Accompanying slides to the blackboard talk

Semi-Baxter and Strong-Baxter: two relatives of the Baxter sequence

Two Baxter families of permutations

Theorem:

- $|Av_n(2\,\underline{41}\,3, 3\,\underline{14}\,2)| = Bax_n$ (Baxter permutations)
- $|Av_n(2\,\underline{41}\,3, 3\,\underline{41}\,2)| = Bax_n$ (twisted Baxter permutations)

where $Bax_n = \frac{2}{n(n+1)^2} \sum_{j=1}^{n} \binom{n+1}{j-1}\binom{n+1}{j}\binom{n+1}{j+1}$.

First few terms: $1, 2, 6, 22, 92, 422, 2\,074, 10\,754$ [A001181]

Selected properties of the Baxter numbers:

- The generating function $\sum_n Bax_n x^n$ is not algebraic but is D-finite.
- There is a recursive formula for Bax_n:

$$Bax_n = \frac{7n^2 + 7n - 2}{(n+3)(n+2)} Bax_{n-1} + \frac{8(n-2)(n-1)}{(n+3)(n+2)} Bax_{n-2}.$$

Schema of proof of $|Av_n(2\underline{41}3, 3\underline{14}2)| = Bax_n$

Generating tree for Baxter permutations

$\downarrow$ (they go together)

Succession rule with two labels: Ω_{Bax}

$\downarrow$ (automatic)

Functional equation for the multivariate generating function $F(x; y, z)$

Coefficient of $x^n y^h z^k$ in $F(x; y, z)$

$=$ number of Baxter permutations of size n with label (h, k)

$\downarrow$ (the hard part, using *obstinate kernel method*)

Expression of the generating function

$\downarrow$ (Lagrange inversion formula)

Formula for the coefficients

Generating function for semi-Baxter permutations

Notation:

- $S(y,z) = S(x;y,z) = \sum_{n,h,k}$ number of semi-Baxter permutations of size n and label $(h,k) \cdot x^n y^h z^k$
- $SB_n = |Av_n(2\underline{41}3)| = [x^n]S(1,1)$, *i.e.* $S(1,1) = \sum_n SB_n x^n$

Theorem: Let W be the unique formal power series in x such that

$$W = \frac{x}{a}(1+a)(W+1+a)(W+a).$$

Then $S(1+a,1+a)$ is obtained by keeping only the terms with non-negative powers of a in

$$Q(a,W) = (1+a)^2\,x + \left(\frac{1}{a^5} + \frac{1}{a^4} + 2 + 2a\right)\,x\,W$$
$$+ \left(-\frac{1}{a^5} - \frac{1}{a^4} + \frac{1}{a^3} - \frac{1}{a^2} - \frac{1}{a} + 1\right)\,x\,W^2 - \left(\frac{1}{a^4} - \frac{1}{a^2}\right)\,x\,W^3.$$

Corollary: $S(1,1)$ is D-finite but not algebraic.

The semi-Baxter sequence SB_n

Explicit formula: $SB_{n+1} =$

$$\frac{1}{n}\sum_{j=0}^{n}\binom{n}{j}\left[2\binom{n+1}{j+2}\binom{n+j+2}{n+2}+\binom{n}{j+1}\binom{n+j+2}{n-3}+3\binom{n}{j+4}\binom{n+j+4}{n+1}\right.$$
$$\left.+2\binom{n}{j+2}\binom{n+j+4}{n}\left(2-\frac{n+j+5}{n+1}-\frac{n}{j+5}\right)+\frac{2n}{j+3}\binom{n}{j+2}\binom{n+j+2}{n}\right].$$

Proof: $SB_n = [x^n]S(1,1) = [a^0x^n]S(1+a,1+a) = [a^0x^n]Q(a,W)$ and use Lagrange inversion formula.

Recursive formula: $SB_n = \frac{11n^2+11n-6}{(n+4)(n+3)}SB_{n-1} + \frac{(n-3)(n-2)}{(n+4)(n+3)}SB_{n-2}$.

Proof: Creative telescoping (an automatic method of Zeilberger).

Asymptotics: $SB_n = A\,\frac{\mu^n}{n^6}\left(1+O\left(\frac{1}{n}\right)\right)$,
where $\mu = \varphi^5 = \frac{11}{2}+\frac{5}{2}\sqrt{5}$, $A = \frac{12\varphi^{-15/2}}{\pi\cdot 5^{1/4}} \approx 94.34$ and $\varphi = \frac{\sqrt{5}-1}{2}$

Proof: Applying a method of M. Bousquet-Mélou and G. Xin.

