Refined enumeration of permutations sorted with two stacks and a D_8 symmetry

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Preview of Permutation Patterns 2012, Glasgow

The little story of the problem, with many characters!

- Questions of Anders, Einar and Mark:
  What are the permutations sorted by the composition of two operators of the form $\mathbf{S} \circ \alpha$ for $\alpha \in D_8$?
  How are they enumerated?
- Answer to the 1st question, with Mike and Michael also:
  Characterization of permutations sorted by $\mathbf{S} \circ \alpha \circ \mathbf{S}$ by (generalized) excluded patterns.
- Conjectures of Anders, Einar and Mark for the 2nd question:
  - $\mathrm{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{S})$ and $\mathrm{Id}(\mathbf{S} \circ \mathbf{S})$ are enumerated by the same sequence, and a tuple of 15 statistics is equidistributed.
  - $\mathrm{Id}(\mathbf{S} \circ \mathbf{i} \circ \mathbf{S})$ and Bax are enumerated by the same sequence, and a tuple of 3 statistics is equidistributed.
- Answer to the 2nd question, by Olivier and myself:
  The conjectures are true, and a few more statistics can be added to the first one.

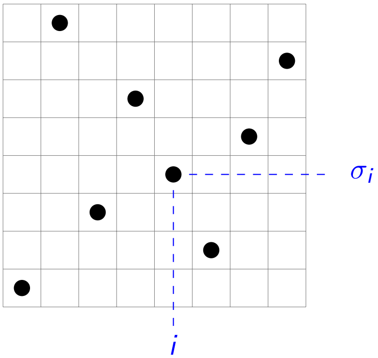
Definitions

Representation of permutations

Permutation: Bijection from $[1..n]$ to itself. Set $\mathfrak{S}_n$.

- Linear representation:
  $\sigma = 1\ 8\ 3\ 6\ 4\ 2\ 5\ 7$
- Two lines representation:
  $$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 8 & 3 & 6 & 4 & 2 & 5 & 7 \end{pmatrix}$$
- Representation as a product of cycles:
  $\sigma = (1)\ (2\ 8\ 7\ 5\ 4\ 6)\ (3)$

- Representation by diagram :



Classical patterns in permutations

Occurrence of a pattern: $\pi \in \mathfrak{S}_k$ is a pattern of $\sigma \in \mathfrak{S}_n$ if $\exists\ i_1 < \ldots < i_k$ such that $\sigma_{i_1} \ldots \sigma_{i_k}$ is order isomorphic ($\equiv$) to π.

Notation: $\pi \preccurlyeq \sigma$.

Equivalently: The normalization of $\sigma_{i_1} \ldots \sigma_{i_k}$ on $[1..k]$ yields π.

Example: $2\,1\,3\,4 \preccurlyeq \mathbf{3}\,\mathbf{1}\,2\,8\,\mathbf{5}\,4\,\mathbf{7}\,9\,6$ since $3\,1\,5\,7 \equiv 2\,1\,3\,4$.

Avoidance: $\mathrm{Av}(\pi, \tau, \ldots)$ = set of permutations that do not contain any occurrence of π or τ or ...

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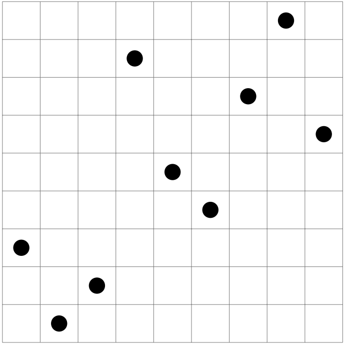

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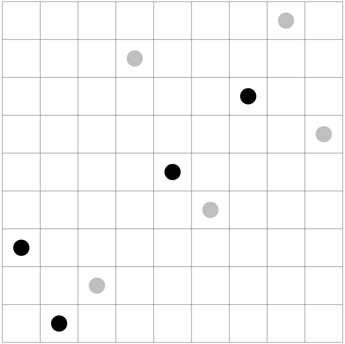

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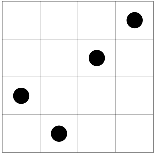

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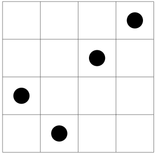
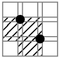
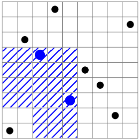


Generalizations of excluded patterns

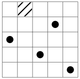
- **Dashed** pattern [Babson, Steingrímsson 2000]:
  Add adjacency constraints between some elements $\sigma_{i_1}, \ldots, \sigma_{i_k}$.
  Example: $\sigma_{i_1}\sigma_{i_2}\sigma_{i_3}\sigma_{i_4}$ occurrence of 2-41-3 $\Rightarrow i_3 = i_2 + 1$.
- **Mesh** pattern [Úlfarsson, Brändén, Claesson 2011]:
  Stretched diagram with shaded cells ▨.

An occurrence of a mesh pattern is a set of points matching the diagram while leaving zones ▨ empty.

Example: $\mu =$  is a pattern of $\sigma =$  .

- **Barred** pattern [West 1990]:
  Mesh pattern with only one cell ▨.

