Between weak and Bruhat: the middle order on permutations

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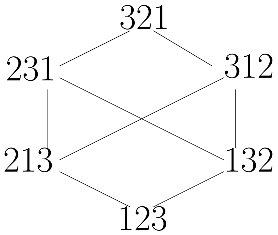
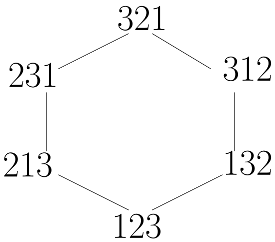
talk based on joint work with
Luca Ferrari and Bridget E. Tenner

Enumerative Combinatorics workshop in Oberwolfach, January 2026.

Familiar pictures

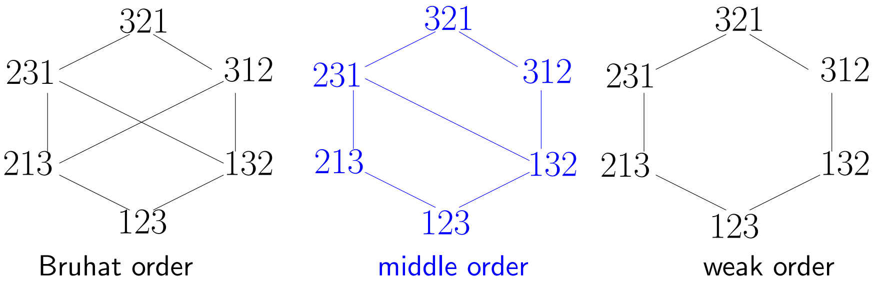


Bruhat order



weak order

Familiar pictures and a third one



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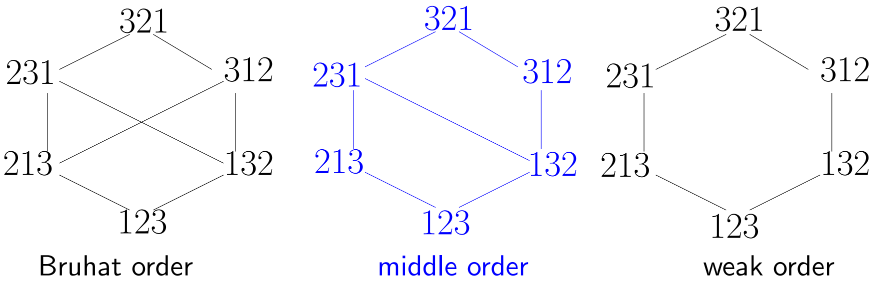


Goals of the talk:

- Define the middle order on S_n
- Give meaning to the property that "it sits between weak and Bruhat"
- Describe some of its properties
- Popularize this new order, hoping you raise new questions about it

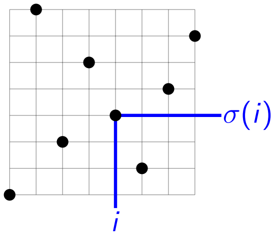
Our 3 orders through “mesh patterns”

Permutation diagrams

Notation:

- S_n = the set of permutations of size n
- $\sigma \in S_n$ is seen as the word $\sigma(1)\sigma(2)\cdots\sigma(n)$
- σ is also seen as its diagram *i.e.* the $n \times n$ grid with points at coordinates $(i, \sigma(i))$

Example: The word 1 8 3 6 4 2 5 7, or equivalently the diagram



, represents $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 8 & 3 & 6 & 4 & 2 & 5 & 7 \end{pmatrix}$

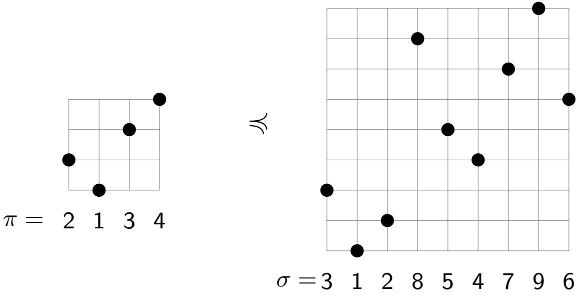
Permutation patterns

A permutation $\pi \in S_k$ is a pattern of a permutation $\sigma \in S_n$ (written $\pi \preccurlyeq \sigma$) when there exist indices $1 \leq i_1 < i_2 \cdots < i_k \leq n$ such that $\sigma(i_a) < \sigma(i_b)$ if and only if $\pi(a) < \pi(b)$

The subsequence $\sigma(i_1)\sigma(i_2)\cdots\sigma(i_k)$ is an occurrence of π in σ

