Wilf-equivalences derived from a bijection between Av(231) and Av(132)

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joint work with Michael Albert

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June 2012, at *groupe de travail CÉA...*

We study permutations sortable by sorting operators which are compositions of stack sorting operators $\mathbf{S}$ and reverse operators $\mathbf{R}$.

From our previous work with O. Guibert, we have:

Theorem

There are as many permutations of $\mathfrak{S}_n$ sortable by $\mathbf{S} \circ \mathbf{S}$ as permutations of $\mathfrak{S}_n$ sortable by $\mathbf{S} \circ \mathbf{R} \circ \mathbf{S}$, and many permutation statistics are equidistributed across these two sets.

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Computer experiments then suggest that:

Conjecture (*The* $(id, \mathbf{R})$ *conjecture*)

For any operator $\mathbf{A}$ which is a composition of operators $\mathbf{S}$ and $\mathbf{R}$, there are as many permutations of $\mathfrak{S}_n$ sortable by $\mathbf{S} \circ id \circ \mathbf{A}$ as permutations of $\mathfrak{S}_n$ sortable by $\mathbf{S} \circ \mathbf{R} \circ \mathbf{A}$. Moreover, many permutation statistics are equidistributed across these two sets.

In 2013: a first talk...

Our primary purpose is to prove the $(id, \mathbf{R})$ conjecture.

Theorem

The $(id, \mathbf{R})$ conjecture holds.

The proof uses:

- The characterization of preimages of permutations by $\mathbf{S}$
- A new bijection (denoted P) between Av(231) and Av(132)

In 2013: a first talk... and a second one

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Theorem

The $(id, \mathbf{R})$ conjecture holds.

The proof uses:

- The characterization of preimages of permutations by **S**
- A new bijection (denoted P) between Av(231) and Av(132)

The bijection P has nice properties, which allow us to derive unexpected enumerative results (Wilf-equivalences). For instance:

Theorem

Av(231, 31254) *and* Av(132, 42351) *have the same enumerative sequence, and their common generating function is*

$$F_5(t) = \frac{t^3 - t^2 - 2t + 1}{2t^3 - 3t + 1}.$$

Definitions

Permutations and patterns

Permutation: Bijection from $[1..n]$ to itself. Set $\mathfrak{S}_n$.

We view permutations as words, $\sigma = \sigma_1 \sigma_2 \dots \sigma_n$
Example: $\sigma = 1\ 8\ 3\ 6\ 4\ 2\ 5\ 7$.

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Occurrence of a pattern: $\pi \in \mathfrak{S}_k$ is a pattern of $\sigma \in \mathfrak{S}_n$ if $\exists\ i_1 < \ldots < i_k$ such that $\sigma_{i_1}\ldots\sigma_{i_k}$ is order isomorphic ($\equiv$) to π.

Notation: $\pi \preccurlyeq \sigma$.

Equivalently: The normalization of $\sigma_{i_1}\ldots\sigma_{i_k}$ on $[1..k]$ yields π.

Example: $2\,1\,3\,4 \preccurlyeq \mathbf{3}\,\mathbf{1}\,2\,8\,\mathbf{5}\,4\,\mathbf{7}\,9\,6$ since $3\,1\,5\,7 \equiv 2\,1\,3\,4$.

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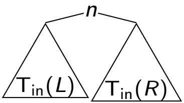
Example: $2134 \preccurlyeq \mathbf{3}\mathbf{1}28\mathbf{5}4\mathbf{7}96$ since $3157 \equiv 2134$.

Avoidance: $\text{Av}(\pi, \tau, \ldots)$ = set of permutations that do not contain any occurrence of π or τ or $\ldots$

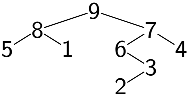
In-order trees of permutations

Recursively defined by $\mathsf{T}_{\text{in}}(LnR) =$



where $n = \max(LnR)$, and $\mathsf{T}_{\text{in}}(\varepsilon) = \emptyset$.

