

Limit shapes of pattern-avoiding permutations

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talk based on joint works with
Frédérique Bassino, Valentin Féray, Lucas Gerin,
Mickaël Maazoun and Adeline Pierrot

Including additional pictures by Jacopo Borga and Carine Pivoteau

Journées combinatoires de Bordeaux, Février 2019

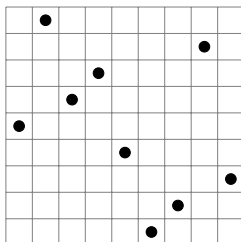
What are permutations? (in this talk)

A **permutation** of size n is a bijection from $\{1, 2, \dots, n\}$ to itself.

We often write a permutation σ of size n as the word $\sigma(1)\sigma(2)\dots\sigma(n)$.

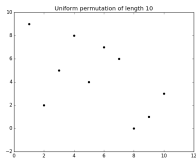
For the purpose of this talk, we represent permutations by their permutation matrices, or rather their **diagram**.

Example: the diagram of $\sigma = 596741283$ is

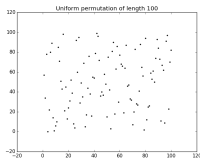


What are random permutations (and their limit shapes)?

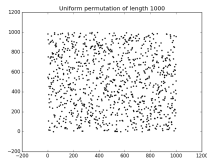
Diagrams of permutations of various sizes picked uniformly at random:



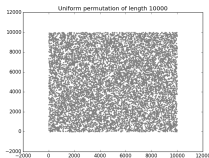
size 10



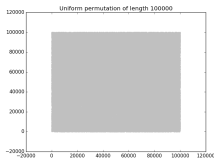
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size 1000



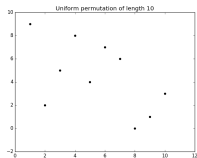
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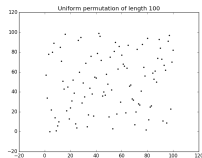
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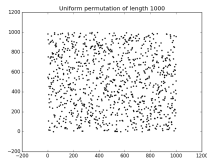
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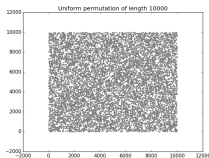
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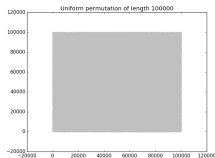
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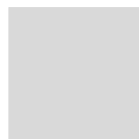
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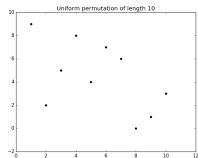
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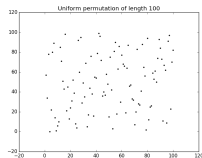
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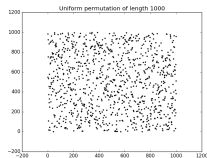
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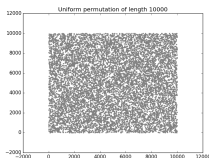
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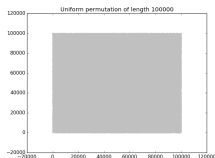
size 100



size 1000



size 10 000



size 100 000



in the limit

Goal of the talk: Describe **limit shapes** of (the diagrams of) **pattern-avoiding permutations**.

Patterns in permutations

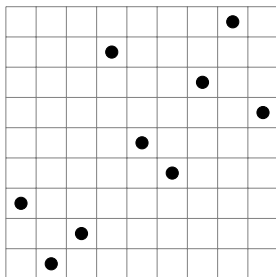
A permutation π of size k is a **pattern** of a permutation σ of size n if there exist $1 \leq i_1 < \dots < i_k \leq n$ such that $\sigma(i_1) \dots \sigma(i_k)$ is in the **same relative order** (\equiv) as π .

Example: 2134 is a pattern of **312854796** since $3157 \equiv 2134$.