Example: 2134 is a pattern of 3 1 2 8 5 4 7 9 6, one occurrence being 3157

This is also seen on diagrams:



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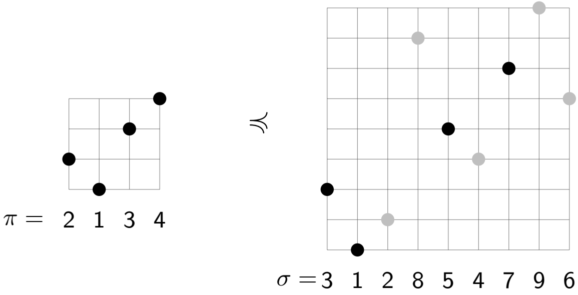
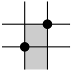
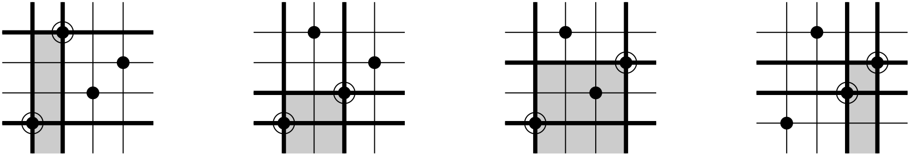


Mesh patterns

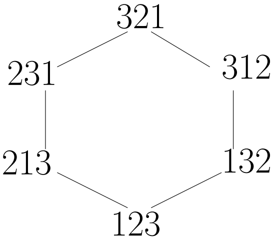
A mesh pattern (π, M) is the data of a pattern π (say, of size k) drawn in the central $k \times k$ square of the grid $[0, k+1]^2$, together with a set M of shaded unit cells in this grid. (M is called the mesh.)

An occurrence of (π, M) in σ is an occurrence of π in σ such that the regions of $[0, n+1]^2$ corresponding to the mesh M contain no points of σ

Example: Consider the mesh pattern $\mu =$ . The permutation 1423 contains four occurrences of 12, but only three of μ.



Weak order, seen through mesh patterns



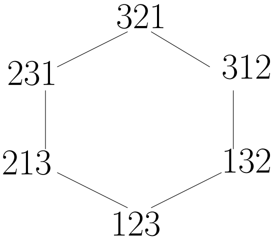
Covering relations are described by

$$\cdots i\, j \cdots \rightsquigarrow \cdots j\, i \cdots$$

i.e., transforming an ascent[(a)] into a descent[(d)] using the same two values.

(a) occurrence of 12 at consecutive positions

(d) occurrence of 21 at consecutive positions



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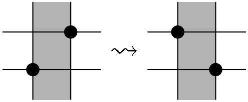
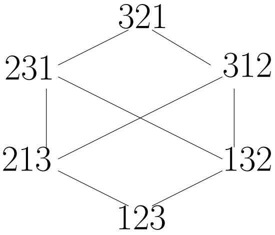


Bruhat order, seen through mesh patterns



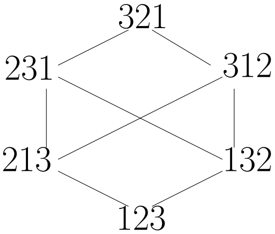
Relations are described by the swaps

$$\cdots i \cdots j \cdots \rightsquigarrow \cdots j \cdots i \cdots$$

i.e., transforming a non-inversion (=occurrence of 12) into an inversion (=occurrence of 21) using the same two values.

Covering relations are the relations that do not create additional inversions.

Bruhat order, seen through mesh patterns



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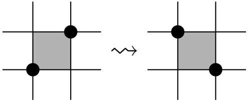

Middle order, defined through mesh patterns

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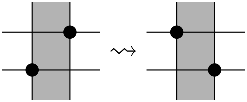
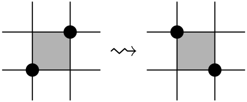


- For the Bruhat order, the covering relations are described by



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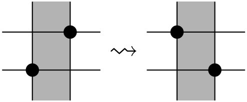

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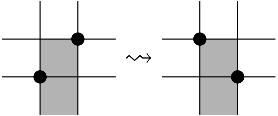

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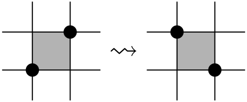
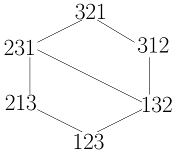


Summary so far, and what's ahead

- The middle order in size 3:



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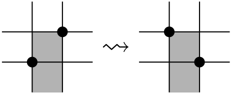

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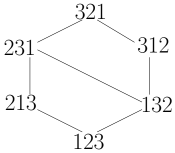

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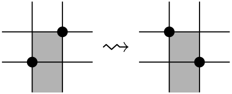


- This interpolates between the weak order and the Bruhat order

What comes next:

- Another combinatorial interpretation of the middle order
- Some of its properties as a poset (in particular: distributive lattice)
- Enumeration of its intervals, and of its boolean intervals
- Implication on its Möbius function
- Combinatorial description of its Euler characteristic
- Restriction to the subset of involutions

The middle order and inversion sequences

Inversion sequences, and bijection with permutations

- Reminder: Inversions are occurrences $\cdots j \cdots i \cdots$ of the pattern 21.
- j is called inversion top.