Example: For $\pi = 5\ 8\ 1\ 9\ 6\ 2\ 3\ 7\ 4$, $\mathsf{T}_{\text{in}}(\pi) =$



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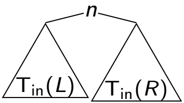
Recursively defined by $\mathrm{T}_{\mathrm{in}}(LnR) =$ 

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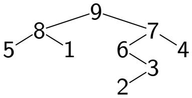
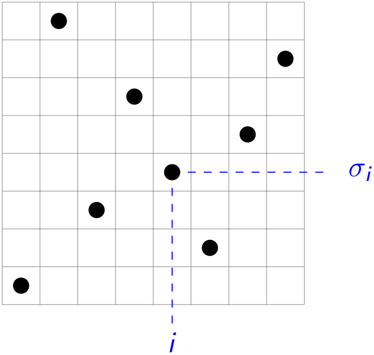
Example: For $\pi = 5\ 8\ 1\ 9\ 6\ 2\ 3\ 7\ 4$, $\mathrm{T}_{\mathrm{in}}(\pi) =$ 

Remark: Many permutation statistics are determined by the shape of in-order trees:

- number and positions of the right-to-left maxima,
- number and positions of the left-to-right maxima,
- up-down word.

Diagrams of permutations; Sum and skew sum

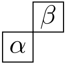
Diagram of $\sigma = 1\ 8\ 3\ 6\ 4\ 2\ 5\ 7$:



α a permutation of $\mathfrak{S}_a$,
β a permutation of $\mathfrak{S}_b$

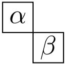
- Sum:

$$\alpha \oplus \beta = \alpha\,(\beta + a) =$$

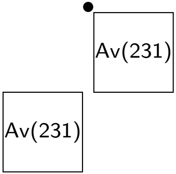


- Skew sum:

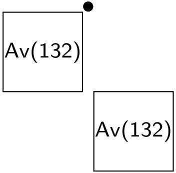
$$\alpha \ominus \beta = (\alpha + b)\,\beta =$$



Describing permutations in Av(231) and Av(132)

- Av(231) $= \varepsilon +$ 

- Any $\pi \neq \varepsilon \in$ Av(231) is decomposed as
$$\pi = \alpha \oplus (1 \ominus \beta)$$
with $\alpha, \beta \in$ Av(231).

- Av(132) $= \varepsilon +$ 

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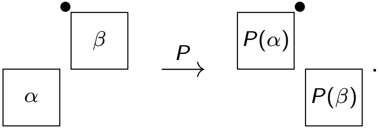
$$\pi = (\alpha \oplus 1) \ominus \beta$$
with $\alpha, \beta \in$ Av(132).

Bijection $\mathsf{Av}(231) \stackrel{P}{\longleftrightarrow} \mathsf{Av}(132)$

Bijection P from Av(231) to Av(132)

P is recursively defined as:

- If $\pi = \alpha \oplus (1 \ominus \beta)$ then $P(\pi) = (P(\alpha) \oplus 1) \ominus P(\beta)$.
- or equivalently, 

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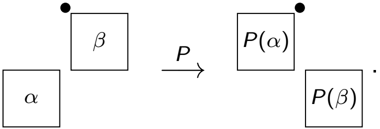

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$P(\pi) = 7\,8\,5\,4\,6\,9\,3\,1\,2$.

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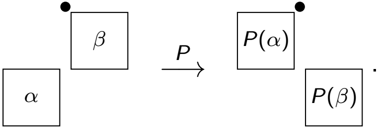
- or equivalently, 

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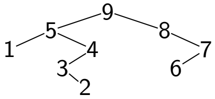
Remark: P is the identity map on Av(231, 132).

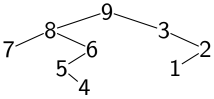
Some properties of P

Proposition: P preserves the shape of in-order trees.

Proof: From the recursive definition of P.

