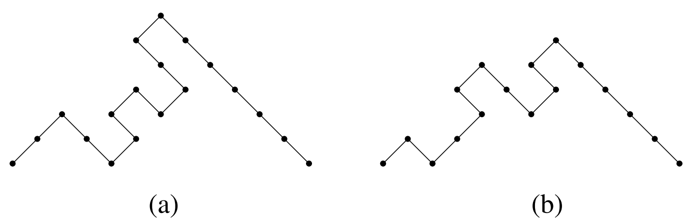


Figure 5.11: (a) A UU-constrained path which is not a steady path; (b) A WU-constrained path which is not a steady path.

Figure 5.11 (a) depicts a UU- but not WU-constrained path, while 5.11 (b) depicts a WU- but not UU-constrained path. It is clear that every steady path is both a UU- and a WU-constrained path (see Figure 5.7 (a)). Moreover, from the growth provided for steady paths in the proof of Proposition 5.3.3, we can easily obtain recursive constructions both for UU-constrained paths and for WU-constrained paths.

Proposition 5.3.11. *The family of UU-constrained paths can be generated by*

$$\Omega_{UU} = \begin{cases} (0,2) \\ \\ (h,k) \quad \rightsquigarrow \quad (h+k-1,2), \dots, (h+1,k), \\ \qquad\qquad\quad (0,k+1), (h-1,k+2), \dots, (0,h+k+1). \end{cases}$$

Proof. The proof of the above statement is provided simply by generalising the growth for the family of steady paths according to Ω_{steady} (see Proposition 5.3.3). Indeed, as steady paths do, UU-constrained paths are uniquely determined by the positions of their up steps, and thus a growth can be defined by adding a new rightmost up step. □

Observe that the first line production is the same as Ω_{steady}, while all the label productions of the second line (except for the first and the last one) differ from Ω_{steady}. By iterating the rule Ω_{UU}, we have been able to obtain the first terms of the sequence enumerating UU-constrained paths, which are

$$1, 2, 6, 23, 107, 586, 3706, 26683, 216221, 1952669, 19483879, 213160098, 2539536946, \dots$$

Yet this number sequence does not match any entry on OEIS [132].

Now, we turn to a recursive construction for WU-constrained paths.

Proposition 5.3.12. *The family of WU-constrained paths can be generated by*

$$\Omega_{WU} = \begin{cases} (0,2) \\ \\ (h,k) \rightsquigarrow & (h+k-1,2),\dots,(h+1,k), \\ & (h,k+1),(0,k+2),\dots,(0,h+k+1). \end{cases}$$

Proof. As UU-constrained paths, WU-constrained paths are uniquely determined by the positions of their up steps, and thus a growth can be defined by adding a new rightmost up step. The proof of the above statement is obtained by generalising the growth of Proposition 5.3.3, analogously to Proposition 5.3.11. □

Observe that Ω_{WU} differs from Ω_{steady} only for the first label of the second line production. By successive iterations of the rule Ω_{WU}, we have been able to obtain the first terms of their enumerative sequence, which are

$$1,2,6,24,118,676,4362,31012,239294,1982336,17487348,163236860,1604203376,\dots$$

Again the above number sequence does not match any entry on OEIS [132].

We tried to study the functional equation obtained by translating the succession rule Ω_{UU} (resp. Ω_{WU}), yet without finding any expression for the ordinary generating function of UU-constrained paths (resp. WU-constrained paths). Nevertheless, we believe that a "nice" recurrence like (5.11) might control the coefficients related to UU-constrained (as well as WU-constrained) paths, although we do not have any hints about it.

Chapter 6

Fighting fish

Plan of the chapter

The aim of this chapter is to introduce and study the properties of a new class of branching surfaces that appears as a generalisation of parallelogram polyominoes. These objects called fighting fish because of their appearance, display remarkable probabilistic and enumerative properties and are strictly related to another well-known combinatorial structure: the family of plane trees. Fighting fish have been introduced for the first time in [G6] and, then their combinatorial properties have been developed in [G7].

This chapter starts in Section 6.1 with a basic definition of fighting fish as a finite set of cells. Then, Section 6.2 continues providing alternative models of the same objects, among them a recursive description called *master decomposition* [G6]. Another recursive decomposition for fighting fish is shown in Section 6.5 and is called the *wasp-waist decomposition* [G7]. The master decomposition and the wasp-waist decomposition are complementary in the sense that they allow us to study the family of fighting fish according to different parameters.

Thanks to these two recursive definitions of Section 6.2 and Section 6.5, two different functional equations having the (multivariate) generating functions of fighting fish as respective solutions can be written down. Then, they are solved respectively in Section 6.3.3 and Section 6.5.3, by using a generalisation of the kernel method described in Section 6.3.2. This shows that the size generating function of fighting fish is algebraic, as well as both multivariate generating functions.

Both equations are useful for enumerative purposes since they provide information on fighting fish according to different parameters. Section 6.4 is a collection of explicit formulas for fighting fish obtained from the master decomposition: the formula for the number of fighting fish (sequence A000139 [132], Section 6.4.1) and for the number of fighting fish with a marked tail (sequence A006013 [132], Section 6.4.2). In Section 6.5.4 we show other remarkable formulas for these objects using the wasp-waist decomposition, rather than the master decomposition.

All the explicit formulas derived for fighting fish are somehow related to plane trees and in Section 6.6 we exploit the link between fighting fish and plane trees, and try to explain combinatorially some of the properties analytically proved.

6.1 Basic definitions

As reported in the `Encyclopædia Britannica` "the *Siamese fighting fish* (*Betta splendens*) is a freshwater tropical fish of the family Osphronemidae (order Perciformes), noted for the pugnacity of the males toward one another. The Siamese fighting fish, a native of Thailand, was domesticated there for use in contests. Combat consists mainly of fin nipping and is accompanied by a display of extended gill covers, spread fins, and intensified colouring".

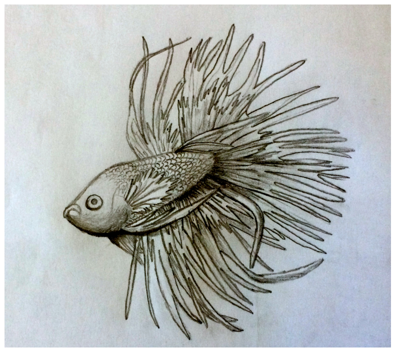


Figure 6.1: Siamese fighting fish.

We are going to introduce a simple model for these objects, which has been inspired on the one hand by the rich literature on polyominoes brilliantly discussed in the recent book "Polygons, Polyominoes and Polycubes" edited by T. Guttmann [91], and on the other hand by the aquatic creatures commonly called *fighting fish* (see Figure 6.1).

We start giving a first description of our combinatorial fighting fish that consists of glueing together unit squares of paper along their edges in a directed way. More precisely, as illustrated by Figure 6.2, we consider 45 degree tilted unit squares, which we call *cells*, and view them as made of two triangular halves, which we briefly call *left scale* and *right scale*. All four edges of each square are distinguished and we refer to them as *left upper edge*, *left lower edge*, *right upper edge* and *right lower edge*.

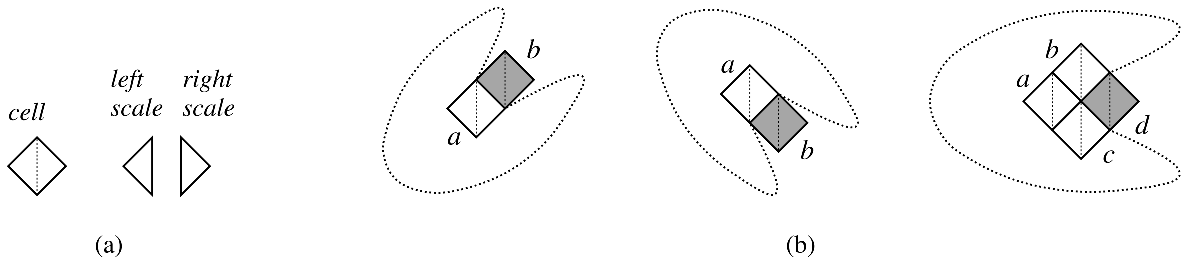


Figure 6.2: (a) The left and right scales of a cell; (b) the three ways to add a cell.

In a fighting fish we define *free* any edge of a cell which is not glued to the edge of another cell. A *vertex* is any point of incidence between two (or more) edges. Then, all fighting fish can be obtained starting from an initial cell, whose both left edges are free, by attaching cells one by one in exactly one of the following ways (see Figure 6.2(b)):

- Let a be a cell in the fish whose right upper edge is free; then glue the left lower edge of a new cell b to the right upper edge of a.

- Let a be a cell in the fish whose right lower edge is free; then glue the left upper edge of a new cell b to the right lower edge of a.

- Let a, b and c be three cells in the fish such that b (resp. c) has its left lower (resp. upper) edge glued to the right upper (resp. lower) edge of a, and b (resp. c) has its right lower (resp. upper) edge free; then simultaneously glue the left upper and left lower edges of a new cell d respectively to the right lower edge of b and to the right upper edge of c.

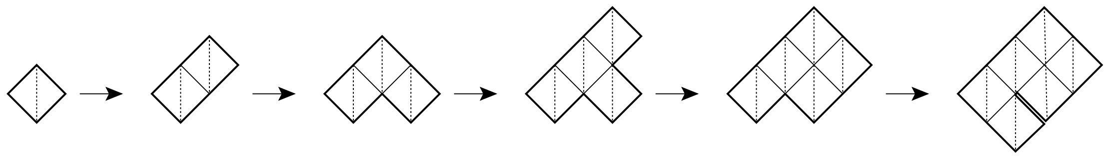


Figure 6.3: One way to construct a fighting fish from an initial cell by using operations of Figure 6.2(b).

Since this description is iterative and an object could be obtained more than once by glueing cells together, in our definition of fighting fish we are interested only in the objects produced disregarding the order in which the cells are added. Indeed, as shown in Figure 6.3 the same object can be constructed by inverting the second and the third cell additions or by anticipating the last cell addition, as that cell is attached only to the right lower edge of the initial cell.

Definition 6.1.1. A finite set of cells glued together edge by edge is a *fighting fish* if and only if it *can* be obtained by the iterative process above or it is empty.

We call the *head* of any fighting fish the left scale of the unique cell with both its left edges free - which is precisely the initial cell of the above construction; whereas, a *tail* is the right scale of any cell having both the right edges free. A *branching point* is any vertex between a free upper edge and a free lower edge, where by travelling the fish boundary from the head counterclockwise the free upper edge precedes the free lower one. According to this definition, the number of branching points in any fighting fish is exactly one less than its number of tails. Figure 6.4(a) shows a fighting fish with two branching points, which are encircled; while the head and the tails of the fighting fish are shaded.

Remark 6.1.2. *According to the cell additions described above and illustrated by Figure 6.2(b), no holes can be generated following this glueing process. Thus, any fighting fish has no holes: Figure 6.4(a) depicts a fighting fish without holes, although its projection onto the plane gives rise to a two-cells hole.*

In addition, starting from the fish head and travelling counterclockwise its boundary, all the free edges of the fighting fish are encountered, thus forming a cycle.

Now, we introduce some parameters on fighting fish: we define the *size* of a fighting fish as the number of its free lower edges and the *area* as the number of its cells.

We observe that vertical edges dividing cells of a fighting fish into scales form vertical segments that cut the fish into *vertical strips* each one consisting of an alternating sequence of connected left and right scales, which start with a free (left or right) lower edge and end with a free (left or right) upper edge.

Remark 6.1.3. *For any fighting fish, the number of its free lower edges is equal to the number of its free upper edges. Indeed, each vertical strip cutting the fighting fish connects a free lower edge to a free upper edge, and no free edges are out of the vertical strips decomposition. Thus, the size of a fighting fish can be equivalently calculated by counting the number of its free lower (resp. upper) edges as well as the number of its vertical strips.*

A simple example of fighting fish is given by parallelogram polyominoes: in particular, parallelogram polyominoes - up to a 45 degree clockwise rotation - are precisely those fighting fish that have only one tail.

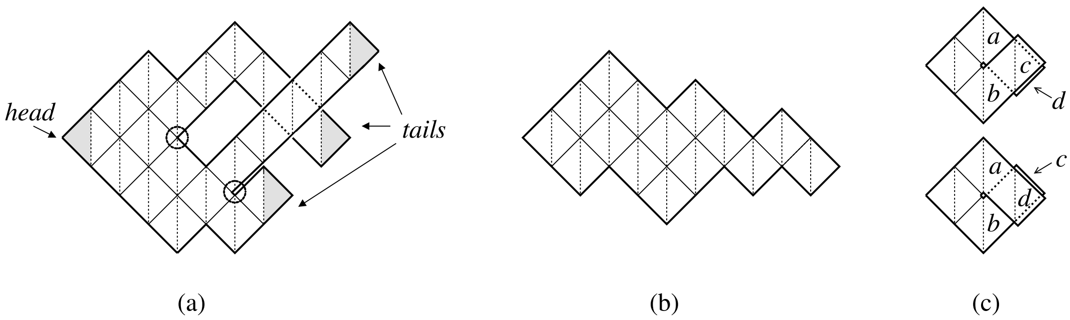


Figure 6.4: (a) A fighting fish which is not a polyomino; (b) a fighting fish with one tail; (c) two different representations of the unique fighting fish with area 5 not fitting in the plane.

Proposition 6.1.4. *Fighting fish with one tail are in one-to-one correspondence with parallelogram polyominoes.*

Proof. Consider a parallelogram polyomino as a set of cells glued together: any parallelogram polyomino is definitely a fighting fish with one tail up to a 45 degree clockwise rotation. Indeed, any parallelogram polyomino can be built by glueing cells column by column, as follows. The cells of the first column are glued together bottom-to-top by iteratively using the first cell addition of Figure 6.2(b), the bottommost cell being the head of the fighting fish. Then, we attach all the cells of the second column by using the second cell addition of Figure 6.2(b) for glueing its bottommost cell to the first column, and the third and first cell additions of Figure 6.2(b) for glueing all the other cells. Finally, recursively repeat this process for the other columns of the parallelogram polyomino from left to right.

Conversely, given a fighting fish having only one cell with both its right edges free, it is not hard to associate a pair of paths that results to define a parallelogram polyomino with it. More precisely, start from the head and end at the unique tail: the upper path is coded by writing a north (resp. east) step for every left (resp. right) free upper edge and the lower path by writing an east (resp. north) step for every left (resp. right) free lower edge. By definition of fighting fish these two paths do not intersect except for the starting and the ending point, thus define a parallelogram polyomino. □

According to the cell additions described in Figure 6.2(b) other important classes of polyominoes can be intuitively constructed: directed convex polyominoes and simply connected (*i.e.* without holes) directed polyominoes - in the sense of [91].

Proposition 6.1.5. *Directed convex polyominoes, and more generally, simply connected directed polyominoes are fighting fish.*

Nevertheless, other important classes of polyominoes are not fighting fish: for instance, convex polyominoes, or directed polyominoes, see [91, Chapter 3].

One should stress also the fact that fighting fish are not necessarily polyominoes because they are not constrained to fit in the plane, as illustrated by Figure 6.4(c).

The smallest fighting fish not fitting in the plane is obtained by glueing a cell a to the right upper edge of the head, a cell b to the right lower edge of the head, a cell c to the right upper edge of b, and a cell d to the right lower edge of a: in the natural projection of this fighting fish onto the plane $\mathbb{R}^2$, cells c and d have the same image. Observe that we do not specify whether c is above or below d; rather we consider that the surface has a branching point at the head vertex between its right upper edge and its right lower edge (see Figure 6.4(c)). A list of all fighting fish of area at most 4 is given in Figure 6.5.

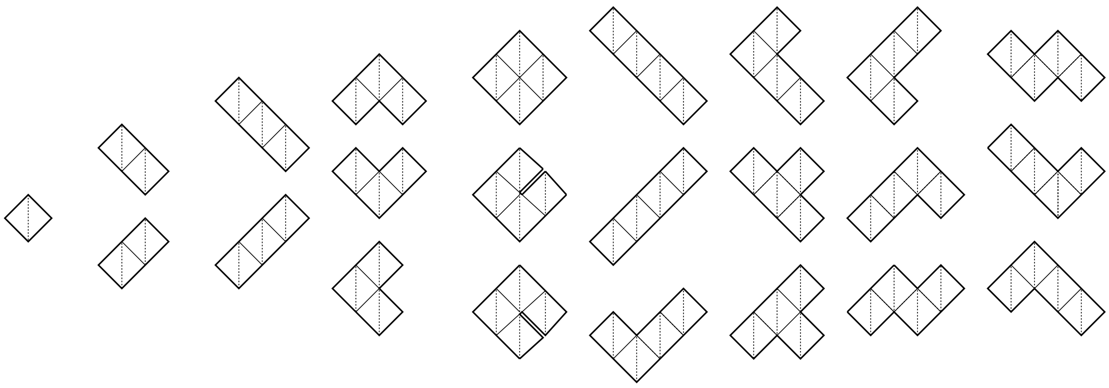


Figure 6.5: Fighting fish of area at most 4.

6.2 Alternative definitions

According to the previous definition fighting fish could remain objects rather mysterious: we cannot recognise a fighting fish without decomposing it and the way to reconstruct it is not unique. In order to present the combinatorial objects subject of this chapter in a clearer way, the following subsections are addressed to provide different equivalent definitions of them.

6.2.1 Topological definition

We can provide an alternative topological definition of fighting fish that follows closely the intuitive definition of the previous section: we construct a branching surface by glueing triangular pieces together. More precisely, let us consider triangular pieces of bicoloured paper (green on one side, and red on the other). If the boundary of the triangle is oriented counterclockwise (resp. clockwise) around the green side of the paper, we call it *left* (resp. *right*) *triangle*. The three edges of each triangle are distinguished and conventionally called the *upper*, *lower* and *vertical* edge, in such a way that the upper edge follows the vertical edge in the boundary circuit - see Figure 6.6.

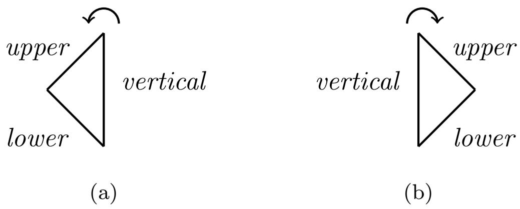


Figure 6.6: A left triangle (a) and a right triangle (b) with their edges named.

We claim that a non-empty fighting fish is a branching surface obtained by glueing together left and right triangles along their oriented edges in such a way that the following conditions are satisfied:

Finiteness condition: The total number of glued triangles is finite.

Local smoothness condition: Each vertical edge of a left triangle is glued to a vertical edge of a right triangle (and conversely), thus forming a unique oriented vertical edge. Each upper (resp. lower) edge of a right triangle can be either free (*i.e.* not glued to anything) or glued to a lower (resp. upper) edge of a left triangle (and conversely).

Local triangular lattice geometry condition: Each vertex is incident to *at most* one outgoing and one incoming vertical edges.

Simple connectedness condition: The resulting surface is simply connected and the set of free edges forms a unique (non oriented) boundary cycle.

Directedness condition: There is *only* one left triangle with both the upper and the lower edges free, which is called the head of the fish. Any triangle is connected to the head by an inner path that alternately crosses vertical and non-vertical edges. Any right triangle with both the upper and the lower edges free is called tail.

By such a definition a non-empty fighting fish is topologically equivalent to a properly coloured disc (with a green side and a red side, for instance).

The definition of fighting fish given in the previous section (Definition 6.1.1) is consistent with the above description: let us think of the left and right triangles above as the left and right scales in which each cell of a non-empty fighting fish can be divided, and associate any such a branching surface described above with a fighting fish of Definition 6.1.1. Hence, there is no abuse of notation in naming the edges of a triangle upper, and lower, and vertical.

More precisely, analysing the above conditions we can notice straightforward correspondences. The finiteness condition on triangles number corresponds to the fact that any fighting fish is a finite set of cells. The local smoothness combined with the local triangular lattice geometry condition ensures that left and right triangles are united to form a cell and two different cells can be glued together only by edges, joining the right upper (resp. right lower) edge of one to the left lower (resp. left upper) edge of the other. The simple connectedness condition means that the surface obtained has no holes, which is a necessary condition to be a fighting fish. The property of being directed is hidden in the iterative construction of fighting fish. Indeed, according to Definition 6.1.1 we start constructing a fighting fish by a single cell, whose left scale has both its upper and lower edges free. By using the iterative construction of Figure 6.2(b) there is no way to glue a new cell whose left scale has both the upper and the lower edges free. Moreover, by construction any inner path between the head and any other scale of the fighting fish can be continuously transformed, without exiting from the fighting fish, into any other inner path linking these two scales. Thus, any triangle is connected to the head by an inner path that alternately crosses vertical and non-vertical edges.

6.2.2 Recursive definition: the master decomposition

For enumerative purpose it is useful to give a recursive description of fighting fish. In order to do this we define the set of *fish tails* and their heights inductively as follows:

Basis. The empty fish is the unique fish tail with height 0.

Inductive step. We define three operations:

- **Operation u:** Given two fish tails T_1 of height $\ell \geq 0$ and T_2 of height $k \geq 0$, then a new fish tail T of height $\ell + 1 + k$ is obtained by glueing a vertical strip of $\ell + 1$ right scales and ℓ left scales to T_1 and by attaching this strip to the topmost point of the leftmost vertical segment of T_2 - see Figure 6.7(a).
- **Operations h, h':** Given two fish tails T_1 of height $\ell \geq 1$ and T_2 of height $k \geq 0$, then a new fish tail T of height $\ell + k$ is obtained by glueing a vertical strip of ℓ right scales and ℓ left scales to T_1 and by attaching this strip to the topmost point of the leftmost vertical segment of T_2. Observe that there are two ways in which this operation can be performed and they depend on whether the added

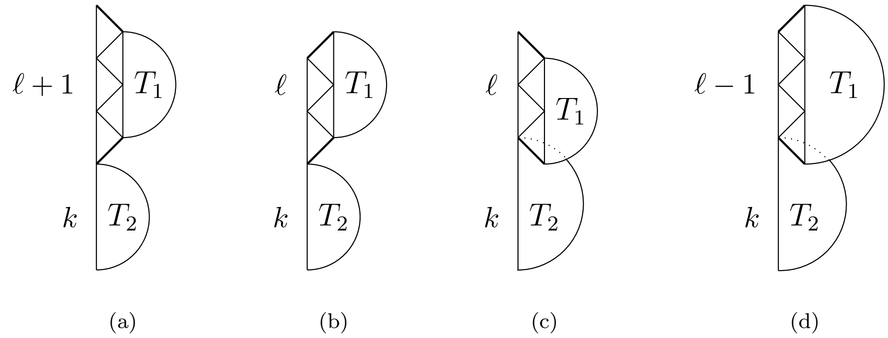


Figure 6.7: Recursive construction of fish tails: (a) operation u, (b) operation h, (c) operation h', (d) operation d.

strip starts with a right scale (operation h in Figure 6.7(b)) or with a left scale (operation h' in Figure 6.7(c)).

Operation d: Given two fish tails T_1 of height $\ell \geq 2$ and T_2 of height $k \geq 0$, then a new fish tail T of height $\ell - 1 + k$ is obtained by glueing a vertical strip of $\ell - 1$ right scales and ℓ left scales to T_1 and by attaching this strip to the topmost point of the leftmost vertical segment of T_2 - see Figure 6.7(d).

Note that any operation, apart from u, admits the empty fish tail as T_2, but not as T_1. There exists a unique fish tail obtained by applying operation u to two empty fish tails: it is a right scale, and it has height 1, and we usually refer to it simply as *tail*.

Proposition 6.2.1. *Every fish tail is uniquely obtained by using operations u, h, h', and d.*

Proof. Given a fish tail T, the proof follows by induction on the numbers of scales which T consists of. There is a unique empty fish tail. Suppose T is non-empty. By definition, T is produced starting from two smaller fish tails T_1 and T_2 and using at least one of the four operations u, h, h' and d. Since each of these operations results in different shapes of the leftmost and topmost vertical strip, T is uniquely obtained from T_1 and T_2 by checking the shape of the connected vertical strip which is the topmost among the leftmost. The result follows by induction on T_1 and T_2. □

Observe that, like fighting fish, fish tails of any height are not constrained to fit into the plane - see, for instance, the fish tails in Figure 6.8(a),(c).

In order to understand better this recursive definition we describe how to decompose the fish tail T of Figure 6.8(a): T is uniquely obtained by operation d with T'_1 of height 2 and T'_2 empty. Then, we decompose T'_1, which is obtained as well as by operation d with L_1 of height 3 and L_2 empty, and in turn L_1 by operation h' starting from a fish tail L'_1 of height 1 and a fish tail L'_2 of heigth 2 - as depicted in Figure 6.8(b). The fish tail L'_1 is obtained from the right scale called tail by a sequence of operations h and h', each time setting the fish tail T_2 as empty. Whereas, the fish tail L'_2 is the result of applying operation u to a fish tail of height 1, which in turn is obtained by applying operation d to

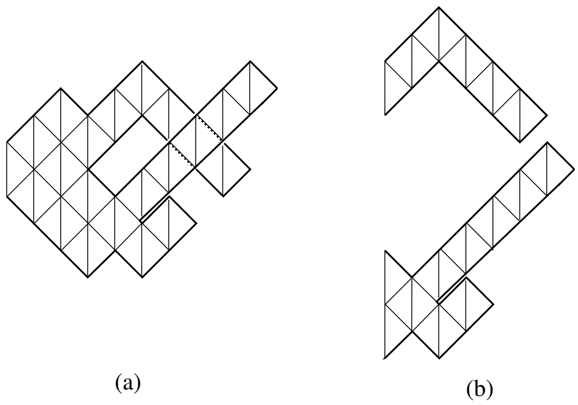


Figure 6.8: (a) Fish tail of height 1, which corresponds to the fighting fish of Figure 6.4(a); (b) fish tails L'_1 (above) and L'_2 (below); (c) a fish tail of height 5 produced by using operation h starting from two smaller fish tails T_1 of height 2 and T_2 of height 3.

two fish tails S_1 and S_2. Finally, both S_1 and S_2 result from the tail by iterating operation h, each time setting T_2 empty.

In the following we retrieve the family of fighting fish as a subfamily of fish tails.

As previously defined, we call free any edge of a scale that is not glued to another edge and *size* of a fish tail its number of free lower edges.

Theorem 6.2.2. *Fighting fish of size* $n+1$ *are in one-to-one correspondence with fish tails having height* 1 *and size* n*, for any* $n \geq 0$*.*

Proof. In order to establish such a correspondence we define, for every k, a set $\mathcal{T}_k$ of objects that are built from an initial sequence of k right scales, whose vertical edges are attached along a vertical segment one by one, by glueing to them cells according to the three types of additions of Figure 6.2(b).

We show that the set of fish tails is exactly $\mathcal{T} = \bigcup_k \mathcal{T}_k$ and the set of elements of $\mathcal{T}_k$ having n free lower edges coincides with the set of fish tails having height k and size n.

Then, obviously the elements of $\mathcal{T}_1$ are in one-to-one correspondence with the family of fighting fish: given a fighting fish, its corresponding element of $\mathcal{T}_1$ is obtained by removing the left scale of its head. Then, any element of $\mathcal{T}_1$ having n free lower edges corresponds uniquely to a fighting fish of size $n+1$.

Now, we proceed by induction on the number of free lower edges of an object of $\mathcal{T}$ to show that the set of fish tails having height k and the set of elements of $\mathcal{T}_k$ having n free lower edges coincide. First, observe that the empty fish tail can be identified with the empty sequence of right scales, that is the unique element of $\mathcal{T}_0$.