- Given $\sigma \in S_n$, let x_j = number of inversions of σ such that j is the inversion top. Observe that $0 \leq x_j < j$.
- Let $\varphi(\sigma) = (x_1, x_2, \cdots, x_n)$ be the inversion sequence of σ.
- Sometimes called Lehmer code. Several (symmetric) variants exist.

- Example: For $\sigma = 4\,1\,5\,6\,2\,3$, we have $\varphi(\sigma) = (0, 0, 0, 3, 2, 2)$
- Remark: $x_j = j - 1$ if and only if j is a LtoR-minimum.

- This is a bijection between S_n and the set I_n of inversion sequences of size n:

$$I_n = [0, 0] \times [0, 1] \times [0, 2] \times \cdots \times [0, n-1]$$

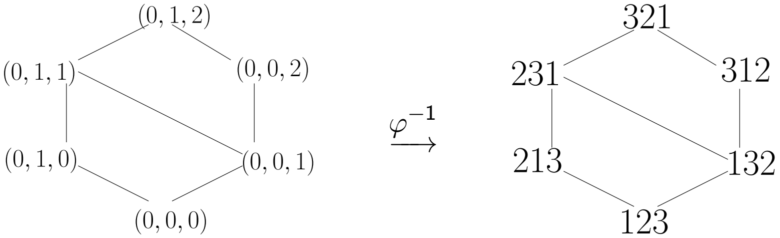
Middle order through inversion sequences

For inversion sequences $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and $\mathbf{y} = (y_1, y_2, \cdots, y_n)$, define the partial order

$$\mathbf{x} \leq \mathbf{y} \text{ when } x_i \leq y_i \text{ for all } i$$

In particular, covering relations correspond to adding 1 to one component (provided we stay among inversion sequences).

Theorem: The middle order is the image of the above by the bijection φ^{-1}.



Proving this characterization of the middle order

Let $\varphi(\sigma) = (x_1, \dots, x_n)$ and $\varphi(\tau) = (x_1, \dots, x_{j-1}, x_j + 1, x_{j+1}, \dots, x_n)$.

- In particular, $x_j < j - 1$.
- So j is not a LtoR-minimum.
- So, we can define i as the rightmost entry to the left of j in σ such that $i < j$, and (i, j) is an occurrence of .
- We check that τ is the permutation obtained swapping i and j, so that τ covers σ in the middle order.

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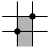
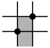
Let τ be obtained from σ by transforming one  into .

- Let j be the largest of the two elements involved in 
- $\varphi(\sigma)$ and $\varphi(\tau)$ differ only at their j-th coordinate
- and the difference is $+1$
- meaning that $\varphi(\tau)$ covers $\varphi(\sigma)$ in the defined order on inversion sequences

Some properties of the middle order

A product of chains

We have seen that the middle order $\mathcal{P}_n$ is isomorphic (with explicit bijection φ) to the product of chains

$$[0,0] \times [0,1] \times [0,2] \times \cdots \times [0, n-1]$$

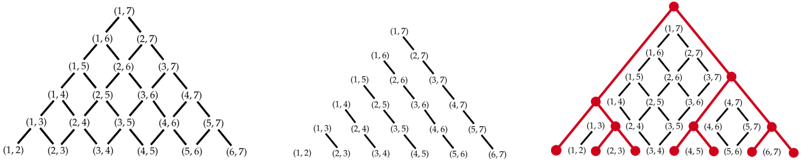
Consequences:

- $\mathcal{P}_n$ is a lattice: any σ and τ have a least upper bound $\sigma \vee \tau$ (called join) and a greatest lower bound, denoted $\sigma \wedge \tau$ (called meet).
  The join (resp. meet) is obtained taking component-wise maximum (resp. minimum) on corresponding inversion sequences.
- In addition, $\mathcal{P}_n$ is a distributive lattice.
  (meaning that $\vee$ is distributive over $\wedge$ and vice-versa).
- $\mathcal{P}_n$ is graded, *i.e.* has a rank function r, meaning that, for any σ, we can define $r(\sigma)$ as the length of any maximal chain from $12 \cdots n$ to σ.
  In $\mathcal{P}_n$, we have $r(\sigma) =$ number of inversions of σ.

