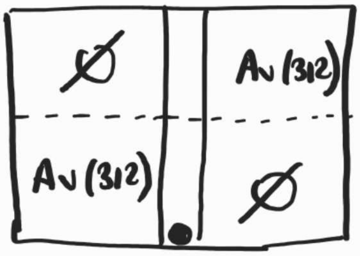
Exo 1.1.1

$Av(312) = \mathcal{E} \uplus$ 

the empty permutation

the min.

Hence the G.F. $C(x)$ of $Av(312)$ satisfies

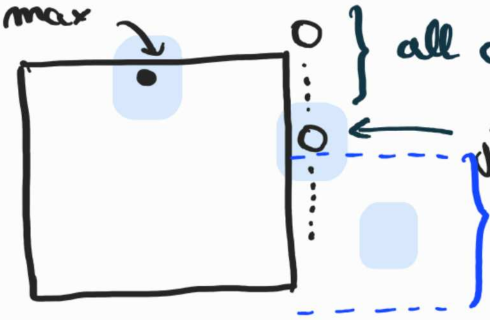
$$C(x) = 1 + x \cdot C(x)^2 .$$

Exo 1.1.2

$Av(321)$

- Insertion in the top site is always possible.
- Insertion in another site, say the j-th from the top: $(j \geq 2)$

max



all o stay o

j-th active site

all o become x

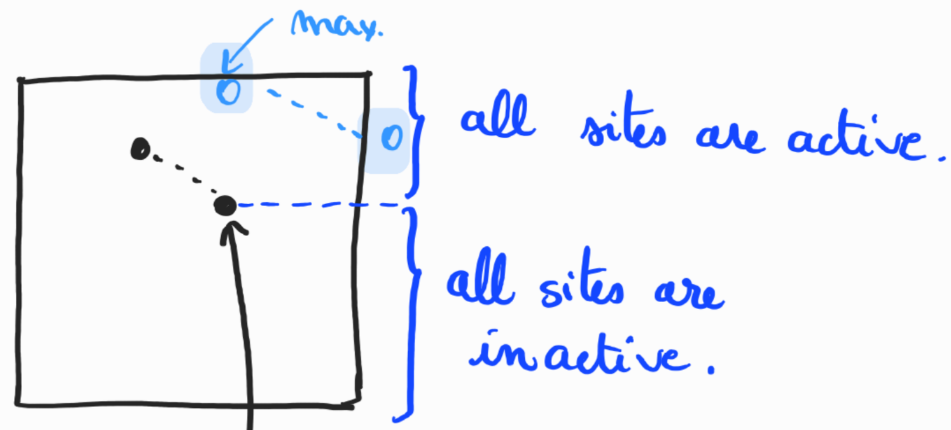
So insertion in the j-th active site produces a child with j active sites.

- The rewriting rule encoding this generating tree is

$$\begin{cases} (2) \\ (k) \rightsquigarrow (k+1), (2), (3), \dots (k). \end{cases}$$

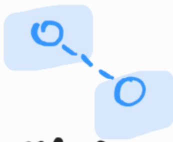
This is Ω_{cat}.

Exo 1.1.2 (another way to understand active sites).



define the highest point
which is the second point
of an inversion
(which exists unless $\sigma = Id$).

$\Rightarrow$ $\begin{cases} \text{• insertion in the top site adds an active site} \\ \text{• insertion in another site creates a new highest inversion} \end{cases}$



. The active sites are those above the new point added.

Hence the rewriting rule is

for $\begin{cases} (2) \\ (k) \rightsquigarrow (k+1), (2), \dots, (k) \end{cases}$

$\blacksquare$

Exo 1.2

1) Last descent = everything which comes after the last up step
= longest suffix of down and double flat steps.

