Operators of equivalent sorting power and related Wilf-equivalences

Mathilde Bouvel (LaBRI, Bordeaux, France)

joint work with
Michael Albert (University of Otago, New Zealand)

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Computer Science

UNIVERSITY of OTAGO

Te Whare Wānanga o Otāgo

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Operators of equivalent sorting power ...

We study permutations sortable by sorting operators which are compositions of stack sorting operators **S** and reverse operators **R**.

Theorem (Bouvel, Guibert 2012)

There are as many permutations of $\mathfrak{S}_n$ sortable by $\mathbf{S} \circ \mathbf{S}$ as permutations of $\mathfrak{S}_n$ sortable by $\mathbf{S} \circ \mathbf{R} \circ \mathbf{S}$, and many permutation statistics are equidistributed across these two sets.

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Theorem (Albert, Bouvel 2013)

*For any operator **A** which is a composition of operators **S** and **R**, there are as many permutations of $\mathfrak{S}_n$ sortable by $\mathbf{S} \circ \mathbf{A}$ as permutations of $\mathfrak{S}_n$ sortable by $\mathbf{S} \circ \mathbf{R} \circ \mathbf{A}$. Moreover, many permutation statistics are equidistributed across these two sets.*

as suggested by the computer experiments of O. Guibert.

...and related Wilf-equivalences

Our proof uses:

- The characterization of preimages of permutations by **S**
  [M. Bousquet-Mélou, 2000]
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Definition: $\{\pi, \pi'\}$ and $\{\tau, \tau'\}$ are Wilf-equivalent when Av(π, π') and Av(τ, τ') are enumerated by the same sequence.

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Specializing, our general result gives for instance:

Proposition

The sets of patterns $\{231, 31254\}$ *and* $\{132, 42351\}$ *are Wilf-equivalent.*
Moreover, the common generating function of the classes Av$(231, 31254)$ *and* Av$(132, 42351)$ *is* $\frac{t^3-t^2-2t+1}{2t^3-3t+1}$.

Definitions

Permutations and patterns

Permutation: Bijection from $[1..n]$ to itself. Set $\mathfrak{S}_n$.

We view permutations as words, $\sigma = \sigma_1\sigma_2\ldots\sigma_n$
Example: $\sigma = 1\ 8\ 3\ 6\ 4\ 2\ 5\ 7$.

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Occurrence of a pattern: $\pi \in \mathfrak{S}_k$ is a pattern of $\sigma \in \mathfrak{S}_n$ if
$\exists\ i_1 < \dots < i_k$ such that $\sigma_{i_1} \dots \sigma_{i_k}$ is order isomorphic $(\equiv)$ to π.

Notation: $\pi \preccurlyeq \sigma$.

Equivalently: The normalization of $\sigma_{i_1} \dots \sigma_{i_k}$ on $[1..k]$ yields π.

Example: $2134 \preccurlyeq \mathbf{3}\,\mathbf{1}\,2\,8\,\mathbf{5}\,4\,\mathbf{7}\,9\,6$ since $3157 \equiv 2134$.

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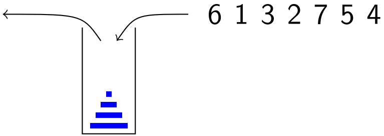
Example: $2134 \preccurlyeq \mathbf{3}\mathbf{1}28\mathbf{5}4\mathbf{7}96$ since $3157 \equiv 2134$.

Avoidance: $Av(\pi, \tau, \dots)$ = set of permutations that do not contain any occurrence of π or τ or ...

The stack sorting operator **S**

Sort (or try to do so) using a stack satisfying the Hanoi condition.



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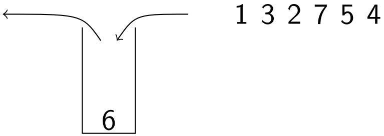

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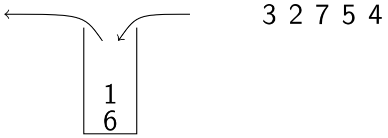

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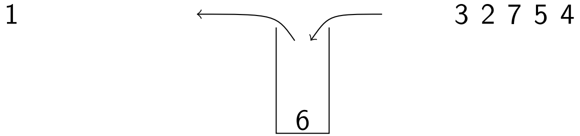

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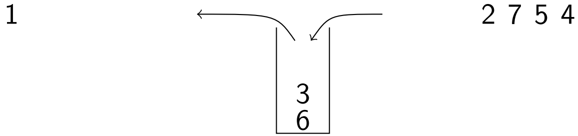

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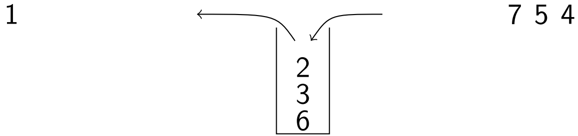