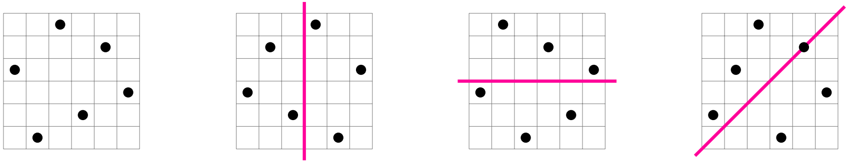
Example:

$3\bar{5}241 =$ 

D_8 symmetries

Symmetries of the square transform permutations *via* their diagrams

Reverse Complement Inverse



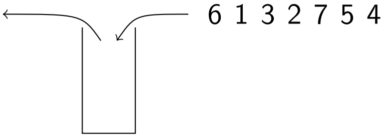
σ $\mathbf{r}(\sigma)$ $\mathbf{c}(\sigma)$ $\mathbf{i}(\sigma)$

These operators generate an 8-element group:

$$D_8 = \{\mathbf{id}, \mathbf{r}, \mathbf{c}, \mathbf{i}, \mathbf{r} \circ \mathbf{c}, \mathbf{i} \circ \mathbf{r}, \mathbf{i} \circ \mathbf{c}, \mathbf{i} \circ \mathbf{c} \circ \mathbf{r}\}$$

The stack sorting operator S

Sort (or try to do so) using a stack satisfying the Hanoi condition.



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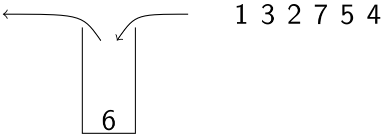

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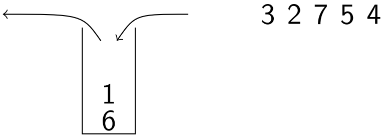

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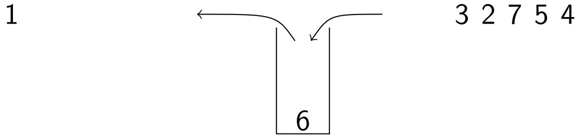

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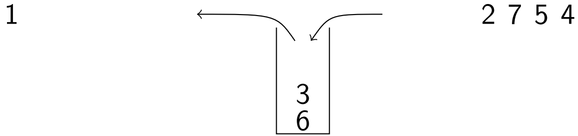

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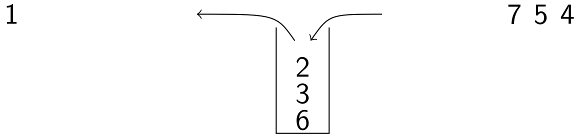

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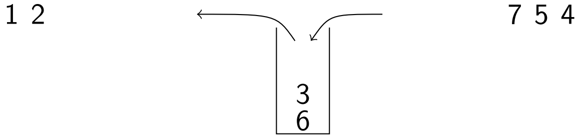

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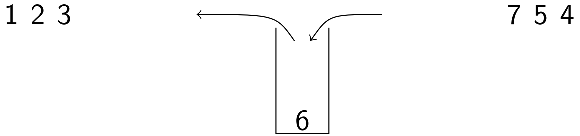

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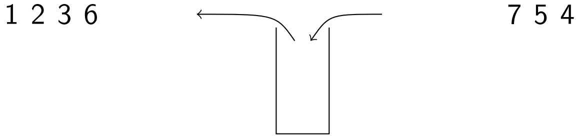

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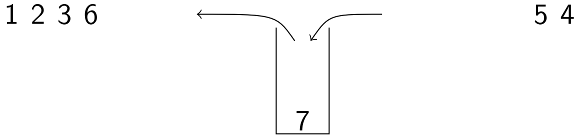

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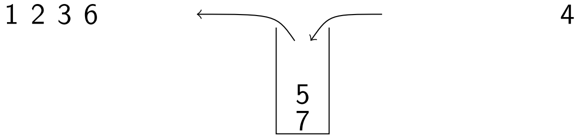

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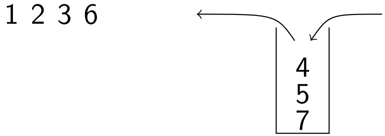

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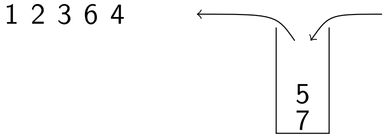

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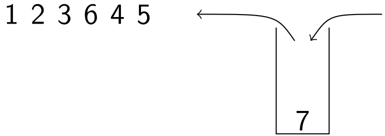

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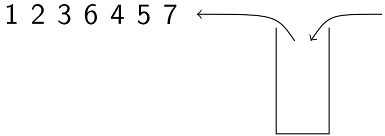
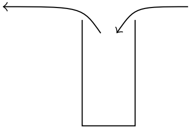


The stack sorting operator **S**

Sort (or try to do so) using a stack satisfying the Hanoi condition.

