Non-uniform permutations
biased according to their records

Mathilde Bouvel
(Loria, CNRS, Univ. Lorraine)

talk based on joint work and work in progress with
Nicolas Auger, Cyril Nicaud and Carine Pivoteau

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Non-uniform permutations

Context:
Analysis of algorithms working on arrays of numbers (sorting, …)

Average-case analysis of algorithms:

- The uniform distribution on the data set is usually assumed.
- It provides a first answer, but it is not always realistic.
  E.g., sorting algorithms are often used on data which is already “almost sorted”. (Ex. of TimSort [Auger, Jugé, Nicaud, Pivoteau, 2018])

⇒ Find non-uniform models with good balance between simplicity (so that we can study it) and accuracy (in terms of modeling data)

Some classical models for non-uniform permutations

- Ewens: $P(\sigma)$ is proportional to $\theta^{\text{number of cycles of } \sigma}$
- Mallows: $P(\sigma)$ is proportional to $\theta^{\text{number of inversions of } \sigma}$
Our record-biased permutations

**Goal:** A non-uniform distribution on permutations, which gives higher probabilities to permutations that are “almost sorted”.

**Record-biased permutations:**
- A record is an element larger than all those preceding it.
  **Example:** $3 4 1 2 6 8 7 9 5$ has 5 records.
- Roughly, a permutation with many records is “almost sorted”. More formally, the number of non-records is a measure of presortedness as defined by [Manilla, 1985], see [Auger, Bouvel, Pivoteau, Nicaud, 2016].
- In our model, $\mathbb{P}(\sigma)$ is proportional to $\theta$ number of records of $\sigma$.

**Remark:** Related to the Ewens distribution via Foata’s *fundamental bijection*, which sends number of cycles to number of records.

**Example:** $2 4 3 1 9 6 8 7 5 = (3)(4 1 2)(6)(8 7)(9 5) \rightarrow 3 4 1 2 6 8 7 9 5$
Outline of the talk

**Goal:** Describe properties of the model of record-biased permutations. Applications to the analysis of algorithms will be discussed only a little.

**Results obtained:**

- Random sampling can be done in **linear time**, in several ways.
  - viewing permutations as words
  - or viewing permutations as *diagrams*

- Behavior of classical permutation statistics:
  - We obtain **precise probabilities** of elementary events.
  - We deduce their **expected values** and **asymptotic distribution**.
  - Applications to analysis of algorithms [ABNP, 2016]:
    - expected running time of **INSERTIONSORT**,
    - expected number of mispredictions in **MINMAXSEARCH**

- What does a large record-biased permutation typically look like?
  - We describe the (deterministic) **permuton limit** for our model.
Before we dive in: Several ways of seeing a permutation

We can represent a permutation of size $n$, say $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\ 1 & 8 & 3 & 6 & 4 & 2 & 5 & 7 \end{pmatrix}$ as

- a **word** (a.k.a. *one-line representation*): $\sigma = 1 \ 8 \ 3 \ 6 \ 4 \ 2 \ 5 \ 7$
- a product of cycles: $\sigma = (1) \ (3) \ (8 \ 7 \ 5 \ 4 \ 6 \ 2)$
- a **diagram**, *i.e.* an $n \times n$ grid with points at coordinates $(i, \sigma(i))$: 

![Diagram](image.png)
Linear random samplers
Random sampling combining Ewens and Foata

- **Ewens-distributed** permutations can be sampled in linear time using a variant of the Chinese restaurant process:
  - Insert $i$ from 1 to $n$.
  - At step $i$, create a new cycle $(i)$ with probability $\frac{\theta}{\theta + i - 1}$, or insert $i$ in an existing cycle, immediately after a previously inserted element, each with probability $\frac{1}{\theta + i - 1}$.

- Using appropriate data structures, we can implement Foata’s transform in linear time, hence sampling record-biased permutation in linear time.

- We can also do it directly, with appropriate data structures.
Another sampling procedure for record-biased permutations of size $n$:

- Start with an empty array of $n$ cells.
- Insert $i$ from 1 to $n$.
- At step $i$,
  - either insert $i$ in the leftmost empty cell (this creates a record): with probability $\frac{\theta}{\theta + n - i}$;
  - or insert $i$ in one of the $n - i$ other empty cells (this does not create a record): with probability $\frac{1}{\theta + n - i}$ for each such cell.