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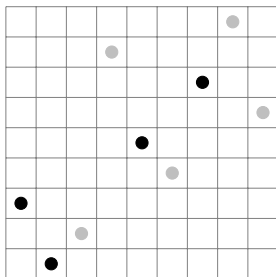
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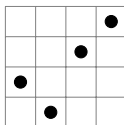
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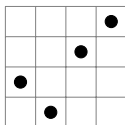
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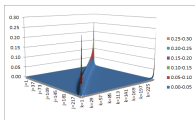
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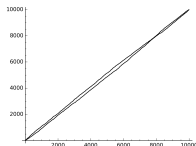
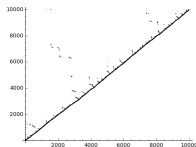
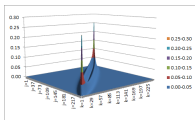
Permutation classes are sets of permutations defined by the **avoidance** of patterns. They are denoted $Av(B)$ for B a set of excluded patterns.

Uniform random permutations in $Av(\tau)$ for τ of size 3

$Av(231)$



$Av(321)$

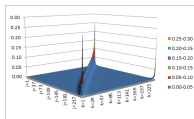


from Miner-Pak (2013)

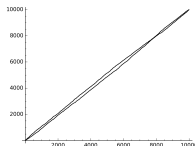
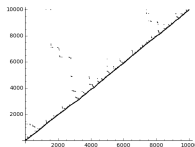
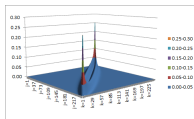
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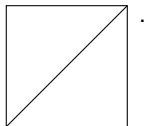


from Miner-Pak (2013) from Hoffman-Rizzolo-Slivken (2015)

- Miner-Pak, also Madras with Atapour, Liu and Pehlivan: very **precise** local description of the **average** asymptotic shape
- Hoffman-Rizzolo-Slivken: **scaling limits** and link with the **Brownian excursion** (for the fluctuations around the main diagonal)

Diagrams of uniform random permutations in classes

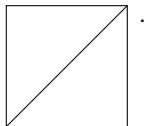
At **first order**, the limit of the diagram of a uniform random permutation in $Av(\tau)$ for $\tau = 231$ or 321 is just



So, is first order interesting ?

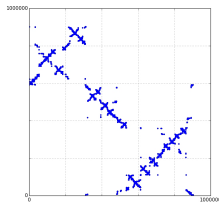
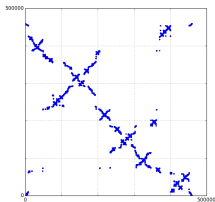
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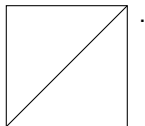
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Typical large permutations in $Av(2413, 3142)$, the class of separable permutations, also described as the substitution-closed class with set of simple permutations \emptyset :



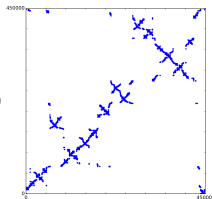
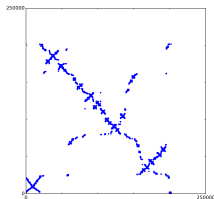
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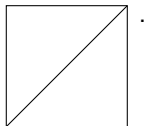
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Typical large permutations in the substitution-closed class with set of simple permutations $\{2413, 3142, 24153\}$:



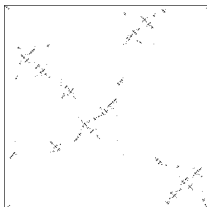
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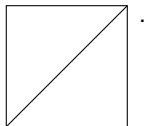
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Typical large permutation in the substitution-closed class with (infinite) set of simple permutations $Av(321) \cap \{Simplex\}$, *i.e.* in the substitution closure of $Av(321)$:



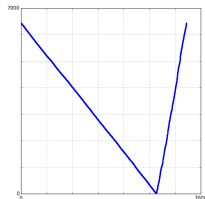
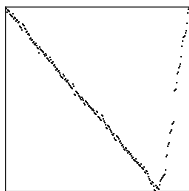
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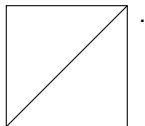
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Typical (large) permutations in $Av(2413, 1243, 2341, 41352, 531642)$:



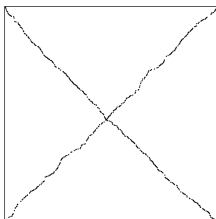
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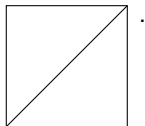
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Typical (large) permutation in $Av(2413, 3142, 2143, 3412)$, called the X-class and denoted \mathcal{X} later:



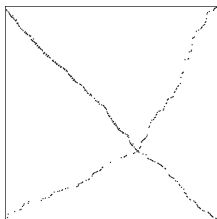
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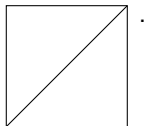
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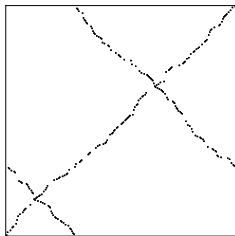
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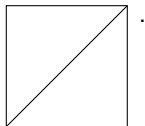
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Typical (large) permutation in the downward closure of $\oplus[\mathcal{X}, \mathcal{X}]$:



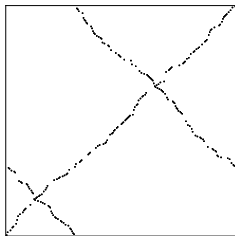
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How can we explain these pictures?

What type of objects are the limiting diagrams?

A **permuton** is a probability measure on the unit square with uniform marginals,

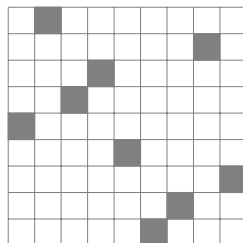
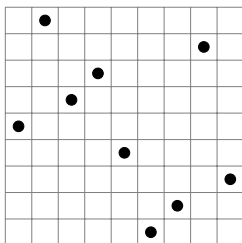
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With its diagram, every permutation σ can be viewed as a permuton μ_σ .

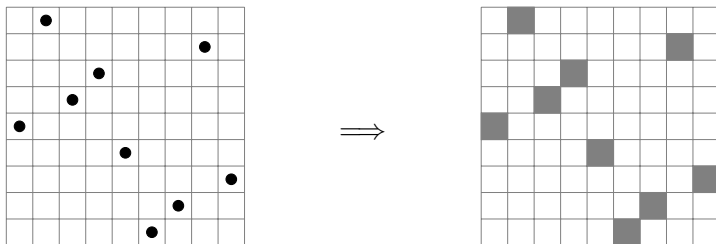


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Informally, permuton can represent permutations of finite size, but also “**permutations of infinite size**”.

Permuton convergence

We say that a sequence of permutations (σ_n) converges to a permuton μ when the sequence of permutons (μ_{σ_n}) converges to μ (for the weak convergence of measures).

This extends to sequences of random permutations (σ_n) , converging to a (*a priori* random) permuton μ .

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Permuton convergence is characterized by the convergence of probabilities of all patterns:

- Define $\widetilde{\text{occ}}(\pi, \sigma)$ as the probability of occurrence of the pattern π in σ :

$$\widetilde{\text{occ}}(\pi, \sigma) = \frac{\text{number of occurrences of } \pi \text{ in } \sigma}{\binom{|\sigma|}{|\pi|}}.$$

- $\widetilde{\text{occ}}(\pi, \mu)$ is similarly defined as the probability that $|\pi|$ points of the unit square picked at random according to μ induce the pattern π .
- (σ_n) converges to $\mu \Leftrightarrow (\widetilde{\text{occ}}(\pi, \sigma_n))_\pi$ converges to $(\widetilde{\text{occ}}(\pi, \mu))_\pi$ in distribution (jointly for all patterns π).

Summary so far, and what comes next

Goal: Prove limit shape results for the diagrams of uniform random permutations in permutation classes.

Framework: Express these limit shape results in the framework of permutons.

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Key tools:

- Permuton convergence is the convergence of all pattern probabilities;
- Thanks to their **substitution decomposition**, permutations are trees and their patterns are subtrees;

- Limit shape results on random trees

OR

- Singularity analysis of generating functions for trees.