$\mathbf{S}(\sigma) = 1\ 2\ 3\ 6\ 4\ 5\ 7$ $\quad$ $6\ 1\ 3\ 2\ 7\ 5\ 4 = \sigma$



Equivalently, $\mathbf{S}(\varepsilon) = \varepsilon$ and $\mathbf{S}(LnR) = \mathbf{S}(L)\mathbf{S}(R)n$, $n = \max(LnR)$

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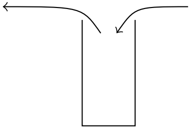
$\mathbf{S}(\sigma) = 1\ 2\ 3\ 6\ 4\ 5\ 7$  $6\ 1\ 3\ 2\ 7\ 5\ 4 = \sigma$

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- Stack sortable permutations: $\text{Id}(\mathbf{S}) = \text{Av}(231)$, enumeration by Catalan numbers [Knuth]
- Two-stack sortable: $\text{Id}(\mathbf{S} \circ \mathbf{S}) = \text{Av}(2341, 3\bar{5}241)$, enumeration by $\frac{2(3n)!}{(n+1)!(2n+1)!}$ [West, Zeilberger,...]
- Many other variants studied, in connection with excluded patterns [Bóna, Bousquet-Mélou, Rossin, ...]

Stating the main results

Characterization by generalized excluded patterns

Theorem

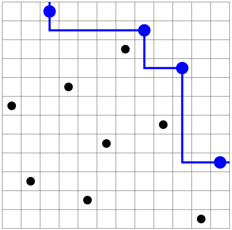
Permutations that are sorted by $\mathbf{S} \circ \alpha \circ \mathbf{S}$*, for* α *in* D_8*, are:*

- $\mathrm{Id}(\mathbf{S} \circ \mathbf{S}) = \mathrm{Id}(\mathbf{S} \circ \mathbf{i} \circ \mathbf{c} \circ \mathbf{r} \circ \mathbf{S}) = \mathrm{Av}(2341, 3\bar{5}241)$;
- $\mathrm{Id}(\mathbf{S} \circ \mathbf{c} \circ \mathbf{S}) = \mathrm{Id}(\mathbf{S} \circ \mathbf{i} \circ \mathbf{r} \circ \mathbf{S}) = \mathrm{Av}(231)$;
- $\mathrm{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{S}) = \mathrm{Id}(\mathbf{S} \circ \mathbf{i} \circ \mathbf{c} \circ \mathbf{S}) = \mathrm{Av}(1342, 31\text{-}4\text{-}2)$ $= \mathrm{Av}(1342, 3\bar{5}142)$;
- $\mathrm{Id}(\mathbf{S} \circ \mathbf{i} \circ \mathbf{S}) = \mathrm{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{c} \circ \mathbf{S}) = \mathrm{Av}(3412, 3\text{-}4\text{-}21)$.

- $\mathrm{Av}(231) = \mathrm{Id}(\mathbf{S})$ is enumerated by Catalan numbers
- $\mathrm{Av}(2341, 3\bar{5}241) = \mathrm{Id}(\mathbf{S} \circ \mathbf{S})$ is enumerated by $\frac{2(3n)!}{(n+1)!(2n+1)!}$

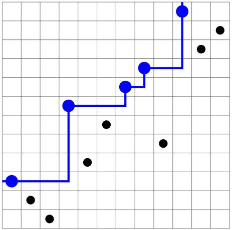
Some permutation statistics. . . and many more

Number of RtoL-max



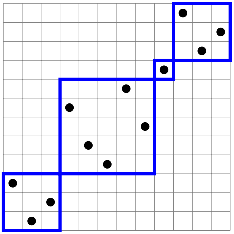
$\text{rmax}(\sigma) = 4$

Number of LtoR-max



$\text{lmax}(\sigma) = 5$

Number of components



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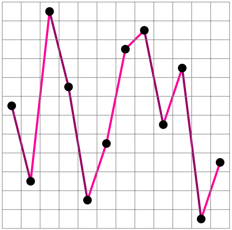

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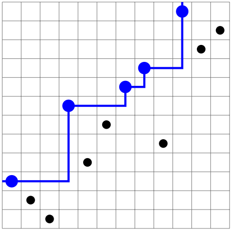

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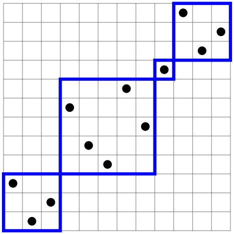


$\text{comp}(\sigma) = 4$

Up-down word of $\sigma \in \mathfrak{S}_n$: $w \in \{u, d\}^{n-1}$, $w_i = \begin{cases} u \text{ if } \sigma_i < \sigma_{i+1} \\ d \text{ if } \sigma_i > \sigma_{i+1} \end{cases}$

$\text{udword}(\sigma) = dudduuudud u$

Refined enumeration of $\mathrm{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{S})$

Theorem

The two sets $\mathrm{Id}(\mathbf{S} \circ \mathbf{S})$ *and* $\mathrm{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{S})$ *are enumerated by the same sequence. Moreover, the tuple of statistics* (udword, rmax, lmax, zeil, indmax, slmax, slmax $\circ\mathbf{r}$) *has the same distribution.*

The underlying bijection actually preserves the joint distribution of these statistics.