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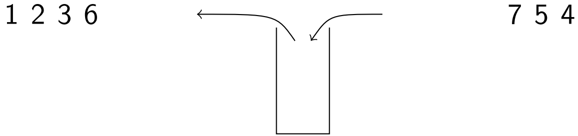

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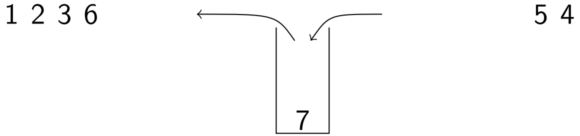

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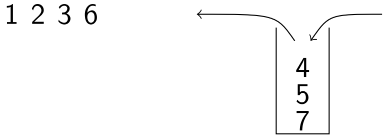

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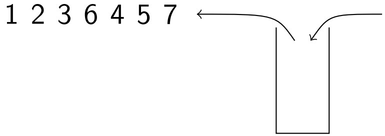

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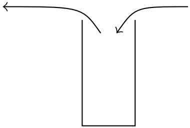
$\mathbf{S}(\sigma) = 1\ 2\ 3\ 6\ 4\ 5\ 7$



$6\ 1\ 3\ 2\ 7\ 5\ 4 = \sigma$

Equivalently, $\mathbf{S}(\varepsilon) = \varepsilon$ and $\mathbf{S}(LnR) = \mathbf{S}(L)\mathbf{S}(R)n$, $n = \max(LnR)$

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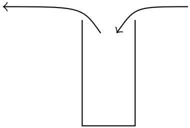

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- Permutations sortable by $\mathbf{S}$: Av(231), enumeration by Catalan numbers [Knuth 1975]
- Sortable by $\mathbf{S} \circ \mathbf{S}$: $\text{Av}(2341, 3\bar{5}241)$[West 1993], enumeration by $\frac{2(3n)!}{(n+1)!(2n+1)!}$ [Zeilberger 1992]
- Sortable by $\mathbf{S} \circ \mathbf{S} \circ \mathbf{S}$: characterization with (generalized) excluded patterns [Claesson, Úlfarsson 2012], no enumeration result

Main result

Reverse operator $\mathbf{R}$: $\mathbf{R}(\sigma_1\sigma_2\cdots\sigma_n) = \sigma_n\cdots\sigma_2\sigma_1$

Theorem

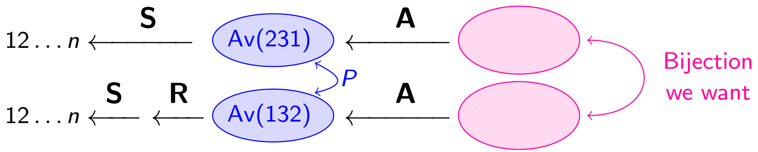
For any operator $\mathbf{A}$ which is a composition of operators $\mathbf{S}$ and $\mathbf{R}$, there are as many permutations of $\mathfrak{S}_n$ sortable by $\mathbf{S}\circ\mathbf{A}$ as permutations of $\mathfrak{S}_n$ sortable by $\mathbf{S}\circ\mathbf{R}\circ\mathbf{A}$.

Main ingredients for the proof:

- the characterization of preimages of permutations by $\mathbf{S}$;
  [M. Bousquet-Mélou, 2000]
- the new bijection P between Av(231) and Av(132).

How does the theorem relate to these ingredients?

Main result, an equivalent statement



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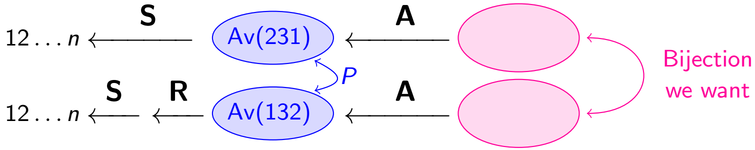


Theorem

For any operator **A** *which is a composition of operators* **S** *and* **R**, *P is a size-preserving bijection between*

- *permutations of* Av(231) *that belong to the image of* **A**, *and*
- *permutations of* Av(132) *that belong to the image of* **A**,

that preserves the number of preimages under **A**.

Proof of the main result

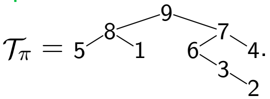
Some ingredients and some ideas

Canonical trees and preimages under **S**

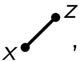
Lemma (Bousquet-Mélou 2000)

For any permutation π in the image of **S**, *there is a unique canonical tree $\mathcal{T}_\pi$ whose post-order reading is π.*

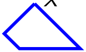
Example: For $\pi = 5\,1\,8\,2\,3\,6\,4\,7\,9$,



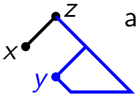
Canonical tree:

For every edge 

,

there exists  $\neq \emptyset$

and y such that



and $y < x$.