Then, given an element T of $\mathcal{T}_k$ having $n > 0$ free lower edges, we show that T is indeed a fish tail of height k and size n. Let us define the cut point x_0 belonging to the initial vertical segment of T: given the lowest right scale such that any two consecutive right scales are glued to a common cell by means of the third addition rule of Figure 6.2(b), the cut point x_0 is its bottommost vertex. If all the pairs of consecutive right scales are glued to a common cell or there is only one right scale attached to the initial vertical segment, then the cut point x_0 is clearly the bottommost point of the initial vertical segment. Suppose x_0 occurs before $1 \leq \ell \leq k$ right scales: by definition, any two consecutive right scales among them have been glued to a common left scale and a vertical strip S of height ℓ has originated. Now, we use S to decompose T into two disconnect objects T_1 and T_2. The shape of S determines uniquely the number of right scales attached to the initial vertical segment of T_1, which could be either $\ell - 1$, or ℓ, or $\ell + 1$, whereas T_2 has always $k - \ell$ right scales attached to its initial vertical segment. Note that T_2 is empty if the cut point x_0 is the bottommost point of the initial vertical segment, while T_1 is empty if the vertical strip S consists only of a right scale. Moreover, since T_1 and T_2 are disconnected from each another, we claim that the sequence of cell additions that produces T can be split into a sequence s_1 of cell additions that produces S and T_1, and a sequence s_2 of cell additions that produces T_2, and again the sequence s_1 can be split in s'_1 and s'_2, with s'_2 producing T_1 from an initial sequence of m right scales. Then, $T_1 \in \mathcal{T}_m$ with n_1 free lower edges and $T_2 \in \mathcal{T}_{k-\ell}$ with n_2 free lower edges, where $n_1 + n_2 + 1 = n$. By inductive hypothesis T_1 is a fish tail of height m and size n_1 and T_2 is a fish tail of height $k - \ell$ and size n_2, and thus T is the fish tail of height k uniquely obtained from T_1 and T_2 by applying one the operations u, h, h' or d suggested by the shape of S.

Conversely, let T be a fish tail of height k and size n obtained using operation u, h, h' or d from two fish tails T_1 and T_2 of smaller size n_1 and n_2, respectively. Then, by the inductive hypothesis there are sequences of cell additions s_1 and s_2 producing T_1 and T_2, such that T_1 has n_1 free lower edges and T_2 has n_2 free lower edges. Then, sequences s_1 and s_2 can be combined together to form a sequence of cell additions applied to the k right scales of the initial vertical segment of T. This implies that T belongs to $\mathcal{T}_k$ and has n free lower edges concluding the proof. □

Note that the fish tail in Figure 6.8(a) corresponds to the fighting fish of Figure 6.4(a) whose size is 19.

6.2.3 Fish bone tree

Recall that any parallelogram polyomino is a fighting fish according to Definition 6.1.1. For the sake of clarity, we precise that any parallelogram polyomino is considered as 45 degree tilted, and so the two paths defining its border can be thought to be with up steps $(1, 1)$ and down steps $(1, -1)$, see Figure 6.4(b).

Now, we propose an alternative characterization of fighting fish based on a well-known correspondence between parallelogram polyominoes and bicoloured Motzkin words (see [11]). More precisely, let $\Sigma = \{u, d, h, h'\}$ be a four letter alphabet and δ be the morphism $(\Sigma^*, \cdot) \rightarrow (\mathbb{Z}, +)$ defined by $\delta(u) = 1$, $\delta(d) = -1$ and $\delta(h) = \delta(h') = 0$. A word w on Σ is a *bicoloured Motzkin word* if and only if $\delta(w) = 1$ and $\delta(v) \geq 1$, for all factorisations $w = zv$ where v is any suffix of length greater than 0.

Proposition 6.2.3. *There is a bijection between parallelogram polyominoes of size $n + 1$ and bicoloured Motzkin words of length n.*

Proof. Let (P, Q) be the pair of paths defining a parallelogram polyomino of size $n + 1$, and P_i (resp. Q_i) be the ith step of P (resp. Q). The word $w = w_1 \dots w_n$ corresponding to (P, Q) is built so that each $w_i \in \Sigma$ describes the pair (P_{i+1}, Q_{i+1}), for any $1 \leq i \leq n$. Precisely, the letter w_i is defined as follows:

- $w_i = u$ if P_{i+1} is a down step and Q_{i+1} is an up step. In this case the width of the polyomino increases by one from right to left: $\delta(d) = +1$;
- $w_i = h$ (resp. $w_i = h'$) if both P_{i+1} and Q_{i+1} are up steps (resp. down steps). In this case the width of the polyomino remains the same: $\delta(w_i) = 0$;
- $w_i = d$ if P_{i+1} is an up step and Q_{i+1} is an down step. In this case the width of the polyomino decreases by one from right to left: $\delta(u) = -1$.

It is straightforward that $\delta(v) \geq 1$, for all factorisations $w = zv$ where v is of length greater than 0, and $\delta(w) = 1$. Conversely, given a Motzkin word w of length n we construct a parallelogram polyomino of size $n + 1$ simply drawing its boundary (P, Q): the first step of P (resp. Q) is up (resp. down) and all the other steps are encoded in w as specified above. □

In order to extend this correspondence to fish tails, we introduce certain trees that can be regarded as an extension of bicoloured Motzkin words.

A *fish bone tree* is a rooted plane tree B where each edge is labelled by a letter of Σ such that:

- the sum of $\delta(e)$ for all edges e of B is positive, where $\delta(e)$ stands for δ applied to the label of e;
- let e be an edge of B of nodes i, j, with i parent of j; then, the sum of $\delta(f)$, where f is running over all edges in the subtree rooted at j, plus $\delta(e)$ is positive.

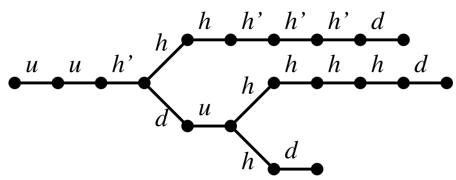


Figure 6.9: The fish bone tree corresponding to the fighting fish in Figure 6.4 (a).

We define the weight $\delta(B)$ of a fish bone tree B as the sum of $\delta(e)$ for all edges e of B,

$$\delta(B) = \sum_{e \in B} \delta(e).$$

Proposition 6.2.4. *Fish tails of height k and size n are in one-to-one correspondence with fish bone trees with weight k and n edges, and in particular, fighting fish of size $n+1$ are in one-to-one correspondence with fish bone trees with weight 1 and n edges.*

Proof. Given a fish tail T, we construct an abstract rooted plane tree B whose edges correspond to the vertical strips of T. Then, the number of edges of the rooted plane tree is the same as free lower edges (*i.e.* the size) of the fish tail. Let i, j be nodes of B such that j is parent of i and (j, i) be their corresponding edge. If the edge (j, i) is associated with a non-empty vertical strip S of connected right and left scales of T and S' is the unique non-empty vertical strip on the left of S that shares a vertical segment or part of it with S (S' is the vertical strip immediately on the left of S), then there exists a node k parent of j such that the edge (k, j) is associated with S'. The label of each edge of B is determined by the shape of its corresponding vertical strip of T, according to the representation given in Figure 6.7. It is easy to check that by using this labelling the height of the fish tail T corresponds to the weight of the fish bone tree B. For instance, any right scale that is a fish tail of height one is associated with a single edge labelled u, which is a fish bone tree of weight one.

Conversely, given a fish bone tree we construct a fighting fish by glueing vertical strips starting from the leaves of the fish bone tree (which correspond to the tails of the fish tail to be constructed) and proceeding bottom-up in the tree by applying operations in Figure 6.7. If the fish bone tree has weight k and n edges, then the fish tail has height k and size n. □

In the special case of fighting fish with one tail, we recover the one-to-one correspondence of Proposition 6.2.3 between parallelogram polyominoes and bicoloured Motzkin words, here presented as a chain (linear fish bone tree) of height 1.

6.3 A first functional equation

In the following we determine and solve a functional equation for the size generating function of fighting fish. This equation is briefly called *master equation*, since it is related to the master decomposition: the first recursive description provided for fighting fish in terms of fish tails (see Section 6.2.2).

6.3.1 The master equation

The parameters on fish tails we take into account are:

- the size, or the total number of free lower edges,
- the number of tails, or right scales having both their right edges free,
- the height, as inductively defined on page 181,

- the area, intended as the total number of left and right scales.

Then, let $R(v,q) \equiv R(t,x;v,q)$ denote the generating function of fish tails with variables t, x, v and q respectively marking size, number of tails, height, and area. The following proposition immediately follows from the inductive definition of fish tails of Section 6.2.2.

Proposition 6.3.1. *The series $R(v,q)$ is the unique power series in t satisfying*

$$\begin{aligned} R(v,q) \;&=\; 1 \,+\, tvq\, R(v,q)\,(R(vq^2,q)-1+x) \,+\, 2t\, R(v,q)\,(R(vq^2,q)-1)\,+ \\ &\quad \frac{t}{vq}\, R(v,q)\left(R(vq^2,q)-1-vq^2\, S(q)\right), \end{aligned} \tag{6.1}$$

where we have denoted $S(q)=[v]R(v,q)$.

Proof. According to the recursive definition of fish tails given in Section 6.2.2, each term of Equation (6.1) for $R(v,q)$ is retrived. The unique fish tail with height 0 is the empty fish tail that gives contribution 1, whereas every non empty fish tail is obtained by applying one of the operation u, h, h' and d to two fish tails. For each of these operations applied we have a different contribution to the expression of $R(v,q)$:

operation u : The contribution is given by the term $tvq\, R(v,q)\,(R(vq^2,q)-1+x)$. In fact, a fish tail T obtained from two fish tails T_1 and T_2 of smaller size by operation u has its size (resp. height) given by the sum of the sizes (resp. heights) of T_1 and T_2 plus 1. Similarly, the area of T is given by summing areas of T_1 and T_2 plus the number $2\ell+1$ of scales glued to T_1, supposing ℓ is the height of T_1. Thus, substitute $v := vq^2$ in the factor accounting for T_1 and multiply it by q. Finally, the number of tails of T is given by the sum of the tails of T_1 and T_2, apart from the case where T_1 is empty. In this case, the number of tails of T is given by the number of tails of T_2 plus 1, as expressed by replacing in the expansion of $R(vq^2,q)$ as power series the constant term 1 for x.

operations h, h' : Their total contribution is given by the term $2t\, R(v,q)\,(R(vq^2,q)-1)$. With the same reasoning as above: any fish tail T obtained from two fish tails T_1 and T_2 of smaller size by operation h or h' has its height given by the sum of the heights of T_1 and T_2 and the factor t comes from the added vertical strip, which T_1 is glued to. Similarly, the area of T is given by summing areas of T_1 and T_2 plus the number 2ℓ of scales glued to T_1, supposing $\ell > 0$ is the height of T_1. Thus, since the fish tail T_1 is not allowed to be empty, first substitute $v := vq^2$ in the factor accounting for T_1, and then subtract 1 from it.

operation d : The contribution is given by the term $\frac{t}{vq}\, R(v,q)\,(R(vq^2,q)-1-vq^2\, S(q))$. The arguments are similar to above: any fish tail T obtained from two fish tails T_1 and T_2 of smaller size by operation d has its height given by the sum of the heights of T_1 and T_2 minus 1 and the factor t comes from the added vertical strip, which T_1 is glued to. Similarly, the area of T is given by summing areas of T_1 and T_2 plus the number $2\ell - 1$ of scales glued to T_1, where $\ell > 1$ is given by the height of T_1. Thus, since the height of T_1 is not allowed to be smaller than 2, first substitute $v := vq^2$ in the factor accounting for T_1, and then subtract $1 + v\,[v]R(vq^2, q)$ from it. Note that $v\,[v]R(vq^2, q)$ corresponds to the case where T_1 has height equal to 1, which can be rewritten also as $vq^2[v]R(v, q)$. □

As explicited in Proposition 6.3.1, the function $S(q)$ of Equation (6.1) indicates the generating function of fish tails of height 1 according to size, area and number of tails. Therefore, we point out that our ultimate aim is to study the function $S(q)$, which is indeed the generating function of fighting fish according to size, area and number of tails. For this purpose, we rewrite Equation (6.1) in polynomial form as

$$\mathcal{M}(R(v, q), R(vq^2, q), S(q), t, v, q) = 0, \tag{6.2}$$

where $\mathcal{M}(w_1, w_2, w_3, t, v, q) \equiv \mathcal{M}(x; w_1, w_2, w_3, t, v, q)$ reads explicitly

$$-vqw_1 + vq \;+\; tv^2q^2\, w_1\,(w_2 - 1 + x) \;+\; 2t\,vqw_1\,(w_2 - 1) + t\,w_1\left(w_2 - 1 - vq^2\, w_3\right).$$

To the best of our knowledge, this type of polynomial catalytic q-equation (6.1) has only been considered in the linear case, and we did not manage to solve it. We will use it in the next Section 6.4 to provide enumerative results about the total and average area of fighting fish.

Nevertheless, by setting $q = 1$ in (6.1) it results a polynomial equation such as those studied by M. Bousquet-Mélou and A. Jehanne in [36]. In this work the authors provide a recipe to solve special functional equations with one catalytic variable that we are going to describe in the next section.

6.3.2 Recipe

The general case studied in [36] assumes that a $(k + 1)$-tuple $(G(u), A_1, \ldots, A_k)$ of power series in t, with $A_1, \ldots, A_k$ not depending on variable u, is completely determined by a polynomial equation of the form

$$\mathcal{P}(G(u), A_1, \ldots, A_k, t, u) = 0.$$

According to D. Zeilberger's terminology, such an equation is said a *polynomial equation with one catalytic variable*. Indeed, $G(u)$ is generally a power series in t with polynomial coefficients in u and each A_i is a specialisation of $G(u)$, which often coincides with $[u^{i-1}]G(u)$, and thus it does not contain the variable u.

The authors of [36] provide a strategy that allows us to conclude that all the $k+1$ series are algebraic and, as a consequence, to compute the polynomial equations they satisfy. The steps of this strategy are listed below as a "recipe" to follow:

1) Start from the non-trivial polynomial equation in $k + 3$ variables

$$\mathcal{P}(G(u), A_1, \ldots, A_k, t, u) = 0, \tag{6.3}$$

which is equal to the general expression $\mathcal{P}(w_0, w_1, \dots, w_k, t, u) = 0$ and *uniquely* defines the $(k+1)$-tuple $(G(u), A_1, \dots, A_k)$. Note that if the polynomial equation (6.3) is combinatorially founded, the $(k+1)$-tuple $(G(u), A_1, \dots, A_k)$ is uniquely defined.

2) Look for the series $U_i \equiv U_i(t)$ that annihilate the derivative of $\mathcal{P}$ with respect to the first variable w_0 evaluated at $(G(u), A_1, \dots, A_k)$

$$\frac{\partial \mathcal{P}}{\partial w_0}(G(u), A_1, \dots, A_k, t, u) = 0. \tag{6.4}$$

Let us suppose we can prove the existence of k such distinct series U_i, $1 \le i \le k$.

3) Derive the original equation (6.3) with respect to u

$$\frac{\partial \mathcal{P}}{\partial w_0}(G(u), A_1, \dots, A_k, t, u) \cdot \frac{\partial G}{\partial u} + \frac{\partial \mathcal{P}}{\partial u}(G(u), A_1, \dots, A_k, t, u) = 0.$$

Since each U_i is a fractional power series in t, the series $G(U_i)$ is a well-defined fractional power series in t, for every i. Hence, substituting each U_i into (6.4) the first term of the previous expression vanishes and for every $1 \le i \le k$, yields

$$\frac{\partial \mathcal{P}}{\partial u}(G(U_i), A_1, \dots, A_k, t, U_i) = 0.$$

4) We build a system of $3k$ polynomial equations in the $3k$ unknowns $U_1, \dots, U_k, G(U_1), \dots, G(U_k), A_1, \dots, A_k$:

$$\begin{cases} \mathcal{P}(G(U_i), A_1, \dots, A_k, t, U_i) = 0, & \\ \dfrac{\partial \mathcal{P}}{\partial w_0}(G(U_i), A_1, \dots, A_k, t, U_i) = 0, & \text{for every } 1 \le i \le k. \\ \dfrac{\partial \mathcal{P}}{\partial u}(G(U_i), A_1, \dots, A_k, t, U_i) = 0, & \end{cases} \tag{6.5}$$

If this system has only a finite number of solutions, it characterises completely the $3k$ unknown series and proves that they are algebraic under the assumption that the series $U_1, \dots, U_k$ are distinct.

The method presented above is shown to work in [36, Section 4], where under a mild hypothesis on the form of Equation (6.3) it is proved that the System (6.5) characterises completely the $3k$ unknowns it involves. Thus, the specialisations $A_1, \dots, A_k$ are proved to be algebraic as well as the complete series $G(u)$. We do not need this general theorem to solve and examine the specific case of fighting fish, thus for the sake of brevity we do not report it here. Nevertheless, it is well worth noticing that this general strategy encapsulates the two well-known cases called *kernel method* and *quadratic method*. In fact, if the polynomial $\mathcal{P}$ is linear in $G(u)$ our recipe and the kernel method coincide - see for instance the case of Dyck paths as a specialisation of Dyck prefixes shown in Section 1.3.7. On the other hand, the so-called quadratic method, which was developed by W. G. Brown around 1965, allows to solve in a systematic way all the equations of type (6.3) that are quadratic in $G(u)$ and involve only one specialisation A_1.

6.3.3 The algebraic solution of the master equation

The above recipe can be applied to a specialisation of the polynomial equations (6.1) in order to prove that the size generating function of fighting fish is algebraic, as well as their multivariate generating function.

Let us start with the master Equation (6.1) and set the variable $q = 1$, since we do not take into account the area parameter. It holds that

$$F(v) = 1 + tv\,F(v)\,(F(v) - 1 + x) + 2t\,F(v)\,(F(v) - 1) + \frac{t}{v}\,F(v)\,(F(v) - 1 - v\,F_1)\,, \quad (6.6)$$

where $F(v) \equiv R(v,1)$ and $F_1 \equiv [v]R(v,q)|_{q=1}$, namely $F_1 = [v]F(v)$. Equivalently, the polynomial Equation (6.2) reduces to

$$\tilde{\mathcal{M}}(x; F(v), F_1, t, v) \equiv \tilde{\mathcal{M}}(F(v), F_1, t, v) = 0,$$

where $\tilde{\mathcal{M}}(w_0, w_1, t, v) \equiv \tilde{\mathcal{M}}(w_0, w_0, w_1, t, v, 1)$, or in explicit form is

$$-vw_0 + v \;+\; tv^2\,w_0\,(w_0 - 1 + x) \;+\; 2t\,vw_0\,(w_0 - 1) + t\,w_0\,(w_0 - 1 - v\,w_1)\,.$$

Equation $\tilde{\mathcal{M}}(F(v), F_1, t, v) = 0$ is now a polynomial equation with one catalytic variable v, which uniquely defines the pair $(F(v), F_1)$, and according to the recipe of Section 6.3.2, it admits an explicitly computable algebraic solution.

Theorem 6.3.2. *Let $V \equiv V(t,x)$ be the unique power series solution of the equation*

$$V = t \cdot \left(1 + V + x \cdot \frac{V^2}{1-V}\right)^2. \quad (6.7)$$

Then, the bivariate generating function of fighting fish according to their size and number of tails is algebraic

$$F_1 \equiv F_1(t,x) = x \cdot V - x^2 \cdot \frac{V^3}{(1-V)^2}. \quad (6.8)$$

Proof. To prove this result we apply step by step the recipe:

- Upon differentiating $\tilde{\mathcal{M}}(F(v), F_1, t, v) = 0$ with respect to w_0 we obtain

$$\frac{\partial \tilde{\mathcal{M}}}{\partial w_0}(F(v), F_1, v) = \big((1+v)^2 \cdot (2\,F(v) - 1) + xv^2 - vF_1\big) \cdot t - v = 0.$$

  Then, $v = t \cdot Q(F(v), F_1, v, x)$, with $Q(w_0, w_1, v, x) = (1+v)^2 \cdot (2w_0 - 1) + xv^2 - vw_1$, and this equation admits a unique power series solution $V \equiv V(t,x)$.

- Now, we differentiate with respect to v

$$\frac{\partial \mathcal{M}'}{\partial w_0}(F(v), F_1, t, v) \cdot F'(v) + \frac{\partial \mathcal{M}'}{\partial v}(F(v), F_1, t, v) = 0.$$

  Since V is a well-defined power series in t, both series $F(V)$ and $F'(V)$ are well-defined power series in t, and under the substitution $v = V$ it holds

$$\frac{\partial \mathcal{M}'}{\partial v}(F(V), F_1, t, V) = 0.$$

- Then the series $V \equiv V(t,x)$, $F_1 \equiv F_1(t,x)$ and $F(V) \equiv F(V,t,x)$ are solutions of the system of polynomial equations

$$\begin{cases} \tilde{\mathcal{M}}(F(V), F_1, t, V) = 0, \\ V = t \cdot Q(F(V), F_1, V, x), \\ \dfrac{\partial \tilde{\mathcal{M}}}{\partial v}(F(V), F_1, t, V) = 0. \end{cases}$$

  Solving the system for t, F_1 and $F(V)$ by elimination, we find that the algebraic curve $(t, V, F_1, F(V))$ admits the following parametrisation

$$\left\{ t = \frac{V \cdot (1-V)^2}{(1-(1-x)\cdot V^2)^2},\ F_1 = x \cdot V - x^2 \frac{V^3}{(1-V)^2},\ F(V) = 1 + x\frac{V^2}{1-V^2} \right\} \tag{6.9}$$

From (6.9) the result follows straightforward. □

An expression for $F(v)$ now follows by Equation (6.6).

6.4 Enumerative results for fighting fish

The next subsections are devoted to provide explicit formulas for the number of fighting fish according to some statistics. Size and number of tails are discussed in Section 6.4.1, where it turns out that fighting fish of size $n+1$ are enumerated by sequence A000139 [132] and those having only one tail are counted by Catalan numbers (sequence A000108 [132]). Fighting fish with a marked tail are studied in Section 6.4.2 and result to be related to another quite famous number sequence (A006013 [132]), which counts some families of plane trees. Finally, we study fighting fish regarding their area. In Section 6.4.4, we asymptotically estimate the average area of a uniform random fighting fish and discover an unusual property for these objects. In fact, fighting fish result to belong to a different universality class if compared to uniform random polyominoes whose results are summarised in [123].

6.4.1 Explicit formulas with respect to size and number of tails

Thanks to the solution of the master equation given in Theorem 6.3.2, we are now in position to prove that fighting fish are enumerated according to their size by sequence A000139 in [132], whose first terms are

$$1, 2, 6, 22, 91, 408, 1938, 9614, 49335, 260130, 1402440, 7702632, 42975796, \dots.$$

Therefore, these objects are equinumerous to other known combinatorial structures, such as West two-stack sortable permutations on n letters [145, 149], rooted non-separable planar maps with n edges [45, 93, 141], and left ternary trees having n nodes [59].

Theorem 6.4.1. *The number f_n of fighting fish with size $n+1$ is*

$$f_n = \frac{2}{(n+1)(2n+1)}\binom{3n}{n}. \tag{6.10}$$

Proof. Let us set $x = 1$. Theorem 6.3.2 reads

$$F_1(t,1) = V(t,1) - \frac{V(t,1)^3}{(1-V(t,1))^2}, \qquad \text{with} \qquad V(t,1) = t \cdot \frac{1}{(1-V(t,1))^2},$$

so that we can apply the Lagrange inversion formula to determine $[t^n]F_1(t,1)$, which gives the number of fighting fish of size $n+1$, owing to Theorem 6.2.2.

For $F_1(t,1) = \psi(V(t,1))$, where $V(t,1) = t\phi(V(t,1))$, the Lagrange inversion formula states that

$$[t^n]F_1(t,1) = \frac{1}{n}[v^{n-1}]\,\psi'(v)\cdot\phi(v)^n, \qquad \text{for all } n.$$

This yields

$$\begin{aligned}
[t^n]F_1(t,1) &= \frac{1}{n}\,[v^{n-1}]\,\psi'(v)\cdot\phi(v)^n = \frac{1}{n}\,[v^{n-1}]\,\frac{\partial}{\partial v}\left(v - \frac{v^3}{(1-v)^2}\right)\cdot\frac{1}{(1-v)^{2n}}\\
&= \frac{1}{n}\,[v^{n-1}]\,\frac{1-3v}{(1-v)^3}\cdot\frac{1}{(1-v)^{2n}} = \frac{1}{n}\,[v^{n-1}]\,\frac{1}{(1-v)^{2n+3}} - \frac{3}{n}\,[v^{n-2}]\,\frac{1}{(1-v)^{2n+3}}\\
&= \frac{1}{n}\binom{3n+1}{2n+2} - \frac{3}{n}\binom{3n}{2n+2} = \frac{4\,(3n)!}{n!\,(2n+2)!}.
\end{aligned}$$

□

Since both Theorems 6.3.2 and 6.5.3 prove that the generating function of fighting fish is algebraic, according to [79, Section VII.7] an asymptotic form of type $C\cdot A^n\cdot n^\gamma$ must hold for the coefficients f_n. Stirling's formula (see Section 1.3.8) provides it by a straightforward calculation.

Proposition 6.4.2. *Let f_n be the number of fighting fish with size $n+1$, Then, as n goes to infinity,*

$$f_n = \frac{2}{(n+1)\cdot(2n+1)}\cdot\frac{(3n)!}{(2n)!\cdot n!} \sim \frac{\sqrt{3}}{2\sqrt{\pi}}\cdot\left(\frac{27}{4}\right)^n\cdot n^{-5/2}. \tag{6.11}$$

This particular asymptotic behaviour explicitly calculated tells something more about fighting fish.

Observation 6.4.3. *Fighting fish cannot be generated by an unambiguous context-free grammar.*

Proof. The above statement is a consequence of C. Banderier and M. Drmota's result in [9, Proposition 7]. In our particular case, it reads as: if the power series $\sum_{n\geq 0} c_n t^n$ is $\mathbb{N}$-algebraic and $c_n \sim C\, A^n n^\gamma$, then the critical exponent γ cannot be $-5/2$. Because of Equation (6.11), the generating function of fighting fish is not $\mathbb{N}$-algebraic, and thus it does not correspond to a generating function of a context-free grammar by the Chomsky-Schützenberger Theorem (see [79, Section I.5.4]). □

The number of fighting fish of size $n+1$ that have only one tail can be also derived by using the Lagrange inversion formula applied to Equations (6.7) defining the series V and Equation (6.8) defining the generating function F_1.

It holds that

$$\begin{aligned}
[x\,t^n]F_1 &= \frac{1}{n}[x\,v^{n-1}]\frac{\partial}{\partial v}\left(xv - x^2\frac{v^3}{(1-v)^2}\right)\cdot\left(1+v+x\frac{v^2}{(1-v)^2}\right)^{2n}\\
&= \frac{1}{n}[x\,v^{n-1}]\left(\frac{x(1-v)^3-3x^2v^2+x^2v^3}{(1-v)^3}\right)\cdot\left(1+v+x\frac{v^2}{(1-v)^2}\right)^{2n}\\
&= \frac{1}{n}[v^{n-1}]\,(1+v)^{2n} = \frac{(2n)!}{n!\,(n+1)!}.
\end{aligned}$$

As expected the coefficient of $x\,t^n$ in F_1 is equal to $C_n = \frac{1}{n+1}\binom{2n}{n}$, which is the nth Catalan number. Indeed, as stated in Proposition 6.1.4, fighting fish of size $n+1$ with only one tail are in one-to-one correspondence with parallelogram polyominoes of semi-perimeter $n+1$.

More generally, in view of the definition of V in Equation (6.7), the coefficient V_ℓ of x^ℓ in V is rational in the Catalan generating function $V_0 = [x^0]V$, where V_0 is the unique power series solution of $V_0 = t(1+V_0)^2$. In addition, thanks to the definition of F_1 of Equation (6.8), the same holds for the generating function of fighting fish with ℓ tails. However, explicit expressions are not particularly simple to express.

6.4.2 Fish with a marked tail

Alternatively, one can consider the total number of tails of a fighting fish and choose to mark one of them - see Figure 6.12(b). The resulting number sequence has been registered on OEIS [132] as A006013 and establishes a further link between fighting fish and plane trees, which will be developed in Section 6.6.1. Indeed, pairs of ternary trees with n nodes as well as bicoloured ordered trees having n nodes are known to be enumerated by sequence A006013, whose first terms are

$$1, 2, 7, 30, 143, 728, 3876, 21318, 120175, 690690, 4032015, 23841480, 142498692, \ldots.$$

Theorem 6.4.4. *The number m_n of fighting fish of size $n+1$ with a marked tail is*

$$m_n = \frac{1}{n}\binom{3n-2}{n-1}. \tag{6.12}$$

Proof. In order to obtain the exact number of fighting fish with a marked tail, we calculate the function $\frac{\partial F_1}{\partial x}$ and then, we use again the Lagrange inversion formula to compute

$$m_n = [t^n]\frac{\partial F_1}{\partial x}(t,1)\,.$$

Let us recall System (6.9)

$$\left\{ t = \frac{V\cdot(1-V)^2}{(1-(1-x)\cdot V^2)^2},\ F_1 = x\cdot V - x^2\frac{V^3}{(1-V)^2},\ F(V) = 1 + x\frac{V^2}{1-V^2} \right\},$$

where both functions F_1 and V depend on t and x.