Consequence

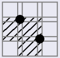
The statistics (asc, des, maj, maj $\circ\mathbf{r}$, maj $\circ\mathbf{c}$, maj $\circ\mathbf{rc}$, valley, peak, ddes, dasc, rir, rdr, lir, ldr) *are also (and jointly) equidistributed.*

Refined enumeration of Id(**S** ∘ **i** ∘ **S**)

Theorem

The set Id(**S** ∘ **i** ∘ **S**) *is enumerated by the Baxter numbers, and the triple of statistics* (lmax, des, comp) *has the same joint distribution on* Id(**S** ∘ **i** ∘ **S**) *and on* Bax.

Lemma

It also has the same distribution than the triple of statistics $(\mathrm{lmax}, \mathrm{occ}_\mu, \mathrm{comp})$ *on* TBax*, where* $\mu =$  .

- Baxter permutations: Bax = Av(2-41-3, 3-14-2)
- Twisted Baxter permutations: TBax = Av(2-41-3, 3-41-2)

Both are enumerated by $Bax_n = \frac{2}{n(n+1)^2} \sum_{k=1}^{n} \binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}$

Elements of proof

$\mathsf{Id}(\mathbf{S}\circ\mathbf{r}\circ\mathbf{S})$: characterization by excluded patterns

Theorem (partial statement)

$$\mathsf{Id}(\mathbf{S}\circ\mathbf{r}\circ\mathbf{S}) = \mathsf{Id}(\mathbf{S}\circ\mathbf{i}\circ\mathbf{c}\circ\mathbf{S}) = \mathsf{Av}(1342, 31\text{-}4\text{-}2) = \mathsf{Av}(1342, 3\bar{5}142)$$

Proof:

Step 1. $31\text{-}4\text{-}2 \preccurlyeq \sigma \Leftrightarrow 3\bar{5}142 \preccurlyeq \sigma$

Step 2. $\sigma \in \mathsf{Id}(\mathbf{S}\circ\mathbf{r}\circ\mathbf{S}) \Leftrightarrow \mathbf{S}\circ\mathbf{r}\circ\mathbf{S}(\sigma) = 12\ldots n$
$\Leftrightarrow \mathbf{r}\circ\mathbf{S}(\sigma) \in \mathsf{Av}(231) \Leftrightarrow \mathbf{S}(\sigma) \in \mathsf{Av}(132)$

$\sigma \in \mathsf{Id}(\mathbf{S}\circ\mathbf{i}\circ\mathbf{c}\circ\mathbf{S}) \Leftrightarrow \mathbf{S}\circ\mathbf{i}\circ\mathbf{c}\circ\mathbf{S}(\sigma) = 12\ldots n$
$\Leftrightarrow \mathbf{i}\circ\mathbf{c}\circ\mathbf{S}(\sigma) \in \mathsf{Av}(231)$
$\Leftrightarrow \mathbf{c}\circ\mathbf{S}(\sigma) \in \mathsf{Av}(312) \Leftrightarrow \mathbf{S}(\sigma) \in \mathsf{Av}(132)$

Step 3. $\mathbf{S}(\sigma) \in \mathsf{Av}(132) \Leftrightarrow \sigma \in \mathsf{Av}(1342, 31\text{-}4\text{-}2)$

Id($\mathbf{S} \circ \mathbf{r} \circ \mathbf{S}$): characterization by excluded patterns

Proof of Step 3.: By contraposition, show that
$\mathbf{S}(\sigma)$ contains 132 $\Leftrightarrow$ σ contains 1342 or 31-4-2.

Id(**S** ∘ **r** ∘ **S**): characterization by excluded patterns

Proof of Step 3.: By contraposition, show that
S(σ) contains 132 $\Leftrightarrow$ σ contains 1342 or 31-4-2.

- If $\sigma_i \sigma_j \sigma_k \sigma_\ell \equiv 1342$, then **S**($\sigma$) contains $\sigma_i \sigma_j \sigma_\ell \equiv 132$.

Id($\mathbf{S} \circ \mathbf{r} \circ \mathbf{S}$): characterization by excluded patterns

Proof of Step 3.: By contraposition, show that
$\mathbf{S}(\sigma)$ contains 132 $\Leftrightarrow$ σ contains 1342 or 31-4-2.

- If $\sigma_i\sigma_j\sigma_k\sigma_\ell \equiv 1342$, then $\mathbf{S}(\sigma)$ contains $\sigma_i\sigma_j\sigma_\ell \equiv 132$.
- If $\sigma_i\sigma_j\sigma_k\sigma_\ell \equiv$ 31-4-2, then $\mathbf{S}(\sigma)$ contains $\sigma_j\sigma_i\sigma_\ell \equiv 132$.

Id($\mathbf{S} \circ \mathbf{r} \circ \mathbf{S}$): characterization by excluded patterns

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  Writing $\sigma = LnR$, we have $\mathbf{S}(\sigma) = \mathbf{S}(L)\mathbf{S}(R)n$.
  Consequently, $132 \preccurlyeq \mathbf{S}(\sigma) \Rightarrow 132 \preccurlyeq \mathbf{S}(L)\mathbf{S}(R)$.

Id(**S** ∘ **r** ∘ **S**): characterization by excluded patterns

Proof of Step 3.: By contraposition, show that
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  Consequently, $132 \preccurlyeq \mathbf{S}(\sigma) \Rightarrow 132 \preccurlyeq \mathbf{S}(L)\mathbf{S}(R)$.
  - If $132 \preccurlyeq \mathbf{S}(L)$ or $\mathbf{S}(R)$, we conclude by induction hypothesis.