A step aside: more middle orders, by Ludovic Schwob

Call **a** (generalized) middle order any distributive lattice that is between weak and Bruhat.

- These are completely characterized (through posets of irreducible elements).



- There are Catalan-many of them.
- Some of this work generalizes to other Coxeter groups.

"J'espère finir de rédiger le préprint d'ici février" – L. Schwob, 15/01/2026

Back to: A product of chains

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Characterizing and counting all intervals in $\mathcal{P}_n$

Intervals of the j-element chain $[0, j-1]$:

- Such intervals are
  - of the form $\{a\}$ for $0 \leq a \leq j-1$,
  - or of the form $[a, b]$ for $0 \leq a < b \leq j-1$.
- Therefore, there are $j + \binom{j}{2} = \binom{j+1}{2}$ such intervals.

Intervals of $\mathcal{P}_n$ (up to isomorphism φ):

- Such intervals correspond to intervals $[(x_1, \ldots, x_n), (y_1, \ldots, y_n)]$ where each $[x_j, y_j]$ is an interval of $[0, j-1]$.
- Therefore, there are $\prod_{j=1}^{n} \binom{j+1}{2} = \frac{n!(n+1)!}{2^n}$ intervals in $\mathcal{P}_n$.

Refined counting of intervals by rank, with a recursive formula for the number $f(n, k)$ of intervals of rank k in $\mathcal{P}_n$:

$$f(n, k) = \sum_{h=0}^{n-1} (n-h) \cdot f(n-1, k-h)$$

Table of $f(n,k)$, which is A139769 on the OEIS

n \ k	**0**	1	2	3	4	5	6	7	8	9	10
1	**1**										
2	**2**	1									
3	**6**	7	4	1							
4	**24**	46	49	36	18	6	1				
5	**120**	326	501	562	497	354	204	94	33	8	1

- $\mathbf{k = 0}$: $f(n,0) = n!$ since it counts elements in $\mathcal{P}_n$
- $k = 1$: $f(n,1)$ counts the covering relations in $\mathcal{P}_n$. From the previous theorem, we have $f(n,1) = n!(n - H_n)$, where $H_n = \sum_{i=1}^{n} \frac{1}{i}$.

  This is sequence A067318 in the OEIS.

  Another interpretation of this sequence is as the sum of reflection lengths of all elements of S_n. Not hard to explain expressing reflection length of σ as (size $-$ number of cycles of σ).

Characterizing and counting boolean intervals in $\mathcal{P}_n$

Dfn: An interval is boolean if it is isomorphic to a boolean algebra.

Characterization and enumeration:

- Boolean intervals of $\mathcal{P}_n$ correspond to pairs $(x_1, \ldots, x_n), (y_1, \ldots, y_n)$ of inversion sequences with $y_j \in \{x_j, x_j + 1\}$ for all j.
- The number of boolean intervals in $\mathcal{P}_n$ is $(2n-1)!!$.
  - Indeed, each pair (x_j, y_j) has j possibilities if $y_j = x_j$, and $j-1$ possibilities if $y_j = x_j + 1$, hence $2j - 1$ possibilities.

The number of boolean intervals of rank k (nec., $k < n$) in $\mathcal{P}_n$ is

$$b(n,k) = \sum_{i=0}^{n} \binom{i}{k} c(n, n-i)$$

where $c(n,j)$ are the signless Stirling numbers of the first kind.

Möbius function

Dfn: The Möbius function μ on any poset $\mathcal{P}$ is defined recursively by

$$\mu(s,u) = \begin{cases} 0 & \text{if } s \not\leq u, \\ 1 & \text{if } s = u, \text{ and} \\ -\sum_{s \leq t < u} \mu(s,t) & \text{for all } s < u. \end{cases}$$

It is typically hard to compute on an ordinary (even combinatorial) poset. But...

Prop: In finite distributive lattices, for any v, w, it holds that

- $\mu(v,w)$ is equal to 0 if the interval $[v,w]$ is not boolean
- and otherwise $\mu(v,w) = (-1)^t$, where t is the rank of $[v,w]$.

Euler characteristic

Let $\mathcal{P}$ be a finite distributive lattice. (Recall that $\mathcal{P}_n$ has this property.)