Differentiating the first and second equations with respect to x and then setting $x=1$, we obtain a system of two equations in two unknowns $\frac{\partial V}{\partial x} \equiv \frac{\partial V}{\partial x}(t,1)$ and $\frac{\partial F_1}{\partial x} \equiv \frac{\partial F_1}{\partial x}(t,1)$

$$\begin{cases} \dfrac{\partial V}{\partial x} = \dfrac{2\,t\cdot\left(\frac{\partial V}{\partial x} + V(t,1)^2 - V(t,1)^3\right)}{(1-V(t,1))^3} \\[2ex] \dfrac{\partial F_1}{\partial x} = \dfrac{\frac{\partial V}{\partial x}\cdot\Big(1-3V(t,1)\Big) + V(t,1)\cdot\Big(1-3V(t,1)\Big) + V(t,1)^3\cdot\Big(1+V(t,1)\Big)}{(1-V(t,1))^3}. \end{cases} \tag{6.13}$$

By substituting $t = V(t,1)\cdot(1-V(t,1))^2$ in the first equation, we simplify System (6.13) that reduces to

$$\begin{cases} \dfrac{\partial V}{\partial x} = \dfrac{2V(t,1)^3(1-V(t,1))}{1-3V(t,1)} \\[2ex] \dfrac{\partial F_1}{\partial x} = V(t,1). \end{cases}$$

Therefore, the Lagrange inversion formula reads

$$\begin{aligned} [t^n]\frac{\partial F_1}{\partial x} &= [t^n]V(t,1) = \frac{1}{n}\,[v^{n-1}]\phi(v)^n \\ &= \frac{1}{n}\,[v^{n-1}]\,\frac{1}{(1-v)^{2n}} = \frac{1}{n}\binom{3n-2}{2n-1}. \end{aligned}$$

□

To conclude, we consider the average number of tails per fighting fish of size $n+1$ that is simply obtained calculating the ratio between m_n (total number of tails in fighting fish of size $n+1$) and f_n (number of fighting fish of size $n+1$). It results the following nice formula for any n.

Corollary 6.4.5. *The average number of tails of a uniform random fighting fish of size* $n+1$ *is*

$$\frac{(n+1)(2n+1)}{3\,(3n-1)}.$$

6.4.3 Total area

The master equation of Section 6.3.1 is needed furthermore in order to consider the area statistic on fighting fish. Remind that the area of any fighting fish has been defined as the number of its cells, whereas the area of a fish tail as the number of its left and right scales. Since a pair of left and right scales gives rise to a cell and fighting fish are nothing else but fish tails of height 1, in the following we choose to consider the "double" area, namely the total number of left and right scales of a fighting fish.

The generating function of the total area of fighting fish is the series $A \equiv A(t,x)$ given by

$$A = \frac{\partial(qS(q))}{\partial q}\Big|_{q=1} = S(1) + \frac{\partial S}{\partial q}(1),$$

which counts fighting fish weighted by their area. Recalling that $S(q)$ counts fish tails of height 1 by their area, the correction factor q needs to take account of the head of the fighting fish which is not included in any fish tail. The series $\frac{\partial S}{\partial q}(1)$ appears in the derivative of the master Equation (6.2) with respect to q

$$\begin{aligned}
&\tfrac{\partial}{\partial q}\mathcal{M}(R(v,q),R(vq^2,q),S(q),t,v,q) = \\
&\qquad \tfrac{\partial R}{\partial q}(v,q)\tfrac{\partial \mathcal{M}}{\partial w_1}(R(v,q),R(vq^2,q),S(q),t,v,q) \\
&\qquad + \left(\tfrac{\partial R}{\partial q}(vq^2,q) + 2vq\tfrac{\partial R}{\partial v}(vq^2,q)\right)\tfrac{\partial \mathcal{M}}{\partial w_2}(R(v,q),R(vq^2,q),S(q),t,v,q) \\
&\qquad + \tfrac{\partial S}{\partial q}(q)\tfrac{\partial \mathcal{M}}{\partial w_3}(R(v,q),R(vq^2,q),S(q),t,v,q) + \tfrac{\partial \mathcal{M}}{\partial q}(R(v,q),R(vq^2,q),S(q),t,v,q) = 0.
\end{aligned}$$

Indeed, for $q=1$ this equation can be rewritten as

$$\begin{aligned}
&\tfrac{\partial}{\partial q}\mathcal{M}(R(v,q),R(vq^2,q),S(q),t,v,q)\ |_{q=1} = \qquad (6.14)\\
&\qquad \tfrac{\partial R}{\partial q}(v,1)\tfrac{\partial \mathcal{M}}{\partial w_1}(F(v),F(v),F_1,t,v,1) \\
&\qquad + \left(\tfrac{\partial R}{\partial q}(v,1) + 2v\tfrac{\partial F}{\partial v}(v)\right)\tfrac{\partial \mathcal{M}}{\partial w_2}(F(v),F(v),F_1,t,v,1) \\
&\qquad + \tfrac{\partial S}{\partial q}(1)\tfrac{\partial \mathcal{M}}{\partial w_3}(F(v),F(v),F_1,t,v,1) + \tfrac{\partial \mathcal{M}}{\partial q}(F(v),F(v),F_1,v,1) = 0,
\end{aligned}$$

where $F(v) \equiv R(v,1)$ and $F_1 \equiv S(1)$, as previously defined.

Recall that the series V introduced in Theorem 6.3.2 is the unique power series solution of

$$\frac{\partial \mathcal{M}}{\partial w_1}(F(v),F(v),F_1,t,v,1) + \frac{\partial \mathcal{M}}{\partial w_2}(F(v),F(v),F_1,t,v,1) = 0.$$

Thus upon setting $v = V$, a simplification occurs as the coefficient of $\frac{\partial R}{\partial q}(v,1)$ is precisely the defining equation for V. Equation (6.14) becomes

$$2v\tfrac{\partial F}{\partial v}(V)\tfrac{\partial \mathcal{M}}{\partial w_2}(F(V),F(V),F_1,t,V,1) \\ +\tfrac{\partial S}{\partial q}(1)\tfrac{\partial \mathcal{M}}{\partial w_3}(F(V),F(V),F_1,t,V,1)+\tfrac{\partial \mathcal{M}}{\partial q}(F(V),F(V),F_1,t,V,1)=0. \quad (6.15)$$

In order to obtain an equation for $\frac{\partial S}{\partial q}(1)$ we need to determine $\frac{\partial F}{\partial v}(V)$.

The derivative of the master Equation (6.2) with respect to v is

$$\tfrac{\partial F}{\partial v}(v)\cdot\left[\tfrac{\partial \mathcal{M}}{\partial w_1}(F(v),F(v),F_1,t,v,1)+\tfrac{\partial \mathcal{M}}{\partial w_2}(F(v),F(v),F_1,t,v,1)\right] \\ +\tfrac{\partial \mathcal{M}}{\partial v}(F(v),F(v),F_1,t,v,1) = 0. \quad (6.16)$$

Now, one cannot simply set $v = V$ in Equation (6.16) to obtain $\frac{\partial F}{\partial v}(V)$ because of the coefficient of $\frac{\partial F}{\partial v}(v)$ that vanishes by definition of V. Then, expand Equation (6.16) at $v = V$ to the second order,

$$\underbrace{\tfrac{\partial F}{\partial v}(V)\cdot\left[\tfrac{\partial \mathcal{M}}{\partial w_1}(.)+\tfrac{\partial \mathcal{M}}{\partial w_2}(.)\right]+\tfrac{\partial \mathcal{M}}{\partial v}(.)}_{=0} \\ +(v-V)\cdot\Bigg(\tfrac{\partial^2 F}{\partial v^2}(V)\cdot\underbrace{\left[\tfrac{\partial \mathcal{M}}{\partial w_1}(.)+\tfrac{\partial \mathcal{M}}{\partial w_2}(.)\right]}_{=0} \\ +\tfrac{\partial F}{\partial v}(V)\cdot\left[\tfrac{\partial F}{\partial v}(V)\cdot\left(\tfrac{\partial}{\partial w_1}+\tfrac{\partial}{\partial w_2}\right)+\tfrac{\partial}{\partial v}\right]\left(\tfrac{\partial \mathcal{M}}{\partial w_1}+\tfrac{\partial \mathcal{M}}{\partial w_2}\right)(.) \\ +\left(\left[\tfrac{\partial F}{\partial v}(V)\cdot\left(\tfrac{\partial}{\partial w_1}+\tfrac{\partial}{\partial w_2}\right)+\tfrac{\partial}{\partial v}\right]\tfrac{\partial \mathcal{M}}{\partial v}\right)(.)\Bigg) \\ = O((v-V)^2),$$

where (.) stands for the evaluation at $(F(V),F(V),F_1,t,V,1)$. Since the coefficients are zero at all orders in this expansion at $v = V$, the coefficient of $(v-V)$ yields a quadratic equation, which turns out to uniquely define $\frac{\partial F}{\partial v}(V)$ in terms of V and x. Indeed, there is a polynomial

$$\mathcal{R}(w,v,x) = -2v(1-v^2)^2(1+v)^2w^2+2(1-v^2)^2(1-v^2+xv^2)w-2xv(1-v^2+xv^2),$$

quadratic in w, such that $\frac{\partial F}{\partial v}(V)$ is the unique power series solution of

$$\mathcal{R}\left(\tfrac{\partial F}{\partial v}(V),V,x\right)=0\,. \quad (6.17)$$

Together with Equation (6.15) it allows us to obtain by elimination a quadratic equation satisfied by $\frac{\partial S}{\partial q}(1)$ over $\mathbb{Q}(V,x)$. Using the expression of F_1 in terms of V (as stated in Theorem 6.3.2) a similar result is obtained for the series A.

Proposition 6.4.6. *The generating function $A \equiv A(t,x)$ for the total area of fighting fish is algebraic of degree 2 over $\mathbb{Q}(V,x)$ and satisfies*

$$-V(1-V)^2A^2 + 2(1-V)^2(1-V^2+xV^2)A - 4xV(1-V^2+xV^2) = 0. \qquad (6.18)$$

Note that extracting the coefficient of x in Equation (6.18) yields

$$2(1-V_0)^2(1-V_0^2)A_1 - 4V_0(1-V_0^2) = 0,$$

where $V_0 = [x^0]V$ and $A_1 = [x^1]A$. According to the definition of V given in Theorem 6.3.2, it holds that

$$V_0 = [x^0]V = t(1+V_0)^2,$$

from which we recover the generating function A_1 for the total area of parallelogram polyominoes, viewed as fighting fish with only one tail

$$A_1 = \frac{2V_0}{(1-V_0)^2} = \frac{2t}{1-4t}.$$

The simplification to a rational function of t is a well-known feature of parallelogram polyominoes [147]. Observe that it implies that the average area of parallelogram polyominoes of size n is $4^n/C_n$, that is of order $n^{3/2}$.

In general upon extracting the coefficient of x^ℓ in Equation (6.18) and again using the rationality of coefficients of x^i in V we obtain that the generating function of the total area of fighting fish with ℓ tails as a rational function of the Catalan generating function V_0, the unique power series solution of $V_0 = t(1+V_0)^2$.

6.4.4 Average area

Now we turn to the estimate of the average area a_n of a fighting fish of size $n+1$, proving that it grows like $n^{5/4}$. Such a result is worth comparing with Table 11.1 in [91]: the area of uniform random polyominoes with perimeter n in all classical non-trivial solvable models of polyominoes behaves like $n^{3/2}$ [123]. Fighting fish thus belong to a different universality class.

Theorem 6.4.7. *Let a_n denote the average area of uniform random fighting fish of size $n+1$. Then, as n goes to infinity,*

$$a_n \sim \frac{3^{3/4}2\sqrt{2\pi}}{\Gamma\left(-\frac{1}{4}\right)} \cdot n^{5/4}.$$

Proof. The series $V(t,1)$ is by definition the unique power series solution of the equation

$$V = t\,\phi(V), \qquad \text{where } \phi(x) = \frac{1}{(1-x)^2}.$$

Standard results in analytic combinatorics [79, Theorem VI.6, p. 404] enable us to estimate coefficients defined implicitly by an equation of the form $y(z) = t\phi(y(z))$. Indeed, it holds that $\phi(x)$ is analytic at $x = 0$ and satisfies

$$\phi(0) \neq 0, \qquad [x^n]\phi(x) \geq 0, \qquad \text{and} \qquad \phi(x) \not\equiv \phi_0 + \phi_1 x.$$

In addition, within the open disc of convergence of ϕ at 0, $|z| < R$, there exists a positive solution V_c to the *characteristic equation*

$$\phi(x) - x\phi'(x) = \frac{1-3x}{(1-x)^3} = 0. \tag{6.19}$$

Then, by [79, Proposition IV.5, p. 278] the radius of convergence of V is the positive value ρ_V obtained from

$$\rho_V = \frac{V_c}{\phi(V_c)}.$$

Therefore, it holds that the radius of convergence ρ_V of the series V is finite and its numerical value is $4/27$, since $V_c = 1/3$ is the positive root of (6.19).

Then, by [79, Theorem VI.6] the singular expansion of V near ρ_V is of the generic square root type

$$V = V_c - \gamma\sqrt{1 - t/\rho_V} + O(1 - t/\rho_V), \tag{6.20}$$

where

$$\gamma := \sqrt{\frac{2\phi(V_c)}{\phi''(V_c)}} = \sqrt{\frac{4}{27}} = \frac{2}{3\sqrt{3}}.$$

Equation (6.18) for $x = 1$ reads

$$V(1-V)^2A^2 - 2(1-V)^2A + 4V = 0,$$

or, equivalently,

$$A = \frac{1}{V} - \frac{\sqrt{(1+V)(1-3V))}}{V(1-V)}.$$

Since A has positive coefficients, by Pringsheim's Theorem [79, Theorem IV.4, p. 240] it has a positive dominant singularity, which is obtained at $t = \rho_V$, that is for $V_c = 1/3$. From the above expression in terms of V and Expansion (6.20), we have the singular expansion

$$\begin{aligned} A &= 3 - \frac{\sqrt{\frac{4}{3}\cdot 3\frac{2}{3\sqrt{3}}\sqrt{1-t/\rho_V})}}{2/9} + O(\sqrt{1-t/\rho_V}) \\ &= 3 - 3^{5/4}\sqrt{2}(1-t/\rho_V)^{1/4} + O((1-t/\rho_V)^{1/2}). \end{aligned}$$

Standard function scale ([79, Theorem VI.1, p. 381]) and transfert Theorem ([79, Theorem VI.3, p. 390]) yield

$$[t^n]A \underset{n\to\infty}{\sim} \frac{3^{5/4}\sqrt{2}}{\Gamma(-\frac{1}{4})} n^{-5/4}\rho_V^{-n}.$$

Now, return to the coefficient of t^n in F_1, as calculated in Theorem 6.4.1,

$$[t^n]F_1 \underset{n\to\infty}{\sim} \frac{\sqrt{3}}{2\sqrt{\pi}} \cdot n^{-5/2} \cdot \rho_V^{-n}.$$

The average area of fighting fish of size n is the ratio of the above two coefficients which, as n goes to infinity, is equivalent to

$$\frac{3^{3/4}\, 2\sqrt{2\pi}}{\Gamma(-\frac{1}{4})} \cdot n^{5/4}.$$

□

6.5 The Wasp-waist decomposition

This section is devoted to providing an alternative description of fighting fish that is self-contained: it does not characterise them as a special case of a larger family of objects, such as fish tails described in Section 6.2.2 or, equivalently, fish bone trees of Section 6.2.3. The description presented allows us to construct these objects recursively - but without ambiguity - only by using fighting fish themselves; indeed, according to the so-called *wasp-waist* decomposition a fighting fish is decomposed into fighting fish of smaller size. In addition, owing to this decomposition it will become possible to refine the enumeration of fighting fish and to discover important properties of these objects.

6.5.1 The wasp-waist definition

In order to describe how to decompose a fighting fish, we introduce a new enumerative parameter: the *fin* of a fighting fish. The fin is the path that starts with the lower edge of its head, follows its border counterclockwise, and ends at the right lower edge of the first tail it meets. The *length* of the fin is given by the number of steps (or edges) it is made of. In Figure 6.10 the fin of a sketched fighting fish has been emphasized.

Theorem 6.5.1. *Let P be a non-empty fighting fish. Then, exactly one of the following cases (A), (B1), (B2), (C1), (C2) or (C3) occurs:*

(A) P consists of a single cell;

(B) P is obtained from a fighting fish P_1 of smaller size:

- *(B1) by glueing the right lower edge of a new cell to the left upper edge of the head of P_1 (Figure 6.10 (B1));*
- *(B2) by glueing every left edge of the fin of P_1 to the right upper edge of a new cell, and glueing the right lower edge and the left upper edge of all pairs of adjacent new cells (Figure 6.10 (B2));*

(C) P is obtained from two fighting fish, P_1 and P_2, of smaller size:

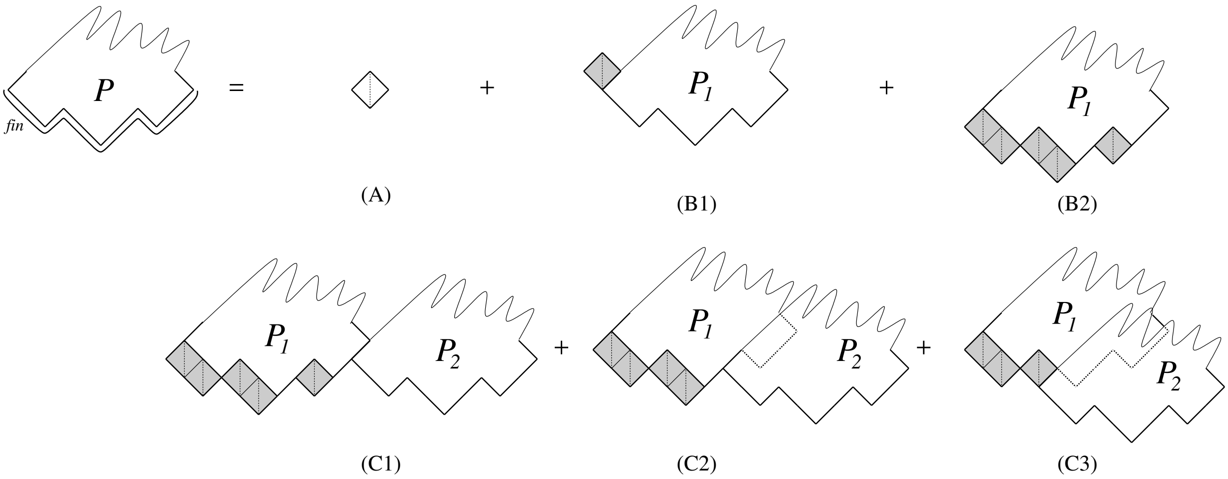


Figure 6.10: The wasp-waist decomposition.

(C1) by performing to P_1 the operation described in (B2) and then glueing the right lower edge of the lowest tail of P_1 to the left upper edge of the head of P_2 (Figure 6.10 (C1));

(C2) by choosing a right edge r on the fin of P_1 (tail excluded) and glueing every left edge of the fish fin preceding r to the right upper edge of a new cell and, as above, glueing the right lower edge and the left upper edge of every pair of two adjacent new cells; Then, glueing the left upper edge of the head of P_2 to r (Figure 6.10 (C2));

(C3) by choosing a left edge ℓ on the fin of P_1 and glueing every left edge of the fish fin preceding ℓ (included) to the right upper edge of a new cell and, as above, glueing the right lower edge and the left upper edge of every pair of adjacent new cells; Then, glueing the left upper edge of the head of P_2 to the right lower edge of the cell glued to ℓ (Figure 6.10 (C3)).

Observe that Cases (A), (B1) and (B2) could be alternatively considered as degenerate cases of Case (C1) where P_1 or P_2 were allowed to be empty.

Proof. The operations above produce valid fighting fish: indeed, inductively given incremental growths of P_1 and P_2 we obtain a valid incremental growth of P upon starting from the new head, and making grow P_1 from the head interleaving its steps with insertions of the new grey cells (each new cell is to be inserted just before the fin cell it will be attached to); when this has been done, the head of P_2 can be glued to P_1 and the incremental growth of P_2 can start. It thus remains to show that every fighting fish of size greater than 2 can be uniquely obtained by applying one of the operations (B) or (C) to fish of smaller size.

In order to prove the result let us describe how to decompose a fish P which is not reduced to a cell. In order to do this we need two further definitions: a *cut edge* of P is any common edge e of two cells of P such that cutting P along e yields two connected components. Second, let the set of *fin cells* of P be the set of cells incident to a left edge of the fin: the head of P is always a fin cell and the other fin cells have non-free left upper edges (since their left lower edges are free and they must be attached by a left edge).

Now the decomposition is as follows:

- First mark the head of P as *removable* and consider the other fin cells iteratively from left to right: mark them as *removable* as long as their left upper edge is not a cut edge of P. Let $\mathsf{R}(P)$ be the set of removable cells of P.
- If all fin cells are marked as removable then removing these cells yields a fighting fish $P_1 = P \setminus \mathsf{R}(P)$, and applying the construction of Case (B2) to P_1 gives P back. Conversely any fish produced as in Case (B2) has all its fin cells removable.
- Otherwise let c be the first fin cell which is not removable. Upon cutting the left upper edge e of c, two components are obtained: let P_2 be the component containing c and let $\bar{P}_1$ be the other component, which contains by construction all the removable cells of P. Using the incremental construction of fighting fish one can easily check that $P_1 = \bar{P}_1 \setminus \mathsf{R}(P)$ is a (possibly empty) fighting fish, and P_2 is a non-empty fighting fish.
  - If P_1 is empty, then only one cell is in $\mathsf{R}(P)$ and applying the construction of Case (B1) to the non-empty fighting fish P_2 yields P back, and conversely all fish produced as in Case (B1) have only one cell removable (*i.e.* the head) and admit a decomposition with P_1 empty.
  - Otherwise the edge e corresponds to the right lower edge of a cell $\bar{c}$ of $\bar{P}_1$. If this cell is in $\mathsf{R}(P)$, then the right upper edge of $\bar{c}$ is a right lower edge $\bar{e}$ on the fin of P_1. Thus, applying the construction of Case (C3) to P_1 and P_2 by choosing the edge $\bar{e}$ on the fin of P_1 yields P back. Conversely, all fish produced as in Case (C3) admit a decomposition with P_1 non-empty and the left upper edge of the head of P_2 glued to a removable fin cell. Else if $\bar{c}$ is not in $\mathsf{R}(P)$, then $\bar{c}$ is in P_1 and e is an edge (resp. the rightmost edge) on the fin of P_1. Thus, applying the construction of Case (C2) (resp. (C1)) to P_1 and P_2 by choosing the edge e on the fin of P_1 yields P back. Conversely, all fish produced as in Case (C2) or (C1) admit a decomposition with P_1 non-empty and the left upper edge of the head of P_2 glued to a fin cell of P_1. □

Observe that parallelogram polyominoes are exactly the fighting fish obtained using only Cases (A), (B1), (B2) and (C1), as proved in [91, Chapter 3].

Figure 6.11 shows the wasp-waist decomposition of the fighting fish of Figure 6.4 whose fin has been re-marked and has length 8. Note that any fighting fish P that is a parallelogram polyomino and has a bar shape is iteratively decomposed either by Case (B1) or by Case (B2) according to the bar direction, as visible in Figure 6.11.

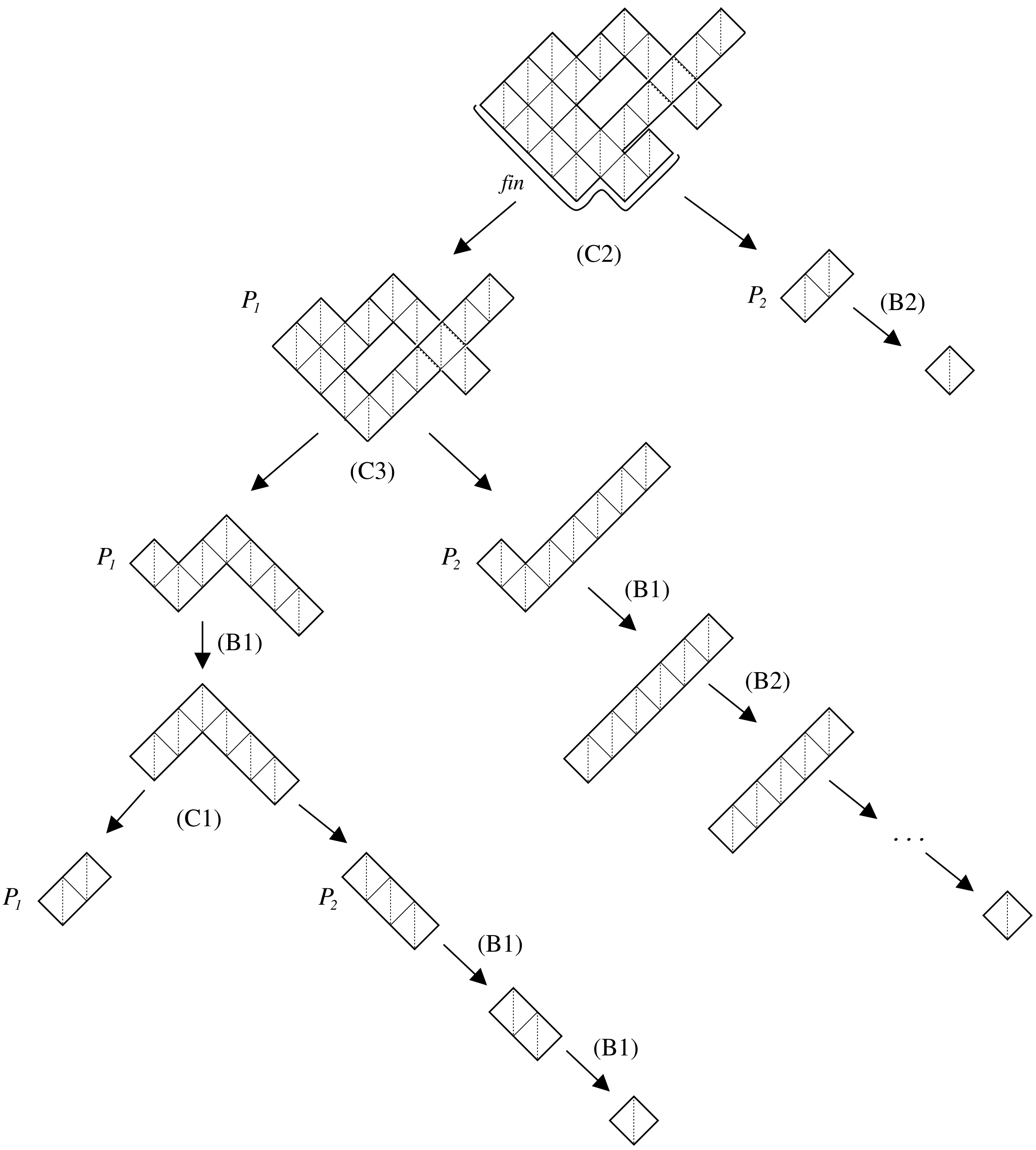


Figure 6.11: The wasp-waist decomposition of the fighting fish of Figure 6.4(a) whose fin length is 8.

6.5.2 A second functional equation

According to the wasp-waist decomposition we can easily write a functional equation whose solution is the generating function of fighting fish. The parameters we choose to take into account in this second decomposition are:

- the size, or the total number of free lower edges,
- the number of tails,
- the *right size*, or the number of free right lower edges,
- the *left size*, or the number of free left lower edges,
- the fin length.

Our aim is now to determine the generating function $P(u) \equiv P(t, x, a, b; u)$, with variables t, x, a, b and u respectively marking size, number of tails, right size, left size, and fin length, **all parameters being decreased by one**. Thus, the smallest fighting fish composed of only one cell gives contribution tu. The reason for decreasing all the parameters is to obtain the neat functional equation showed in the following proposition.