Id($\mathbf{S} \circ \mathbf{r} \circ \mathbf{S}$): characterization by excluded patterns

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$\mathrm{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{S})$: characterization by excluded patterns

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  - If $\mathbf{S}(\sigma)$ contains $acb \equiv 132$, with $a \in \mathbf{S}(L)$, $cb \in \mathbf{S}(R)$, then there exists $d \in R$ such that R contains $cdb \equiv 231$, and σ contains $acdb \equiv 1342$.

Id($\mathbf{S} \circ \mathbf{r} \circ \mathbf{S}$): characterization by excluded patterns

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  - If $\mathbf{S}(\sigma)$ contains $acb \equiv 132$, with $a \in \mathbf{S}(L)$, $cb \in \mathbf{S}(R)$, then there exists $d \in R$ such that R contains $cdb \equiv 231$, and σ contains $acdb \equiv 1342$.
  - If $\mathbf{S}(\sigma)$ contains $acb \equiv 132$, with $ac \in \mathbf{S}(L)$, $b \in \mathbf{S}(R)$, then σ contains either $acnb \equiv 1342$ or $canb \equiv 3\bar{5}142$.

Id(S ∘ r ∘ S): characterization by excluded patterns

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$\mathbf{S}(\sigma)$ contains 132 $\Leftrightarrow$ σ contains 1342 or 31-4-2.

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Enumeration of Id($\mathbf{S} \circ \mathbf{r} \circ \mathbf{S}$)

Theorem (partial statement)

The two sets Id($\mathbf{S} \circ \mathbf{S}$) *and* Id($\mathbf{S} \circ \mathbf{r} \circ \mathbf{S}$) *are enumerated by the same sequence.*

Method of proof:

- Id($\mathbf{S} \circ \mathbf{S}$) $= \mathrm{Av}(2341, 3\bar{5}241)$
- Id($\mathbf{S} \circ \mathbf{r} \circ \mathbf{S}$) $= \mathrm{Av}(1342, 3\bar{5}142)$
- Provide a common rewriting system (encoding isomorphic **generating trees**) for Id($\mathbf{S} \circ \mathbf{S}$) and Id($\mathbf{S} \circ \mathbf{r} \circ \mathbf{S}$)

Refinement according to statistics: introduce each statistics into the rewriting system

Generating trees

- Generating tree for $\text{Av}(\pi, \tau, \ldots)$: an infinite tree where vertices at level n are permutations of $\mathfrak{S}_n$ avoiding $\pi, \tau, \ldots$.
- The children σ' of σ are obtained by insertion of a new element in the active sites of σ.
  - Sites are often on one of the four sides of the diagram (e.g. on the right).
  - Sites are active when $\sigma' \in \text{Av}(\pi, \tau, \ldots)$ *i.e.*, when the insertion does not create a pattern π or τ ...

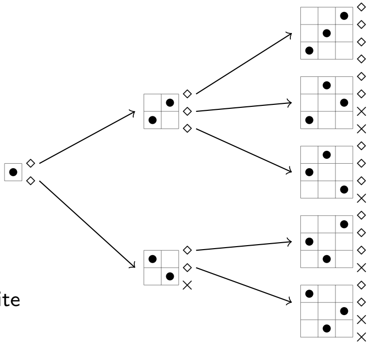
Theorem

Two classes having isomorphic generating trees are in bijection.

...eventhough the bijection is not explicit.

Generating trees

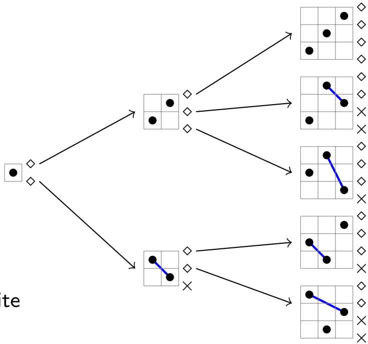
Example: Av(321) with insertion on the right:



Legend: { ◇ for an active site
× for an inactive site

Generating trees

Example: Av(321) with insertion on the right:



Legend: ◇ for an active site; × for an inactive site

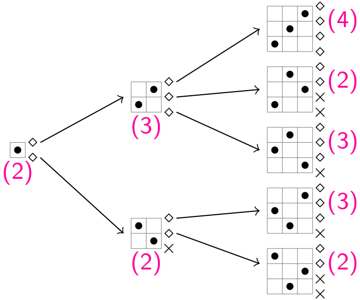
Remark: Active sites are the one above the higher inversion of σ (higher an above are intended w.r.t. the second element)

Associated rewriting systems

Idea: Describe the tree compactly by a rewriting rule/systems

- Associate labels to permutations (e.g. number of active sites)
- From the label of σ, describe the labels of the children of σ

Example: The generating tree of Av(321) with labels



Associated rewriting systems

Idea: Describe the tree compactly by a rewriting rule/systems

- Associate labels to permutations (e.g. number of active sites)
- From the label of σ, describe the labels of the children of σ

Example: For Av(321), we obtained

$$\begin{cases} (2) & \\ (k) & \rightsquigarrow \quad (k+1)(2)(3)\dots(k) \end{cases}$$

Proof:

- Labels record the number of sites above the higher inversion.
- Insertion in the topmost site creates one new active site.
- Insertion in any other site creates an inversion with $\max(\sigma)$.

