The Brownian limit of separable permutations

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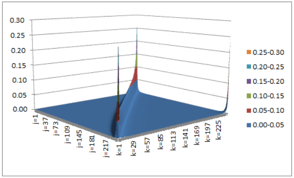
talk based on a joint work with
Frédérique Bassino, Valentin Féray, Lucas Gerin and Adeline Pierrot

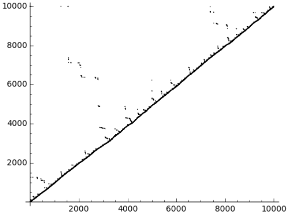
Arxiv: 1602.04960

Permutation Patterns, June 2016

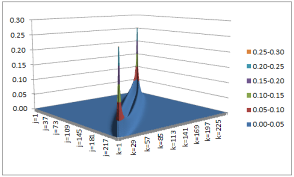
Random permutations in $Av(\tau)$ for τ of size 3

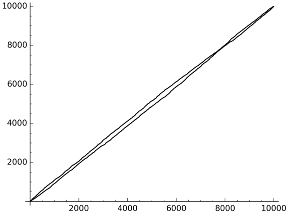
$Av(231)$





$Av(321)$



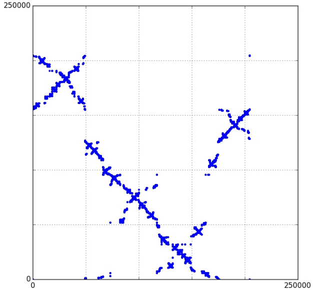


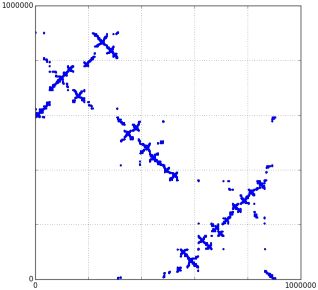
from Miner-Pak presented at PP 2013

from Hoffman-Rizzolo-Slivken presented at PP 2015

Random separable permutations (*i.e.*, in $Av(2413, 3142)$)

Separable permutations of size 204523 and 903073, drawn uniformly at random among those of the same size.
(Figures generated with a Boltzmann sampler by Carine Pivoteau.)





Goal: Explain these diagrams, by describing the "limit shape" of random separable permutations of size $n \to +\infty$.

More about $Av(\tau)$ for τ of size 3

- Madras with Atapour, Liu and Pehlivan and Miner-Pak:
  $\hookrightarrow$ very precise local description of the asymptotic shape
- Hoffman-Rizzolo-Slivken:
  $\hookrightarrow$ scaling limits and link with the Brownian excursion (for the fluctuations around the main diagonal)
- Janson, following earlier works by Bóna, Cheng-Eu-Fu, Homberger, Janson-Nakamura-Zeilberger, Rudolph:
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Main result of Janson: For any pattern π, the quantity

$$\frac{\text{number of occurrences of } \pi \text{ in uniform } \sigma \in Av_n(132)}{n^{|\pi| + \text{ number of descents of } \pi + 1}}$$

converges in distribution to a strictly positive random variable.

Our main result: the limit of separable permutations

Notation:

- $\widetilde{\mathrm{occ}}(\pi,\sigma) = \frac{\text{number of occurrences of } \pi \text{ in } \sigma}{\binom{n}{k}}$ for $n = |\sigma|$ and $k = |\pi|$
- $\boldsymbol{\sigma}_n$ = a uniform random separable permutation of size n

Theorem

There exist random variables (Λ_π), π ranging over all permutations, such that for all π, $0 \leq \Lambda_\pi \leq 1$ and when $n \to +\infty$, $\widetilde{\mathrm{occ}}(\pi,\boldsymbol{\sigma}_n)$ converges in distribution to Λ_π.

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Moreover,

- We describe a construction of Λ_π.
- This holds jointly for patterns $\pi_1, \ldots, \pi_r$.
- If π is separable of size at least 2, Λ_π is non-deterministic.
- Combinatorial formula for all moments of Λ_π.

Why separable permutations?

Separable permutations are $Av(2413, 3142)$. But more importantly, they:

- form one of the most studied class after $Av(\tau)$ for $|\tau| = 3$;
- are the smallest family closed under $\oplus$ and $\ominus$
  (and hence form the simplest non-trivial substitution-closed class);
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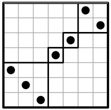
Example: $\sigma = 3214576 = \oplus[\ominus[1,1,1],1,1,\ominus[1,1]] =$ 

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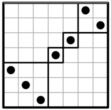
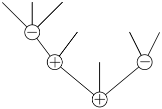
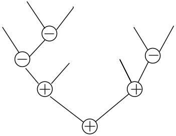
Example: $\sigma = 3214576 = \oplus[\ominus[1,1,1],1,1,\ominus[1,1]] =$ 

σ corresponds to  or  , among other trees.