- Using appropriate data structures (one linked-list and two auxiliary arrays), we can implement this sampling procedure in linear time.
Yet another sampling procedure for record-biased permutations of size $n$:

- Start with an empty diagram.
- For $i$ from 1 to $n$, insert an $i$-th column and a new row, with a new point at their intersection:
  - with probability $\frac{\theta}{\theta + i - 1}$, the new row is the topmost one (hence the new point a record);
  - for each $j < i$, with probability $\frac{1}{\theta + i - 1}$, the new row is just under the point in column $j$ (hence not a record).

- Using appropriate data structures (a linked list with direct access to its cells), we can implement this sampling procedure in linear time.
Histograms are for $10^6$ permutations, of size $n = 100$, and for $\theta = 1, 50, 100$ and 500 (resp. $\theta = 0.2, 0.5, 1, 10$ and 50).
Playing with the samplers: a typical diagram arises

Recall that the diagram of a permutation $\sigma$ of size $n$ is the set of points at coordinates $(i, \sigma(i))$ for $1 \leq i \leq n$.

The normalized diagram of $\sigma$ is the same picture, rescaled to the unit square.

Pictures obtained overlapping 10 000 permutations of size 100 sampled according to the record-biased model with $\theta = 1, 50, 100$ and 500:

We explain it by describing the permutoon limit of record-biased permutations.
Behavior of statistics
Number of records

Recall that a record of a permutation $\sigma$ is given by an index $i$ such that $\sigma(i) > \sigma(j)$ for all $j < i$.

**Results:**

- The expected number of records in record-biased permutations of size $n$ is $\sum_{i=1}^{n} \frac{\theta}{\theta+i-1}$.
- For fixed $\theta$, it is $\sim \theta \log(n)$ as $n \to \infty$.
- For fixed $\theta$, the distribution of the number of records in record-biased permutations is asymptotically Gaussian.

**Proof idea:** Via the Foata bijection, records in record-biased permutations correspond to cycles in Ewens-distributed permutations.

**Remark:** Expectation can also be derived from $\mathbb{P}($record at $i$) = $\frac{\theta}{\theta+i-1}$, which is obvious from the random sampler of diagrams.
A descent of a permutation $\sigma$ is given by an index $i$ s.t. $\sigma(i-1) > \sigma(i)$.

**Results:**

- The expected number of descents in record-biased permutations of size $n$ is $\frac{n(n-1)}{2(\theta+n-1)}$.
- For fixed $\theta$, it is $\sim \frac{n}{2}$ as $n \to \infty$.
- For fixed $\theta$, the distribution of the number of descents in record-biased permutations is asymptotically Gaussian.

**Proof idea:** Descents in record-biased permutations correspond to anti-exceedances in Ewens-distributed permutations. These are closely related to weak exceedances studied by [Féray, 2013].

**Remark:** $\mathbb{P}(\text{descent at } i)$ and hence the expectation can also be derived from the random sampler of diagrams.
An inversion of $\sigma$ is given by a pair $(i, j)$ s.t. $i < j$ and $\sigma(i) > \sigma(j)$.

**Results:**

- The expected number of inversions in record-biased permutations of size $n$ is
  
  $$\frac{n(n+1-2\theta)}{4} + \frac{\theta(\theta-1)}{2} \sum_{i=1}^{n} \frac{1}{\theta+i-1}$$

- For fixed $\theta$, it is $\sim \frac{n^2}{4}$ as $n \to \infty$.

- For fixed $\theta$, the distribution of the number of inversions in record-biased permutations is asymptotically Gaussian.

Histogram for $10^6$ permutations, of size $n = 100$, and for $\theta = 1, 50, 100$ and 500.

**Remark:** No known natural analogue on Ewens-distributed permutations.
Number of inversions: proof sketch

Let $\text{inv}_j$ be the number of inversions of the form $(i, j)$, and $\text{inv} = \sum_j \text{inv}_j$ be the number of inversions.

**Remarks:** With the sampling procedure as diagrams

- $\text{inv}_j$ is completely determined by step $j$ of the procedure, and depends only on the height of the $j$-th point inserted;
- in particular, for $j \neq j'$, $\text{inv}_j$ and $\text{inv}_{j'}$ are independent.