Rewriting system for $\mathrm{Id}(\mathbf{S} \circ \mathbf{S})$ and $\mathrm{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{S})$

Lemma

A rewriting system for both $\mathrm{Id}(\mathbf{S} \circ \mathbf{S})$ *and* $\mathrm{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{S})$ *is*

$$\mathcal{R}_\Phi \begin{cases} (2,1,(1)) & \\ (x,k,(p_1,\dots,p_k)) & \rightsquigarrow \quad (2+p_j, j, (p_1,\dots,p_{j-1},i)) \\ & \qquad\qquad \text{for } 1 \le j \le k \text{ and } p_{j-1} < i \le p_j \\ & \quad\;\; (x+1, k+1, (p_1,\dots,p_k,i)) \\ & \qquad\qquad \text{for } p_k < i \le x \end{cases}$$

Interpretation of the labels:

- $x =$ the number of active sites of σ,
- $k =$ the number of RtoL-max in σ
- $p_\ell =$ the number of active sites above the ℓ-th RtoL-max in σ

Construction of the rewriting system $\mathcal{R}_\Phi$

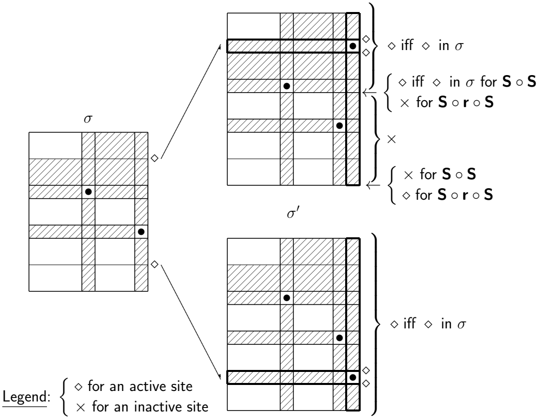
Adapted from [Dulucq, Gire, Guibert] by application of $\mathbf{c} \circ \mathbf{i}$.

Proof:
$\text{Id}(\mathbf{S} \circ \mathbf{S}) = \text{Av}(2341, 3\bar{5}241)$
$\text{Id}(\mathbf{S} \circ \mathbf{r} \circ \mathbf{S}) = \text{Av}(1342, 3\bar{5}142)$

Examine when insertion in ◇ creates an excluded pattern.

Case 1:
Insertion above σ_n
Need to consider the higher RtoL-max below ◇.

Case 2:
Insertion below σ_n



Refinement according to the statistics rmax

Recall the common rewriting system for Id(**S** ∘ **S**) and Id(**S** ∘ **r** ∘ **S**):

$$\mathcal{R}_\Phi \begin{cases} (2, 1, (1)) & \\ (x, k, (p_1, \dots, p_k)) & \rightsquigarrow \quad (2 + p_j, j, (p_1, \dots, p_{j-1}, i)) \\ & \qquad \text{for } 1 \leq j \leq k \text{ and } p_{j-1} < i \leq p_j \\ & \quad (x+1, k+1, (p_1, \dots, p_k, i)) \\ & \qquad \text{for } p_k < i \leq x \end{cases}$$

- Id(**S** ∘ **S**) and Id(**S** ∘ **r** ∘ **S**) have isomorphic generating trees.

⇒ At any level n, there is the same number of vertices labeled $(x, k, (p_1, \dots, p_k))$ in both trees.

- In the label $(x, k, (p_1, \dots, p_k))$ of σ we have $k = \text{rmax}(\sigma)$.

⇒ The statistics rmax is equidistributed in Id(**S** ∘ **S**) and Id(**S** ∘ **r** ∘ **S**)

Refinement according to the statistics lmax

Lemma

The rewriting system can be refined to account for the statistics lmax *as follows:*

$$\mathcal{R}_{\Phi}^{\mathrm{lmax}} \begin{cases} (2,1,(1),1) & \\ (x,k,(p_1,\dots,p_k),q) \quad \rightsquigarrow & (2+p_1,1,(1),q+1) \\ & (2+p_j,j,(p_1,\dots,p_{j-1},i),q) \\ & \qquad \text{for } 1 \le j \le k \text{ and } p_{j-1} < i \le p_j, i \ne 1 \\ & (x+1,k+1,(p_1,\dots,p_k,i),q) \\ & \qquad \text{for } p_k < i \le x \end{cases}$$

Proof: The number of LtoR-max does not change when inserting a new element on the right, except when inserting a maximal element ($+1$ in this case).