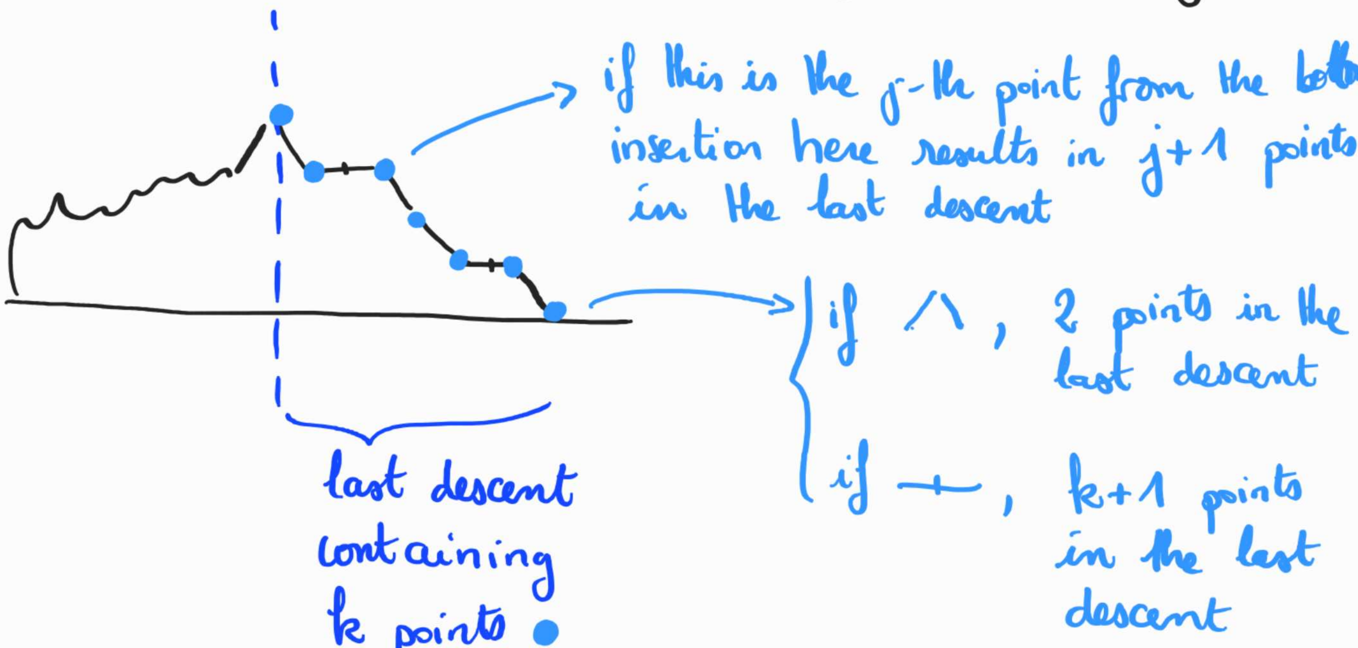
What happens if we remove the final step of a Schröder path?
→ if it is flat, we just remove it.
→ if it is a down step, we can remove it together with the last up step of the path.

So, the "good choice" to define insertions is:
* insertion of an up step at any point of the last descent + of a final down step;
* insertion of a double flat step at the end of the path

This ensures that all Schröder paths are generated exactly once.

2) Label = number of points in the last descent $\underbrace{+1}$

the last point "counts twice", for insertion of $\wedge$ and of —+—.



Hence the production is

$$(*) \quad (k) \rightsquigarrow (k+1), (3), (4), \ldots, (k+1)$$

The starting points would be

$\wedge$ of label (3) and —+— of label (3).

We can consider the empty path ε and assign to it the label (2).

The general rule $(*)$ for $k = 2$ gives

$$\underset{\varepsilon}{(2)} \rightsquigarrow \underset{\text{—+—}}{(3)} \; \underset{\wedge}{(3)}.$$

So we can take (2) for the root label.

This gives $\begin{cases} (2) \\ (k) \rightsquigarrow (k+1), (3), \dots, (k+1). \end{cases}$

3)

$$S(x,y) = \underset{\varepsilon}{y^2} + \sum_{\substack{n \geq 0 \\ k \geq 2}} s_{n,k}\, x^{n+1} \left(y^{k+1} + y^3 + \dots + y^{k+1} \right)$$

contribution of the children of each path

$$= y^2 + xy\,S(x,y) + x y^3 \sum_{\substack{n \geq 0 \\ k \geq 2}} s_{n,k}\, x^n \underbrace{\left(1 + \dots + y^{k-2}\right)}_{\frac{1-y^{k-1}}{1-y}}$$

$$= y^2 + xy\,S(x,y) + \frac{xy^3}{1-y} \left(\sum_{\substack{n \geq 0 \\ k \geq 2}} s_{n,k} x^n - \sum_{\substack{n \geq 0 \\ k \geq 2}} s_{n,k} x^n y^{k-1} \right)$$

$$= y^2 + xy\,S(x,y) + \frac{xy^3}{1-y} S(x,1) - \frac{xy^2}{1-y} S(x,y)$$

$$\Rightarrow S(x,y) \cdot \left(1 - xy + \frac{xy^2}{1-y} \right) = y^2 + \frac{xy^3}{1-y} S(x,1)$$