Proposition 6.5.2. *The generating function $P(u)$ of fighting fish satisfies*

$$P(u) = tu(1 + a\,P(u))(1 + b\,P(u)) + txabu\,P(u)\frac{P(1) - P(u)}{1 - u}. \tag{6.21}$$

Proof. The wasp-waist decomposition of Theorem 6.5.1 is readily translated into the following functional equation - see Figure 6.10:

$$\begin{aligned} P(u) &= tu + tub\,P(u) + tua\,P(u) + tuab\,P(u)^2 + \\ &\quad + txab\,P(u) \sum_{P_1} t^{\mathrm{size}(P_1)-1} x^{\mathrm{tails}(P_1)-1} a^{\mathrm{rsize}(P_1)-1} b^{\mathrm{lsize}(P_1)-1} \left(u + \ldots + u^{\mathrm{fin}(P_1)-1}\right) = \\ &= tu(1 + a\,P(u))(1 + b\,P(u)) + txabu\,P(u)\frac{P(1) - P(u)}{1 - u}, \end{aligned}$$

where the only difficult point lies in the discrete derivative that comes from Cases (C2) and (C3). In fact, given a pair (P_1, P_2) of non-empty fighting fish such that P_1 has its fin of length $k + 1$ and P_2 of length m, Cases (C2) and (C3) together produce k fighting fish of fin length $2 + m$, $3 + m$, …, $k + 1 + m$ respectively. □

The above equation is a combinatorially funded polynomial equation with one catalytic variable: indeed, Equation (6.21) can be rewritten in a polynomial form as

$$\mathcal{W}(P(u), P(1), t, u) = 0,$$

where $\mathcal{W}(w_0, w_1, t, u) \equiv \mathcal{W}(x, a, b; w_0, w_1, t, u)$ reads explicitly

$$(u-1)w_0 - tu(u-1)(1 + a\,w_0)(1 + b\,w_0) - txabu\,w_0(w_0 - w_1).$$

The function $P(1)$ is a series in t, which does not depend on the catalytic variable u. This class of equations was thoroughly studied by M. Bousquet-Mélou and A. Jehanne in [36], as already described previously in Section 6.3.2.

6.5.3 The algebraic solution of the wasp-waist equation

Now, we turn to solve Equation (6.21) that in polynomial form reads

$$(u-1)P(u) - tu(u-1)(1 + a\,P(u))(1 + b\,P(u)) - txabu\,P(u)\,(P(u) - P_1) = 0, \quad (6.22)$$

where $P_1 \equiv P(1)$. Since it is a polynomial equation in one catalytic variable u that uniquely defines the pair of series $(P(u), P_1)$, we make use of the recipe presented in Section 6.3.2 also in this second case in order to solve (6.22). It is well worth stressing that the function $P(u)$ enumerates fighting fish not only with respect to their size and their number of tails, but also with respect to their left and right size and their fin length. This provides further enumerative results on this family of objects, which will be examined in the next sections. Therefore, it results useful to provide the algebraic equations $P(u)$ and P_1 satisfy.

Theorem 6.5.3. *Let $B \equiv B(t; x, a, b)$ denote the unique power series solution of the equation*

$$B = t \cdot \left(1 + x\frac{abB^2}{1 - abB^2}\right)^2 (1 + aB)(1 + bB). \quad (6.23)$$

Then the generating function $P_1 \equiv P(t; x, a, b, 1)$ of fighting fish can be expressed as

$$P_1 = B - \frac{xabB^3(1 + aB)(1 + bB)}{(1 - abB^2)^2}. \quad (6.24)$$

Proof. We apply the recipe, starting from the polynomial equation (6.22) with polynomial $\mathcal{W}(w_0, w_1, t, u)$ given by

$$(u-1)w_0 - tu(u-1)(1 + a\,w_0)(1 + b\,w_0) - txabu\,w_0(w_0 - w_1).$$

- By differentiating $\mathcal{W}(w_0, w_1, t, u) = 0$ with respect to the first variable w_0 and evaluating at $(P(u), P_1, t, u)$, we obtain

  $$(u-1) - tu(u-1)(a + b + 2abP(u)) - txabu(2P(u) - P_1) = 0,$$

  which has a unique well-defined power series solution $U \equiv U(t, x, a, b)$ defined by

  $$U = 1 + tU(U-1)(a + b + 2abP(U)) + txabU(2P(U) - P_1). \quad (6.25)$$

- By differentiating $\mathcal{W}(P(u), P_1, t, u) = 0$ with respect to u, it holds

$$\frac{\partial \mathcal{W}}{\partial u}(P(u), P_1, t, u) + \frac{\partial \mathcal{W}}{\partial w_0}(P(u), P_1, t, u) \cdot \frac{\partial}{\partial u} P(u) = 0,$$

which explicitly reads as

$$\begin{aligned} &P(u) - t(2u-1)(1+aP(u))(1+bP(u)) - txabP(u)(P(u)-P_1) \\ &\quad = -\frac{\partial}{\partial u} P(u) \cdot (u - 1 - tu(u-1)(a+b+2abP(u)) - txabu(2P(u)-P_1)). \end{aligned}$$

Now, by substituting $u = U$ the right-hand side vanishes, and then U must cancel the left-hand side. So it holds that

$$P(U) = t(2U-1)(1+aP(U))(1+bP(U)) + txabP(U)(P(U)-P_1). \tag{6.26}$$

- Finally, for $u = U$ Equation (6.22) reads

$$(U-1)P(U) = tU(U-1)(1+aP(U))(1+bP(U)) + txabUP(U)(P(U)-P_1). \tag{6.27}$$

Equations (6.25), (6.26), and (6.27) form a system that admits as series solutions U, $P(U)$, and P_1.

Now, we prove that the power series B in the statement of Theorem 6.5.3 is indeed the power series $P(U)$, which allows us to write an expression for the generating function of fighting fish P_1 in terms of the series B.

By comparing Equation (6.27) and Equation (6.26) multiplied by U, a simpler relation is immediately deduced

$$P(U) = tU^2(1+aP(U))(1+bP(U)). \tag{6.28}$$

Then, comparing Equation (6.27) and Equation (6.25) multiplied by $P(U)$, up to cancelling a factor tU, yields

$$(U-1)(1+(a+b)P(U)+abP(U)^2) = (U-1)P(U)(a+b+2abP(U)) + xabP(U)^2,$$

that is

$$U = 1 + x \cdot \frac{abP(U)^2}{1-abP(U)^2}. \tag{6.29}$$

In view of Equations (6.28) and (6.29), the function $P(U)$ is thus the unique formal power series solution of the equation

$$P(U) = t\left(1 + x \cdot \frac{abP(U)^2}{1-abP(U)^2}\right)^2 (1+aP(U))(1+bP(U)). \tag{6.30}$$

Then, by setting $B := P(U)$ Equation (6.23) and Equation (6.30) coincide.

Now, we are able to provide an expression for the generating function P_1, which involves only the formal power series B and the variables x, a and b.

We use Equation (6.29) rewritten as

$$U - 1 = x \cdot \frac{abB^2}{1 - abB^2},$$

to eliminate the factors $(U - 1)$ in Equation (6.27), and after cancelling a factor $xabB$, it yields

$$\frac{B^2}{1 - abB^2} = t\,U \frac{B}{1 - abB^2}(1 + aB)(1 + bB) + tU(B - P_1).$$

Equation (6.28) is used to expand a factor B in the left-hand side, and after cancelling a factor $t\,U$, the previous equation becomes

$$\frac{B\,U(1 + aB)(1 + bB)}{1 - abB^2} = \frac{B}{1 - abB^2}(1 + aB)(1 + bB) + (B - P_1).$$

In other words,

$$B - P_1 = (U - 1)\frac{B(1 + aB)(1 + bB)}{1 - abB^2}$$

and again using Equation (6.29),

$$B - P_1 = \frac{xabB^3(1 + aB)(1 + bB)}{(1 - abB^2)^2}.$$

Finally,

$$P_1 = B - \frac{xabB^3(1 + aB)(1 + bB)}{(1 - abB^2)^2}.$$

□

The full generating function $P(u) \equiv P(t, x, a, b; u)$ of fighting fish, where the variable u marks the fin length decreased by one, results to be algebraic of degree at most 2 over $\mathbb{Q}(x, a, b, u, B)$ by Equations (6.22) and (6.24). It admits, in fact, a parametrisation directly extending the one of the Theorem 6.5.3, which will be visible later in Corollary 6.6.11 (Section 6.6.4).

6.5.4 Enumerative results: left and right size

In this section we specialise the explicit formulas determined for fighting fish in Section 6.4 by taking into account the more accurate statistics of left and right size. Note that in order to deal with the area of fighting fish by using the wasp-waist decomposition, one should refine Equation (6.21) introducing new parameters for counting the fish area. Nonetheless, we have already studied such a statistic on fighting fish thoroughly by using the master equation in Section 6.4.3 and Section 6.4.4 and we do not refine Equation (6.21) here.

Thanks to the wasp-waist decomposition and the generating function given in Theorem 6.5.3, we are able to enumerate fighting fish according to their number of left (resp.

right) free lower edges - *i.e.* left (resp. right) size. Such a specialisation on the number of fighting fish can similarly be seen as the way Narayana numbers refine Catalan numbers, as Proposition 6.5.5 illustrates.

Moreover, the following results confirm the apparently close relation between fighting fish and other well-studied combinatorial structures enumerated by sequence A000139 [132]. In fact, the explicit formula given in Theorem 6.5.4 is known to count rooted non-separable planar maps with respect to vertices and faces [45, 93, 141], or West two-stack sortable permutations with respect to ascents and descents [145, 149], or left ternary trees with respect to nodes with even and odd abscissa [59], as will be developed in Section 6.6.3.

Theorem 6.5.4. *The number of fighting fish of left size j and right size i is*

$$\frac{1}{ij}\binom{2i+j-2}{j-1}\binom{2j+i-2}{i-1}. \tag{6.31}$$

Proof. Our aim is to derive the explicit formula for fighting fish of size n, and left and right size j and i, respectively, where $n = i + j$. Then, let us implicitly **set** $x = 1$ **in all functions**, and in order to apply the bivariate Lagrange inversion formula, rewrite Equation (6.24) for P_1 in terms of the series

$$\bar{R} = \frac{aB(1+bB)}{1-abB^2}, \qquad \text{and} \qquad \bar{S} = \frac{bB(1+aB)}{1-abB^2}. \tag{6.32}$$

Indeed, it results that

$$B = t\frac{(1+aB)(1+bB)}{(1-abB^2)^2} = t(1+\bar{R})(1+\bar{S}), \tag{6.33}$$

so that $\bar{R}$ and $\bar{S}$ satisfy

$$\begin{cases} \bar{R} &= ta(1+\bar{R})(1+\bar{S})^2 \\ \bar{S} &= tb(1+\bar{R})^2(1+\bar{S}). \end{cases} \tag{6.34}$$

From (6.32) and (6.33), Equation (6.24) rewrites as

$$P_1 = B - abB^3\frac{(1+aB)(1+bB)}{(1-abB^2)^2} = t(1+\bar{R})(1+\bar{S})(1-\bar{R}\bar{S}). \tag{6.35}$$

Given a system $\{A_1 = a_1\Phi_1(A_1,A_2), A_2 = a_2\Phi_2(A_1,A_2)\}$ the bivariate Lagrange Inversion theorem [124] states that for any function $F(x_1,x_2)$,

$$\begin{aligned}
[a_1^{n_1}a_2^{n_2}]F(A_1,A_2) = \frac{1}{n_1n_2}[x_1^{n_1-1}x_2^{n_2-1}]\Bigg(&\frac{\partial^2 F(x_1,x_2)}{\partial x_1\partial x_2}\Phi_1(x_1,x_2)^{n_1}\Phi_2(x_1,x_2)^{n_2} \\
&+\frac{\partial F(x_1,x_2)}{\partial x_1}\frac{\partial(\Phi_1(x_1,x_2)^{n_1})}{\partial x_2}\Phi_2(x_1,x_2)^{n_2} \\
&+\frac{\partial F(x_1,x_2)}{\partial x_2}\frac{\partial(\Phi_2(x_1,x_2)^{n_2})}{\partial x_1}\Phi_1(x_1,x_2)^{n_1}\Bigg).
\end{aligned}$$

In other words, $[a_1^{n_1} a_2^{n_2}]F(A_1, A_2) = 1/(n_1 n_2)\left[x_1^{n_1-1} x_2^{n_2-1}\right] \Phi_1(x_1,x_2)^{n_1}\Phi_2(x_1,x_2)^{n_2} H$, where

$$H = \frac{\partial^2 F(x_1,x_2)}{\partial x_1 \partial x_2} + n_1 \frac{\partial F(x_1,x_2)}{\partial x_1}\frac{\partial \Phi_1(x_1,x_2)}{\partial x_2}\frac{1}{\Phi_1(x_1,x_2)} + n_2 \frac{\partial F(x_1,x_2)}{\partial x_2}\frac{\partial \Phi_2(x_1,x_2)}{\partial x_1}\frac{1}{\Phi_2(x_1,x_2)}.$$

Then, by setting $t = 1$ and applying the bivariate Lagrange inversion formula to the function $P_1(\bar{R}, \bar{S})$ in Equation (6.35), where $\bar{R} = a\,\Phi_1(\bar{R}, \bar{S})$ and $\bar{S} = b\,\Phi_2(\bar{R}, \bar{S})$ as defined in System (6.34), yields

$$\begin{aligned}[a^{i-1} b^{j-1}]P_1 = \frac{1}{(i-1)(j-1)}[\bar{R}^{i-2}\bar{S}^{j-2}]\big((1+\bar{R})^{i+2j-3}(1+\bar{S})^{2i+j-3}(&-4+4\bar{R}\bar{S}\\ &+2i(1-2\bar{R}\bar{S}-\bar{S})\\ &+2j(1-2\bar{R}\bar{S}-\bar{R}))\big).\end{aligned}$$

By extracting coefficients of $\bar{R}^{i-2}\bar{S}^{j-2}$ yields

$$\begin{aligned}[a^{i-1} b^{j-1}]P_1 = \frac{1}{(i-1)(j-1)}\Bigg[& 2(i+j-2)\binom{2j+i-3}{i-2}\binom{2i+j-3}{j-2}\\ & -4(i+j-1)\binom{2j+i-3}{i-3}\binom{2i+j-3}{j-3}\\ & -2j\binom{2j+i-3}{i-3}\binom{2i+j-3}{j-2} - 2i\binom{2j+i-3}{i-2}\binom{2j+i-3}{j-3}\Bigg].\end{aligned}$$

Manipulating and summing all the binomial coefficients, it results

$$[a^{i-1} b^{j-1}]P_1 = \frac{1}{ij}\binom{2j+i-2}{i-1}\binom{2i+j-2}{j-1}.$$

□

Proposition 6.5.5. *Fighting fish of size $n+1$ and right (or, equivalently, left) size k that have only one tail are counted by Narayana numbers*

$$\frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

Proof. The generating function of fighting fish with only one tail is easily obtained by setting $x = 0$ in Equation (6.24) defining P_1. Hence, it yields $[x^0\, t^n\, a^{k-1}]P_1 = [x^0\, t^n\, a^{k-1}]B$ (or, equivalently, $[x^0\, t^n\, b^{k-1}]P_1 = [x^0\, t^n\, b^{k-1}]B$). Since the series B defined in Equation (6.23) is symmetric in a and b, we only prove the statement of Proposition 6.5.5 by calculating the number of fighting fish of size $n+1$ and right size k, and discard the variable b in Equation (6.23) by setting $b = 1$. Then, by substituting $x = 0$ and $b = 1$ in Equation (6.23), it holds that

$$B(t,0,a,1) = t\Big(1 + aB(t,0,a,1)\Big)\Big(1 + B(t,0,a,1)\Big).$$

Now, simply by applying Lagrange inversion formula, it holds that

$$\begin{aligned}[t^n\, a^{k-1}]B(t,0,a,1) &= \frac{1}{n}[y^{n-1}\, a^{k-1}]\,(1+ay)^n \cdot (1+y)^n\\ &= \frac{1}{n}\binom{n}{k-1}[y^{n-k}]\,(1+y)^n = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.\end{aligned}$$

□

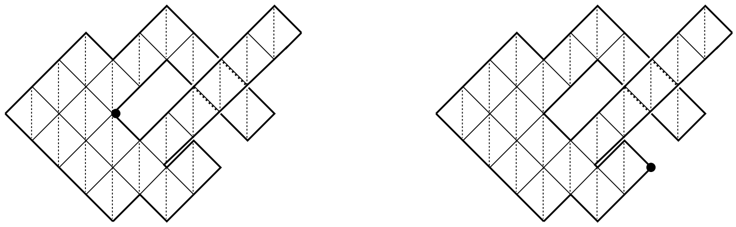


Figure 6.12: Fish with marked points: (a) branching point, (b) tail.

Now, we specialise the result of Theorem 6.5.4 for fighting fish with a marked tail. To this purpose, let $P^< \equiv P^<(t,x,a,b)$ (resp. $P^> \equiv P^>(t,x,a,b)$) denote the size generating function of fighting fish with a marked branching point (resp. with a marked tail). Figure 6.12 shows an example of markings of the same fighting fish: a branching point (on the left) and a tail (on the right).

It holds that

$$P^> = \frac{x\partial}{\partial x}(xP_1), \qquad P^< = \frac{x\partial}{\partial x}P_1, \qquad \text{and} \qquad P^> - P^< = P_1.$$

Moreover, differentiating with respect to x Equation (6.22) coming from the wasp waist decomposition, it yields

$$-tuabP(u)\left(P(u) - P_1 - \frac{x\partial}{\partial x}P_1\right) =$$
$$-\frac{\partial}{\partial x}P(u)\cdot(u-1-tu(u-1)(a+b+2abP(u)) - txabu(2P(u)-P_1))\,.$$

The right-hand side of the above equation annihilates if $u = U$, where U has been defined in the proof of Theorem 6.5.3. Therefore, under $u = U$ the left-hand side must vanish and it holds that

$$P(U) - P_1 - \frac{x\partial}{\partial x}P_1 = 0,$$

which means that

$$P(U) - P_1 - P^< = P(U) - P^> = 0.$$

Then, the generating function $P^>(t,x,a,b)$ of fighting fish with a marked tail and the power series $B(t,x,a,b)$ as defined in Theorem 6.5.3 coincide. A bijective proof of this fact will be shown in Section 6.6.2.

As in the univariate case, fighting fish with a marked tail satisfy a nice bivariate formula.

Theorem 6.5.6. *The number of fighting fish of left size j and right size i having a marked tail is*

$$\frac{(2i+2j-3)}{(2i-1)(2j-1)}\binom{2i+j-3}{j-1}\binom{2j+i-3}{i-1}. \tag{6.36}$$

Proof. In order to prove this result, we use the same parametrisation of Theorem 6.5.4 so as to apply the Lagrange inversion formula. Thus, from Equation (6.33)

$$B = t(1+\bar{R})(1+\bar{S}),$$

where $\bar{R}$ and $\bar{S}$ are defined in System (6.34). Setting $t = 1$ and applying the bivariate Lagrange inversion formula to the function $B(\bar{R}, \bar{S})$, with $\bar{R}$ and $\bar{S}$ defined by System (6.34), yields

$$\begin{aligned}
[a^{i-1}b^{j-1}]B &= \frac{1}{(i-1)(j-1)}[\bar{R}^{i-2}\bar{S}^{j-2}]\left((1+\bar{R})^{i+2j-3}(1+\bar{S})^{2i+j-3}(2i+2j-3)\right)\\
&= \frac{(2i+2j-3)}{(i-1)(j-1)}\binom{2j+i-3}{i-2}\binom{2i+j-3}{j-2}\\
&= \frac{(2i+2j-3)}{(2i-1)(2j-1)}\binom{2j+i-3}{i-1}\binom{2i+j-3}{j-1}.
\end{aligned}$$

□

6.6 Bijective interpretations

The final section of this chapter is a summary of all the existing relations investigated between fighting fish (with or without a marked tail) and plane trees. Some of them have only an analytic explanation and lack a bijective interpretation that seems to remain quite obscure.

6.6.1 Fish with a marked tail and bicoloured ordered trees: the tails/cherries relation

In Section 6.4.2 we anticipated that fighting fish with a marked tail are equinumerous to pairs of ternary trees, or equivalently, bicoloured ordered trees - all three families being enumerated by sequence A006013 [132]. Now, we strengthen this relation between these families by proving that there exist some statistics on trees sharing the same distribution as the number of tails in fighting fish with a marked tail.

First, recall the definition of a ternary tree - as in Figure 6.13(a).

Definition 6.6.1. A plane tree is called *ternary tree*, if it is either empty or contains a root and three disjoint ternary trees called the left, middle and right subtree of the root.

The above definition has a straightforward consequence: let $T \equiv T(t)$ be the generating function of ternary trees according to the number of nodes, T satisfies

$$T = 1 + t\,T^3. \tag{6.37}$$

Now, consider pairs of ternary trees and enumerate them according to the total number of nodes and *right branches*: a right branch of a ternary tree is given by a maximal non-empty sequence of consecutive right edges. Figure 6.13(b) depicts a pair of ternary trees with 2 right branches for each tree.

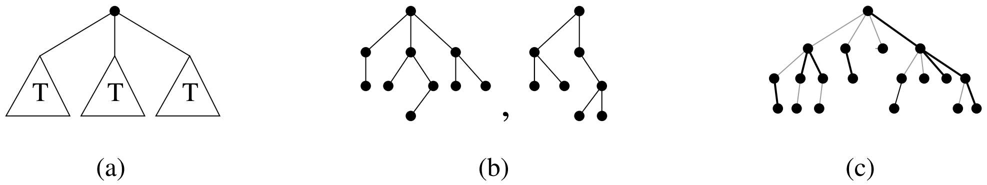


Figure 6.13: (a) The decomposition of a non-empty ternary tree; (b) a pair of ternary trees with 18 nodes and 4 right branches; (c) the bicoloured ordered tree corresponding to the pair in (b).

Proposition 6.6.2. *The generating function $P_{tt} \equiv P_{tt}(t,x)$ of pairs of ternary trees, where t counts the number of nodes and x the number of right branches, satisfies*

$$P_{tt} = \left(1 + t\,P_{tt} + \frac{t^2\,P_{tt}^2\,x}{1 - t\,P_{tt}}\right)^2.$$

Proof. It follows from a refinement of Equation (6.37). Indeed, each ternary tree of the pair is either empty (providing the contribution 1) or it has a root and, possibly, subtrees. We divide the second case in two subcases: if the root has or not its right subtree. If it does not have a right subtree, then it contributes $t\,P_{tt}$, where P_{tt} is the contribution for the left and the middle subtrees of the root (they might be empty). On the other hand, the case in which the root has a non-empty right subtree contributes $t\,P_{tt} \cdot t\,P_{tt}\,x/(1 - t\,P_{tt})$, where in addition to the factor $t\,P_{tt}$ for the root it is counted a non-empty sequence of consecutive right edges each one carrying its left and middle subtrees contributing for $t\,P_{tt}\,x/(1 - t\,P_{tt})$. □

The first terms of the series expansion of P_{tt} confirm that pairs of ternary trees without any right branches - that are trivially in bijection with pairs of binary trees - are counted by Catalan numbers

$$P_{tt} = 1 + 2\,t + (5 + 2x)\,t^2 + (14 + 16x)\,t^3 + (42 + 92x + 9x^2)\,t^4 + O(t^5).$$

A pair of ternary tree can be easily mapped into a plane tree whose edges have been coloured in a specific way - this representation appears to be more compact and for this reason we prefer it and use it in the remainder of this subsection.

A *bicoloured ordered tree* is a plane tree with gray and black edges ordered so that in any node a gray edge cannot be on the right of a black edge. A *cherry* of a bicoloured ordered tree is a set of edges that all start from the same node and share the same colour provided that for each node the set is maximal and counts at least two edges; for instance, the bicoloured ordered tree in Figure 6.13(c) has 2 cherries of each colour.

Proposition 6.6.3. *Pairs of ternary trees with n nodes and k right branches are in bijection with bicoloured ordered trees with $n + 1$ nodes and k cherries.*

Proof. To simplify the description of this bijection, we can think of a bicoloured ordered tree as a compound of two subtrees: a bicoloured tree can indeed be split at the root so that one tree has only gray edges starting from the root and the other only black edges. By merging their roots we obtain a unique bicoloured ordered tree, since in any node gray edges must precede black edges.

Then, the idea is to encode the ternary trees forming a pair separately. Each ternary tree is mapped in a bicoloured ordered tree having the same colour in all the edges starting from its root, and we choose to map the ternary tree on the left (resp. right) into a bicoloured ordered tree having all gray (resp. black) edges starting from the root.

The encoding used to map a ternary tree τ into a bicoloured ordered tree β is a generalisation of the one known to map binary trees into ordered trees:

- if τ is empty, then β has only one node;

- if τ has root r, then β has an edge starting from the root. The colour of this edge is determined according to which ternary tree of the pair is being encoded.

- Let τ_1, τ_2 and τ_3 be the left, middle and right subtrees of a node s in τ and e be the edge of β corresponding to s. Then, τ_1, τ_2 and τ_3 are recursively encoded by pre-order visiting as follows: a node s' left (resp. middle) child of s is mapped into a gray (resp. black) edge child of e, whereas a node right child of s is mapped into a twin of e, namely an edge that shares colour and origin with e, and lies on its right.

This mapping can be reversed with ease - an example is shown in Figure 6.13(b),(c).

Note that the pair of empty ternary trees corresponds to the unique bicoloured ordered tree with only one node, and any right branch is mapped into a cherry, and vice versa. □

Proposition 6.6.4. *Fighting fish with a marked tail of size $n+1$ and $k+1$ tails are as many as bicoloured ordered trees with n nodes and k cherries.*

Proof. First of all, recall that the generating function $P^{>}(t,x,a,b)$ of fighting fish with a marked tail coincides with function $B(t;x,a,b)$ defined in Theorem 6.5.3 - a bijective proof will be given in the next subsection.

The proof of our statement is analytical: the bivariate generating function $B(t;x,1,1)$ of fighting fish with a marked tail and the bivariate generating function $F_{bt}(t,x) := t \cdot P_{tt}$ of bicoloured ordered trees satisfy the same functional equation. Indeed, by Proposition 6.6.2 and 6.6.3 the generating function $F_{bt} \equiv F_{bt}(t,x)$ satisfy

$$F_{bt} = t \cdot \left(1 + F_{bt} + x \cdot \frac{F_{bt}^2}{1 - F_{bt}}\right)^2 = t \cdot \left(1 + x \cdot \frac{F_{bt}^2}{1 - F_{bt}^2}\right)^2 \cdot (1 + F_{bt})^2,$$

which is exactly Equation (6.23) by substitution $a = b = 1$. □

As previously noticed, bicoloured ordered trees without cherries are counted by Catalan numbers (sequence A000108 [132]) as well as fighting fish with a unique marked tail.

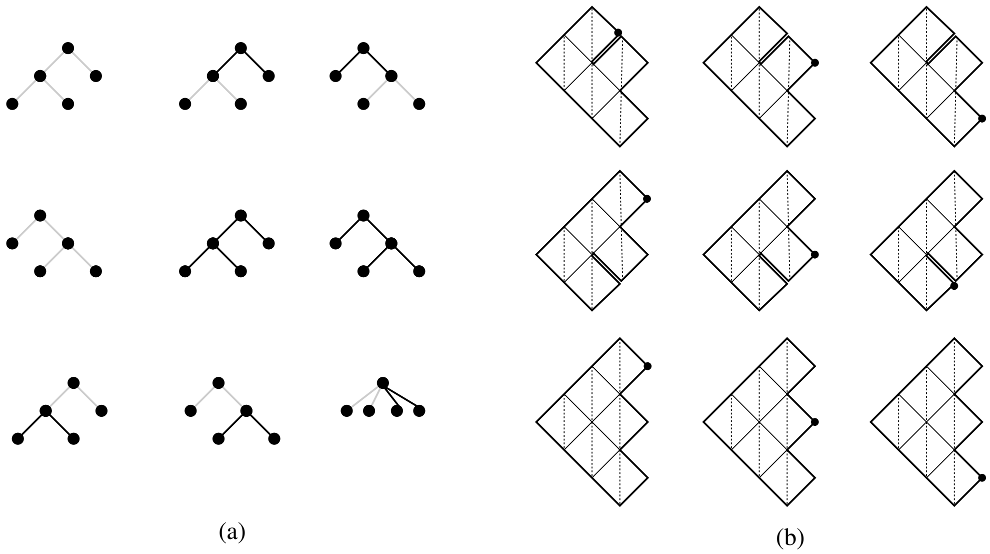


Figure 6.14: (a) All the bicoloured ordered trees with 5 nodes and 2 cherries; (b) all the fighting fish with a marked tail of size 6 having 3 tails.