Refinement according to the statistics udword

Lemma

The rewriting system can be refined to account for the statistics udword *as follows:*

$$\mathcal{R}_{\Phi}^{\text{udword}} \begin{cases} (2, 1, (1), \varepsilon) & \\ (x, k, (p_1, \ldots, p_k), w) & \rightsquigarrow \quad (2 + p_j, j, (p_1, \ldots, p_{j-1}, i), w \cdot u) \\ & \qquad \textit{for } 1 \leq j \leq k \textit{ and } p_{j-1} < i \leq p_j \\ & \quad\;\; (x + 1, k + 1, (p_1, \ldots, p_k, i), w \cdot d) \\ & \qquad \textit{for } p_k < i \leq x \end{cases}$$

Proof: In the first (resp. second) case of the rewriting rule, a new element on the right is inserted above (resp. below) the rightmost one.

From Id(S ∘ i ∘ S) to Bax... via TBax and twin binary trees

Id(**S** ∘ **i** ∘ **S**)	$\longleftrightarrow$	TBax	$\longleftrightarrow$	Pairs of twin binary trees	$\longleftrightarrow$	Bax
lmax	$\longleftrightarrow$	lmax	$\longleftrightarrow$	rightmost branch	$\longleftrightarrow$	lmax
des	$\longleftrightarrow$	occ_μ	$\longleftrightarrow$	left edges	$\longleftrightarrow$	des
comp	$\longleftrightarrow$	comp	$\longleftrightarrow$	?	$\longleftrightarrow$	comp

From Id(S ∘ i ∘ S) to Bax... via TBax and twin binary trees

Id(S ∘ i ∘ S)	⟷	TBax	⟷	Pairs of twin binary trees	⟷	Bax
lmax	⟷	lmax	⟷	rightmost branch	⟷	lmax
des	⟷	occ_μ	⟷	left edges	⟷	des
comp	⟷	comp	⟷	?	⟷	comp

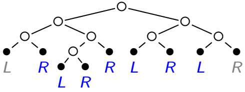
Bijection between Id(S ∘ i ∘ S) and TBax: Rewriting system, refined according to the three statistics.

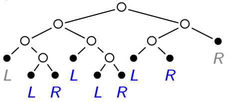
From Id(**S** ∘ **i** ∘ **S**) to Bax... via TBax and twin binary trees

Id(**S** ∘ **i** ∘ **S**)	$\longleftrightarrow$	TBax	$\longleftrightarrow$	Pairs of twin binary trees	$\longleftrightarrow$	Bax
lmax	$\longleftrightarrow$	lmax	$\longleftrightarrow$	rightmost branch	$\longleftrightarrow$	lmax
des	$\longleftrightarrow$	occ_μ	$\longleftrightarrow$	left edges	$\longleftrightarrow$	des
comp	$\longleftrightarrow$	comp	$\longleftrightarrow$	?	$\longleftrightarrow$	comp

Bijection between Id(**S** ∘ **i** ∘ **S**) and TBax: Rewriting system, refined according to the three statistics.

Bijection between TBax and Bax: One recently described by S. Giraudo, that goes through pairs of twin binary trees *i.e.*, trees of complementary canopies

Example: 

and 

Enumeration of $\mathrm{Id}(\mathbf{S} \circ \mathbf{i} \circ \mathbf{S})$

Theorem (partial statement)

The two sets $\mathrm{Id}(\mathbf{S} \circ \mathbf{i} \circ \mathbf{S})$ *and* TBax *are enumerated by the same sequence.*

Method of proof: again rewriting system

Lemma

A rewriting system for both $\mathrm{Id}(\mathbf{S} \circ \mathbf{i} \circ \mathbf{S})$ *and* TBax *is*

$$\mathcal{R}_\Psi \begin{cases} (2,0) & \\ (q,r) & \rightsquigarrow \quad (i+1, q+r-i) \text{ for } 1 \le i \le q \\ & \quad (q, r-j) \text{ for } 1 \le j \le r \end{cases}$$

It is obtained inserting elements respectively under and to the right of the diagram.

Refinement according to the statistics lmax

Lemma

The rewriting system can be refined to account for the statistics lmax *as follows:*

$$\mathcal{R}_{\Psi}^{\mathrm{lmax}} \begin{cases} (2,0,1) & \\ (q,r,m) & \rightsquigarrow \quad (2, q+r-1, m+1) \\ & \quad (i+1, q+r-i, m) \text{ for } 2 \le i \le q \\ & \quad (q, r-j, m) \text{ for } 1 \le j \le r \end{cases}$$

Refinement according to the statistics comp

Lemma

The rewriting system can be refined to account for the statistics comp *as follows:*

$$\mathcal{R}_{\Psi}^{\text{comp}} \begin{cases} (2,0,1,(2)) & \\ (q,r,c,(z_1,\ldots,z_c)) & \rightsquigarrow \quad (2,q+r-1,c+1,(2,z_1-1,z_2,\ldots,z_c)) \\ & \quad (i+1,q+r-i,c,(z_1+1,z_2,\ldots,z_c)) \\ & \qquad \textit{for } 2 \leq i \leq q \\ & \quad (q,r-j,c-k+1,(\sum_{i=1}^{k} z_i - j, z_{k+1},\ldots,z_c)) \\ & \qquad \textit{for } 1 \leq j \leq r, \textit{ where } k \textit{ is the} \\ & \qquad \textit{minimal } h \textit{ such that } \sum_{i=1}^{h} z_i \geq q+j \end{cases}$$

z_i = number of active sites in component i

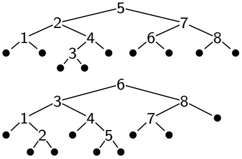
S. Giraudo's bijection between TBax and Bax

- To any $\sigma \in \mathfrak{S}_n$, associate $T_\rightarrow$ the (unlabelled) binary search tree obtained by insertion of $\sigma_1, \sigma_2, \dots, \sigma_n$.
- Similarly for $T_\leftarrow$ by insertion of $\sigma_n, \dots, \sigma_2, \sigma_1$.