$\times (1-y)$

$$S(x,y) \cdot \underbrace{\left(1 - xy - y + 2xy^2\right)}_{K(x,y)} = y^2 - y^3 + xy^3\, S(x,1)$$

4) $K(x,y) = 2x\,y^2 - (1+x)\,y + 1.$

Solving $K(x,y) = 0$

$$\Delta = (1+x)^2 - 8x = 1 - 6x + x^2$$

Solutions: $\dfrac{1+x \pm \sqrt{1-6x+x^2}}{4x}$

The only formal power series solution is

$$Y(x) = \frac{1+x - \sqrt{1-6x+x^2}}{4x}.$$

Substituting $y = Y(x)$ in the Kernel equation gives:

$$S(x,1) = \frac{Y(x)^3 - Y(x)^2}{x\,Y(x)^3} = \frac{1 - 1/Y(x)}{x}$$

$$\frac{1}{Y(x)} = \frac{4x\left(1+x+\sqrt{1-6x+x^2}\right)}{(1+x)^2 - (1-6x+x^2)}$$

$$= \frac{4x\left(1+x+\sqrt{1-6x+x^2}\right)}{8x}$$

$$= \frac{1+x+\sqrt{1-6x+x^2}}{2}$$

$$S(x,1) = \frac{2-\left(1+x+\sqrt{1-6x+x^2}\right)}{2x}$$

$$= \frac{1-x-\sqrt{1-6x+x^2}}{2x}$$

Exo 1.3

1) Finding the parent :

* if the last step is flat, remove it.
* if the last step is a down step, remove it and transform the last up step into a flat step.

So, to define the children, we consider the last descent (= longest suffix of flat and down steps).

We can add a flat step at the end.

We can also change each flat step of the last descent into an up step, adding a down step at the end of the path.

Defining the labels as the number of flat steps in the last descent +1, this yields the rewriting rule

for the path ⟼

$$\begin{cases} (2) \\ (k) \rightsquigarrow (k+1), (1), \dots, (k-1) \end{cases}$$

adding a flat step at the end

2) Let $m_{n,k}$ be the number of Motzkin paths of size n and label k, and let

$$M(x,y) = \sum_{\substack{n\geq 1\\ k\geq 1}} m_{n,k}\, x^n y^k .$$

The rewriting rule gives

$$M(x,y) = xy^2 + \sum_{\substack{n\geq 1\\ k\geq 1}} m_{n,k}\, x^{n+1} \Big(\underbrace{y+\dots+y^{k-1}}_{\frac{y-y^k}{1-y}} + y^{k+1} \Big)$$

$$= xy^2 + \frac{xy}{1-y} M(x,1) - \frac{x}{1-y} M(x,y) + xy M(x,y)$$

In kernel form:

$$M(x,y)\left(1 + \frac{x}{1-y} - xy\right) = xy^2 + \frac{xy}{1-y} M(x,1)$$

or

$$(\ast)\ M(x,y)\left(1-y+x-xy+xy^2\right) = xy^2 - xy^3 + xy\, M(x,1)$$

Let $K(x,y) = xy^2 - (1+x)y + (1+x)$.
The F.P.S. solution of $K(x,y)=0$ is

$$\varphi(x) = \frac{1+x - \sqrt{1-2x-3x^2}}{2x}.$$

$\Delta = 1+x^2+2x$
$-4x-4x^2$

Substituting into $(*)$ gives

$$M(x,1) = \frac{x\,\Psi(x)^3 - x\,\Psi(x)^2}{x\,\Psi(x)}$$

$$= \Psi(x)\,\big(\Psi(x) - 1\big)$$

$$= \frac{1+x-\sqrt{1-2x-3x^2}}{2x} \cdot \frac{1-x-\sqrt{1-2x-3x^2}}{2x}$$

$$= \frac{1-x^2+1-2x-3x^2 - 2\sqrt{1-2x-3x^2}}{4x^2}$$

$$= \frac{1-x-2x^2-\sqrt{1-2x-3x^2}}{2x^2}.$$

$\blacksquare$

Exo 1.4.1 $\quad Av(2\underline{41}3,\ 3\underline{14}2)$.