Proposition 6.6.4 calls for a direct bijection between these two families - see Figure 6.14 - that despite the proved equidistribution still appears unclear.

A further remarkable result is stated in the following proposition, that unfortunately misses an interpretation on fighting fish with a marked tail. This fact makes even more intriguing finding a direct bijection between fighting fish and bicoloured ordered trees.

Proposition 6.6.5. *Bicoloured ordered trees can be generated by the following succession rule*

$$\Omega_{trees} = \begin{cases} (2) \\ (h) \rightsquigarrow \quad (3), (4), \ldots, (h+2). \end{cases}$$

Proof. In order to describe a possible growth according to Ω_{trees}, we define the *rightmost path* of a bicoloured ordered tree as the branch that starts from the root and, by choosing the rightmost child among all the children of each node it meets, stops when it reaches a leaf.

Note that by removing the last edge of the rightmost path, whichever colour it has, we obtain a unique bicoloured ordered tree.

We perform edge additions to the nodes of the rightmost path of a bicoloured ordered tree as follows: if the node is a leaf or it has only gray edges as children, both a gray and a black edge can be added as a new child, otherwise only a black edge can be added. For brevity, a leaf or a node without black children is said to have power 2, while a node with at least a black child to have power 1.

We assign to a bicoloured ordered tree a label (h) given by summing the powers of all the nodes in its rightmost path. The one-node tree has only one leaf in the rightmost path, so it has label (2). Assume τ is a bicoloured ordered tree with label (h) and such that its root has no black children. Then, by adding a new black edge to the root of τ a bicoloured ordered tree with label (3) is generated, whereas by adding a new gray edge to its root we generate a bicoloured ordered tree with label (4). Going on visiting all the nodes of the rightmost path and adding edges we generate bicoloured ordered trees with labels $(5), \ldots, (h+1), (h+2)$. In particular, the label $(h+1)$ (resp. $(h+2)$) is assigned to that bicoloured ordered tree obtained by adding a black (resp. gray) edge as first child of the last node visited - which is a leaf of τ. A similar reasoning can be repeated supposing the root of τ has at least one black child completing the proof. □

6.6.2 A bijective proof of $P^{>} = P(U)$

In Section 6.5.4, it has analytically been proved that $P^{>} = P(U) = B$, where

$$B = t\left(1 + x \cdot \frac{abB^2}{1 - abB^2}\right)(1 + aB)(1 + bB).$$

Although we are not able to combinatorially describe the validity of the above formula for $P^{>}$, we can give a combinatorial explanation of $P^{>} = P(U)$ by providing a bijection between fighting fish with a marked tail and objects counted by $P(U)$.

To this purpose, we need a combinatorial interpretation of the power series U, which has been defined in the proof of Theorem 6.5.3. Returning to Equation (6.22) and differentiating it with respect to t, yields

$$xabuP(u)\left(P(u) - P_1 - \frac{t\partial}{\partial t}P_1\right) + u(u-1)(1 + bP(u))(1 + aP(u)) =$$
$$-\frac{\partial}{\partial t}P(u) \cdot (u - 1 - tu(u-1)(a + b + 2abP(u)) - txabu(2P(u) - P_1)).$$

By definition of the power series U, the right-hand side of this equation annihilates, if $u = U$. Then, by setting $u = U$ the left-hand side must vanish and, after manipulating it, it results

$$xabP(U)\frac{t\partial}{\partial t}P_1 + U(1 + aP(U))(1 + bP(U))(1 - U) = 0.$$

Now, eliminating $P(U)$ from the above expression by means of Equation (6.28), yields

$$U \cdot \left(1 - txab\frac{t\partial}{\partial t}P_1\right) = 1. \tag{6.38}$$

The derivative of P_1 with respect to t can be combinatorially interpreted, as t is the variable counting the fish size minus one. Then, the function $t\partial P_1/\partial t$ is the generating function of fighting fish having a mark on their boundary: we choose to mark the end point of a free upper edge travelling the boundary counterclockwise, apart from the upper edge of the fish head - see Figure 6.15. It is well worth noticing that fighting fish with a marked tail are not contained in this family, whereas those with a marked branching point form a subfamily in it (Figure 6.12). Hence,

$$\frac{t\partial}{\partial t}P_1 = P^< + P^-,$$

where by difference P^- is the generating function of fighting fish with a marked *flat* point (*i.e.* a point of a free upper edge that is neither a branching point, nor the tip of the head or of the tails) - see Figure 6.15(b)(c).

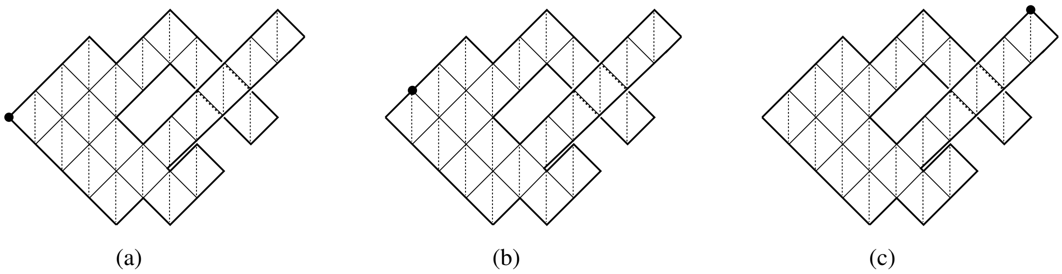


Figure 6.15: (a) An unauthorised marking for a fighting fish; (b)-(c) possible markings for the same fighting fish.

Then, according to (6.38), the series U satisfies

$$U = \frac{1}{1 - txab\,(P^< + P^-)}.$$

In terms of fighting fish with a marked point, $U \equiv U(t; x, a, b)$ is the generating function counting sequences of fighting fish having a flat or a branching point marked, where the variables t, x, a, b exactly mark their size, number of tails, left and right size, respectively.

We are now able to describe combinatorially objects counted by $P(U)$ and establish a bijection with those counted by $P^>$.

Proposition 6.6.6. *There is a bijection between*

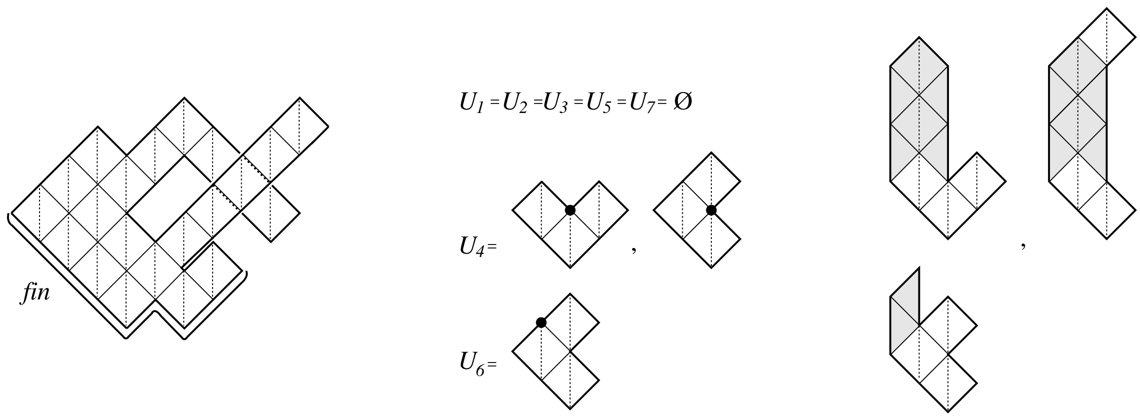
1. *fighting fish with a marked tail having left size $j+1$, and right size $i+1$, and $h+1$ tails,*

2. *and pairs (P, S) where P is a fighting fish with fin length $k+1$ and S is a k-tuple $(U_1, \ldots, U_k)$ of sequences $U_i = (V_{i,1}, \ldots, V_{i,j_i})$ of fighting fish having a flat point or a branching point marked, such that the total number of free left lower edges (resp. right lower edges, tails) of P and S is $j+1$ (resp. $i+1$, $h+1$).*

Proof. Let (P, S) be a pair as described in *2.* (see Figure 6.16(a),(b)). Starting from the head and travelling along the boundary of P counterclockwise, mark the first tail encountered. Then, slice P along the vertical segments starting at the end of each free lower edge of its fin and stopping at the first free upper edge incident the vertical segment. There are k of these *vertical cuts*, which are numbered left-to-right and each one has its own height h_i, with $i = 1, \ldots, k$, given by the number of vertical edges chopped.

Now, consider S and, in particular, the fighting fish $V_{i,1}$, for some i. Cut the boundary of $V_{i,1}$ first at its marked point and then, on the tip of its head. Inflate it vertically by adding auxiliary left and right scales so as to reach the height h_i in both cut-points -see Figure 6.16(c). Glue the inflated fighting fish to the right border of the ith vertical cut of P matching its height and, repeat this proceeding for all fighting fish $V_{i,j}$, where $1 < j \leq j_i$. We precise that not only is the last fighting fish of the sequence V_{i,j_i} glued to the inflated fish V_{i,j_i-1} on the right, but also to the left border of the ith vertical cut of P on the left - see Figure 6.17. The fact that the marked point of each $V_{i,j}$ is a flat point or a branching point ensures that the inflated fish forms a branching point with the boundary of P once they are glued together.

Then, for all i, sequences U_i are inserted into P producing a fighting fish with a marked tail and left (resp. right) size $j + 1$ (resp. $i + 1$).



(a) (b) (c)

Figure 6.16: (a) A fighting fish P of fin length 8, whose vertical cuts have heights left-to-right $1, 2, 3, 2, 1, 1, 1, 1$; (b) a sequence $S = (U_1, U_2, U_3, U_4, U_5, U_6, U_7)$; (c) fighting fish $V_{4,1}, V_{4,2}, V_{6,1}$ inflated according to $h_4 = 2$ and $h_6 = 1$.

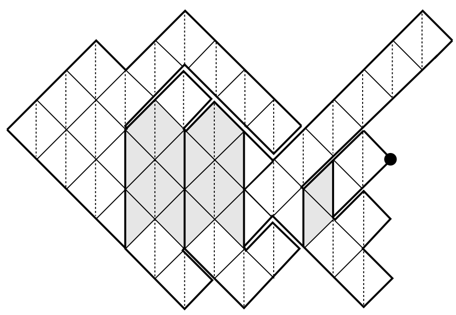


Figure 6.17: The fighting fish with a marked tail coming from P of Figure 6.16(a) and sequence S of Figure 6.16(b).

Conversely, let M be a fighting fish with a marked tail. We choose to decompose M both by travelling along its boundary and its *spine*: by definition of fighting fish, there exists a sequence of connected left/right scales that links the marked tail of M to its head and we call it spine.

Then, starting from the marked tail of M, after removing its mark, travel the boundary clockwise and cut M in slices as follows:

1. name x_i the end point of the free lower edge encountered clockwise by travelling step-by-step the boundary of M. Note that at the beginning x_1 is the end point of the free lower edge of the tail;

2. if x_i is not a branching point (namely it is followed by a free lower edge), step forward and go back to point 1., setting $i := i + 1$;

3. if x_i is a branching point, then let h_i be the height of the vertical cut C_i starting from x_i. Now, set $j := 1$ and travel the spine of M as follows:

   (a) check right-to-left the heights of the vertical segments on the left of C_i - note that at least one must have height h_i;

   (b) let D_j be the rightmost vertical segment that returns to value h_i and $\bar{x}_j$ its starting point. We remove all the part of M included between D_j and C_i and deflate it by h_i. The ending point of D_j and $\bar{x}_j$ joint give rise to the head of a fighting fish $E_{i,j}$ that carries a mark in place of C_i. Note that $E_{i,j}$ is marked at the end of a free upper edge because of x_i being a branching point;

   (c) consider the point $\bar{x}_j$ of M. If it is a branching point of M, then resume at point 3.(a) and set $j := j + 1$. Otherwise, the factor $(E_{i,1}, \dots, E_{i,j})$ is completed, thus step forward and go back to point 1., setting $i := i + 1$;

4. if x_i is the tip of the head, then stop. The fighting fish remaining has fin length $s := i$. Call it P and $U_i := (E_{s-i,1}, \dots, E_{s-i,j_{s-i}})$, for any $1 \leq i < s$.

Then, the pair (P, S), where S is $(U_1, \dots, U_{s-1})$, is as described in *2.* - see Figure 6.18.

The proof that the above constructions are inverse one another is not hard to verify. □

6.6.3 Fighting fish and left ternary tree: the fin/core relation and a refined conjecture

In this subsection we try to give an explanation of the relation existing between fighting fish and left ternary trees - both families are counted by sequence A000139 [132].

Any ternary tree (see Definition 6.6.1) can be naturally embedded in the plane in a deterministic way: the root has abscissa j and the left (resp. middle, right) child of a node in abscissa $i \in \mathbb{Z}$ takes abscissa $i - 1$ (resp, i, $i + 1$) - see Figure 6.19(a). A *j-positive tree* is defined as a ternary tree whose nodes all have nonnegative abscissa; 0-positive trees

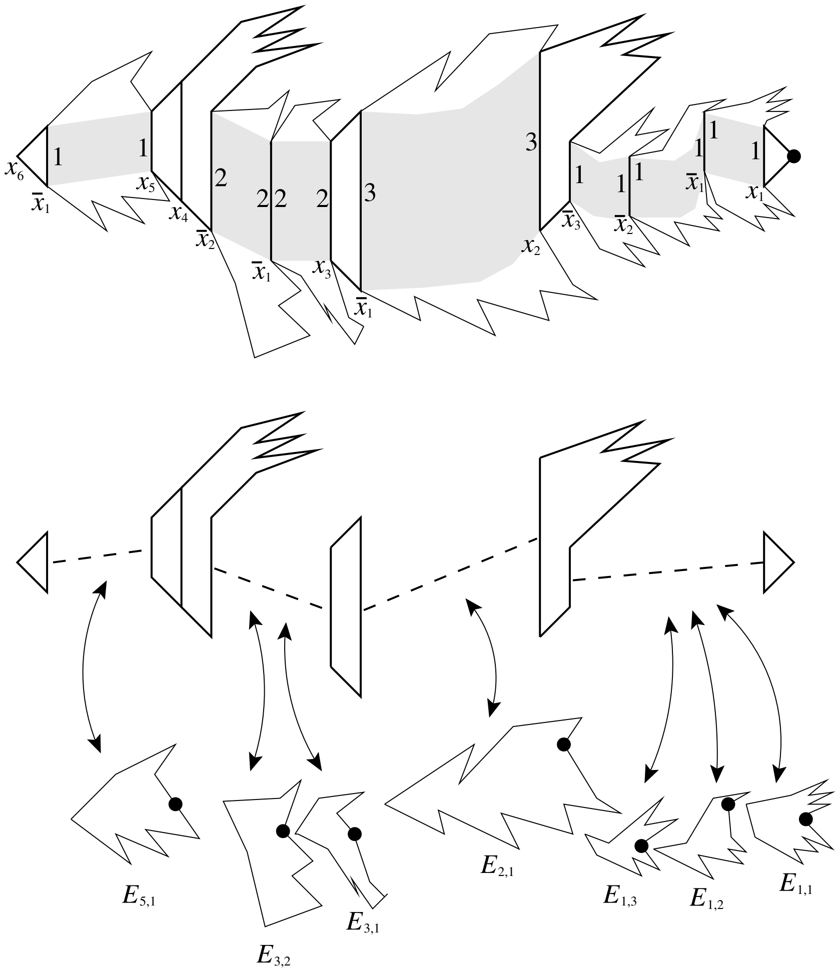


Figure 6.18: Decomposition of a sketched fighting fish with a marked tail into a fighting fish P and sequences of fighting fish marked in a flat or a branching point.

were first introduced in the literature with the name of *left ternary trees* [59, 93] and in order to be coherent with their notation we choose to orient the x-axis toward the left - see Figure 6.19(b).

It is known that left ternary trees with i nodes at even abscissas and j nodes at odd abscissas are enumerated by the explicit formula (6.31) of Theorem 6.5.4 on page 207 (see [59, 93]), which is the formula counting fighting fish according to their left and right size.

It turns out that this equidistribution can be refined further upon considering a new parameter on ternary trees: the *core size*. Let the *core* of a ternary tree be the largest subtree including its root and consisting only of its left and middle edges. The size of the core is given by the number of nodes belonging to the core. Equivalently, the core of a ternary tree can be obtained by removing all its right edges and their corresponding subtrees - see Figure 6.19(c).

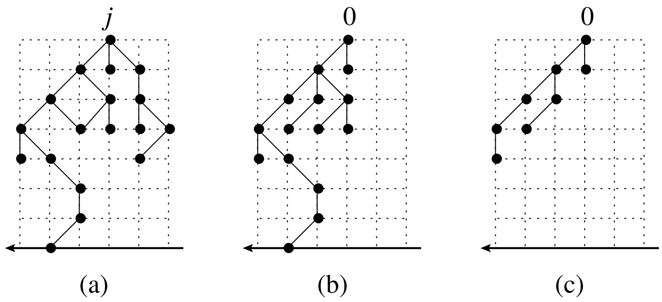


Figure 6.19: (a) A ternary tree embedded in the plane - at point $(j+1, 4)$ there are two nodes; (b) a 0-positive tree, or a left ternary tree; (c) the core of the left ternary tree depicted in (b).

Therefore, the relevant parameters we take into account in a left ternary tree are:

- number of nodes (*i.e.* size),
- number of nodes belonging to its core (*i.e.* core size),
- number of right branches (*i.e.* maximal sequences of right edges),
- number of non-root nodes in an even (resp. odd) abscissa.

All these parameters appear to share their distribution with those found on fighting fish.

Conjecture 6.6.7. *The number of fighting fish with size $n+1$, fin length $k+1$, having $h+1$ tails, of right size $i+1$ and left size $j+1$ is equal to the number of left ternary trees with n nodes, core size k, having h right branches, with i non-root nodes in even abscissa and j nodes in odd abscissa.*

This conjecture naturally calls for a bijective proof, however we have not been able to provide such a proof yet - see Figure 6.20. Instead, we are able to prove a weaker version of Conjecture 6.6.7 that involves the distribution of the core size and of the fin length: the number of left ternary trees of size n and core size k and the number of fighting fish of size $n+1$ and fin length $k+1$ coincide.

Theorem 6.6.8. *The number of fighting fish with size $n+1$ and fin length $k+1$ is equal to the number of left ternary trees with n nodes, k of which are accessible from the root using only left and middle edges.*

The proof of the above theorem, which is postponed at the end of the next subsection, is analytical and follows from checking that the generating function of left ternary trees with respect to their size and their core size coincides with the generating function of fighting fish with respect to their size and their fin length up to a constant term given by the empty tree.

Enumerating left ternary trees had been a hard problem tackled in [59, 93, 105]. Let $F_j \equiv F_j(t; x, a, b, u)$ denote the generating function of ternary trees such that, when the

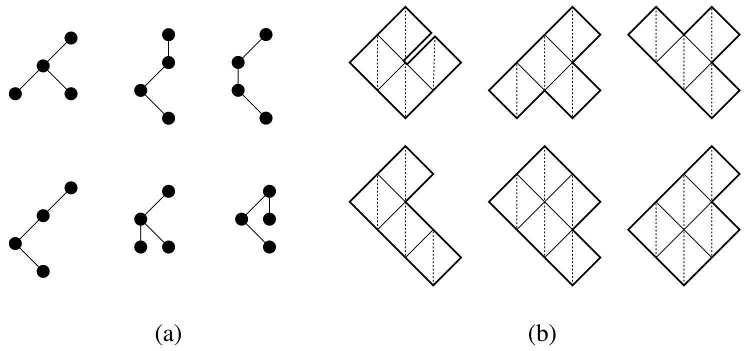


Figure 6.20: (a) All left ternary trees of size 4 and core size 3; (c) all fighting fish of size 5 and fin length 4.

root is embedded at abscissa j, all nodes have non negative abscissa, where t marks the number of nodes, x the number of right branches, a (resp. b) the number of non-root nodes in an even (resp. odd) abscissa, and u the core size. The generating function of left ternary trees $F_0(t; 1, a, b, 1)$ was first computed bijectively in [59, 93], and the more general generating function $T_j \equiv F_j(t; 1, 1, 1, 1)$ was computed by M. Kuba [105] following the educated guess and check approach by P. Di Francesco [68], as the following subsection illustrates.

Remark 6.6.9. *Although we proved the weaker version of Conjecture 6.6.7 in Theorem 6.6.8, we do not have this result combinatorially explained. The only combinatorial interpretation we have for formulas of the next subsection is in terms of ternary trees.*

In fact, in the next subsection we will make use of a function $D(u) \equiv D(t; x, a, b, u)$*, which is defined as the unique power series solution of*

$$D(u) = tu\left(1 + aD(u) + xaD(u)\frac{bB(1 + aB)}{1 - abB^2}\right)\left(1 + bD(u) + xbD(u)\frac{aB(1 + bB)}{1 - abB^2}\right),$$

where $B = D(1)$*. Although* $D(u)$ *has not been interpreted in terms of fighting fish, apart from the case* $D(1) = B$ *(see Section 6.6.2), it results that* $D(u)/tu$ *is the generating function of pairs of ternary trees with respect to the announced parameters. More precisely,* $D(u)/tu$ *is the generating function of pairs of ternary trees, where* t *marks the number of nodes,* x *the number of right branches,* a *(resp.* b*) the number of nodes in an even (resp. odd) abscissa, and* u *the core size, assuming that for any pair the ternary tree the one on the left (resp. right) has root in an even (resp. odd) abscissa. Then, following the decomposition in Proposition 6.6.2, each factor corresponding to one ternary tree of the pair comprises the contribution* 1 *of the empty ternary tree and the contributions given by a non-empty ternary tree. A non-empty ternary tree whose root has no right edges gives contribution* $aD(u)$ *(resp.* $bD(u)$*) if it is the tree on the left (resp. right) of the pair. Whereas, a non-empty ternary tree whose root has a right branch gives contribution* $xaD(u)(bB{+}abB^2)/(1{-}abB^2)$ *(resp.* $xbD(u)(aB{+}abB^2)/(1{-}abB^2)$*), where we distinguish the two cases in which the sequence of right edges ends in an even or odd abscissa.*

6.6.4 Analytic proof of the fin/core relation: Theorem 6.6.8

Let $B_1 \equiv B(t,1,1,1)$ be the specialisation of the unique power series solution of Equation (6.23) for $x=a=b=1$, so that

$$B_1 = \frac{t}{(1-B_1)^2}.$$

Let furthermore $T \equiv T(t) = 1/(1-B_1)$ and $X \equiv X(t)$ be the unique formal power series solution of the equation

$$X = (1+X+X^2)B_1.$$

The series T satisfies $T = 1 + tT^3$, thus there is no abuse of notation since it is the size generating function of ternary trees.

Building on P. Di Francesco's educated guess and check approach [68], M. Kuba obtained a formula for the generating functions of j-positive trees:

Theorem 6.6.10 ([68, 105])**.** *The generating functions $T_j \equiv T_j(t)$ of j-positive trees with respect to the number of nodes is given for all $j \geq 0$ by the explicit expression:*

$$T_j = T\frac{(1-X^{j+5})(1-X^{j+2})}{(1-X^{j+4})(1-X^{j+3})}.$$

Now, the generating function of ternary trees is easily refined to take into account the number of nodes in the core. Let

$$J(u) = 1 + D_1(u)T \quad \text{and} \quad D_1(u) = tuJ(u)^2,$$

where $J(1) = T$ and $J(u)$ is the generating function of ternary trees according to the size and the number of nodes in the core. One could compare the definition of the series $D_1(u)$ to the specialisation for $x=a=b=1$ of the generating function $D(u)$ defined in Remark 6.6.9 and notice that $D_1(u) \equiv D(t,1,1,1,u)$.

Furthermore, the full generating function of fighting fish $P(u) \equiv P(t,x,a,b;u)$ yields the following striking parametrisation.

Corollary 6.6.11. *Let $D(u) \equiv D(t;x,a,b,u)$ be the unique power series solution of equation*

$$D(u) = tu\left(1 + aD(u) + xaD(u)\frac{bB(1+aB)}{1-abB^2}\right)\left(1 + bD(u) + xbD(u)\frac{aB(1+bB)}{1-abB^2}\right) \tag{6.39}$$

so that $D(1) = B$, as defined in Theorem 6.5.3, then

$$P(u) = D(u) - xabD(u)^2B\frac{(1+aB)(1+bB)(1-abB^2+xabB^2)}{(1-abB^2)^2(1-abD(u)B+xabD(u)B)}.$$

Proof. From Theorem 6.5.3 the generating function $P(u)$ is algebraic of degree at most 2 over $\mathbb{Q}(x, a, b, u, B)$. The above parametrisation can be obtained extending the one of Theorem 6.5.3. □

As revealed in advance in Remark 6.6.9, the function $D(u)$ has not been interpreted combinatorially on fighting fish, except for $D(1) = B = P^{>}$ in Section 6.6.2.

Then, the generating function of fighting fish according to the size and the fin length, given by Corollary 6.6.11, can be written as

$$\begin{aligned} P(t,1,1,1,u) &= D_1(u) - D_1(u)^2 \frac{B_1}{(1-B_1)^2} = D_1(u) - D_1(u)^2 B_1 T^2 \\ &= (J(u)-1)(1-B_1) - (J(u)-1)^2 B_1 \\ &= J(u)(1+B_1) - J(u)^2 B_1 - 1. \end{aligned} \tag{6.40}$$

By refining Theorem 6.6.10, we can prove a formula for the generating function of j-positive trees with respect to the number of nodes and the core size, which involves the function $J(u)$. Hence, by comparing Equation (6.40) and the expression obtained for 0-positive trees (*i.e.* left ternary trees) we are able to prove the equidistribution stated in Theorem 6.6.8.

Theorem 6.6.12. *The generating functions $J_j(u) \equiv J_j(t,u)$ of j-positive trees with respect to the number of nodes and size of the core is given for $j \geq -1$ by*

$$J_j(u) = J(u) \frac{H_j(u)}{H_{j-1}(u)} \frac{1 - X^{j+2}}{1 - X^{j+3}} \tag{6.41}$$

where for all $j \geq -2$,

$$H_j(u) = (1 - X^{j+1}) X J(u) - (1+X)(1 - X^{j+2}).$$

Proof. In order to prove the theorem it is sufficient to show that the series given by the right-hand side of Equation (6.41) satisfies for all $j \geq -1$ the equation

$$J_j(u) = 1 + tu J_{j+1}(u) J_j(u) T_{j-1} \tag{6.42}$$

where T_j is given by Theorem 6.6.10, with the convention that $T_{-2} = 0$. Indeed, the system of Equations (6.42) clearly admits the generating function of j-positive ternary trees as its unique power series solution. The case $j = -1$ is immediate, since

$$J_{-1}(u) = J(u) \frac{H_{-1}(u)}{H_{-2}(u)} \frac{1 - X}{1 - X^2} = 1,$$

where by definition, $H_{-1} = -(1+X)(1-X)$ and $H_{-2}(u) = (X-1)J(u)$. Let now $j \geq 0$, then the right-hand side of Equation (6.42) reads

$$\begin{aligned} &1 + tu \left(J(u) \frac{H_{j+1}(u)}{H_j(u)} \frac{1 - X^{j+3}}{1 - X^{j+4}} \right) \left(J(u) \frac{H_j(u)}{H_{j-1}(u)} \frac{1 - X^{j+2}}{1 - X^{j+3}} \right) T \frac{1 - X^{j+4}}{1 - X^{j+3}} \frac{1 - X^{j+1}}{1 - X^{j+2}} \\ &= \frac{H_{j-1}(u)(1 - X^{j+3}) + (J(u)-1) H_{j+1}(u)(1 - X^{j+1})}{H_{j-1}(u)(1 - X^{j+3})} \end{aligned} \tag{6.43}$$

and we want to show that this expression is equal to the right-hand side of Equation (6.41), which has the same denominator as (6.43). Then, expanding the numerator of Equation (6.43), the following terms are obtained

$$H_{j-1}(u)(1-X^{j+3}) = (1-X^{j+3})(1-X^{j})XJ(u) - (1+X)(1-X^{j+1})(1-X^{j+3}),$$

$$-H_{j+1}(u)(1-X^{j+1}) = -(1-X^{j+1})(1-X^{j+2})XJ(u) + (1+X)(1-X^{j+3})(1-X^{j+1}),$$

$$J(u)H_{j+1}(u)(1-X^{j+1}) = (1-X^{j+1})(1-X^{j+2})XJ(u)^2 - (1+X)(1-X^{j+3})(1-X^{j+1})J(u),$$

while the numerator of Equation (6.41) reads as

$$J(u)H_j(u)(1-X^{j+2}) = (1-X^{j+2})(1-X^{j+1})XJ(u)^2 - (1+X)(1-X^{j+2})^2 J(u).$$

The coefficients of $J(u)^2$ and $J(u)^0$ in the numerators of (6.43) and of (6.41) are clearly matching. Upon expanding all contributions to the coefficient of $J(u)$ in these numerators in powers of X, the various terms are seen to match as well completing the proof. □

Theorem 6.6.12, together with Corollary 6.6.11 as rewritten in Equation (6.40), directly implies Theorem 6.6.8.