Example: For $\pi = 1\,5\,3\,2\,4\,9\,8\,6\,7$ (and $P(\pi) = 7\,8\,5\,4\,6\,9\,3\,1\,2$):

$T_{in}(\pi) =$ 

$T_{in}(P(\pi)) =$ 

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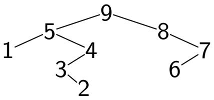
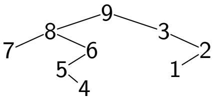
$T_{\text{in}}(\pi) =$ 

$T_{\text{in}}(P(\pi)) =$ 

Consequence: P preserves the following statistics:

- number and positions of the right-to-left maxima,
- number and positions of the left-to-right maxima,
- up-down word.

Proof: These are determined by the shape of in-order trees.

More about the bijection

$$\mathrm{Av}(231) \stackrel{P}{\longleftrightarrow} \mathrm{Av}(132)$$

Related Wilf-equivalences

P and Wilf-equivalences

Two classes Av$(\pi, \pi', \dots, \pi'')$ and Av$(\tau, \tau', \dots, \tau'')$ are Wilf-equivalent when they are enumerated by the same sequence.

Examples:
- Av(231) and Av(123);
- trivial Wilf-equivalences: Av$(\pi, \pi', \dots, \pi'')$ and Av$(\mathbf{Z}(\pi), \mathbf{Z}(\pi'), \dots, \mathbf{Z}(\pi''))$ for every symmetry $\mathbf{Z}$;

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Theorem: Description of the patterns $\pi \in \text{Av}(231)$ such that P provides a bijection between $\text{Av}(231, \pi)$ and $\text{Av}(132, P(\pi))$
$\Rightarrow$ Many Wilf-equivalences (most of them not trivial)

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Theorem: Description of the patterns $\pi \in$ Av(231) such that P provides a bijection between Av$(231, \pi)$ and Av$(132, P(\pi))$
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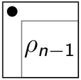
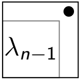
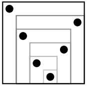
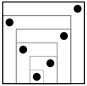
Theorem: Computation of the generating function of such classes Av$(231, \pi)$ ... and it depends only on $|\pi|$.
$\Rightarrow$ Even more Wilf-equivalences!

The families of patterns (λ_n) and (ρ_n)

- $\lambda_0 = \rho_0 = \varepsilon$ (or $\lambda_1 = \rho_1 = 1$)
- $\lambda_n = 1 \ominus \rho_{n-1}$
- $\rho_n = \lambda_{n-1} \oplus 1$

$\lambda_n =$ 

, $\rho_n =$ 

; $\lambda_6 =$ 
, $\rho_6 =$ 

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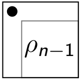
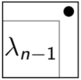
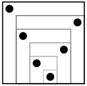
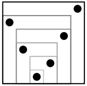
$\lambda_n =$ 

, $\rho_n =$ 

; $\lambda_6 =$ 
, $\rho_6 =$ 

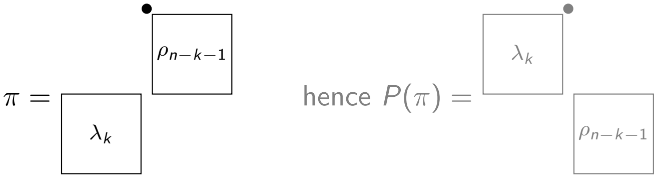
Remarks: for all n,

- λ_n starts with its maximum, and ρ_n ends with its maximum;
- λ_n is $\oplus$-indecomposable and ρ_n is $\ominus$-indecomposable;
- λ_n and ρ_n are in Av(231, 132), hence are fixed by P.

Patterns π such that $\mathrm{Av}(231,\pi) \stackrel{P}{\longleftrightarrow} \mathrm{Av}(132, P(\pi))$

Theorem

A pattern $\pi \in \mathrm{Av}(231)$ is such that P provides a bijection between $\mathrm{Av}(231,\pi)$ and $\mathrm{Av}(132, P(\pi))$ if and only if $\pi = \lambda_k \oplus (1 \ominus \rho_{n-k-1})$.



Remark: $\mathrm{Av}(231,\pi) \stackrel{P}{\longleftrightarrow} \mathrm{Av}(132, P(\pi))$ is equivalent to $\forall \sigma \in \mathrm{Av}(231), \pi \preccurlyeq \sigma$ iff $P(\pi) \preccurlyeq P(\sigma)$.