Example: $\sigma = 5\,2\,4\,7\,1\,8\,3\,6$

$T_\rightarrow =$

$T_\leftarrow =$



S. Giraudo's bijection between TBax and Bax

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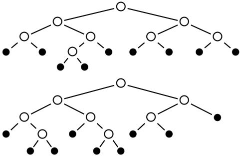

Lemma

$(T_{\rightarrow}, T_{\leftarrow})$ is a pair of twin binary trees (with $n+1$ leaves).

S. Giraudo's bijection between TBax and Bax

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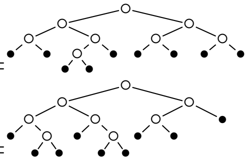
- Similarly for $T_\leftarrow$ by insertion of $\sigma_n, \ldots, \sigma_2, \sigma_1$.

Example: $\sigma = 5\,2\,4\,7\,1\,8\,3\,6$

$T_\rightarrow =$ $T_\leftarrow =$



Lemma

$(T_\rightarrow, T_\leftarrow)$ is a pair of twin binary trees (with $n+1$ leaves).

Theorem ([Giraudo])

A pair $(T_\rightarrow, T_\leftarrow)$ corresponds to a set of permutations containing exactly one Baxter and exactly one Twisted Baxter permutation. This provides a bijection between Bax *and* TBax.

The statistics lmax into S. Giraudo's bijection

Lemma

The elements on the rightmost branch from the root of $T_{\rightarrow}$ are the LtoR-max of σ.

This holds in particular when $\sigma \in$ Bax or TBax.

Theorem (partial statement)

The bijection of S. Giraudo between Bax *and* TBax *preserves the number of LtoR-max.*

The statistics comp into S. Giraudo's bijection

Lemma ([Giraudo])

If $\sigma \in$ Bax has exactly one component, then so does every τ sharing $(T_{\rightarrow}, T_{\leftarrow})$ with σ.

This holds in particular for $\tau \in$ TBax.

The statistics comp into S. Giraudo's bijection

Lemma ([Giraudo])

If $\sigma \in \mathrm{Bax}$ has exactly one component, then so does every τ sharing $(T_{\rightarrow}, T_{\leftarrow})$ with σ.

This holds in particular for $\tau \in \mathrm{TBax}$.

Lemma

If $\sigma \in \mathrm{Bax}$ and $\tau \in \mathrm{TBax}$ are in correspondance by S. Giraudo's bijection, then $\mathrm{comp}(\sigma) = \mathrm{comp}(\tau)$.

This does not hold in general, but only for $\tau \in \mathrm{TBax}$!

Proof: The above lemma and $\mathrm{TBax} = \mathrm{Av}(2\text{-}41\text{-}3, 3\text{-}41\text{-}2)$.
In particular, no interpretation of comp on $(T_{\rightarrow}, T_{\leftarrow})$...

From computer experiments to open questions

- For any $\alpha, \beta \in D_8$, describe the permutations sorted by $\mathbf{S} \circ \alpha \circ \mathbf{S} \circ \beta \circ \mathbf{S}$ by excluded patterns.

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Or when computers provide conjectures (Schröder numbers)...

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Or when computers provide conjectures (Schröder numbers)...

- And keep composing: $\mathbf{S} \circ \alpha \circ \mathbf{S} \circ \beta \circ \mathbf{S} \circ \gamma \circ \mathbf{S} \ldots$

And new conjectures

Conjecture

Fix $k \geq 1$. For any k-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_k) \in \{\mathrm{id}, \mathbf{r}\}^k$, permutations sorted by $\mathbf{S} \circ \alpha_1 \circ \mathbf{S} \circ \alpha_2 \circ \ldots \circ \mathbf{S} \circ \alpha_k \circ \mathbf{S}$ are enumerated by the same sequence.

⇒New approach to the study of k-stack sortable permutations.

And new conjectures

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⇒New approach to the study of k-stack sortable permutations.

Stronger conjecture

For any $(\alpha_1, \alpha_2, \dots, \alpha_k)$ and $(\beta_1, \beta_2, \dots, \beta_k)$, we have either

- $\mathrm{Id}(\mathbf{S} \circ \alpha_1 \circ \mathbf{S} \circ \alpha_2 \circ \dots \circ \mathbf{S} \circ \alpha_k \circ \mathbf{S}) = \mathrm{Id}(\mathbf{S} \circ \beta_1 \circ \mathbf{S} \circ \beta_2 \circ \dots \circ \mathbf{S} \circ \beta_k \circ \mathbf{S})$;
- *or these sets are not enumerated by the same sequence;*
- *or they fall into the first conjecture.*