**Expectation:** The first remark gives $\mathbb{P}(\text{inv}_j = k) = \begin{cases} \frac{\theta}{\theta + j - 1} & \text{if } k = 0 \\ \frac{1}{\theta + j - 1} & \text{if } k \neq 0 \end{cases}$, from which we deduce expressions for $\mathbb{E}(\text{inv}_j) = \sum_k k \cdot \mathbb{P}(\text{inv}_j = k)$ and $\mathbb{E}(\text{inv}) = \sum_j \mathbb{E}(\text{inv}_j)$.

**Asymptotic normality:** Follows from independence comparing the order of $\sum_j \mathbb{E}(\text{inv}_j^3) = \Theta(n^4)$ and $\sqrt{\text{Var}(\text{inv})^3} = \Theta(n^{3/2})$. 
Results:

- The expected value of $\sigma(1)$ in record-biased permutations of size $n$ is $\frac{\theta + n}{\theta + 1}$.
- For fixed $\theta$, it is $\sim \frac{n}{\theta + 1}$ as $n \to \infty$.
- For fixed $\theta$, asymptotically, the rescaled first value $\sigma(1)/n$ in a record-biased permutation of size $n$ follows a beta distribution of parameters $(1, \theta)$.

Remark: Corresponds to the minimum over all cycles of the maximal value in a cycle for Ewens-distributed permutations. This statistics was not studied so far.
Value of the first element: proof sketch

**Expectation:** We use the sampling procedure as **words**.

- The first element is $k$ when the first $k-1$ insertions do **not** create records but the $k$-th insertion creates a record.
- Therefore $\mathbb{P}(\sigma(1) = k) = \prod_{i=1}^{k-1} \frac{n-i}{\theta+n-i} \cdot \frac{\theta}{\theta+n-k} = \frac{(n-1)! \theta^{n-k} \theta}{(n-k)! \theta^{n}}$, where $x^{(m)} = x(x+1)\ldots(x+m-1)$ is the rising factorial.
- (Magical?) simplifications arise giving $\mathbb{E}(\sigma(1)) = \frac{\theta+n}{\theta+1}$.

**Asymptotic distribution:** We compute **moments** of $\sigma(1)$ similarly.

- The computation of $\mathbb{E}(\sigma(1)^r)$ uses similar simplifications and involves Eulerian polynomials $A_r(z)$ (because $\sum_n n^r z^n = \frac{z A_r(z)}{(1-z)^{r+1}}$).
- We obtain $\mathbb{E}(\sigma(1)^r) \sim_{n \rightarrow \infty} \frac{r! n^r}{(\theta+1)^{(r)}}$.
- After normalization, we recognize the $r$-th moment $\frac{r!}{(\theta+1)^{(r)}}$ of a **beta** distribution of parameter $(1, \theta)$. 
One remark: Various regimes for $\theta$

For our four statistics, we have:

- formula (depending on $\theta$ and $n$) for its expectation, valid for $\theta$ fixed and $\theta = \theta(n)$;
- the asymptotic behavior of these expectations when $\theta$ is fixed;
- the limiting distribution when $\theta$ is fixed.

**Asymptotic behavior of expectations in various regimes for $\theta$:**

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>fixed $\theta &gt; 0$</th>
<th>$\theta = n^\epsilon$, $0 &lt; \epsilon &lt; 1$</th>
<th>$\theta = \lambda n$, $\lambda &gt; 0$</th>
<th>$\theta = n^\delta$, $\delta &gt; 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>records</td>
<td>$\log n$</td>
<td>$\theta \cdot \log n$</td>
<td>$(1 - \epsilon) \cdot n^\epsilon \log n$</td>
<td>$\lambda \log(1 + 1/\lambda) \cdot n$</td>
</tr>
<tr>
<td>descents</td>
<td>$n/2$</td>
<td>$n/2$</td>
<td>$n/2$</td>
<td>$n/2(\lambda + 1)$</td>
</tr>
<tr>
<td>inversions</td>
<td>$n^2/4$</td>
<td>$n^2/4$</td>
<td>$n^2/4$</td>
<td>$n^2/4 \cdot f(\lambda)$</td>
</tr>
<tr>
<td>first value</td>
<td>$n/2$</td>
<td>$n/(\theta + 1)$</td>
<td>$n^{1-\epsilon}$</td>
<td>$(\lambda + 1)/\lambda$</td>
</tr>
</tbody>
</table>

where $f(\lambda) = 1 - 2\lambda + 2\lambda^2 \log (1 + 1/\lambda)$.

In the last part of the talk, we will focus on the regime $\theta = \lambda n$. 