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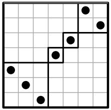
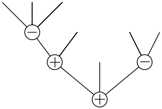
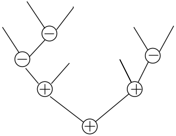
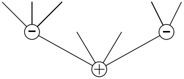
Example: $\sigma = 3214576 = \oplus[\ominus[1,1,1],1,1,\ominus[1,1]] =$ 

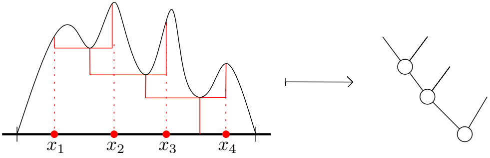
σ corresponds to  or  , among other trees.

The correspondence can be made one-to-one imposing alternating signs.

Here, $\sigma \leftrightarrow$  .

Excursion = continuous function f from $[0,1]$ to $[0,+\infty)$ with $f(0)=f(1)=0$.

With $\mathbf{x}=\{x_1,\dots,x_k\}$ a set of points in $[0,1]$, and f an excursion, we classically associate a tree looking at the minima of f between the x_i's.



Excursion = continuous function f from $[0,1]$ to $[0,+\infty)$ with $f(0)=f(1)=0$.

Signed excursion = pair (f,s) where f is an excursion and s a sign function giving a $+$ or – sign to each local minimum of f.

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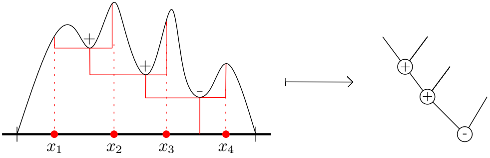
Signed variant associating a signed Schröder tree $\mathrm{Tree}_\pm(f,s,\mathbf{x})$ with (f,s) and $\mathbf{x}$.



Construction of Λ_π (2/2)

For π a pattern and (f, s) a signed excursion,

$$\Psi_\pi(f, s) = \text{probability that } \mathrm{Tree}_\pm(f, s, \mathbf{X}) \text{ is a signed Schröder tree of } \pi$$

when $\mathbf{X}$ consists of $k = |\pi|$ uniform and independent points in $[0, 1]$.

Remark: Ψ_π is identically 0 if π is not separable.

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The signed Brownian excursion is (e,S)
where e is the Brownian excursion and S assigns signs to the local minima of e in a balanced and independent manner.

$$\text{For all } \pi, \quad \Lambda_\pi = \Psi_\pi(e,S).$$

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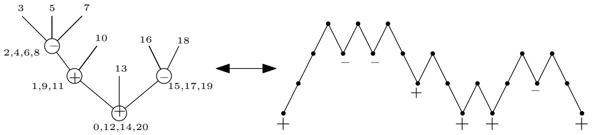
You may ask:
What is the link between trees extracted from the signed Brownian excursion and occurrences of patterns in separable permutations?

Contours of trees

Classical case: tree → excursion called contour (*via* depth-first search).
Signed variant: signed tree → signed excursion called signed contour.



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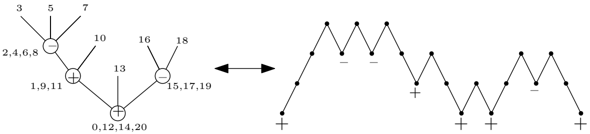

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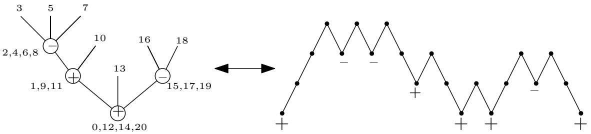

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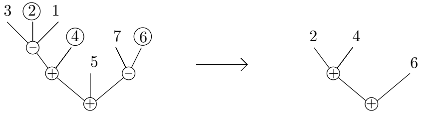
Open: Do signed Schröder trees converge to the signed Brownian excursion?

Extracting a pattern π from a separable permutation σ,

$$\sigma = 3214576 \longmapsto \pi = 123$$

$\equiv$ extracting a subtree (induced by leaves) in a signed Schröder tree of σ



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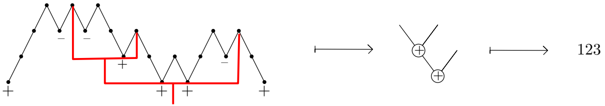

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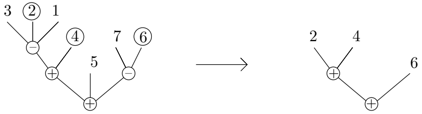

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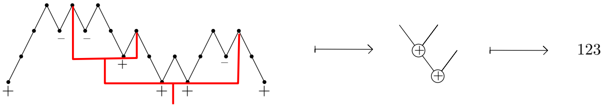

Patterns in separable permutations $\approx \text{Tree}_{\pm}(f, s, \mathbf{x}) \to \Lambda_\pi \approx \text{Tree}_{\pm}(e, S, \mathbf{X})$.

