Non-uniform permutations
biased according to their records

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talk based on joint work and work in progress with
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Non-uniform permutations

Context:
Analysis of algorithms working on arrays of numbers (sorting, ...)

Average-case analysis of algorithms:
- The uniform distribution on the data set is usually assumed.
- It provides a first answer, but it is not always realistic. E.g., sorting algorithms are often used on data which is already “almost sorted”. (Ex. of TimSort, wait 30 minutes to know more!)

⇒ Find non-uniform models with good balance between simplicity (so that we can study it) and accuracy (in terms of modeling data)

Some classical models for non-uniform permutations
- Ewens: \( P(\sigma) \) is proportional to \( \theta^{\text{number of cycles of } \sigma} \)
- Mallows: \( P(\sigma) \) is proportional to \( \theta^{\text{number of inversions of } \sigma} \)
Our record-biased permutations

**Goal:** A non-uniform distribution on permutations, which gives higher probabilities to permutations that are “almost sorted”.

**Record-biased permutations:**
- A record is an element larger than all those preceding it.  
  **Example:** $3 4 1 2 6 8 7 9 5$ has 5 records.
- Roughly, a permutation with many records is “almost sorted”. More formally, the number of non-records is a measure of presortededness.
- In our model, $\mathbb{P}(\sigma)$ is proportional to $\theta^{\text{number of records of } \sigma}$.
- We focus on the regime where $\theta = \lambda \cdot n$, $n$ being the size of $\sigma$.

**Remark:** Related to the Ewens distribution via Foata’s *fundamental bijection*, which sends number of cycles to number of records.
**Example:** $2 4 3 1 9 6 8 7 5 = (3)(4 1 2)(6)(8 7)(9 5) \rightarrow 3 4 1 2 6 8 7 9 5$
Outline of the talk

Goal: Describe properties of the model of record-biased permutations. Applications to the analysis of algorithms won’t be discussed so much.

Results obtained:

- Random sampling can be done in linear time.
- Behavior of classical permutation statistics:
  - We obtain their expected values and precise probabilities.
    Applications to analysis of algorithms were presented at AofA 2016:
    - expected running time of InsertionSort,
    - expected number of mispredictions in MinMaxSearch
  - We plan to study their distribution.
- What does a large record-biased permutation typically look like?
  - We describe the (deterministic) permuton limit for our model.
Linear random samplers
Ewens-distributed permutations can be sampled in linear time using a variant of the Chinese restaurant process:

- Insert $i$ from 1 to $n$.
- At step $i$, create a new cycle $(i)$ with probability $\frac{\theta}{\theta + i - 1}$, or insert $i$ in an existing cycle, immediately after a previously inserted element, each with probability $\frac{1}{\theta + i - 1}$.

Using appropriate data structures, we can implement Foata’s transform in linear time, hence sampling record-biased permutation in linear time.

We can also do it directly, with appropriate data structures.
Playing with the sampler: a typical diagram arises

The **diagram** of a permutation $\sigma$ of size $n$ is the set of points at coordinates $(i, \sigma(i))$ for $1 \leq i \leq n$.

The **normalized diagram** of $\sigma$ is the same picture, rescaled to the unit square.

Pictures obtained overlapping 10 000 permutations of size 100 sampled according to the record-biased model with $\theta = 1, 50, 100$ and 500:

We explain it by describing the **permuton limit** of record-biased permutations.
Playing with the sampler: number of records

A record of a permutation $\sigma$ is given by an index $i$ such that $\sigma(i) > \sigma(j)$ for all $j < i$.

Empirical distribution of the number of records in record-biased permutations:

Histograms for $10^6$ permutations, of size $n = 100, 200$ and $500$, and for $\theta = 1, \frac{n}{2}, n$ and $5n$.

Remark: Corresponds to number of cycles for Ewens distribution, known to be Gaussian for fixed $\theta$. 
Playing with the sampler: number of descents

A descent of a permutation $\sigma$ is given by an index $i$ s.t. $\sigma(i-1) > \sigma(i)$.

Empirical distribution of the number of descents in record-biased permutations:

Histograms for $10^6$ permutations, of size $n = 100, 200$ and $500$, and for $\theta = 1, \frac{n}{2}, n$ and $5n$.

Remark: Corresponds to number of anti-exceedances (given by $i$ s.t. $\sigma(i) < i$) for Ewens distribution, which can be proved Gaussian for fixed $\theta$. 

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Playing with the sampler: number of inversions

An inversion of $\sigma$ is given by a pair $i, j$ s.t. $i < j$ and $\sigma(i) > \sigma(j)$.

Empirical distribution of the number of inversions in record-biased permutations:

Histograms for $10^6$ permutations, of size $n = 100, 200$ and $500$, and for $\theta = 1, \frac{n}{2}, n$ and $5n$.

Remark: No known natural analogue for Ewens distribution
Playing with the sampler: first value

Empirical distribution of the first value $\sigma(1)$ in record-biased permutations:

Histogram for $10^6$ permutations, of size $n = 100$, and for $\theta = 0.2, 0.5, 1, 10$ and $50$.