1) Let h = number of active sites below the final value
and k = " " " above ".
Take the labels as (h,k).
Avoidance of $2\underline{41}3$: the non-empty descents (occ. of $2\underline{31}$) determine the inactive sites.

Avoidance of $3\underline{14}2$: the non-empty ascents (occ. of $2\underline{13}$) determine the inactive sites, in a symmetric fashion

For $Av(2\underline{41}3)$, we had :

$$\Omega_{semi}\begin{cases}(1,1)\\ (h,k) \rightsquigarrow \overbrace{(1,k+1),\ldots,(h-1,k+1)}^{\text{below the final value}}\overbrace{(h,k+1)}^{\text{immediately below it}}\\ \qquad\qquad \underbrace{(h+k,1),\ \ldots\ (h+1,k)}_{\text{above the final value.}}\end{cases}$$

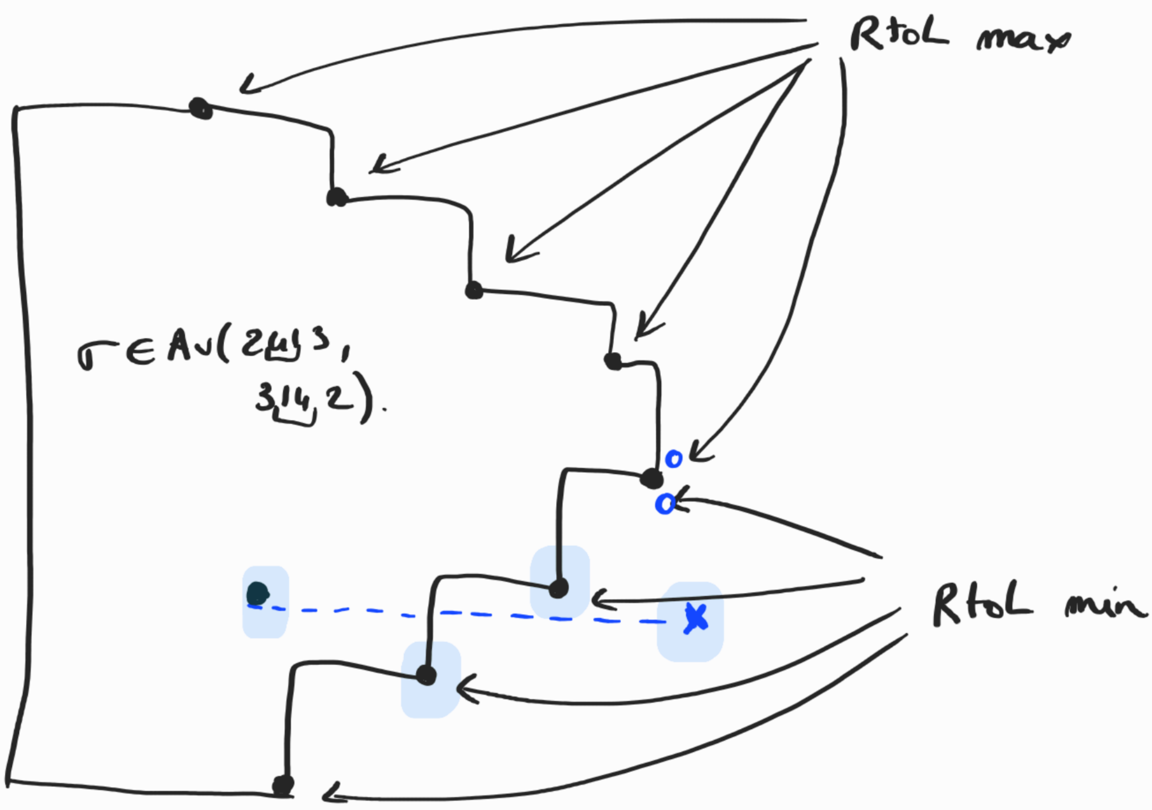
For $Av(2\underline{41}3,\ 3\underline{14}2)$, we have :