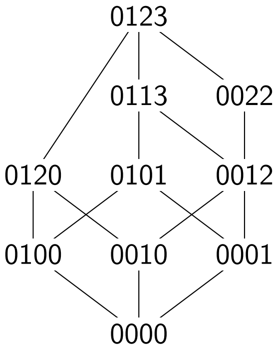
- Dfn: A valuation on $\mathcal{P}$ is a function ν that satisfies $\nu(\min(\mathcal{P})) = 0$ and for all x, y, $\nu(x) + \nu(y) = \nu(x \wedge y) + \nu(x \vee y)$.
- Prop.: A valuation is determined by its values on the join-irreducibles.
- Dfn: An element of $\mathcal{P}$ is join-irreducible if it covers exactly one element of $\mathcal{P}$.
- Dfn: The Euler characteristic is the unique valuation χ such that $\chi(a) = 1$ for every join-irreducible a.

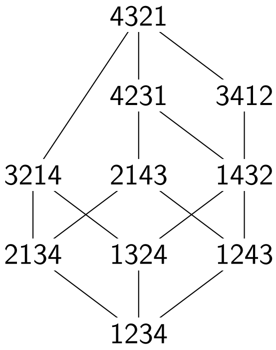
We can characterize the join-irreducible elements of $\mathcal{P}_n$, and subsequently prove that the Euler characteristic χ on $\mathcal{P}_n$ is given by

$$\chi(\sigma) = \text{ number of RtoL-non-minima of } \sigma.$$

Finding something not-so-nice in something too-beautiful

$\mathcal{P}_n$ is extremely well behaved. What about its restriction to involutions?





The subsequent poset $\mathcal{I}_n$ is not a lattice, not graded, and not an interval-closed subposet of $\mathcal{P}_n$.

But ... we can still compute the Möbius function in $\mathcal{I}_n$.

Möbius function on $\mathcal{I}_n$

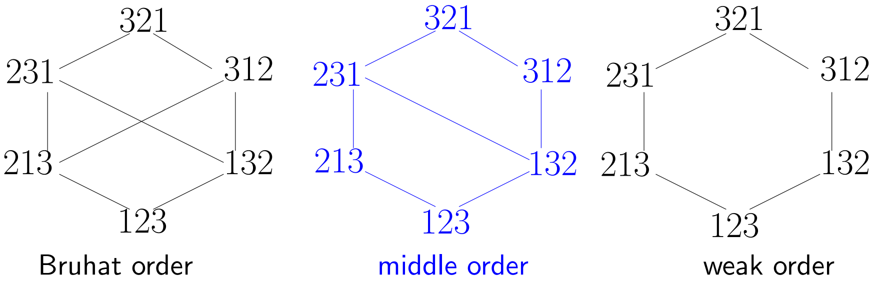
We can characterize inversion sequences of involutions, using the classical decomposition that proves the relation $i(n) = i(n-1) + (n-1) \cdot i(n-2)$ for $n \geq 2$.

- We say that $(x_1, \dots, x_n)$ is slow-climbing when it does not contain large ascents, defined as factors (x_i, x_{i+1}) with $x_{i+1} > x_i + 1$.
- Lemma: The inversion sequence of an involution is slow-climbing if and only if it is the concatenation of factors $(0, 1, \cdots, h)$ for some (possibly different) $h \geq 0$.

Theorem: For any involution $\sigma \in \mathcal{I}_n$, let α be the number of non-zero entries in $\varphi(\sigma)$. The Möbius function in $\mathcal{I}_n$ is given by

$$\mu(12\dots n, \sigma) = \begin{cases} (-1)^\alpha & \text{if } \varphi(\sigma) \text{ is slow-climbing, and} \\ 0 & \text{otherwise.} \end{cases}$$

Recap



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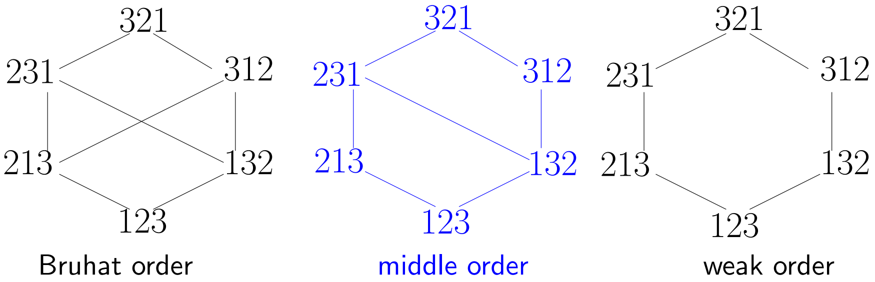

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