Proof for the GF result (1/2)

1. $S_{h,k} = \sum_n \sharp$semi-Baxter perm. of size n and label $(h,k) \cdot x^n$.
From Ω_{semi}, we get $S(y,z)$

$$= xyz + x \sum_{h,k \geq 1} S_{h,k} \left((y + y^2 + \ldots + y^h) z^{k+1} + (y^{h+k} z + y^{h+k-1} z^2 + \ldots + y^{h+1} z^k) \right)$$

$$= xyz + x \sum_{h,k \geq 1} S_{h,k} \left(\frac{1 - y^h}{1 - y} y\, z^{k+1} + \frac{1 - \left(\frac{y}{z}\right)^k}{1 - \frac{y}{z}} y^{h+1} z^k \right)$$

$$= xyz + \frac{xyz}{1-y} \left(S(1,z) - S(y,z) \right) + \frac{xyz}{z-y} \left(S(y,z) - S(y,y) \right) .$$

2. Write this functional equation in kernel form, for $S(1+a, z)$:

$$(\star) \quad K(a,z) S(1+a,z) = xz(1+a) + \frac{xz(1+a)}{a} S(1,z) - \frac{xz(1+a)}{z-1-a} S(1+a, 1+a),$$

with kernel $K(a,z) = 1 - \frac{xz(1+a)}{a} - \frac{xz(1+a)}{z-1-a}$.

3. Solve $K(a,z) = 0$ for z.
Of the two roots, only one (denoted Z) is a f.p.s. in x.

4. Substitute $z = Z$ in $(\star)$: LHS is 0.

Proof for the GF result (2/2)

5. *The obstinate trick:* Find rational transformations $(a, z) \mapsto (f(a, z), g(a, z))$ that leave $K(a, z)$ unchanged. Here, we find two, which generate a group of order 10.

6. Identify those such that $f(a, Z)$ and $g(a, Z)$ are f.p.s. in x. Here, we find 5.

7. Substituting in $(\star)$ gives a system of 5 equations (with LHS 0).

8. Eliminate $S(1, Z)$ and similar unknowns from the system. Here, we obtain $S(1+a, 1+a) - \frac{(1+a)^2 x}{a^4} S\left(1, 1+\frac{1}{a}\right) + Q(a, W) = 0$ where $W = Z - (1+a)$.

9. $W = Z - (1+a)$ and $K(a, Z) = 0$ define W as claimed.

10. Extract the non-negative powers of a in the above equation for $S(1+a, 1+a)$.

(short blackboard break . . .)

Generating function for strong-Baxter permutations

$I(y,z) = I(x;y,z) = \sum_{n,h,k}$ number of strong-Baxter permutations of size n and label $(h,k) \cdot x^n y^h z^k$

1. From Ω_{strong}, functional equation for $I(y,z)$:
$I(y,z) = xyz + \frac{x}{1-y}(y\, I(1,z) - I(y,z)) + xz\, I(y,z) + \frac{xyz}{1-z}(I(y,1) - I(y,z)).$

2. Kernel form of the equation, for $J(a,b) = I(1+a, 1+b)$:

$$K(a,b)J(a,b) = x(1+a)(1+b) - x\,\frac{1+a}{a}\,J(0,b) - x\,\frac{(1+a)(1+b)}{b}\,J(a,0),$$

with $K(a,b) = (1 - xQ(a,b))$ where $Q(a,b) = \frac{1}{a} + \frac{1}{b} + \frac{a}{b} + a + b + 2.$

5. There are two rational transformations that leave the kernel unchanged. Here, they generate a group which seems to be infinite.

$\Rightarrow$ The obstinate kernel method fails ...

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... but the kernel equation is reminiscent of walks in the quarter plane.

Connection to walks in the quarter plane

Notation:

- $W_{n,h,k}$ = number of walks in $\mathbb{N}^2$, starting at $(0,0)$, on step set $\mathfrak{S} = \{(-1,0), (0,-1), (1,-1), (1,0), (0,1)\}$, with n steps, and ending in (h,k).
- $Y_{n,h,k}$ = same, with step set $\mathfrak{S} \cup \{(0,0), (0,0)\}$
- $W(t;a,b)$ and $Y(t;a,b)$ their generating functions.

Theorem [Bostan, Raschel, Salvy]:
The generating function $W(t;a,b) = W(a,b)$ satisfies
$W(a,b) = 1 + t\left(\frac{1}{a} + \frac{1}{b} + \frac{a}{b} + a + b\right) W(a,b) - \frac{t}{a} W(0,b) - t\,\frac{(1+a)}{b} W(a,0)$.
Moreover neither $W(a,b)$ nor $W(0,0)$ are D-finite.

Fact 1: $Y(x;a,b) = W\left(\frac{x}{1-2x}; a, b\right) \frac{1}{1-2x}$ (combinatorial argument).
Fact 2: $J(x;a,b) = (1+a)(1+b)\,x\,Y(x;a,b)$ (same kernel eq.).

Corollary: The generating function $I(1,1) = J(0,0)$ of strong-Baxter numbers is not D-finite (and neither is $J(a,b)$).