Substitution decomposition

Ingredients:

- A way of building bigger permutations from smaller ones
 \rightsquigarrow substitution or inflation;
- “Building blocks” allowing to build all permutations
 \rightsquigarrow simple permutations.

Essential property:

- For every permutation σ , there exists a unique way of obtaining it recursively using inflations of simple permutations.

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Outcome:

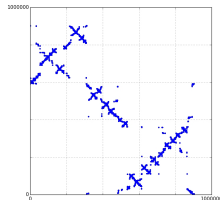
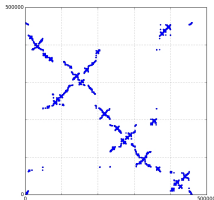
- Bijection between permutations and **decomposition trees**.
- (Some) permutation classes are (nice) families of trees:
 - easiest case: **substitution-closed classes**;
 - beyond those: classes with a **finite combinatorial specification**.

**Separable permutations
and the Brownian separable permuton**

Separable permutations

They are equivalently described as

- $Av(2413, 3142)$;
- the **substitution-closed** class with set of simple permutations \emptyset .

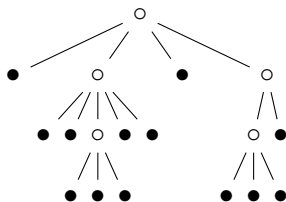


Theorem:

Uniform random separable permutations converge to a genuinely random permutation: the **Brownian separable permutation**.

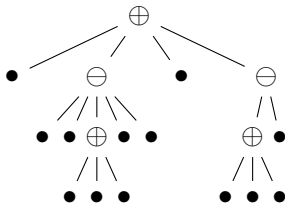
Decomposition trees of separable permutations

- A **Schröder tree** of size n is a rooted plane tree with n leaves whose internal vertices have at least two children.



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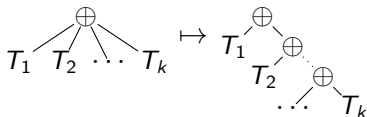
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- Decomposition trees of separable permutations are **signed Schröder trees**, as above with additional signs \oplus and \ominus on the internal vertices, which alternate on any path from the root to a leaf (*i.e.* the sign of the root determines all others).

The combinatorial specification of separable permutations

Up to the “right binarization”



(and same with \ominus),

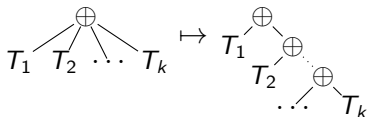
the **decomposition trees of separable permutations** are generated by the following combinatorial specification:

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Starting point of the “analytic combinatorics” proof, discussed later.

The combinatorial specification of separable permutations

Up to the “right binarization”



(and same with \ominus),

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Starting point of the “analytic combinatorics” proof, discussed later.
For now, we present the “random trees” proof.

Contours and their limits

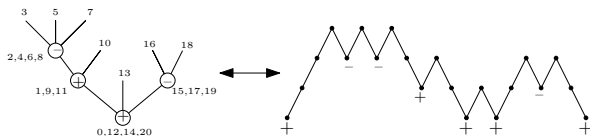
The contour of a uniform random [Schröder tree](#) converges to the [Brownian excursion](#).

Contours and their limits

The contour of a uniform random **Schröder tree** converges to the **Brownian excursion**.

We can define **signed contours** of signed Schröder trees:

- Peaks \leftrightarrow leaves.
- Local minima with signs \leftrightarrow signed internal nodes.

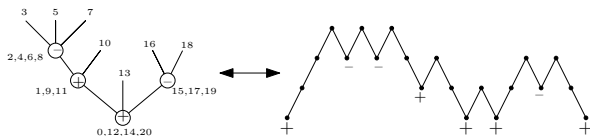


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We can define a **signed version of the Brownian excursion**:

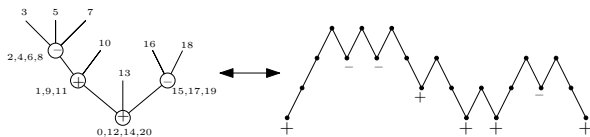
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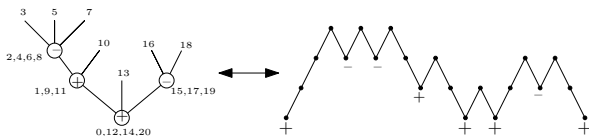
Not known: Do signed contours of signed Schröder trees converge to the signed Brownian excursion?