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$\Leftarrow$ If $\pi = \lambda_k \oplus (1 \ominus \rho_{n-k-1})$ then $\mathrm{Av}(231,\pi) \overset{P}{\longleftrightarrow} \mathrm{Av}(132, P(\pi))$

- Proof by induction on n.
- Examine how $\lambda_k \oplus (1 \ominus \rho_{n-k-1})$ can occur in $\sigma \in \mathrm{Av}(231)$.
- Examine how $P(\lambda_k \oplus (1 \ominus \rho_{n-k-1}))$ can occur in $P(\sigma) \in \mathrm{Av}(132)$.

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$\Leftarrow$ If $\pi = \lambda_k \oplus (1 \ominus \rho_{n-k-1})$ then $\mathrm{Av}(231,\pi) \overset{P}{\longleftrightarrow} \mathrm{Av}(132,P(\pi))$

$\Rightarrow$ If $\mathrm{Av}(231,\pi) \overset{P}{\longleftrightarrow} \mathrm{Av}(132,P(\pi))$ then $\pi = \lambda_k \oplus (1 \ominus \rho_{n-k-1})$

- Such a π may be written $\pi = \alpha \oplus (1 \ominus \beta)$.
- Claim 1: α and β are also such that $\mathrm{Av}(231,\gamma) \overset{P}{\longleftrightarrow} \mathrm{Av}(132,P(\gamma))$.
- Claim 2: α starts with its maximum, and β ends with its maximum.
- By induction, every γ such that $\mathrm{Av}(231,\gamma) \overset{P}{\longleftrightarrow} \mathrm{Av}(132,P(\gamma))$ starting (resp. ending) with its maximum is equal to λ_ℓ (resp. ρ_ℓ) for some ℓ.

Patterns π such that $Av(231, \pi) \stackrel{P}{\longleftrightarrow} Av(132, P(\pi))$

Theorem

A pattern $\pi \in Av(231)$ is such that P provides a bijection between $Av(231, \pi)$ and $Av(132, P(\pi))$ if and only if $\pi = \lambda_k \oplus (1 \ominus \rho_{n-k-1})$.

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$\Rightarrow$ If $Av(231, \pi) \stackrel{P}{\longleftrightarrow} Av(132, P(\pi))$ then $\pi = \lambda_k \oplus (1 \ominus \rho_{n-k-1})$

Consequence: For all $\pi = \lambda_k \oplus (1 \ominus \rho_{n-k-1})$, $Av(231, \pi)$ and $Av(132, P(\pi))$ are Wilf-equivalent.

Example:

- $\lambda_3 \oplus (1 \ominus \rho_1) = 31254 \in Av(231)$ and $P(31254) = 42351$.

$\Rightarrow$ P is a bijection between $Av(231, 31254)$ and $Av(132, 42351)$
$\Rightarrow$ $Av(231, 31254)$ and $Av(132, 42351)$ are Wilf-equivalent

Known Wilf-equivalences that we recover (or not)

☺ We recover

- for $\pi = 312$, $\mathrm{Av}(231, 312) \sim_{Wilf} \mathrm{Av}(132, 312)$,
- for $\pi = 3124$, $\mathrm{Av}(231, 3124) \sim_{Wilf} \mathrm{Av}(132, 3124)$,
- for $\pi = 1423$, $\mathrm{Av}(231, 1423) \sim_{Wilf} \mathrm{Av}(132, 3412)$,

which are (up to symmetry) referenced in Wikipedia.

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With $|\pi| = 3$ or 4, there are five more non-trivial Wilf-equivalence of the form $\text{Av}(231, \pi) \sim_{Wilf} \text{Av}(132, \pi')$ (up to symmetry).
☹ We do not recover them.