Proof of Theorem 6.6.8. We want to show that the generating function $J_0(u)$ of left ternary trees according to the size and the number of nodes in the core is equal to the generating function $P(t,1,1,1,u)$ of fighting fish according to the size and the fin length up to a constant term

$$J_0(u) = 1 + P(t,1,1,1,u).$$

From Theorem 6.6.12,

$$J_0(u) = J(u)\frac{H_0(u)}{H_{-1}(u)}\frac{1-X^2}{1-X^3},$$

and by definition $H_{-1} = -(1+X)(1-X)$ and $H_0(u) = (1-X)XJ(u) - (1+X)(1-X^2)$, so that

$$J_0(u) = J(u)\frac{XJ(u)-(1+X)^2}{-(1+X+X^2)} = -J(u)^2B_1 + J(u)(1+B_1),$$

which coincides with Equation (6.40) up to a constant term given by the empty left ternary tree. □

Appendix A

Semi-Baxter permutations

A.1 Generating function of semi-Baxter permutations

The semi-Baxter functional equation and its kernel form:

```
> Eq:= -S(y,z) + x*y*z + x*y*z*(S(1,z)-S(y,z))/ (1-y) + x*y*z*(S(y,z) -
S(y,y))/ (z-y);
> Ker:=-(coeff(collect(%,S(y,z)),S(y,z)));
> EqK:=Ker*S(y,z)=x*z*y+x*z*y*S(1,z)/(1-y)-x*z*y*S(y,y)/(z-y);
```

$$Eq := -S(y,z) + xyz + \frac{xyz\,(S(1,z) - S(y,z))}{1-y} + \frac{xyz\,(S(y,z) - S(y,y))}{z-y}$$

$$Ker := \frac{xyz}{1-y} + 1 - \frac{xyz}{z-y}$$

$$EqK := \left(\frac{xyz}{1-y} + 1 - \frac{xyz}{z-y}\right) S(y,z) = -\frac{xyzS(y,y)}{z-y} + xyz + \frac{xyzS(1,z)}{1-y}$$

Substitution $y = 1 + a$:

```
> Kera:=subs(y=1+a,Ker);
> EqA:=subs(y=1+a,EqK);
> Ker1:=collect(subs(a=A,Kera),x,factor):
> Ker2:=collect(subs(z=Z,Kera),x,factor):
```

$$Kera := -\frac{x(1+a)z}{a} + 1 - \frac{x(1+a)z}{z-1-a}$$

$$EqA := \left(-\frac{x(1+a)z}{a} + 1 - \frac{x(1+a)z}{z-1-a}\right) S(1+a,z) \quad = \\ -\frac{x(1+a)zS(1+a,1+a)}{z-1-a} + x(1+a)z - \frac{x(1+a)zS(1,z)}{a}$$

Kernel symmetry transformations generate a group of order 10:

```
> solve(Ker1/Kera=1,A);
> solve(Ker2/Kera=1,Z);
```

$$a, -\frac{-z+1+a}{1+a}$$

$$z, -\frac{-a+az+z-1}{-z+1+a}$$

```
> [a,z];
> factor([(%[2]-1-%[1])/(1+%[1]),%[2]]);
> factor([%[1],(1+%[1]-%[2]-%[1]*%[2])/(1+%[1]-%[2])]);
> factor([(%[2]-1-%[1])/(1+%[1]),%[2]]);
> factor([%[1],(1+%[1]-%[2]-%[1]*%[2])/(1+%[1]-%[2])]);
> factor([(%[2]-1-%[1])/(1+%[1]),%[2]]);
> factor([%[1],(1+%[1]-%[2]-%[1]*%[2])/(1+%[1]-%[2])]);
> factor([(%[2]-1-%[1])/(1+%[1]),%[2]]);
> factor([%[1],(1+%[1]-%[2]-%[1]*%[2])/(1+%[1]-%[2])]);
> factor([(%[2]-1-%[1])/(1+%[1]),%[2]]);
> factor([%[1],(1+%[1]-%[2]-%[1]*%[2])/(1+%[1]-%[2])]);
```

$$[a, z]$$

$$\left[-\frac{-z+1+a}{1+a}, z\right]$$

$$\left[-\frac{-z+1+a}{1+a}, \frac{z-1}{a}\right]$$

$$\left[-\frac{-z+1+a}{az}, \frac{z-1}{a}\right]$$

$$\left[-\frac{-z+1+a}{az}, \frac{1+a}{a}\right]$$

$$\left[(z-1)^{-1}, \frac{1+a}{a}\right]$$

$$\left[(z-1)^{-1}, -\frac{z}{-z+1+a}\right]$$

$$\left[-\frac{a}{-z+1+a}, -\frac{z}{-z+1+a}\right]$$

$$\left[-\frac{a}{-z+1+a}, -\frac{(z-1)(1+a)}{-z+1+a}\right]$$

$$\left[a, -\frac{(z-1)(1+a)}{-z+1+a}\right]$$

$$[a, z]$$

Kernel root Z_+:

```
> collect(Kera,x,factor);
> [solve(Kera=0,z)]:
> factor(series(%[2],x,10)) assuming a>0;
> sol:=%%[2];
```

$$1 + \frac{z(z-1)(1+a)x}{a(-z+1+a)}$$

$$1+a+(1+a)^2 x + \frac{(1+a)^3(2a+1)}{a}x^2 + \frac{(5a^2+5a+1)(1+a)^4}{a^2}x^3 + \frac{(7a^2+7a+1)(1+a)^5(2a+1)}{a^3}x^4 + \frac{(42a^4+84a^3+56a^2+14a+1)(1+a)^6}{a^4}x^5 + \frac{(66a^4+132a^3+84a^2+18a+1)(1+a)^7(2a+1)}{a^5}x^6 + \frac{(429a^6+1287a^5+1485a^4+825a^3+225a^2+27a+1)(1+a)^8}{a^6}x^7 + \frac{(715a^6+2145a^5+2431a^4+1287a^3+319a^2+33a+1)(1+a)^9(2a+1)}{a^7}x^8 + O\left(x^9\right)$$

$$sol := -\frac{1}{2}\,\frac{-a-x-ax+\sqrt{a^2-2ax-6a^2x+x^2+2ax^2+a^2x^2-4a^3x}}{x(1+a)}$$

Check that all the substitutions are well-defined power series in x if $z = Z_+$:

```
> factor(series(subs(z=sol,-(-z+1+a)/(1+a)),x,6)) assuming a>0;
> factor(series(subs(z=sol,(z-1)/a),x,6)) assuming a>0;
> factor(series(subs(z=sol,-(-z+1+a)/(1+a)),x,6)) assuming a>0;
```

$$x(1+a) + \frac{(1+a)^2(2a+1)x^2}{a} + \frac{(5a^2+5a+1)(1+a)^3x^3}{a^2} + \frac{(7a^2+7a+1)(1+a)^4(2a+1)x^4}{a^3} + O\left(x^5\right)$$

$$1 + \frac{(1+a)^2x}{a} + \frac{(1+a)^3(2a+1)x^2}{a^2} + \frac{(5a^2+5a+1)(1+a)^4x^3}{a^3} + \frac{(7a^2+7a+1)(1+a)^5(2a+1)x^4}{a^4} + O\left(x^5\right)$$

$$x\left(1+a\right)+\frac{\left(1+a\right)^2\left(2\,a+1\right)x^2}{a}+\frac{\left(5\,a^2+5\,a+1\right)\left(1+a\right)^3x^3}{a^2}+\frac{\left(7\,a^2+7\,a+1\right)\left(1+a\right)^4\left(2\,a+1\right)x^4}{a^3}+O\left(x^5\right)$$

System of 5 equations and 6 overlapping unknowns that results in a unique equation Eq8 (Equation (3.5)):

```
> EqA:
> Eq1:= collect(rhs(EqA),S,factor);
> Eq2:= collect(subs({a=-(-z+1+a)/(1+a)}, Eq1), S, factor);
> Eq3:= collect(subs({a=-(-z+1+a)/(1+a), z=(z-1)/a}, Eq1), S, factor);
> Eq4:= collect(subs({a=-(-z+1+a)/(a*z), z=(z-1)/a}, Eq1), S, factor);
> Eq5:= collect(subs({a=-(-z+1+a)/(a*z), z=(1+a)/a}, Eq1), S, factor);
> collect(Eq1-B*Eq2-C*Eq3-D*Eq4-E*Eq5, S, factor):
> {coeff( %, S(1,z) ),coeff( %, S(1-(-z+1+a)/(1+a),1-(-z+1+a)/(1+a)) ),
coeff(%, S(1,(z-1)/a) ), coeff( %, S(1-(-z+1+a)/(a*z),1-(-z+1+a)/(a*z)) )}:
> subs(solve(%, {B, C, D, E}), %%):
> Eq6:= collect(%, S, factor);
> Eq7:= collect(%*(-z+1+a)/((1+a)*z*x), S, factor);
> coeff(%, S(1, (1+a)/a)):
> factor(expand(radsimp(subs(z =sol, %), ratdenom)));
> Eq8:= collect(%*S(1,(1+a)/a)+S(1+a,1+a)+Eq7-(coeff(Eq7,S(1,(1+a)/a)) *
S(1, (1+a)/a)), S, factor);
> P:=Eq8-S(1+a,1+a)-coeff(Eq8,S(1, (1+a)/a))*S(1, (1+a)/a);
```

$$\mathit{Eq1}\ :=\ -\frac{x\left(1+a\right)zS\left(1,z\right)}{a}+\frac{x\left(1+a\right)zS\left(1+a,1+a\right)}{-z+1+a}+x\left(1+a\right)z$$

$$\mathit{Eq2}\ :=\ \frac{z^2xS\left(1,z\right)}{-z+1+a}-zxS\left(1-\frac{-z+1+a}{1+a},1-\frac{-z+1+a}{1+a}\right)a^{-1}+\frac{z^2x}{1+a}$$

$$\mathit{Eq3}\ :=\frac{zx\left(z-1\right)}{a\left(-z+1+a\right)}S\left(1,\frac{z-1}{a}\right)+\frac{zx\left(z-1\right)}{\left(-z+1+a\right)}S\left(1-\frac{-z+1+a}{1+a},1-\frac{-z+1+a}{1+a}\right)+\frac{zx\left(z-1\right)}{\left(1+a\right)a}$$

$$\mathit{Eq4}\ :=\frac{\left(z-1\right)^2\left(1+a\right)x}{a\left(-z+1+a\right)}S\left(1,\frac{z-1}{a}\right)+\frac{x\left(z-1\right)\left(1+a\right)}{a\left(-z+1+a\right)}S\left(1-\frac{-z+1+a}{az},1-\frac{-z+1+a}{az}\right)+\frac{\left(z-1\right)^2\left(1+a\right)x}{za^2}$$

$$Eq5 := \frac{(1+a)^2(z-1)x}{a(-z+1+a)} S\left(1, \frac{1+a}{a}\right) - \frac{x(z-1)(1+a)}{a} S\left(1 - \frac{-z+1+a}{az}, 1 - \frac{-z+1+a}{az}\right) + \frac{(1+a)^2(z-1)x}{za^2}$$

$$Eq6 := -\frac{(1+a)^2 x}{a^3(z-1)} S\left(1, \frac{1+a}{a}\right) + \frac{x(1+a)z}{-z+1+a} S(1+a, 1+a) + \frac{1}{(z-1)a^4}\left(x(1+a)(-za^4 + z^2a^4 - a^3z + z^2a^3 - 2a^2 - z^3a^2 + z^2a^2 + za^2 - 4a + 5az - 3az^2 + z^3a - z^2 + 3z - 2)\right)$$

$$Eq7 := -\frac{(-z+1+a)(1+a)}{za^3(z-1)} S\left(1, \frac{1+a}{a}\right) + S(1+a, 1+a) + \frac{(-z+1+a)}{z(z-1)a^4}\left(-za^4 + z^2a^4 - a^3z + z^2a^3 - 2a^2 - z^3a^2 + z^2a^2 + za^2 - 4a + 5az - 3az^2 + z^3a - z^2 + 3z - 2\right)$$

$$\frac{(1+a)^2 x}{a^4}$$

$$Eq8 := \frac{(1+a)^2 x}{a^4} S\left(1, \frac{1+a}{a}\right) + S(1+a, 1+a) + \frac{(-z+1+a)}{z(z-1)a^4}\left(-za^4 + z^2a^4 - a^3z + z^2a^3 - 2a^2 - z^3a^2 + z^2a^2 + za^2 - 4a + 5az - 3az^2 + z^3a - z^2 + 3z - 2\right)$$

$$P := \frac{1}{z(z-1)a^4}\left((-z+1+a)\left(-za^4 + z^2a^4 - a^3z + z^2a^3 - 2a^2 - z^3a^2 + z^2a^2 + za^2 - 4a + 5az - 3az^2 + z^3a - z^2 + 3z - 2\right)\right)$$

Substitution $z = Z_+$:

```
>  subs(z = sol, -P):
>  factor(series(%, x, 8)) assuming a > 0;
>  map(coeff, %, a, 0);
```

$$(1+a)^2 x + \frac{(2a^5+1)(1+a)^3}{a^5 x^2} + \frac{(5a^6 + a^5 - a^4 + a^3 - a^2 + a + 1)(1+a)^4}{a^6 x^3} + \frac{(14a^7 + 7a^6 - 5a^5 + 3a^4 - a^3 - a^2 + 3a + 1)(1+a)^5}{a^7 x^4} + O\left(x^5\right)$$

$$x + 2x^2 + 6x^3 + 23x^4 + O\left(x^5\right)$$

Further substitution to apply Lagrange inversion:

```
>  collect(subs(z=W+1+a,Kera),x,factor);
>  factor(subs(z=W+1+a,-P));
```

```
> collect(%/W*(W+1+a)*(1+a)*(W+a)*x/a,{x,W},expand);
> factor(subs(W = sol-1-a,%)):
> factor(series(%, x, 20)) assuming a > 0:
> map(coeff, %, a, 0);
```

$$1-\frac{(W+1+a)(1+a)(W+a)\;x}{aW}$$

$$-\frac{W\left(-a^5-a^6+W^2-a^4W^2-2\,a^5W+2\,a^3W^2+a^2W^3-a^2W^2-aW^3-W\right)}{(W+1+a)(W+a)\,a^4}$$

$$\left(-\frac{1}{a^2}+\frac{1}{a^4}\right)x\,W^3+\left(\frac{1}{a^3}-\frac{1}{a^2}-\frac{1}{a}+1-\frac{1}{a^5}-\frac{1}{a^4}\right)x\,W^2+\left(2+2a+\frac{1}{a^5}+\frac{1}{a^4}\right)x\,W$$
$$+(1+2a+a^2)\,x$$

$$x+2\,x^2+6\,x^3+23\,x^4+104\,x^5+530\,x^6+2958\,x^7+17734\,x^8+112657\,x^9+750726\,x^{10}+$$
$$5207910\,x^{11}+37387881\,x^{12}+276467208\,x^{13}+2097763554\,x^{14}+16282567502\,x^{15}+O\left(x^{16}\right)$$

A.2 Formulas for semi-Baxter numbers

Applying Langrange inversion formula:

```
> (2*binomial(n+j,j)+2*binomial(n+j,j-1)+binomial(n+j,j+4)+binomial(n+j,
j+5)) * binomial(n-1,j+1) + ( -binomial(n+2+j,j+2)+binomial(n+2+j,j+4)) * 3
* binomial(n-1,j+3) + (binomial(n+1+j,j)-binomial(n+1+j,j+1)-binomial(n+1+j,
j+2) + binomial(n+1+j,j+3) - binomial(n+1+j,j+4) - binomial(n+1+j,j+5)) * 2
* binomial(n-1,j+2):
> simplify(%):
> factor(expand(%)):
> %/(n-1)*binomial(n-1,j);
> F:= unapply(%,n,j):
```

$$\frac{(-n+1+j)(-n+j)(n+1+j)}{n\,(j+1)^2\,(n+1)\,(j+4)\,(j+3)^2\,(j+2)^2\,(j+5)\,(n-1)}\binom{n}{j}\binom{n+j}{j}\binom{n-1}{j}(-720$$
$$+792\,n-1524\,j-j^6+n^6j+j^6n-113\,n^4j+2\,n^3j^4-2\,n^5j^2-15\,n^5j+311\,n^2j^3$$
$$+965\,n^2j^2-2\,n^4j^3+16\,n^3j^3+9\,n^3j+1456\,n^2j-30\,n^4j^2+38\,n^3j^2+1710\,jn$$
$$+1481\,j^2n+160\,j^4n+660\,j^3n+20\,j^5n+3\,j^5n^2+49\,j^4n^2-135\,n^4-45\,n^3$$
$$-147\,j^4-27\,n^5-593\,j^3-19\,j^5+9\,n^6-1316\,j^2+846\,n^2)$$

```
> sb := n-> add(F(n,j),j=0..n-1);
> seq(sb(k),k=2..30);
```

2, 6, 23, 104, 530, 2958, 17734, 112657, 750726, 5207910, 37387881, 276467208, 2097763554, 16282567502, 128951419810, 1039752642231, 8520041699078, 70840843420234, 596860116487097, 5089815866230374, 43886435477701502, 382269003235832006, 3361054683237796748, 29808870409714471629, 266506375018970260798, 2400594944788679086246, 21775140746921451807813, 198809340676892441106504, 1826282268703405468306242

Applying the method of creative telescoping:

```
> with(SumTools[Hypergeometric]):
> Zeilberger(F(n,j),n,j,En):
> rec:= add( factor(coeff(%[1],En,k))*c(n+k),k=0..2);
> cert := factor(%[2]);
```

$$rec := -(n-1)\,nc(n) + \left(-11\,n^2 - 55\,n - 60\right)c(n+1) + (n+6)(5+n)\,c(n+2)$$

$$cert := \frac{j\,(n+1+j)}{(j+2)^2(j+4)(n+1)(j+1)^2(3+n)(-n-1+j)(n+2)(-n+j)(j+3)(n+4)}$$

$$\binom{n}{j}\binom{n+j}{j}\binom{n-1}{j}\left(-69120 - 8493\,j^4n^5 - 189504\,n + 22684\,n^7j - 214272\,j + 649\,j^2n^8 - 88\,n^{10} + 43041\,j^5n^2 - 11638\,j^4n^2 + 5854\,j^2n^7 + 24\,j^8 - 13012\,n^7 + 1008\,j^7 - 7024\,n^8 - 1286\,n^9 - 64\,j^3n^7 + 33248\,n^6 + 12096\,j^6 + 114491\,n^6j + 17460\,j^6n + 264656\,n^4j - 74382\,n^3j^4 - 3590\,n^5j^2 + 290827\,n^5j - 610362\,n^2j^3 - 1542088\,n^2j^2 - 217328\,n^4j^3 - 532024\,n^3j^3 - 387636\,n^3j - 1161024\,n^2j - 307195\,n^4j^2 - 1052152\,n^3j^2 - 945552\,jn - 916304\,j^2n + 163194\,j^4n - 160440\,j^3n + 90096\,j^5n + j^8n^3 + 7014\,n^3j^5 + 633\,n^2j^7 + 21838\,n^6j^2 - 37715\,n^4j^4 - 45610\,n^5j^3 - 11\,jn^{10} + 28\,j^2n^9 + 11\,j^3n^8 - 36\,j^4n^7 - 23\,j^5n^6 + 9\,j^6n^5 + 9\,j^7n^4 + 9\,j^8n^2 + 26\,j^8n - 906\,j^4n^6 + 1344\,j^7n + 2008\,jn^8 - 11\,jn^9 + 126\,n^3j^7 + 2217\,j^6n^3 + 241\,j^6n^4 - 282\,j^5n^5 - 538\,j^5n^4 + 465576\,n^4 + 400600\,n^3 + 157896\,j^4 + 222818\,n^5 + 9197\,j^6n^2 + 140832\,j^3 + 63792\,j^5 - 92256\,j^2 + 2784\,n^2 - 4371\,j^3n^6\right)$$

Check Equation (3.13):

```
> factor(expand(subs(c(n)=F(n,j),c(n+1)=F(n+1,j),c(n+2)=F(n+2,j),rec))):
```

```
> factor(expand( subs(j=j+1,cert)-cert)):
> factor(%%-%);
```

$$0$$

Check sum over j:

```
> factor(expand(simplify( F(n+1,n+1) ))) assuming n>0;
> factor(expand(simplify( F(n,n+1) ))) assuming n>0;
> factor(expand(simplify( F(n,n) ))) assuming n>0;
> factor(expand(simplify( subs(j=0,cert) ))) assuming n>0;
> factor(expand(simplify( subs(j=n+2,cert) ))) assuming n>0;
```

$$0$$
$$0$$
$$0$$
$$0$$
$$0$$

Shift $n \to n-2$:

```
> f{collect(subs(n=n-2,rec),c,factor),c(0)=0,c(1)=1};
> SB:= rectoproc(%,c(n),remember):
> seq(SB(k),k=0..20);
```

$$\left\{(n+4)(3+n)\,c(n) - (n-3)(n-2)\,c(n-2) + \left(-11\,n^2 - 11\,n + 6\right)c(n-1)\,,\; c(0)=0, c(1)=1\right\}$$

0, 1, 2, 6, 23, 104, 530, 2958, 17734, 112657, 750726, 5207910, 37387881, 276467208, 2097763554, 16282567502, 128951419810, 1039752642231, 8520041699078, 70840843420234, 596860116487097

Explict expressions of Theorem 3.4.3 and Proposition 3.4.4:

```
> 24 / (n-1) / n^2 / (n+1) / (n+2) * binomial(n,j+2) * binomial(n+2,j) *
binomial(n+j+2,j+1);
> A:= unapply(%,n,j):
> An:= n-> add(A(n,j),j=0..n+1):
> seq(An(k),k=2..10);
> 24 / (n-1) / n^2 / (n+1) / (n+2) * binomial(n,j+2) * binomial(n+1,j) *
binomial(n+j+2,j+3);
> A2:= unapply(%,n,j):
> A2n:= n-> add(A2(n,j),j=0..n+1):
> seq(A2n(k),k=2..10);
> 24 / (n-1) / n^2 / (n+1) / (n+2) * binomial(n+1,j+3) * binomial(n+2,j+1)
* binomial(n+j+3,j);
```

```
> A3:= unapply(%,n,j):
> A3n:= n-> add(A3(n,j),j=0..n+1):
> seq(A3n(k),k=2..10);
```

$$\frac{24}{(n-1)\,n^2\,(n+1)\,(n+2)}\binom{n}{j+2}\binom{n+2}{j}\binom{n+j+2}{j+1}$$

2, 6, 23, 104, 530, 2958, 17734, 112657, 750726

$$\frac{24}{(n-1)\,n^2\,(n+1)\,(n+2)}\binom{n}{j+2}\binom{n+1}{j}\binom{n+j+2}{j+3}$$

2, 6, 23, 104, 530, 2958, 17734, 112657, 750726

$$\frac{24}{(n-1)\,n^2\,(n+1)\,(n+2)}\binom{n+1}{j+3}\binom{n+2}{j+1}\binom{n+j+3}{j}$$

2, 6, 23, 104, 530, 2958, 17734, 112657, 750726

```
> 24 * ( (5*n^3-5*n+6)*binomial(n+1,j)^2*binomial(n+1+j,j) -   ((5*n^2+
15*n+18)*binomial(n,j)^2*binomial(n+j,j))) / (5*(n-1)) / (n^2*(n+2)^2) /
((n+3)^2*(n+4));
> Ap:= unapply(%, n, j):
> Apn:= n->add(Ap(n,j),j=0..n+1):
> seq(Apn(k),k=2..10);
```

$$\frac{24}{5}\,\frac{(5\,n^3-5\,n+6)\left(\binom{n+1}{j}\right)^2\binom{n+1+j}{j}-(5\,n^2+15\,n+18)\left(\binom{n}{j}\right)^2\binom{n+j}{j}}{(n-1)\,n^2\,(n+2)^2\,(3+n)^2\,(n+4)}$$

2, 6, 23, 104, 530, 2958, 17734, 112657, 750726

```
> Zeilberger(F(n,j),n,j,En)[1];
> Zeilberger(Ap(n,j),n,j,En)[1];
> Zeilberger(A(n,j),n,j,En)[1];
> Zeilberger(A2(n,j),n,j,En)[1];
> Zeilberger(A3(n,j),n,j,En)[1];
```

$$En^2\left(n^2+11\,n+30\right)+\left(-11\,n^2-55\,n-60\right)En-n^2+n$$
$$\left(-n^2-11\,n-30\right)En^2+\left(11\,n^2+55\,n+60\right)En+n^2-n$$
$$\left(-n^2-11\,n-30\right)En^2+\left(11\,n^2+55\,n+60\right)En+n^2-n$$
$$\left(-n^2-11\,n-30\right)En^2+\left(11\,n^2+55\,n+60\right)En+n^2-n$$

$$\left(-n^2 - 11\,n - 30\right) En^2 + \left(11\,n^2 + 55\,n + 60\right) En + n^2 - n$$

A.3 Asymptotics of the semi-Baxter numbers

```
> unapply(A(n,j),j);
> Ratio:=simplify(expand(%(j)/%(j+1)));
> simplify(diff((j+2)/(n+2-j),j));
> simplify(diff((j+1)/(n-2-j),j));
> simplify(diff((j+3)/(n+j+3),j));
```

$$j \mapsto \frac{24}{(n-1)\,n^2\,(n+1)\,(n+2)} \binom{n}{j+2} \binom{n+2}{j} \binom{n+j+2}{j+1}$$

$$Ratio := \frac{(j+2)\,(j+1)\,(j+3)}{(-n-2+j)\,(-n+j+2)\,(n+j+3)}$$

$$\frac{n+4}{(-n-2+j)^2}$$

$$\frac{n-1}{(-n+j+2)^2}$$

$$\frac{n}{(n+j+3)^2}$$

```
> solve(Ratio=1,j):
> series(%[1],n=infinity,5);
```

$$\left(1/2\,\sqrt{5} - 1/2\right) n - 3/2 + 3/10\,\sqrt{5} + \frac{-3 + \frac{4}{25}\sqrt{5}}{n} + \frac{9 - \frac{162}{125}\sqrt{5}}{n^2} + O\left(n^{-3}\right)$$