Remark: Corresponds to the minimum over all cycles of the maximal value in a cycle for Ewens distribution.
Behavior of classical statistics
**Proposition:** In record-biased permutations of size \( n \), for any \( i \in \{2, \ldots, n\} \), the probability that there is a descent at position \( i \) is

\[
\mathbb{P}(\sigma(i-1) > \sigma(i)) = \frac{(i-1)(2\theta+i-2)}{2(\theta+i-1)(\theta+i-2)}.
\]

**Corollary:** In record-biased permutations of size \( n \), the expected value of the number of descents is \( \mathbb{E}[\text{number of descents}] = \frac{n(n-1)}{2(\theta+n-1)}. \)

In particular, when \( \theta = \lambda n \), \( \mathbb{E}[\text{number of descents}] \sim n/2(\lambda + 1). \)

**Application** (using the precise probabilities given by the proposition): Average number of mispredictions in algorithms solving MinMaxSearch.

**Question:** Can we say more than just the expectation when \( \theta = \lambda n \)? Can we find the limiting distribution?

**More examples:** Similar statements for number of records, number of inversions and first value, with applications to the analysis of InsertionSort and MinMaxSearch.
Permuton limit
of record-biased permutations
Definition: A permuton $\mu$ is a probability measure on the unit square with uniform projections (or marginals):

$$\text{for all } a < b \text{ in } [0,1], \quad \mu([a, b] \times [0,1]) = \mu([0, 1] \times [a, b]) = b - a.$$ 

Remark: The normalized diagrams of permutations (denoted $\sigma$) are essentially permutons (denoted $\mu_\sigma$)

Replacing each point $(i/n, \sigma(i)/n)$ by a little square $[(i - 1)/n, i/n] \times [\sigma(i) - 1/n, \sigma(i)/n]$, and distributing the mass 1 uniformly on these little squares

Convergence of a sequence of permutations $(\sigma_n)$ to a permuton $\mu$:

- inherited from the weak convergence of measures, namely:
- $\sigma_n \rightarrow \mu$ when $\sup_{R \text{ rectangle } \subset [0,1]^2} |\mu_{\sigma_n}(R) - \mu(R)| \rightarrow 0$ as $n \rightarrow +\infty$.
- If each $\sigma_n$ has size $n$, taking $R$ of the form $[0, i/n] \times [0, j/n]$ is enough.
Theorem:
Let $\sigma_n$ be a random record-biased permutation of size $n$ for $\theta = \lambda n$. $\mu_{\sigma_n}$ converges in probability to $\mu = \mu_c + \mu_u$ defined below.

Letting $f_\lambda(x) = \frac{x(\lambda+1)}{\lambda+x}$, we define

- $\mu_u$ is the uniform measure of total mass $c_\lambda \int_0^1 f_\lambda$ for $c_\lambda = \frac{1}{\lambda+1}$ on the area under the curve $y = f_\lambda(x)$;
- $\mu_c$ is the measure supported by the curve $y = f_\lambda(x)$ with density $\frac{\lambda}{\lambda+x}$ with respect to $\text{Leb}_c$, defined by $\text{Leb}_c(x, f_\lambda(x)) = \text{Lebesgue}(x)$.

Two steps towards this statement: guessing $\mu$ and proving convergence.
Guessing the limit $\mu$

The pictures suggest to decompose $\mu$ as $\mu_u + \mu_c$, with $\mu_c$ on a curve, and $\mu_u$ uniform under the curve. To determine are:

- the equation $y = f_\lambda(x)$ of the curve,
- how to distribute the mass between $\mu_c$ and $\mu_u$.

To find the equation $y = f_\lambda(x)$ of the curve,

- we estimate $\mathbb{P}(\text{max before position } i \text{ is } j)$ for $i \approx xn$ and $j \approx yn$;
- we find the relation between $x$ and $y$ which makes this probability not larger than 1, and non-zero once summed over $j$.

To find the relative measures on the curve and below,

- we compute the measure of the records in $\sigma_n$ and take the limit in $n$: this gives the measure $\int_0^1 \frac{\lambda}{\lambda+x} dx$ on the curve;
- we distribute uniformly the mass $c_\lambda \int_0^1 f_\lambda(x) dx$ below the curve, for $c_\lambda$ s.t. $\int_a^b \left( \frac{\lambda}{\lambda+x} + c_\lambda f_\lambda(x) \right) dx = b - a$. 

Stay tuned

What was done:
- Definition of the record-biased model
- Behavior of statistics (precise probabilities, expectation)
- Applications to the analysis of algorithms

What is new:
(but needs to be written down...)
- Efficient random samplers
- Permuton limit

What is left to do:
- Is the number of inversions Gaussian?
- Do the Gaussian limiting distributions hold also in the $\theta = \lambda n$ regime?

!! Thank you !!