Our main theorem

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This gives the proportion of occurrences of any pattern π in a uniform separable permutation of size n, as $n \to +\infty$.

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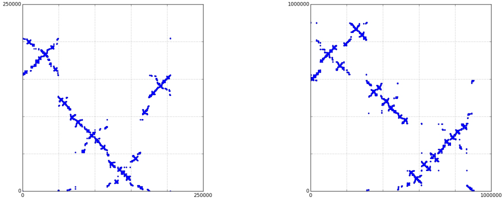
But how does this relate to the limit diagram of large uniform separable permutations?



Permuton interpretation of our result

- A permuton is a measure on $[0,1]^2$ with uniform marginals.
- The diagram of any permutation σ is a permuton, denoted μ_σ (up to normalizing and filling in uniformly the cells containing dots).
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There exists a random permuton μ such that μ_{σ_n} tends to μ in distribution (in the weak convergence topology).

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The proof uses a result of Hoppen-Kohayakawa-Moreira-Rath-Sampaio: For $(\sigma_n)_{n\geq 1}$ a deterministic sequence of permutations of increasing size n, assuming that $\widetilde{\mathrm{occ}}(\pi,\sigma_n)$ has a limit as $n \to +\infty$ for every pattern π, it holds that the sequence of permutons $(\mu_{\sigma_n})_n$ has a limit.

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Some hints about the proof
(that you were spared)

Proof, step 1: Convergence in expectation in enough

Our main theorem: $\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)$ converges in distribution to Λ_π.

It is enough to prove: $\mathbb{E}\,[\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)] \longrightarrow \mathbb{E}\,[\Lambda_\pi]$,
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- Our random variables are bounded (they take values in $[0, 1]$), so convergence in distribution $\Leftrightarrow$ convergence of all moments.
- Expectation determines all moments, since we can write:

$$\prod_{i=1}^{r} \Lambda_{\pi_i} = \sum_{\rho \in \mathfrak{S}_K} c^{\rho}_{\pi_1,\dots,\pi_r} \Lambda_\rho \text{ and}$$

$$\prod_{i=1}^{r} \widetilde{\mathrm{occ}}(\pi_i, \sigma) = \sum_{\rho \in \mathfrak{S}_K} c^{\rho}_{\pi_1,\dots,\pi_r} \widetilde{\mathrm{occ}}(\rho, \sigma) \; + \; \mathcal{O}(1/n).$$

Remark: The above is the combinatorial formula for computing moments of Λ_π, knowing a combinatorial formula for $\mathbb{E}[\Lambda_\pi]$ (omitted but easy).

Proof, step 2: Convergence in the unsigned case

For t_0 a tree, f an excursion, and d a probability distribution on $[0,1]$, let

$$\Psi_{t_0}(f,d) = \mathbb{P}(\text{the tree extracted from the set of points } \mathbf{X} \text{ in } f \text{ is } t_0)$$

where $\mathbf{X}$ consists of $k = |t_0|$ points in $[0,1]$ drawn independently along d.

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- C_n = the (normalized) contour of a uniform Schröder tree with n leaves;
- d_n = the uniform distribution on the peaks of C_n;
- e = the Brownian excursion;
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Theorem

For all t_0, $\Psi_{t_0}(C_n, d_n)$ converges in distribution to $\Psi_{t_0}(e,u)$ when $n \to +\infty$.

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- $C_n \to e$: Pitman-Rizzolo or Kortchemski (with Galton-Watson trees)
- $d_n \to u$: similar to Marckert-Mokkadem (concentration inequalities)
- continuity of Ψ_{t_0}: exercise (using nice properties of e)

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Proof idea:

1. Taking k leaves uniformly in a uniform Schröder tree with n leaves, the distances between their common ancestors tend to $+\infty$ with n.
2. With a subtree exchangeability argument, this implies that the parities of the height of these common ancestors are balanced and independent.

Proof, step 3: Re-introducing signs

- In the signed Brownian excursion (e, S), the signs are balanced and independent.
  So, this also holds in the signed trees extracted from (e, S).
- We prove that, in the limit when $n \to +\infty$, the signs are also balanced and independent in the trees extracted from the signed contours of separable permutations of size n.

Proof idea:

1. Taking k leaves uniformly in a uniform Schröder tree with n leaves, the distances between their common ancestors tend to $+\infty$ with n.
2. With a subtree exchangeability argument, this implies that the parities of the height of these common ancestors are balanced and independent.

Conclusion of the proof:
relate the expectation in the signed and in the unsigned cases.