Expansions of the logarithms involved in the proof of Lemma 3.5.5:

```
> log(1/(1-s/(1-phi)/(sqrt(n))-2/(1-phi)/n));
> gdev(%,n=infinity,2):
> map(simplify,%):
> map(X -> factor(expand(radsimp(X,ratdenom))), %);
> collect(simplify(convert(%,polynom)*n),n,factor);
```

$$\ln\left(\left(1 - \frac{s}{(1-\phi)\sqrt{n}} - 2\,\frac{1}{(1-\phi)\,n}\right)^{-1}\right)$$

$$-\frac{s}{\sqrt{n}\,(-1+\phi)} - 1/2\,\frac{-s^2 - 4 + 4\,\phi}{(-1+\phi)^2\,n} + O\left(n^{-3/2}\right)$$

$$-\frac{s\sqrt{n}}{-1+\phi} - 1/2\,\frac{-s^2-4+4\,\phi}{(-1+\phi)^2}$$

```
> log((1+s/sqrt(n)/phi+2/phi/n)/(1-(s/sqrt(n)/(1-phi))-2/(1-phi)/n));
> gdev(%,n=infinity,2):
> map(simplify,%):
> map(X -> factor(expand(radsimp(X,ratdenom))), %);
> collect(simplify(convert(%,polynom)*(n*phi+s*sqrt(n)+2)),n,factor);
```

$$\ln\left(\left(1+\frac{s}{\sqrt{n}\phi}+2\,\frac{1}{\phi\,n}\right)\left(1-\frac{s}{(1-\phi)\sqrt{n}}-2\,\frac{1}{(1-\phi)\,n}\right)^{-1}\right)$$

$$-\frac{s}{\phi\,(-1+\phi)\sqrt{n}} - 1/2\,\frac{s^2-4\,\phi+4\,\phi^2-2\,s^2\phi}{\phi^2\,(-1+\phi)^2\,n} + O\left(n^{-3/2}\right)$$

$$-\frac{s\sqrt{n}}{-1+\phi} - 1/2\,\frac{-s^2-4\,\phi+4\,\phi^2}{\phi\,(-1+\phi)^2} - 1/2\,\frac{s\,(s^2-8\,\phi+8\,\phi^2-2\,s^2\phi)}{\phi^2\,(-1+\phi)^2\sqrt{n}} - \frac{s^2-4\,\phi+4\,\phi^2-2\,s^2\phi}{\phi^2\,(-1+\phi)^2\,n}$$

```
> log((1+2/n)/(1-s/(1-phi)/(sqrt(n))+2/(1-phi)/n));
> gdev(%,n=infinity,2):
> map(simplify,%):
> map(X -> factor(expand(radsimp(X,ratdenom))), %);
> collect(simplify(convert(%,polynom)*(n+2)),n,factor);
```

$$\ln\left(\left(1+2\,n^{-1}\right)\left(1-\frac{s}{(1-\phi)\sqrt{n}}+2\,\frac{1}{(1-\phi)\,n}\right)^{-1}\right)$$

$$-\frac{s}{\sqrt{n}\,(-1+\phi)} + 1/2\,\frac{s^2-4\,\phi+4\,\phi^2}{(-1+\phi)^2\,n} + O\left(n^{-3/2}\right)$$

$$-\frac{s\sqrt{n}}{-1+\phi} + 1/2\,\frac{s^2-4\,\phi+4\,\phi^2}{(-1+\phi)^2} - 2\,\frac{s}{\sqrt{n}\,(-1+\phi)} + \frac{s^2-4\,\phi+4\,\phi^2}{(-1+\phi)^2\,n}$$

```
> log((1+s/sqrt(n)/phi)/(1-(s/sqrt(n)/(1-phi))+2/(1-phi)/n));
> gdev(%,n=infinity,2):
> map(simplify,%):
> map(X -> factor(expand(radsimp(X,ratdenom))), %);
> collect(simplify(convert(%,polynom)*(n*phi+s*sqrt(n))),n,factor);
```

$$\ln\left(\left(1+\frac{s}{\sqrt{n}\phi}\right)\left(1-\frac{s}{(1-\phi)\sqrt{n}}+2\,\frac{1}{(1-\phi)\,n}\right)^{-1}\right)$$

$$-\frac{s}{\phi\,(-1+\phi)\sqrt{n}} + 1/2\,\frac{-s^2+2\,s^2\phi-4\,\phi^2+4\,\phi^3}{\phi^2\,(-1+\phi)^2\,n} + O\left(n^{-3/2}\right)$$

$$-\frac{s\sqrt{n}}{-1+\phi}+1/2\,\frac{s^2-4\,\phi^2+4\,\phi^3}{\phi\,\left(-1+\phi\right)^2}+1/2\,\frac{\left(-s^2+2\,s^2\phi-4\,\phi^2+4\,\phi^3\right)s}{\phi^2\left(-1+\phi\right)^2\sqrt{n}}$$

```
> log((1+s/sqrt(n)/(1+phi)+2/(1+phi)/n)/(1+(s/sqrt(n)/phi)+1/phi/n));
> gdev(%,n=infinity,2):
> map(simplify,%):
> map(X -> factor(expand(radsimp(X,ratdenom))), %);
> collect(simplify(convert(%,polynom)*(n*phi+s*sqrt(n)+1)),n,factor);
```

$$\ln\left(\left(1+\frac{s}{\sqrt{n}\,(1+\phi)}+2\,\frac{1}{(1+\phi)\,n}\right)\left(1+\frac{s}{\sqrt{n}\phi}+\frac{1}{\phi\,n}\right)^{-1}\right)$$

$$-\frac{s}{(1+\phi)\,\phi\,\sqrt{n}}+1/2\,\frac{s^2+2\,\phi^3+2\,s^2\phi-2\,\phi}{\left(1+\phi\right)^2\phi^2 n}+O\left(n^{-3/2}\right)$$

$$-\frac{s\sqrt{n}}{1+\phi}+1/2\,\frac{-s^2+2\,\phi^3-2\,\phi}{\phi\,\left(1+\phi\right)^2}+1/2\,\frac{s\left(s^2+2\,\phi^3+2\,s^2\phi-4\,\phi-2\,\phi^2\right)}{\left(1+\phi\right)^2\phi^2\sqrt{n}}$$
$$+1/2\,\frac{s^2+2\,\phi^3+2\,s^2\phi-2\,\phi}{\left(1+\phi\right)^2\phi^2 n}$$

```
> log((1+1/n)/(1+(s/sqrt(n)/(1+phi)+2/(1+phi)/n)));
> gdev(%,n=infinity,2):
> map(simplify,%):
> map(X -> factor(expand(radsimp(X,ratdenom))), %);
> collect(simplify(convert(%,polynom)*(n+1)),n,factor);
```

$$\ln\left(\left(1+n^{-1}\right)\left(1+\frac{s}{\sqrt{n}\,(1+\phi)}+2\,\frac{1}{(1+\phi)\,n}\right)^{-1}\right)$$

$$-\frac{s}{\sqrt{n}\,(1+\phi)}+1/2\,\frac{s^2-2+2\,\phi^2}{\left(1+\phi\right)^2 n}+O\left(n^{-3/2}\right)$$

$$-\frac{s\sqrt{n}}{1+\phi}+1/2\,\frac{s^2-2+2\,\phi^2}{\left(1+\phi\right)^2}-\frac{s}{\sqrt{n}\,(1+\phi)}+1/2\,\frac{s^2-2+2\,\phi^2}{\left(1+\phi\right)^2 n}$$

```
> Digits := 20:
> phi := (sqrt(5)-1)/2:
> evalf(12/Pi*5^(-1/4)*phi^(-15/2));
```

$$94.340065653208596487$$

Appendix B

Inversion sequences $\mathbf{I}(\geq, \geq, \geq)$

B.1 Generating function of $\mathbf{I}(\geq, \geq, \geq)$

The functional equation associated with $\Omega_{\mathbf{I}(\geq,\geq,\geq)}$:

```
> eqA:=-A+x*y*z+x*z/(1-y)*(A1-A)+x*y*z/(z-y)*(A-Ay);
```

$$eqA := -A + xyz + \frac{xz\,(A1 - A)}{1-y} + \frac{xyz\,(A - Ay)}{z-y}$$

```
> Aser:=proc(n)
> if n=0 then x*y*z
> else normal(series(subs(A=Aser(n-1),A1=subs(y=1,Aser(n-1)),Ay=subs(z=y,
Aser(n-1)),eqA+A),x,n+1)): fi: end:
```

```
> Aser(4);
```

$$xyz + z\left(z + y^2\right)x^2 + z\left(y^3 + y^2 + 2\,yz + z\right)x^3 + z(y^4 + 3\,y^3 + y^2 + 3\,zy^2 + 3\,yz$$
$$+2\,z^2 + 2\,z\,)\,x^4 + O\left(x^5\right)$$

```
> ser:=subs(y=1,z=1,Aser(14));
```

$$ser := x + 2\,x^2 + 5\,x^3 + 15\,x^4 + 51\,x^5 + 191\,x^6 + 772\,x^7 + 3320\,x^8 + 15032\,x^9$$
$$+71084\,x^{10} + 348889\,x^{11} + 1768483\,x^{12} + 9220655\,x^{13} + 49286863\,x^{14} + O\left(x^{15}\right)$$

The kernel and its symmetry transformations, setting $y = 1 + a$:

```
> coeff(eqA,A);
> Ker:=(subs(y=1+a,-%));
> simplify(factor(subs(z=Z,Ker))):
> solve(Ker/%=1,Z);
> simplify(factor(subs(a=A,Ker))):
> solve(Ker/%=1,A);
```

$$-1 - \frac{xz}{1-y} + \frac{xyz}{z-y}$$

$$Ker := 1 - \frac{xz}{a} - \frac{x\,(1+a)\,z}{z-1-a}$$

$$z,\ -\frac{-1+a^2-a+a^3+z+az}{-z+1+a}$$

$$a,\ -\frac{-z+1+a}{1+a}$$

```
>  [a,z];
>  [factor(-(1+%[1]-%[2])/(1+%[1])),%[2]];
>  [%[1],factor(-(-1+%[1]^2+%[1]^3+%[2]*%[1]-%[1]+%[2])/(1+%[1]-%[2]))];
>  [factor(-(1+%[1]-%[2])/(1+%[1])),%[2]];
>  [%[1],factor(-(-1+%[1]^2+%[1]^3+%[2]*%[1]-%[1]+%[2])/(1+%[1]-%[2]))];
>  [factor(-(1+%[1]-%[2])/(1+%[1])),%[2]];
>  [%[1],factor(-(-1+%[1]^2+%[1]^3+%[2]*%[1]-%[1]+%[2])/(1+%[1]-%[2]))];
>  [factor(-(1+%[1]-%[2])/(1+%[1])),%[2]];
>  [%[1],factor(-(-1+%[1]^2+%[1]^3+%[2]*%[1]-%[1]+%[2])/(1+%[1]-%[2]))];
>  [factor(-(1+%[1]-%[2])/(1+%[1])),%[2]];
>  [%[1],factor(-(-1+%[1]^2+%[1]^3+%[2]*%[1]-%[1]+%[2])/(1+%[1]-%[2]))];
>  [factor(-(1+%[1]-%[2])/(1+%[1])),%[2]];
>  [%[1],factor(-(-1+%[1]^2+%[1]^3+%[2]*%[1]-%[1]+%[2])/(1+%[1]-%[2]))];
```

$$[a, z]$$

$$\left[-\frac{-z+1+a}{1+a}, z\right]$$

$$\left[-\frac{-z+1+a}{1+a}, \frac{z\,(a^2-1+z)}{a\,(1+a)^2}\right]$$

$$\left[-\frac{-z+1+a}{(1+a)\,a}, \frac{z\,(a^2-1+z)}{a\,(1+a)^2}\right]$$

$$\left[-\frac{-z+1+a}{(1+a)\,a}, \frac{a^2-1+z}{a^2}\right]$$

$$\left[\frac{1}{a}, \frac{a^2-1+z}{a^2}\right]$$

$$\left[\frac{1}{a}, -\frac{z\,(1+a)}{a\,(-z+1+a)}\right]$$

$$\left[-\frac{1+a}{-z+1+a}, -\frac{z\,(1+a)}{a\,(-z+1+a)}\right]$$

$$\left[-\frac{1+a}{-z+1+a}, \frac{z\,(a^2-1+z)}{(-z+1+a)^2}\right]$$

$$\left[-\frac{(1+a)\,a}{-z+1+a},\frac{z\,(a^2-1+z)}{(-z+1+a)^2}\right]$$

$$\left[-\frac{(1+a)\,a}{-z+1+a},-\frac{(a^2-1+z)\,(1+a)}{-z+1+a}\right]$$

$$\left[a,-\frac{(a^2-1+z)\,(1+a)}{-z+1+a}\right]$$

$$[a,z]$$

The kernel root Z_+:

```
> solve(Ker,z):
> map(factor,series(%[1],x,5) assuming a>0);
> map(factor,series(%%[2],x,5) assuming a>0);
> Sol:=%%%[2];
> series(subs(a=1,Sol),x,8);
```

$$\frac{a}{x}-(1+a)\,a-(1+a)^2\,x-\frac{(1+a)^4\,x^2}{a}-\frac{(a^2+3\,a+1)\,(1+a)^4\,x^3}{a^2}+O\left(x^4\right)$$

$$1+a+(1+a)^2\,x+\frac{(1+a)^4\,x^2}{a}+\frac{(a^2+3\,a+1)\,(1+a)^4\,x^3}{a^2}+O\left(x^4\right)$$

$$Sol\;:=\;-1/2\,\frac{-a-x+xa^2+\sqrt{a^2-2\,xa-2\,xa^3+x^2-2\,x^2a^2+x^2a^4-4\,xa^2}}{x}$$

$$2+4\,x+16\,x^2+80\,x^3+448\,x^4+2688\,x^5+16896\,x^6+O\left(x^7\right)$$

Check which substitutions are well-defined power series in x if $z=Z_+$:

```
> map(factor,series(subs( z=Sol, -(1+a-z)/(1+a)),x,5) assuming a>0);
> map(factor,series(subs( z=Sol, z*(-1+z+a^2)/((1+a)^2*a)),x,5) assuming
a>0);
> map(factor,series(subs( z=Sol, -(1+a-z)/(a*(1+a))),x,5) assuming a>0);
> map(factor,series(subs( z=Sol, (-1+z+a^2)/a^2),x,5) assuming a>0);
> map(factor,series(subs( z=Sol, -z*(1+a)/(a*(1+a-z)) ),x,5)   assuming
a>0);
> map(factor,series(subs( z=Sol, -(1+a)/(1+a-z)),x,5) assuming a>0);
> map(factor,series(subs( z=Sol, z*(-1+z+a^2)/(1+a-z)^2),x,5)  assuming
a>0);
> map(factor,series(subs( z=Sol, -a*(1+a)/(1+a-z)),x,5) assuming a>0);
> map(factor,series(subs( z=Sol, -(-1+z+a^2)*(1+a)/(1+a-z)),x,5) assuming
a>0);
```

$$x\,(1+a)+\frac{(1+a)^3\,x^2}{a}+\frac{(a^2+3\,a+1)\,(1+a)^3\,x^3}{a^2}+O\left(x^4\right)$$

$$1+\frac{(1+a)^2\,x}{a}+\frac{(a^2+3\,a+1)\,(1+a)^2\,x^2}{a^2}+\frac{(a^2+5\,a+1)\,(1+a)^4\,x^3}{a^3}+O\left(x^4\right)$$

$$\frac{x(1+a)}{a}+\frac{(1+a)^3x^2}{a^2}+\frac{(a^2+3a+1)(1+a)^3x^3}{a^3}+O\left(x^4\right)$$

$$\frac{1+a}{a}+\frac{(1+a)^2x}{a^2}+\frac{(1+a)^4x^2}{a^3}+\frac{(a^2+3a+1)(1+a)^4x^3}{a^4}+O\left(x^4\right)$$

$$\frac{1}{xa}-\frac{1+a}{a^2}-\frac{(1+a)^2x}{a^2}+O\left(x^2\right)$$

$$\frac{1}{x(1+a)}-\frac{1+a}{a}-\frac{x(1+a)}{a}+O\left(x^2\right)$$

$$\frac{a}{(1+a)^2x^2}-x^{-1}-1-\frac{(1+a)^2x}{a}+O\left(x^2\right)$$

$$\frac{a}{x(1+a)}-1-a+(-1-a)x+O\left(x^2\right)$$

$$\frac{a}{x}-(1+a)a-(1+a)^2x+O\left(x^2\right)$$

System of 6 equations and 7 overlapping unknowns that results in a unique equation Eq7 (Equation (5.5):

```
> eqA;
> eq1:= x*(1+a)*z + factor( subs(y=(1+a),coeff(eqA,Ay)) ) * A(1+a,1+a) +
factor( subs(y=1+a,coeff(eqA,A1)) ) * A(1,z);
```

$$-A+xyz+\frac{xz(A1-A)}{1-y}+\frac{xyz(A-Ay)}{z-y}$$

$$eq1 := x(1+a)z+\frac{x(1+a)zA(1+a,1+a)}{-z+1+a}-\frac{xzA(1,z)}{a}$$

```
> eq1:= collect( eq1, A, factor);
> eq2:= collect( subs(a=(z-1-a)/(1+a), eq1), A, factor);
> eq3:= collect( subs({a=(z-(1+a))/(1+a), z=z*(-1+z+a^2)/((1+a)^2*a) },
eq1), A,factor);
> eq4:= collect( subs({a=-(1+a-z)/(a*(1+a)), z=z*(-1+z+a^2)/((1+a)^2*a)},
eq1), A, factor);
> eq5:= collect( subs({a=-(1+a-z)/(a*(1+a)), z=(-1+z+a^2)/a^2}, eq1), A,
factor);
> eq6:= collect( subs({a=1/a,z=(-1+z+a^2)/a^2},eq1), A, factor);
```

$$eq1 := x(1+a)z+\frac{x(1+a)zA(1+a,1+a)}{-z+1+a}-\frac{xzA(1,z)}{a}$$

$$eq2 := \frac{x(1+a)zA(1,z)}{-z+1+a}-xzA\left(1+\frac{z-1-a}{1+a},1+\frac{z-1-a}{1+a}\right)a^{-1}+\frac{xz^2}{1+a}$$

$$eq3 := xz\left(a^2-1+z\right)A\left(1,\frac{z\left(a^2-1+z\right)}{a\left(1+a\right)^2}\right)(-z+1+a)^{-1}a^{-1}(1+a)^{-1}+\left(a^2-1+z\right)xz\,A\left(1+\frac{z-1-a}{1+a},1+\frac{z-1-a}{1+a}\right)(1+a)^{-1}(-z+1+a)^{-1}+\frac{\left(a^2-1+z\right)xz^2}{(1+a)^3a}$$

$$eq4 := xz\left(a^2-1+z\right)A\left(1,\frac{z\left(a^2-1+z\right)}{a\left(1+a\right)^2}\right)(1+a)^{-1}(-z+1+a)^{-1}+xz\left(a^2-1+z\right)A\left(1-\frac{-z+1+a}{(1+a)a},1-\frac{-z+1+a}{(1+a)a}\right)(-z+1+a)^{-1}a^{-1}(1+a)^{-1}+\frac{z\left(a^2-1+z\right)^2x}{a^2(1+a)^3}$$

$$eq5 := x\left(a^2-1+z\right)(1+a)A\left(1,\frac{a^2-1+z}{a^2}\right)a^{-1}(-z+1+a)^{-1}-x\left(a^2-1+z\right)A\left(1-\frac{-z+1+a}{(1+a)a},1-\frac{-z+1+a}{(1+a)a}\right)a^{-1}+\frac{\left(a^2-1+z\right)^2x}{(1+a)a^3}$$

$$eq6 := -x\left(a^2-1+z\right)A\left(1,\frac{a^2-1+z}{a^2}\right)a^{-1}+\frac{x\left(a^2-1+z\right)(1+a)}{a(-z+1+a)}A\left(1+\frac{1}{a},1+\frac{1}{a}\right)+\frac{x\left(a^2-1+z\right)(1+a)}{a^3}$$

```
> collect(eq1-B*eq2-C*eq3-D*eq4-E*eq5-F*eq6,A,factor):
> { coeff( %,A(1,z)), coeff( %,A(1+(z-(1+a))/(1+a),1+(z-(1+a))/(1+a))),
coeff( %,A(1, z*(-1+z+a^2)/((1+a)^2*a))), coeff( %,A(1-(1+a-z)/(a*(1+a)),
1-(1+a-z)/(a*(1+a)))), coeff( %,A(1,(-1+z+a^2)/a^2))}:
> map(factor,subs(solve(%,{B,C,D,E,F}),%%)):
> Eq7:= collect(%*(-z+1+a)*a^6/x/z*(1+a)^2/a^6,A,factor);
> expand(coeff(%,A((1+a)/a, (1+a)/a)));
> P:= factor( -Eq7 + coeff(Eq7,A((1+a)/a,(1+a)/a)) * A((1+a)/a,(1+a)/a)
+ coeff(Eq7,A(1+a,1+a)) * A(1+a,1+a));
```

$$Eq7 := -(1+a)^3A\left(\frac{1+a}{a},\frac{1+a}{a}\right)a^{-4}+(1+a)^3A(1+a,1+a)+\frac{(a-1)(-z+1+a)}{a^6}\left(a^8+4a^7+7a^6+a^5z+8a^5-z^2a^4+2za^4+8a^4+3za^3+7a^3-a^3z^2+4a^2+6a^2z-3z^2a^2+a+az^3-4z^2a+6az+2z-z^2\right)$$

$$-\frac{1}{a^4}-3\frac{1}{a^3}-3\frac{1}{a^2}-\frac{1}{a^1}$$

$$P := -\frac{(a-1)(-z+1+a)}{a^6}\left(a^8+4a^7+7a^6+a^5z+8a^5-z^2a^4+2za^4+8a^4+3za^3+7a^3-a^3z^2+4a^2+6a^2z-3z^2a^2+a+az^3-4z^2a+6az+2z-z^2\right)$$

Substitution $z = Z_+$:

```
> factor(series(subs(z=Sol,P),x,18)) assuming a>0:
> map(coeff,%,a,0);
```

$$x+2\,x^2+5\,x^3+15\,x^4+51\,x^5+191\,x^6+772\,x^7+3320\,x^8+15032\,x^9+71084\,x^{10}$$
$$+348889\,x^{11}+1768483\,x^{12}+9220655\,x^{13}+49286863\,x^{14}+O\left(x^{15}\right)$$

Further substitution to apply Lagrange inversion:

```
> Ker;
> KerW:=collect(subs({z=W+(1+a)},%),x,factor);
```

$$1-\frac{xz}{a}-\frac{x\,(1+a)\,z}{z-1-a}$$

$$KerW\;:=\;1-\frac{(W+1+a)\,(W+a+a^2)\,x}{aW}$$

```
> Q:=collect(simplify(subs(z=W+(1+a),P)),W,factor);
> factor(subs(W=Sol-(1+a),%)):
> factor(series(%, x, 14)) assuming a > 0:
> map(coeff,%, a, 0);
> a1:=expand(coeff(Q,W));
> a2:=expand(coeff(Q,W^2));
> a3:=expand(coeff(Q,W^3));
> a4:=expand(coeff(Q,W^4));
```

$$Q\;:=\;\frac{(a-1)\,W^4}{a^5}-\frac{(a-1)\,(a^2-a+1)\,(1+a)^2\,W^3}{a^6}-\frac{(a-1)\,(a^2+1)\,(1+a)^2\,W^2}{a^5}$$
$$+\frac{(a-1)\,(a^2+a+1)\,(a^2-a+1)\,(1+a)^4\,W}{a^6}$$

$$x+2\,x^2+5\,x^3+15\,x^4+51\,x^5+191\,x^6+772\,x^7+3320\,x^8+15032\,x^9+O\left(x^{10}\right)$$

$$a1\;:=\;-\frac{1}{a^6}-3\,\frac{1}{a^5}+a^3-3\,\frac{1}{a^4}-\frac{1}{a^3}+1+3\,a^2+3\,a$$

$$a2\;:=\;-\frac{1}{a}-1+\frac{1}{a^4}+\frac{1}{a^5}$$

$$a3\;:=\;\frac{1}{a^3}-\frac{1}{a}-\frac{1}{a^4}+\frac{1}{a^6}$$

$$a4\;:=\;\frac{1}{a^4}-\frac{1}{a^5}$$

B.2 Formulas

Applying Lagrange inversion formula:

```
> a1:= binomial(n,k+1)*(-binomial(n+1,k-4)-3*binomial(n+2,k-2)-binomial(
n+1,k-1)+binomial(n+1,k+2)+3*binomial(n+2,k+4)+binomial(n+1,k+5));
> a2:= binomial(n,k+2)*(binomial(n+3,k)-binomial(n+3,k+4));
```

```
> a3:= binomial(n,k+3)*(binomial(n+3,k)-binomial(n+3,k+2)+binomial(n+3,
k+3)-binomial(n+3,k+5));
> a4:= binomial(n,k+4)*(-binomial(n+4,k+3)+binomial(n+4,k+4));
> aa:= a1+2*a2+3*a3+4*a4:
```

$$a1 := \binom{n}{k+1}\left(-\binom{n+1}{k-4} - 3\binom{n+2}{k-2} - \binom{n+1}{k-1} + \binom{n+1}{k+2} + 3\binom{n+2}{k+4} + \binom{n+1}{k+5}\right)$$

$$a2 := \binom{n}{k+2}\left(\binom{n+3}{k} - \binom{n+3}{k+4}\right)$$

$$a3 := \binom{n}{k+3}\left(\binom{n+3}{k} - \binom{n+3}{k+2} + \binom{n+3}{k+3} - \binom{n+3}{k+5}\right)$$

$$a4 := \binom{n}{k+4}\left(-\binom{n+4}{k+3} + \binom{n+4}{k+4}\right)$$

```
> an:= binomial(n,k)*aa/n;
> seq(sum(an,k=0..n),n=1..29);
```

$$an := \frac{1}{n}\binom{n}{k}\binom{n}{k+1}\left(-\binom{n+1}{k-4} - 3\binom{n+2}{k-2} - \binom{n+1}{k-1} + \binom{n+1}{k+2} + 3\binom{n+2}{k+4} + \binom{n+1}{k+5}\right) + 2\binom{n}{k+2}\left(\binom{n+3}{k} - \binom{n+3}{k+4}\right) + 3\binom{n}{k+3}\left(\binom{n+3}{k} - \binom{n+3}{k+2} + \binom{n+3}{k+3} - \binom{n+3}{k+5}\right) + 4\binom{n}{k+4}\left(-\binom{n+4}{k+3} + \binom{n+4}{k+4}\right)$$

1, 2, 5, 15, 51, 191, 772, 3320, 15032, 71084, 348889, 1768483, 9220655, 49286863, 269346822, 1501400222, 8519796094, 49133373040, 287544553912, 1705548000296, 10241669069576, 62201517142632, 381749896129920, 2365758616886432, 14793705539872672, 93289069357036472, 592912570551842369, 3796109485501600235, 24472444947142838215

Applying the method of creative telescoping:

```
> with(SumTools[Hypergeometric]):
> In:= unapply(an,n,k):
> collect(Zeilberger(In(n,k),n,k,En)[1],En,factor);
> recA:=add( factor(coeff(%,En,k))*c(n+k),k=0..4);
> cert:=Zeilberger(In(n,k),n,k,En)[2]:
```

$$-(n+9)(n+8)(n+6)\,En^3 + \left(6n^3 + 464n + 776 + 92n^2\right)En^2 + (n+2)\left(15n^2 + 133n + 280\right)En + 8\,(n+3)(n+2)(n+1)$$

$$recA := 8\,(n+3)(n+2)(n+1)\,c(n) + (n+2)\left(15n^2 + 133n + 280\right)c(n+1) + \left(6n^3 + 464n + 776 + 92n^2\right)c(n+2) - (n+9)(n+8)(n+6)\,c(n+3)$$

Check Equation (5.9):

```
> factor(expand( subs( c(n)=In(n,k), c(n+1)=In(n+1,k), c(n+2)=In(n+2,k),
c(n+3)=In(n+3,k), recA) )):
> factor(expand( subs(k=k+1,cert)-cert ) ):
> factor(%%-%);
> factor(expand(subs(k=0,cert)));
> factor(expand(subs(k=n+9,cert)));
```

$$0$$

$$0$$

$$0$$

Proof of Proposition 5.1.15:

```
> cros:=8*(n+3)*(n+1)*a(n)+(7*n^2+53*n+88)*a(n+1)-(n+8)*(n+7)*a(n+2);
> collect(cros*(n+2),a,factor):
> collect(subs(n=n+1,cros)*(n+6),a,factor):
> collect(%+%%,a,factor);
> subs(a=c,%)-recA;
```

$$cros := 8\,(n+3)\,(n+1)\,a\,(n) + \left(7\,n^2+53\,n+88\right) a\,(n+1) - (n+8)\,(n+7)\,a\,(n+2)$$

$$8\,(n+3)\,(n+1)\,(n+2)\,a\,(n) + (n+2)\left(15\,n^2+133\,n+280\right) a\,(n+1) \\ + \left(6\,n^3+464\,n+776+92\,n^2\right) a\,(n+2) - (n+9)\,(n+8)\,a\,(n+3)\,(n+6)$$

$$0$$

List of Figures

List of Tables

Bibliography

[1] E. Ackerman, G. Barequet, and R. Y. Pinter. "A bijection between permutations and floorplans, and its applications". In: *Discrete Applied Mathematics* 154.12 (2006), pp. 1674–1684.