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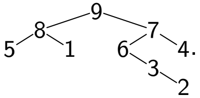

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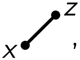

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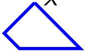
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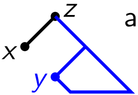

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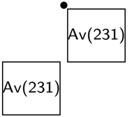
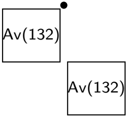
Theorem (Bousquet-Mélou 2000)

$\mathcal{T}_\pi$ determines $\mathbf{S}^{-1}(\pi)$.
Moreover $|\mathbf{S}^{-1}(\pi)|$ *is determined only by the shape of* $\mathcal{T}_\pi$.

Bijection $\mathrm{Av}(231) \overset{P}{\longleftrightarrow} \mathrm{Av}(132)$

Representing permutations as diagrams, we have

$\mathrm{Av}(231) = \varepsilon +$ 

and $\mathrm{Av}(132) = \varepsilon +$ 

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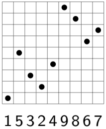
Example: $\pi =$ 

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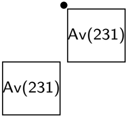
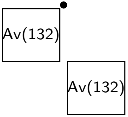
Av(231) $= \varepsilon +$ 

and Av(132) $= \varepsilon +$ 

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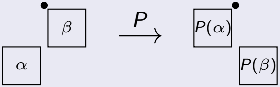
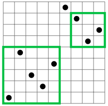
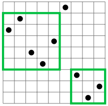
Definition

We define P : Av(231) $\rightarrow$ Av(132) recursively as follows:



, with $\alpha, \beta \in$ Av(231)

Example: For $\pi =$ 
1 5 3 2 4 9 8 6 7
, we obtain $P(\pi) =$ 
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Bijection $\Phi_{\mathbf{A}}$ between $\mathbf{S} \circ \mathbf{A}$- and $\mathbf{S} \circ \mathbf{R} \circ \mathbf{A}$-sortables

For $\pi \in \text{Av}(231)$, write $P(\pi) \in \text{Av}(132)$ as $P(\pi) = \lambda_\pi \circ \pi$.

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$\Phi_{\mathbf{A}}$ *is a size-preserving bijection between permutation sortable by* $\mathbf{S} \circ \mathbf{A}$ *and those sortable by* $\mathbf{S} \circ \mathbf{R} \circ \mathbf{A}$.

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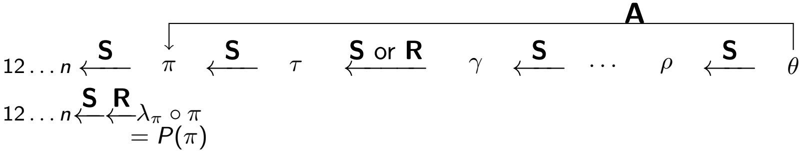

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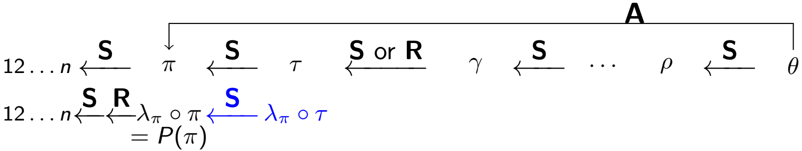

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$$\pi \xleftarrow{\mathbf{A}} \theta$$

$$12\dots n \xleftarrow{\mathbf{S}} \pi \xleftarrow{\mathbf{S}} \tau \xleftarrow{\mathbf{S} \text{ or } \mathbf{R}} \gamma \xleftarrow{\mathbf{S}} \cdots \quad \rho \xleftarrow{\mathbf{S}} \theta$$

$$12\dots n \xleftarrow{\mathbf{S}} \xleftarrow{\mathbf{R}} \lambda_\pi \circ \pi \xleftarrow{\mathbf{S}} \lambda_\pi \circ \tau \xleftarrow{\mathbf{S} \text{ or } \mathbf{R}} \lambda_\pi \circ \gamma$$

$$= P(\pi)$$

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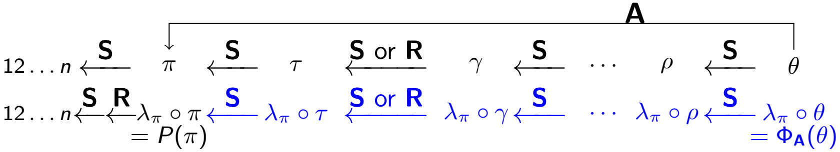

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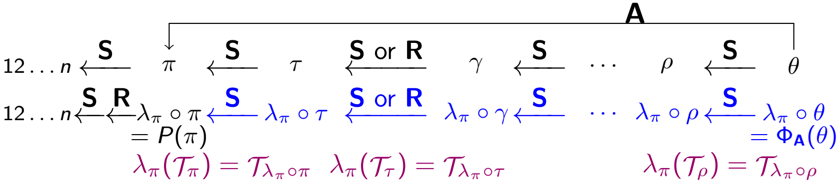
$\Phi_{\mathbf{A}}$ *is a size-preserving bijection between permutation sortable by* $\mathbf{S} \circ \mathbf{A}$ *and those sortable by* $\mathbf{S} \circ \mathbf{R} \circ \mathbf{A}$.