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Another remark: analysis of algorithms

**InsertionSort:**
- For $i = 1, 2, \ldots, n$, swap $i$ with the elements to its left until $i$ reaches the $i$-th cell.
- The **number of swaps** is the **number of inversions**, whose expected behavior is known from the previous table.

**MinMaxSearch:**
- Several algorithms to **find the min and the max** in an array: **naive** version with $2n$ comparisons, **clever** version with $\frac{3}{2}n$ comparisons.
- But the **naive** algorithm is typically **more efficient on uniform data**! Why? Not only the comparisons count in practice.
- The **branch predictors** cause **mispredictions**, hence a slow-down. We quantify this by computing the **average number of mispredictions**.
- This also explains why the **clever** algorithm is **more efficient on “almost sorted” data** (in some regimes for $\theta$).
Permuton limit
of record-biased permutations
(in the regime $\theta = \lambda n$)
**Definition:** A *permuton* $\mu$ is a probability measure on the unit square with uniform projections (or marginals):

for all $a < b$ in $[0, 1]$, $\mu([a, b] \times [0, 1]) = \mu([0, 1] \times [a, b]) = b - a$.

**Remark:** The normalized diagrams of permutations (denoted $\sigma$) are essentially permutons (denoted $\mu_{\sigma}$)

Replacing each point $(i/n, \sigma(i)/n)$ by a little square $[((i-1)/n, i/n) \times [\sigma(i)-1/n, \sigma(i)/n]$, and distributing the mass 1 uniformly on these little squares

**Convergence** of a sequence of permutations ($\sigma_n$) to a permuton $\mu$:

- inherited from the weak convergence of measures, namely:
  - $\sigma_n \to \mu$ when $\sup_{R \text{ rectangle } \subset [0,1]^2} |\mu_{\sigma_n}(R) - \mu(R)| \to 0$ as $n \to +\infty$.
  - If each $\sigma_n$ has size $n$, taking $R$ of the form $[0, i/n] \times [0, j/n]$ is enough.
Theorem:
Let $\sigma_n$ be a random record-biased permutation of size $n$ for $\theta = \lambda n$. $\mu_{\sigma_n}$ converges in probability to $\mu = \mu_c + \mu_u$ defined below.

Letting $f_\lambda(x) = \frac{x(\lambda+1)}{\lambda+x}$, we define

- $\mu_u$ is the uniform measure of total mass $c_\lambda \int_0^1 f_\lambda$ for $c_\lambda = \frac{1}{\lambda+1}$ on the area under the curve $y = f_\lambda(x)$;
- $\mu_c$ is the measure supported by the curve $y = f_\lambda(x)$ with density $\frac{\lambda}{\lambda+x}$ with respect to $Leb_c$, defined by $Leb_c(x, f_\lambda(x)) = Lebesgue(x)$

Two steps towards this statement:
guessing $\mu$ and proving convergence.
Guessing the limit $\mu$

The pictures suggest to decompose $\mu$ as $\mu_u + \mu_c$, with $\mu_c$ on a curve, and $\mu_u$ uniform under the curve. To determine are:

- the equation $y = f_\lambda(x)$ of the curve,
- how to distribute the mass between $\mu_c$ and $\mu_u$.

To find the equation $y = f_\lambda(x)$ of the curve,

- we estimate $\mathbb{P}(\text{max before position } i \text{ is } j)$ for $i \approx xn$ and $j \approx yn$;
- we find the relation between $x$ and $y$ which makes this probability not larger than 1, and non-zero once summed over $j$.

To find the relative measures on the curve and below,

- we compute the measure of the records in $\sigma_n$ and take the limit in $n$: this gives the measure $\int_0^1 \frac{\lambda}{\lambda + x} dx$ on the curve;
- we distribute uniformly the mass $c_\lambda \int_0^1 f_\lambda(x)dx$ below the curve, for $c_\lambda$ s.t. $\int_a^b (\frac{\lambda}{\lambda + x} + c_\lambda f_\lambda(x))dx = b - a$. 
Wrapping up

- We introduced a new model of non-uniform random permutations
  - with a bias toward sortedness via their records,
  - motivated by the analysis of algorithms,
  - and with applications there.

- Our model is however closely related to the Ewens model by Foata’s bijection.

- We have several efficient procedures for sampling our record-biased permutations.

- We described properties of this model, namely
  - the behavior of some classical statistics
  - and the permuton limit

!! Thank you !!

Any questions or suggestions?