$$\Omega_{Bax}\begin{cases}(1,1)\\ (h,k) \rightsquigarrow (1,k+1),\ldots\ldots,(h-1,k+1)(h,k+1)\\ \qquad\qquad \underbrace{(h+1,1),\ldots(h+1,k-1)}_{\text{above}}\underbrace{(h+1,k)}_{\text{immediately above}}\end{cases}$$

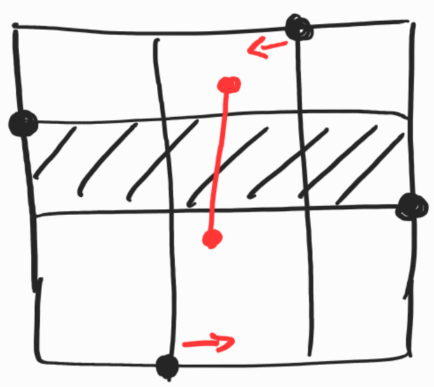
(see slide 33 "flipped ↕" for a justification).

2) Active sites are immediately above the RtoL max and immediately below the RtoL min.

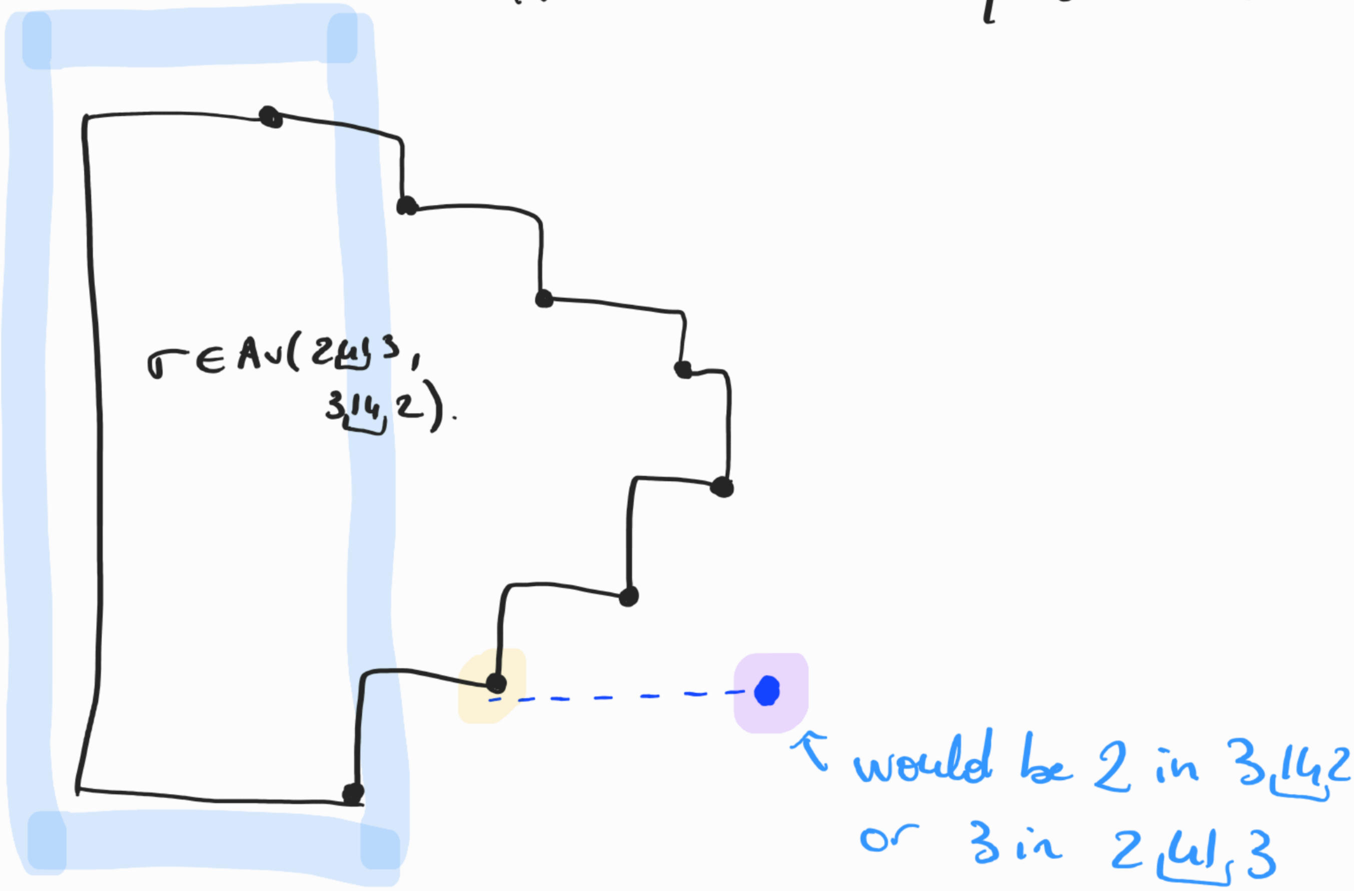
(*) Not OK below • which is not a RtoL min:



Here, we find a 3142, s.t. the 3 and the 2 have consecutive values. This implies the existence of a $3\underbrace{14}2$:



④ OK below a RtL min: By contradiction.



→ for $2\underbrace{41}3$, the $\underbrace{41}$ of the pattern would need to appear here, so we could replace with and still get a $2\underbrace{41}3$, contradiction.

→ for $3\underbrace{14}2$, the 31 is here; thus the 4 is also here or in the next column, and again we can replace with to get a contradiction.

3) $$B(x,y,z) = xyz + \sum_{\substack{n\geq 1\\ h\geq 1\\ k\geq 1}} b_{n,h,k}\, x^{n+1}\left((y+\cdots+y^h)\,z^{k+1} + y^{h+1}(z+\cdots+z^k)\right)$$

$$= xyz + xz \sum b_{n,h,k}\, x^n \frac{y-y^{h+1}}{1-y} z^h + xy \sum b_{n,h,k}\, x^n y^h \frac{z-z^{k+1}}{1-z}$$

$$= xyz + \frac{xyz}{1-y}\left(B(x,1,z) - B(x,y,z)\right) + \frac{xyz}{1-z}\left(B(x,y,1) - B(x,y,z)\right)$$

In kernel form:

$$B(x,y,z)\,\underbrace{\left(1 + \frac{xyz}{1-y} + \frac{xyz}{1-z}\right)}_{:=K(x,y,z)} = xyz + \frac{xyz}{1-y} B(x,1,z) + \frac{xyz}{1-z} B(x,y,1)$$

Original notation from [BM02]: $x \longleftrightarrow t$

$y \longleftrightarrow u$

$z \longleftrightarrow v$

$B(x,y,z) \longleftrightarrow G(t,u,v) \equiv G(u,v)$

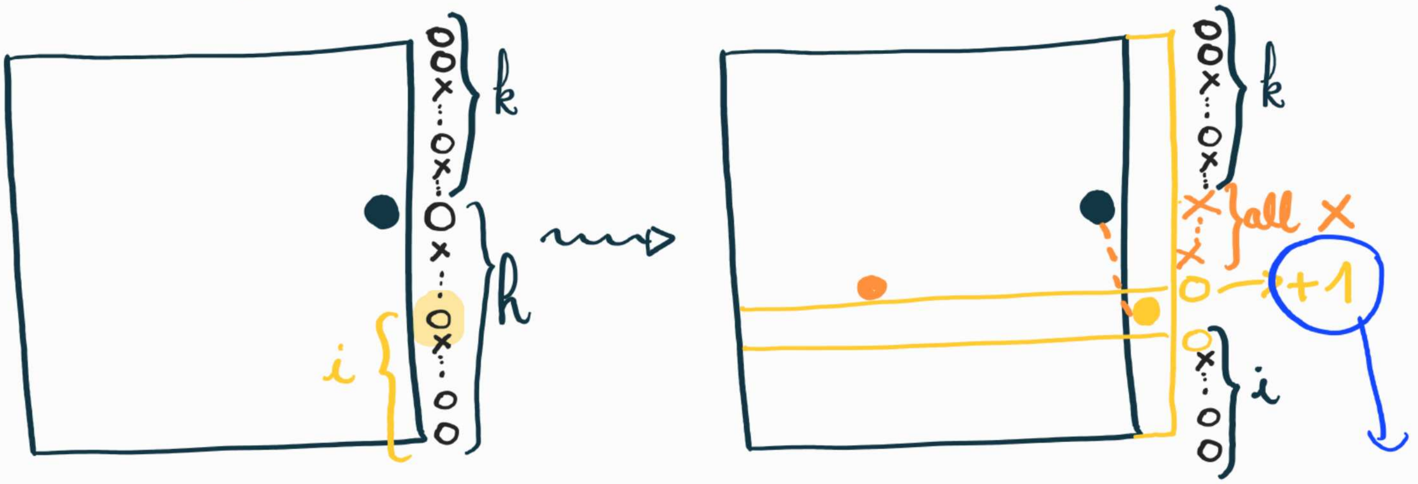
And now see Sect° 2.2 in [BM02].