More about the bijection

$$\mathrm{Av}(231) \overset{P}{\longleftrightarrow} \mathrm{Av}(132)$$

Related Wilf-equivalences

P and Wilf-equivalences

$\{\pi, \pi', \ldots\}$ and $\{\tau, \tau', \ldots\}$ are Wilf-equivalent when $\mathsf{Av}(\pi, \pi', \ldots)$ and $\mathsf{Av}(\tau, \tau', \ldots)$ are enumerated by the same sequence.

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Description of the patterns $\pi \in \mathrm{Av}(231)$ *such that* P *provides a bijection between* $\mathrm{Av}(231, \pi)$ *and* $\mathrm{Av}(132, P(\pi))$

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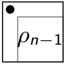
Theorem

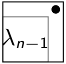
Computation of the generating function of such classes $\mathrm{Av}(231, \pi)$ *…and it depends only on* $|\pi|$.

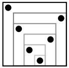
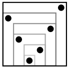
$\Rightarrow$ Even more Wilf-equivalences!

(λ_n), (ρ_n) and patterns π such that $\mathrm{Av}(231, \pi) \overset{P}{\longleftrightarrow} \mathrm{Av}(132, P(\pi))$

From $\lambda_0 = \rho_0 = \varepsilon$, define recursively

$\lambda_n =$ 

and $\rho_n =$ 

. Ex.: $\lambda_6 =$ 
, $\rho_6 =$ 
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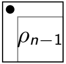
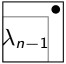
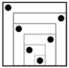
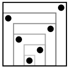
$\lambda_n =$ 

and $\rho_n =$ 

. Ex.: $\lambda_6 =$  , $\rho_6 =$  .

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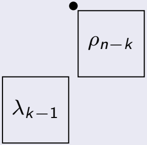
$\pi =$ 

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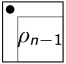
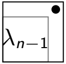
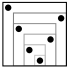
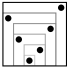
$\lambda_n =$ 

and $\rho_n =$ 

. Ex.: $\lambda_6 =$ 
, $\rho_6 =$ 
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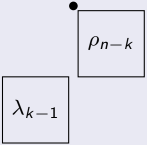
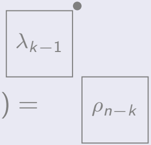
$\pi =$ 

thus $P(\pi) =$ 

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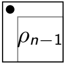
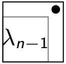
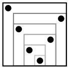
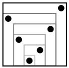
$\lambda_n =$ 

and $\rho_n =$ 

. Ex.: $\lambda_6 =$ 
, $\rho_6 =$ 
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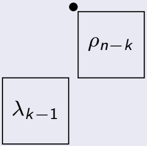
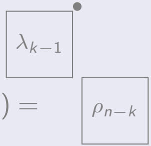
$\pi =$ 

thus $P(\pi) =$ 

$\Rightarrow$ For all such π, $\{231, \pi\}$ and $\{132, P(\pi)\}$ are Wilf-equivalent.

Example: $\{231, 31254\}$ and $\{132, 42351\}$ are Wilf-equivalent

Common generating function when $\mathrm{Av}(231,\pi) \overset{P}{\longleftrightarrow} \mathrm{Av}(132, P(\pi))$

Definition: $F_1(t) = 1$ and $F_{n+1}(t) = \frac{1}{1-tF_n(t)}$.

Theorem

For $\pi \in \mathrm{Av}(231)$ *such that* $\mathrm{Av}(231,\pi) \overset{P}{\longleftrightarrow} \mathrm{Av}(132, P(\pi))$, *denoting* $n = |\pi|$, *the generating function of* $\mathrm{Av}(231,\pi)$ *is* F_n.

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For $\pi \in \mathrm{Av}(231)$ such that $\mathrm{Av}(231,\pi) \stackrel{P}{\longleftrightarrow} \mathrm{Av}(132, P(\pi))$, denoting $n = |\pi|$, the generating function of $\mathrm{Av}(231,\pi)$ is F_n.

Theorem

$\{231,\pi\}$ and $\{132, P(\pi)\}$ are all Wilf-equivalent when $|\pi| = |\pi'| = n$ and π and π' are of the form described earlier. Moreover, the generating function of $\mathrm{Av}(231,\pi)$ is F_n.