More Wilf-equivalences that we obtain

Patterns π such that $\mathrm{Av}(231, \pi) \sim_{Wilf} \mathrm{Av}(132, P(\pi))$
and $\mathrm{Av}(231, \pi) \overset{P}{\longleftrightarrow} \mathrm{Av}(132, P(\pi))$:

π	$P(\pi)$
42135	42135
21534	43512
53124	53124
31254	42351
15324	45213

π	$P(\pi)$
216435	546213
531246	531246
312645	534612
642135	642135
421365	532461
164235	563124

π	$P(\pi)$
6421357	6421357
3127546	6457213
7531246	7531246
4213756	6435712
1753246	6742135
5312476	6423571
2175346	6573124

π	$P(\pi)$
31286457	75683124
75312468	75312468
64213587	75324681
53124867	75346812
86421357	86421357
21864357	76842135
42138657	75468213
18642357	78531246

Except two they are non-trivial.

More Wilf-equivalences that we obtain

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216435	546213
531246	531246
312645	534612
642135	642135
421365	532461
164235	563124

π	$P(\pi)$
6421357	6421357
3127546	6457213
7531246	7531246
4213756	6435712
1753246	6742135
5312476	6423571
2175346	6573124

π	$P(\pi)$
31286457	75683124
75312468	75312468
64213587	75324681
53124867	75346812
86421357	86421357
21864357	76842135
42138657	75468213
18642357	78531246

Except two they are non-trivial.
But because of symmetries, there are some redundancies.

Conjectural Wilf-equivalences that we miss

Patterns π such that conjecturally $\text{Av}(231, \pi) \sim_{Wilf} \text{Av}(132, P(\pi))$

π	$P(\pi)$
2137465	5467231
1327645	5647312

π	$P(\pi)$
63125478	64235178
87153246	87452136
65312478	65312478
87421356	87421356

For all of those, $\text{Av}(231, \pi) \overset{P}{\longleftrightarrow} \text{Av}(132, P(\pi))$ does not hold.

Common generating function when $\mathrm{Av}(231,\pi) \stackrel{P}{\longleftrightarrow} \mathrm{Av}(132, P(\pi))$

Definition: $F_1(t) = 1$ and $F_{n+1}(t) = \frac{1}{1-tF_n(t)}$.

Theorem

For $\pi \in \mathrm{Av}(231)$ such that $\mathrm{Av}(231,\pi) \stackrel{P}{\longleftrightarrow} \mathrm{Av}(132, P(\pi))$, denoting $n = |\pi|$, the generating function of $\mathrm{Av}(231,\pi)$ is F_n.

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Proof:

- Recall that $\pi = \lambda_k \oplus (1 \ominus \rho_{n-k-1})$.
- For $\pi = \lambda_n$ or ρ_n, proof by induction.
- For the general case, use a nice property of (F_n): setting $g(x, y) = \frac{1-txy}{1-tx-ty}$, we have $F_n = g(F_j, F_k)$ for any j, k, n such that $j + k = n - 1$.

Many Wilf-equivalent classes

Theorem

The classes Av$(231, \pi)$ *and* Av$(132, P(\pi))$ *are all Wilf-equivalent when* $|\pi| = n$ *and* π *is of the form* $\lambda_k \oplus (1 \ominus \rho_{n-k-1})$. *Moreover, their generating function is* F_n.

Remark: For other patterns π of size $n = 7$ or 8 such that Av$(231, \pi)$ and Av$(132, P(\pi))$ are conjecturally Wilf-equivalent, the generating function of class Av$(231, \pi)$ is not F_n.

Many Wilf-equivalent classes ... and even more?

Theorem

The classes $\mathrm{Av}(231, \pi)$ *and* $\mathrm{Av}(132, P(\pi))$ *are all Wilf-equivalent when* $|\pi| = n$ *and* π *is of the form* $\lambda_k \oplus (1 \ominus \rho_{n-k-1})$. *Moreover, their generating function is* F_n.

Remark: For other patterns π of size $n = 7$ or 8 such that $\mathrm{Av}(231, \pi)$ and $\mathrm{Av}(132, P(\pi))$ are conjecturally Wilf-equivalent, the generating function of class $\mathrm{Av}(231, \pi)$ is not F_n.

In future?: For classes recursively described (like Av(231) and Av(132)), define recursive bijections (like P), to find or explain more Wilf-equivalences.