Exo 1.4.2 $Av(2\underbrace{41}3,\ 3\underbrace{14}2,\ 3\underbrace{41}2)$

↑ one additional excluded pattern w.r.t. Exo 1.4.1

We go back to the reasoning determining the active sites of $Av(2\underbrace{41}3,\ 3\underbrace{14}2)$, and see which active sites become inactive, because they create a $3\underbrace{41}2$.

The difference is for the insertion below, which may create a $\underbrace{41}$ in a $3\underbrace{41}2$:



now inactive for $Av(2\underbrace{41}3,\ 3\underbrace{14}2,\ 3\underbrace{41}2)$.

So, the rewriting rule becomes

$$\Omega_{\text{strong}} \left\{ \begin{array}{l} (1,1) \\ (h,k) \rightsquigarrow (1,k\cancel{+1}), \ldots, (h-1,k\cancel{+1})\,(h,k+1) \\ \underbrace{(h+1,1), \ldots (h+1,k-1)}_{\text{above}}\ \underbrace{(h+1,k)}_{\text{immediately above}} \end{array} \right.$$

We then follow [BGRR18, Section 5.2].

→ kernel equation

→ the group of transformations leaving the kernel unchanged is not of small order. ∎

Exo 1.5

Since σ_n is uniform, for $n \geqslant 2$

$F(\sigma_n) = (\underset{2}{\underset{\parallel}{\ell_1}}, \dots, \ell_n, \underset{3}{\underset{\parallel}{\ell_{n+1}}})$ is uniform among

sequences of labels produced according to Ω_{Sch} (because of the "generating tree bijection").

So, we just need to prove that the conditioned random walk given gives the same probability to every sequence $(\underset{2}{\underset{\parallel}{\ell_1}}, \dots, \ell_n, \underset{3}{\underset{\parallel}{\ell_{n+1}}})$ consistent with Ω_{Sch}.

Fix such a sequence $(\underset{2}{\underset{\parallel}{\ell_1}}, \dots, \ell_n, \underset{3}{\underset{\parallel}{\ell_{n+1}}})$.

$$\mathbb{P}\left((X_i)_{i \in [1,\dots,n]} = (\overset{2}{\overset{\parallel}{\ell_1}}, \dots, \ell_n) \text{ et } X_{n+1} = 3 \right)$$

$$= \mathbb{P}(X_1 = \overset{2}{\overset{\parallel}{\ell_1}}) \cdot \mathbb{P}(X_2 = \ell_2 \mid X_1 = \ell_1) \cdot \dots$$

$$\dots \mathbb{P}(X_n = \ell_n \mid X_{n-1} = \ell_{n-1}) \cdot \mathbb{P}(X_n = \ell_n \mid X_{n+1} = 3)$$

$$= 1 \cdot \left(p \cdot q^{l_2 - l_1}\right) \cdot \ \cdots \ \left(p \cdot q^{l_n - l_{n-1}}\right) \cdot \left(p \cdot q^{3 - l_n}\right)$$

$$= p^n \cdot q^{3 - \overbrace{l_1}^{=2}} = \underbrace{p^n \cdot q}.$$

The result is independent of $(l_1, \dots, l_{n+1})$. $\blacksquare$