[2] M. H. Albert, M. Elder, A. Rechnitzer, P. Westcott, and M. Zabrocki. "On the Stanley-Wilf limit of 4231-avoiding permutations and a conjecture of Arratia". In: *Advances in Applied Mathematics* 36.2 (2006), pp. 96–105.

[3] M. H. Albert, C. Homberger, J. Pantone, N. Shar, and V. Vatter. "Generating permutations with restricted containers". preprint, `https://arxiv.org/abs/1510.00269`, to appear in Journal of Combinatorial Theory, Series A.

[4] J. C. Aval, A. Boussicault, M. Bouvel, O. Guibert, and M. Silimbani. "Baxter Tree-like tableaux". in preparation.

[5] D. Avis and M. Newborn. "On pop-stacks in series". In: *Utilitas Math.* 19 (1981), pp. 129–140.

[6] E. Babson and E. Steingrímsson. "Generalized permutation patterns and a classification of the Mahonian statistics". In: *Séminaire Lotharingien de Combinatoire* 44.B44b (2000).

[7] J. Backelin, J. West, and G. Xin. "Wilf-equivalence for singleton classes". In: *Advances in Applied Mathematics* 38 (2007), pp. 133–148.

[8] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, and D. Gouyou-Beauchamps. "Generating functions for generating trees". In: *Discrete Mathematics* 246 (2002), pp. 29–55.

[9] C. Banderier and M. Drmota. "Formulae and Asymptotics for Coefficients of Algebraic Functions". In: *Combinatorics, Probability and Computing* 24.1 (2015), pp. 1–53.

[10] C. Banderier and P. Flajolet. "Basic analytic combinatorics of directed lattice paths". In: *Theoretical Computer Science* 281 (2002), pp. 37–80.

[11] E. Barcucci, A. Del Lungo, E. Pergola, and R. Pinzani. "A construction for enumerating k-coloured Motzkin paths". In: *Computing and Combinatorics. COCOON 1995. Lecture Notes in Computer Science*. Vol. 959. Springer, 1995, pp. 254–263.

[12] E. Barcucci, A. Del Lungo, E. Pergola, and R. Pinzani. "A methodology for plane trees enumeration". In: *Discrete Mathematics* 180 (1998), pp. 45–64.

[13] E. Barcucci, A. Del Lungo, E. Pergola, and R. Pinzani. "ECO: a methodology for the enumeration of combinatorial objects". In: *Journal of Difference Equations and Applications* 5 (1999), pp. 435–490.

[14] E. Barcucci, A. Del Lungo, E. Pergola, and R. Pinzani. "Random generation of trees and other combinatorial objects". In: *Theoretical Computer Science* 218 (1999), pp. 219–232.

[15] E. Barcucci, A. Del Lungo, E. Pergola, and R. Pinzani. "Some combinatorial interpretations of q-Analogs of Schröder numbers". In: *Annals of Combinatorics* 3 (1999), pp. 171–190.

[16] E. Barcucci, E. Pergola, R. Pinzani, and S. Rinaldi. "A Bijection for Some Paths on the Slit Plane". In: *Advances in Applied Mathematics* 26.2 (2001), pp. 89–96.

[17] E. Barcucci, E. Pergola, R. Pinzani, and S. Rinaldi. "ECO method and hill-free generalized Motzkin paths". In: *Séminaire Lotharingien de Combinatoire* 46.B46b (2001).

[18] A. M. Baxter and L. Pudwell. "Enumeration schemes for vincular patterns". In: *Discrete Mathematics* 312 (2012), pp. 1699–1712.

[19] A. M. Baxter and M. Shattuck. "Some Wilf-equivalences for vincular patterns". In: *Journal of Combinatorics* 6 (2015), pp. 19–45.

[20] G. Baxter. "On fixed points of the composite of commuting functions". In: *Proceedings of the American Mathematical Society* 15 (1964), pp. 851–855.

[21] S. Benchekroun and P. Moszkowski. "A new bijection between ordered trees and legal bracketings". In: *European Journal of Combinatorics* 17 (1996), pp. 605–611.

[22] D. I. Bevan. "On the growth of permutation classes". PhD thesis. `https://arxiv.org/abs/1506.06688`: The Open University, 2015.

[23] D. I. Bevan. *Private communication*. 2017.

[24] M. Bóna. *Combinatorics of permutations*. Discrete Mathematics and Its Applications. Chapman and Hall/CRC, 2004.

[25] M. Bóna, ed. *Handbook of Enumerative Combinatorics*. Discrete Mathematics and Its Applications. Chapman and Hall/CRC, 2015.

[26] M. Bóna. "New records in Stanley-Wilf limits". In: *European journal of Combinatorics* 28.1 (2007), pp. 75–85.

[27] N. Bonichon, M. Bousquet-Mélou, and E. Fusy. "Baxter permutations and plane bipolar orientations". In: *Séminaire Lotharingien de Combinatoire* 61A.B61Ah (2010).

[28] J. E. Bonin, L. W. Shapiro, and R. Simion. "Some q-analogues of the Schröder numbers arising from combinatorial statistics on lattice paths". In: *J. Statistical Planning and Inference* 34 (1993), pp. 35–55.

[29] P. Bose, J. F. Buss, and A. Lubiw. "Patten matching for permutations". In: *Information Processing Letters* 65.5 (1998), pp. 277–283.

[30] A. Bostan, K. Raschel, and B. Salvy. "Non D-finite excursions in the quarter plane". In: *J. Combin. Theory Ser. A* 121 (2014), pp. 45–63.

[31] M. Bousquet-Mélou. "A method for the enumeration of various classes of column-convex polygons". In: *Discrete Mathematics* 154 (1996), pp. 1–25.

[32] M. Bousquet-Mélou. "Four Classes of Pattern-Avoiding Permutations Under One Roof: Generating Trees with Two Labels". In: *The Electronic Journal of Combinatorics* 9.R19 (2003).

[33] M. Bousquet-Mélou. "Walks in the quarter plane: Kreweras' algebraic model". In: *Annals of Applied Probability* 15 (2005), pp. 1451–1491.

[34] M. Bousquet-Mélou and R. Brak. "Exactly Solved Models". In: *Polygons, Polyominoes and Polycubes.* Vol. 775. Lecture Notes in Phys. Springer, Dordrecht, 2009, pp. 43–78.

[35] M. Bousquet-Mélou and S. Butler. "Forest-like permutations". In: *Annals of Combinatorics* 11 (2007), pp. 335–354.

[36] M. Bousquet-Mélou and A. Jehanne. "Polynomial equations with one catalytic variable, algebraic series and map enumeration". In: *J. Combin. Theory Ser. B* 96.5 (2006), pp. 623–672.

[37] M. Bousquet-Mélou and M. Mishna. "Walks with small steps in the quarter plane". In: *Algorithmic Probability and Combinatorics* 520 (2010), pp. 1–39.

[38] M. Bousquet-Mélou and M. Petkovšek. "Linear recurrences with constant coefficients: the multivariate case". In: *Discrete Mathematics* 225 (2000), pp. 51–75.

[39] M. Bousquet-Mélou and G. Schaeffer. "Walks on the slit plane". In: *Probability Theory and Related Fields* 124.3 (2002), pp. 305–344.

[40] M. Bousquet-Mélou and G. Xin. "On partitions avoiding 3-crossing". In: *Séminaire Lotharingien de Combinatoire* 54.article B54e (2005).

[41] M. Bouvel and O. Guibert. "Refined enumeration of permutations sorted with two stacks and a D_8-simmetry". In: *Annals of Combinatorics* 18.2 (2014), pp. 199–232.

[42] W. M. Boyce. "Generation of a class of permutations associated with commuting functions". In: *Mathematical Algorithms* 2 (1967), pp. 19–26.

[43] P. Brändén and A. Claesson. "Mesh patterns and the expansion of permutation statistics as sums of permutation patterns". In: *Electron. J. Combin.* 18.2 (2011), p. 14.

[44] S. Brlek, E. Duchi, E. Pergola, and S. Rinaldi. "On the equivalence problem for succession rules". In: *Discrete Mathematics* 298 (2005), pp. 142–154.

[45] W. G. Brown. "Enumeration of non-separable planar maps". In: *Canadian Journal of Mathematics* 15 (1963), pp. 526–545.

[46] S. Burrill, J. Courtiel, E. Fusy, S. Melczer, and M. Mishna. "Tableau sequences, open diagrams, and Baxter families". In: *European Journal of Combinatorics* 58 (2016), pp. 144–165.

[47] S. Burrill, S. Elizalde, M. Mishna, and L. Yen. "A Generating Tree Approach to k-Nonnesting Partitions and Permutations". In: *Annals of Combinatorics* 20.3 (2016), pp. 453–485.

[48] S. Burrill, S. Melczer, and M. Mishna. "A Baxter class of a different kind, and other bijective results using tableau sequences ending with a row shape". In: *Discrete Mathematics and Theoretical Computer Science* (2015), pp. 369–380.

[49] D. Callan. "A bijection to count (1-23-4)-avoiding permutations". preprint, `https://arxiv.org/abs/1008.2375`. 2010.

[50] D. Callan. "A combinatorial survey of identities for the double factorial". preprint, `https://arxiv.org/abs/0906.1317`. 2009.

[51] G. Castiglione, A. Frosini, A. Restivo, and S. Rinaldi. "Enumeration of L-convex polyominoes by rows and columns". In: *Theoretical Computer Science* 347 (2005), pp. 336–352.

[52] E. C. Catalan. "Note sur une équation aux différences finies". In: *Journal de mathématiques pures et appliquées 1re série* 3 (1838), pp. 508–516.

[53] F. R. K. Chung, R. L. Graham, V. E. Hoggatt, and M. Kleiman. "The number of Baxter permutations". In: *J. Combin. Theory Ser. A* 24 (1978), pp. 382–394.

[54] J. Cibulka. "On constants in the Füredi–Hajnal and the Stanley-Wilf conjecture". In: *J. Combin. Theory Ser. A* 116.2 (2009), pp. 290–302.

[55] A. Claesson, V. Jelìnek, and E. Steingrímsson. "Upper bounds for the Stanley-Wilf limit of 1324 and other layered patterns". In: *J. Combin. Theory Ser. A* 119 (2012), pp. 1680–1691.

[56] A. Claesson and S. Kitaev. "Classification of bijections between 321- and 132-avoiding permutations". In: *Proceedings of FPSAC 2008 - Discrete Mathematics and Theoretical Computer Science*. 2008, pp. 495–506.

[57] S. Corteel. "Sèries gènèratrices exponentielles pour les eco-systémes signès". In: *Proceedings of the 12th FPSAC*. 2000, pp. 655–665.

[58] S. Corteel, M. A. Martinez, C. D. Savage, and M. Weselcouch. "Patterns in Inversion Sequences I". In: *Discrete Mathematics and Theoretical Computer Science* 18.2 (2016).

[59] A. Del Lungo, F. Del Ristoro, and J.-G. Penaud. "Left ternary trees and non-separable rooted planar maps". In: *Theoretical Computer Science* 233.1-2 (2000), pp. 201–215.

[60] M.-P. Delest, D. Gouyou-Beauchamps, and B. Vauquelin. "Enumeration of parallelogram polyominoes with given bond and site perimeter". In: *Graphs and Combinatorics* 3 (1987), pp. 325–339.

[61] M.-P. Delest and G. Viennot. "Algebraic languages and polyominoes enumeration". In: *Theoretical Computer Science* 34 (1984), pp. 169–206.

[62] D. Denisov and V. Wachtel. "Random walk in cones". In: *The Annals of Probability* 43 (2015), pp. 992–1044.

[63] N. Dershowitz and S. Zaks. "Enumerations of ordered trees". In: *Discrete Mathematics* 31 (1980), pp. 9–28.

[64] E. Deutsch. "A bijection on Dyck paths and its consequences". In: *Discrete Mathematics* 179 (1998), pp. 253–256.

[65] E. Deutsch. "A bijective proof of the equation linking the schröder numbers, large and small". In: *Discrete Mathematics* 241 (2001), pp. 235–240.

[66] E. Deutsch. "Dyck path enumeration". In: *Discrete Mathematics* 204 (1999), pp. 167–202.

[67] E. Deutsch, L. Ferrari, and S. Rinaldi. "Production matrices". In: *Advances in Applied Mathematics* 34 (2005), pp. 101–122.

[68] P. Di Francesco. "Geodesic distance in planar graphs: an integrable approach". In: *Ramanujan J.* 10.2 (2005), pp. 153–186.

[69] E. Duchi, J. M. Fedou, and S. Rinaldi. "From object grammars to ECO method". In: *Theoretical Computer Science* 314 (2004), pp. 57–95.

[70] S. Dulucq, S. Gire, and O. Guibert. "A combinatorial proof of J. West's conjecture". In: *Discrete Mathematics* 187 (1998), pp. 71–96.

[71] S. Dulucq and O. Guibert. "Baxter permutations". In: *Discrete Mathematics* 180 (1998), pp. 143–156.

[72] S. Elizalde. "Asymptotic enumeration of permutations avoiding generalised patterns". In: *Advances in Applied Mathematics* 36 (2006), pp. 138–155.

[73] S. Elizalde and M. Noy. "Consecutive patterns in permutations". In: *Advances in Applied Mathematics* 30 (2003), pp. 110–125.

[74] G. Fayolle, R. Iasgorodski, and V. Malyshev. *Random walks in the quarter-plane: Algebraic methods, boundary value problems and applications.* Vol. 40. Applications of Mathematics. Springer-Verlag, Belin, 1999.

[75] S. Felsner, E. Fusy, M. Noy, and D. Orden. "Bijections for Baxter families and related objects". In: *J. Combin. Theory Ser. A* 118.3 (2011), pp. 993–1020.

[76] L. Ferrari. "On the foundations of the ECO method". PhD thesis. University of Florence, 2003.

[77] L. Ferrari, E. Pergola, R. Pinzani, and S. Rinaldi. "An algebraic characterization of the set of succession rules". In: *Theoretical Computer Science* 281 (2002), pp. 351–367.

[78] L. Ferrari, E. Pergola, R. Pinzani, and S. Rinaldi. "Jumping succession rules and their generating functions". In: *Discrete Mathematics* 271 (2003), pp. 29–50.

[79] P. Flajolet and R. Sedgewick. *Analytic Combinatorics*. Cambridge University Press, Cambridge, 2009.

[80] D. Foata and D. Zeilberger. "A Classic Proof of a Recurrence for a Very Classical Sequence". In: *J. Combin. Theory Ser. A* 80.2 (1997), pp. 380–384.

[81] J. Fox. "Stanley-Wilf limits are typically exponential". preprint, `https://arxiv.org/abs/1310.8378`. 2013.

[82] E. Fusy, D. Poulalhon, and G. Schaeffer. "Bijective counting of plane bipolar orientations and schnyder woods". In: *European Journal of Combinatorics* 30 (2009), pp. 1646–1658.

[83] S. Garrabrant and I. Pak. "Pattern avoidance is not P-recursive". preprint, `https://arxiv.org/abs/1505.06508`.

[84] A. M. Garsia and H. Mark. "A remarkable q, t-Catalan sequence and q-Lagrange Inversion". In: *Journal of Algebraic Combinatorics* 5.3 (1996), pp. 191–244.

[85] I. Gessel and G. Viennot. "Binomail determinants, paths, and hook length formulae". In: *Advances in Mathematics* 58.3 (1985), pp. 300–321.

[86] S. Giraudo. "Algebraic and combinatorial structures on pairs of twin binary trees". In: *Journal of Algebra* 360 (2012), pp. 115–157.

[87] S. Gire. "Arbres, permutations à motifs exclus at cartes planaires: quelques problèmes algorithniques et combinatoires". PhD thesis. LaBRI - Université Bordeaux 1, 1993.

[88] H. W. Gould. *Bell and Catalan numbers: A reasearch bibliography of two special number sequences*. 1978.

[89] N. S. S. Gu, N. Y. Li, and T. Mansour. "2-Binary trees: Bijections and related issues". In: *Discrete Mathematics* 308 (2008), pp. 1209–1221.

[90] A. J. Guttmann. "Indicators of solvability for lattice models". In: *Discrete Mathematics* 217 (2000), pp. 167–189.

[91] A. J. Guttmann, ed. *Polygons, Polyominoes and Polycubes*. Vol. 775. Lecture Notes in Phys. Springer, Dordrecht, 2009.

[92] X. Hong, G. Huang, Y. Cai, J. Gu, S. Dong, C. K. Cheng, and J. Gu. "Corner block list: an effective and efficient topological representation of non-slicing floorplan". In: *Proceedings of the 2000 IEEE/ACM international conference on Computer-aided design*. 2000, pp. 8–12.

[93] B. Jacquard and G. Schaeffer. "A Bijective Census of Nonseparable Planar Maps". In: *J. Combin. Theory Ser. A* 83.1 (1998), pp. 1–20.

[94] E. J. Janse van Rensburg, T. Prellberg, and A. Rechnitzer. "Partially directed paths in a wedge". In: *J. Combin. Theory Ser. A* 115 (2008), pp. 623–650.

[95] M. Kac, G.-C. Rota, and J. Schwartz. “Combinatorics”. In: *Discrete Thoughts*. Birkhäuser, Boston, MA, 1992.

[96] A. Kasraoui. “New Wilf-equivalence results for vincular patterns”. In: *European Journal of Combinatorics* 34.2 (2013), pp. 322–337.

[97] D. Kim and Z. Lin. “Refined restricted inversion sequences”. In: *Séminaire Lotharingien de Combinatoire - FPSAC 2017*. Vol. 78B. 52. 2017.

[98] S. Kitaev. *Patterns in Permutations and Words*. Monographs in Theoretical Computer Science. Springer, Heidelberg, 2011.

[99] D. E. Knuth. *The Art of Computer Programming. Volume 1*. Addison-Wesley, 1968.

[100] A. G. Konheim and B. Weiss. “An Occupancy Discipline and Applications”. In: *SIAM Journal on Applied Mathematics* 14 (1966), pp. 1266–1274.

[101] C. Krattenthaler. “Permutations with restricted patterns and Dyck paths”. In: *Advances in Applied Mathematics* 27 (2001), pp. 510–530.

[102] C. Krattenthaler and S. G. Mohanty. *Lattice paths combinatorics - applications to probability and statistics*. Encyclopedia of Statistical Sciences. Wiley, New York, 2003.

[103] D. Kremer. “Permutations with forbidden subsequences and a generalized Schröder number”. In: *Discrete Mathematics* 218 (2000), pp. 121–130.

[104] G. Kreweras and P. Moszkowski. “A new enumerative property of the Narayana numbers”. In: *Elsevier Journal of Statistical Planning and Inference* 14 (1986), pp. 63–67.

[105] M. Kuba. “A note on naturally embedded ternary trees”. In: *Electron. J. Combin.* 18.1 (2011), Paper 142, 20.

[106] L. Lipshitz. “D-finite power series”. In: *Journal of Algebra* 122 (1989), pp. 353–373.

[107] C. L. Mallows. “Baxter permutations rise again”. In: *J. Combin. Theory Ser. A* 27 (1979), pp. 394–396.

[108] T. Mansour and M. Shattuck. “Pattern avoidance in inversion sequences”. In: *PU. M. A.* 25.2 (2015), pp. 157–176.

[109] A. Marcus and G. Tardos. “Excluded permutation matrices and the Stanley-Wilf conjecture”. In: *J. Combin. Theory Ser. A* 107 (2004), pp. 153–160.

[110] M. A. Martinez and C. D. Savage. “Patterns in Inversion Sequences II: Inversion sequences avoiding triples of relations”. preprint, `https://arxiv.org/abs/1609.08106`. 2016.

[111] R. J. McIntosh. “An asymptotic formula for binomial sums”. In: *Journal of Number Theory* 58.1 (1996), pp. 158–172.

[112] S. Melczer and M. Mishna. “Singularity analysis via the iterated kernel method”. In: *Combinatorics, Probability and Computing* 23 (2014), pp. 861–888.

[113] A. d. Mier and M. Noy. “A solution to the tennis ball problem”. In: *Theoretical Computer Science* 346 (2005), pp. 254–264.

[114] M. Mishna and A. Rechnitzer. “Two non-holonomic lattice walks in the quarter plane”. In: *Theoretical Computer Science* 410 (2009), pp. 3616–3630.

[115] S. G. Mohanty. *Lattice paths counting and applications*. Academic Press, 1979.

[116] E. Pergola. “ECO: a Methodology for Enumerating Combinatorial Objects”. PhD thesis. University of Florence, 1999.

[117] E. Pergola and R. A. Sulanke. “Schröder triangles, paths, and parallelogram polyominoes”. In: *Journal of Integer sequences* 1.98.1.7 (1998).

[118] M. Petkovšek, H. S. Wilf, and D. Zeilberger. *A=B*. AK Peters, Wellesley, 1996.

[119] G. Polya. “On the number of certain polygons”. In: *Journal of Combinatorial Theory* 6.1 (1969), pp. 102–105.

[120] H. Prodinger. “The kernel method: a collection of examples”. In: *Séminaire Lotharingien de Combinatoire* 50.B50f (2004).

[121] L. Pudwell. “Enumeration schemes for permutations avoiding barred patterns”. In: *Journal of Combinatorics* 17.Peper R29 (2010).

[122] N. Reading. “Lattice congruence, fans and Hopf algebras”. In: *J. Combin. Theory Ser. A* 110.2 (2005), pp. 237–273.

[123] C. Richard. “Limit Distributions and Scaling Functions”. In: *Polygons, Polyominoes and Polycubes*. Vol. 775. Lecture Notes in Phys. Springer, Dordrecht, 2009, pp. 247–299.

[124] B. Richmond. “Multivariate Lagrange Inversion”. In: *Algorithms seminar, 1997-1998*. 1998, pp. 23–26.

[125] J. Riordan. “Ballots and trees”. In: *Journal of Combinatorial Theory* 6.4 (1969), pp. 408–411.

[126] D. G. Rogers and L. W. Shapiro. “Some correspondences involving the Schröder numbers and relations”. In: *Lecture Notes in Mathematics*. Vol. 686. Springer-Verlag, Belin, 1978, pp. 267–276.

[127] C. D. Savage and M. J. Schuster. “Ehrhart series of lecture hall polytopes and eulerian polynomials for inversion sequences”. In: *J. Combin. Theory Ser. A* 119 (2012), pp. 850–870.

[128] L. W. Shapiro. “A Catalan triangle”. In: *Discrete Mathematics* 14 (1976), pp. 83–90.

[129] L. W. Shapiro and A. B. Stephens. “Bootstrap Percolation, the Schröder Numbers, and the N-Kings Problem”. In: *SIAM Journal on Discrete Mathematics* 4 (1991), pp. 275–280.

[130] L. W. Shapiro and R. A. Sulanke. "Bijections for the Schröder numbers". In: *Mathematics Magazine* 73 (2000), pp. 369–376.

[131] R. Simion and F. W. Schmidt. "Restricted permutations". In: *European Journal of Combinatorics* 6.4 (1985), pp. 383–406.

[132] N. J. A. Sloane. *The On-line Encyclopedia of Integer Sequences*. 2011.

[133] R. P. Stanley. *Catalan numbers*. Cambridge University Press, Cambridge, 2015.

[134] R. P. Stanley. *Enumerative combinatorics*. Vol. 1. Cambridge University Press, Cambridge, 1997.

[135] R. P. Stanley. *Enumerative combinatorics*. Vol. 2. Cambridge University Press, Cambridge, 1999.

[136] R. P. Stanley. "Hipparchus, Plutarch, Schröder, and Hough". In: *Amer. Math. Monthly* 104 (1997), pp. 344–350.

[137] E. Steingrímsson. "Generalized permutation patterns - a short survey". In: *Permutation Patterns*. Ed. by S. Linton, N. Ruskuc, and V. Vatter. Vol. 376. London Mathematical Society Lecture Note Series. Cambridge University Press, 2010, pp. 137–152.

[138] E. Steingrímsson. "Some open problems on permutation patterns". In: *Surveys in Combinatorics 2013*. Ed. by S. R. Blackburn, S. Gerke, and M. Wildon. Vol. 409. London Mathematical Society Lecture Note Series. Cambridge University Press, 2013, pp. 239–263.

[139] E. Steingrímsson and B. E. Tenner. "The Möbius Function of the Permutation Pattern Poset". In: *Journal of Combinatorics* 1 (2010), pp. 39–52.

[140] R. A. Sulanke. "Three recurrences for parallelogram polyominoes". In: *Journal of Difference Equations and Applications* 5 (1999), pp. 155–176.

[141] W. T. Tutte. "A census of planar maps". In: *Canadian Journal of Mathematics* 15 (1963), pp. 249–271.

[142] J. West. "Enumeration of Reading's twisted Baxter permutations". Local Proceedings of Permutation Patterns 2006, Reykjavìk University. 2006.

[143] J. West. "Generating trees and forbidden subsequences". In: *Discrete Mathematics* 157 (1996), pp. 363–374.

[144] J. West. "Generating trees and the Catalan and Schröder numbers". In: *Discrete Mathematics* 146 (1995), pp. 247–262.

[145] J. West. "Sorting twice through a stack". In: *Theoretical Computer Science* 117.1-2 (1993), pp. 303–313.

[146] H. S. Wilf. *Generatingfunctionology*. Boston: Academic Press, 1994.

[147] W.-J. Woan, L. Shapiro, and D. G. Rogers. "The Catalan numbers, the Lebesgue integral, and 4^{n-2}". In: *Amer. Math. Monthly* 104.10 (1997), pp. 926–931.

[148] B. Yao, H. Chen, C. K. Cheng, and R. L. Graham. "Floorplan representations: Complexity and connections". In: *ACM Transacyion on Design Automation of Electonic Systems* 8 (2003), pp. 55–80.

[149] D. Zeilberger. "A proof of Julian West's conjecture that the number of two-stack sortable permutations of length n is $2(3n)!/((n+1)!(2n+1)!)$". In: *Discrete Mathematics* 102 (1992), pp. 85–93.

[150] D. Zeilberger. "The method of creative telescoping". In: *Journal of Symbolic Computation* 11 (1991), pp. 195–204.

[151] D. Zeilberger. "The umbral transfer-matrix method: I. Foundations". In: *J. Combin. Theory Ser. A* 91 (2000), pp. 451–463.

Author's contribution

[G1] M. Bouvel, V. Guerrini, and S. Rinaldi. "Slicings of parallelogram polyominoes, or how Baxter and Schroder can be reconciled". In: *Discrete Mathematics and Theoretical Computer Science* (2016), pp. 287–298.

[G2] N. R. Beaton, M. Bouvel, V. Guerrini, and S. Rinaldi. "Slicings of parallelogram polyominoes: Catalan, Schröder, Baxter, and other sequences". preprint, https://arxiv.org/abs/1511.04864.

[G3] R. Cori, E. Duchi, V. Guerrini, and S. Rinaldi. "Families of parking functions counted by the Schröder and Baxter numbers". In: *Lattice paths Combinatorics 2015*. Ed. by G. Andrews, C. Krattenthaler, and A. Krinik. Springer, 2016.

[G4] M. Bouvel, V. Guerrini, A. Rechnitzer, and S. Rinaldi. "Semi-Baxter and Strong-Baxter Permutations". In: *Séminaire Lotharingien de Combinatoire - FPSAC 2017*. Vol. 78B. 19. 2017.

[G5] M. Bouvel, V. Guerrini, A. Rechnitzer, and S. Rinaldi. "Semi-Baxter and Strong-Baxter: two relatives of the Baxter sequence". preprint, https://arxiv.org/abs/1702.04529.

[G6] E. Duchi, V. Guerrini, S. Rinaldi, and G. Schaeffer. "Fighting Fish". In: *Journal of Physics A: Mathematical and Theoretical* 50.2 (2016), p. 024002.

[G7] E. Duchi, V. Guerrini, S. Rinaldi, and G. Schaeffer. "Fighting Fish: Enumerative Properties". In: *Séminaire Lotharingien de Combinatoire - FPSAC 2017*. Vol. 78B. 43. 2017.