Contours and their limits

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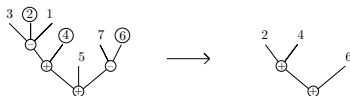
Convergence of the extracted patterns/subtrees

On **finite** objects:

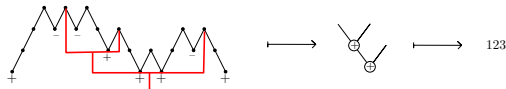
Extracting a pattern π of size k from a separable permutation σ

$$\sigma = 3214576 \mapsto \pi = 123$$

\equiv Extracting a signed subtree (induced by k leaves) in a signed Schröder tree of σ



\equiv Extracting a signed tree from a set of k **peaks** in a signed contour of σ



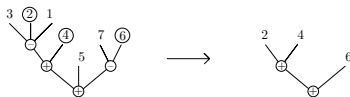
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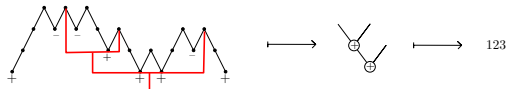
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Extracting a signed tree from a set of k **uniformly chosen points** in the signed Brownian excursion \mathbf{e}_{\pm}

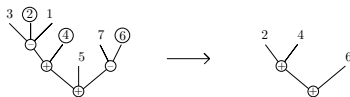
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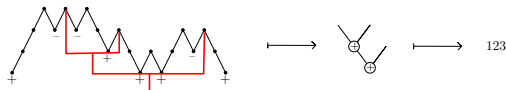
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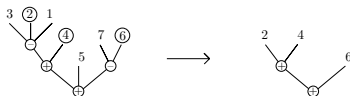
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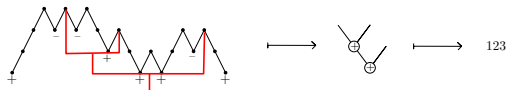
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In the limit:

This characterizes the probabilities of all subtrees extracted from \mathbf{e}_{\pm} , hence of all patterns extracted from the Brownian separable permutation (so defined).



Extracting a signed tree from a set of k uniformly chosen points in the signed Brownian excursion \mathbf{e}_{\pm}

**Substitution-closed classes
and universality
of the Brownian separable permuton**

Separable permutations and substitution-closed classes

The class of **separable permutations** is the one whose decomposition trees are described by the combinatorial specification

$$\left\{ \begin{array}{l}
 \mathcal{T}_{\text{sep}} = \{\bullet\} \uplus \left(\mathcal{T}_{\text{sep}}^{\text{not}\oplus} \uplus \mathcal{T}_{\text{sep}} \right) \uplus \left(\mathcal{T}_{\text{sep}}^{\text{not}\ominus} \uplus \mathcal{T}_{\text{sep}} \right) ; \\
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Separable permutations and substitution-closed classes

Substitution-closed classes are those whose decomposition trees described by a combinatorial specification of the form

$$\left\{ \begin{array}{l} \mathcal{T} = \{\bullet\} \uplus \begin{array}{c} \oplus \\ \swarrow \quad \searrow \\ \mathcal{T}_{\text{not}\oplus} \quad \mathcal{T} \end{array} \uplus \begin{array}{c} \ominus \\ \swarrow \quad \searrow \\ \mathcal{T}_{\text{not}\ominus} \quad \mathcal{T} \end{array} \uplus \begin{array}{c} \pi \\ \swarrow \quad \searrow \\ \mathcal{T} \quad \mathcal{T} \end{array} \uplus \dots \uplus \begin{array}{c} \pi \\ \swarrow \quad \searrow \\ \mathcal{T} \quad \mathcal{T} \end{array} ; \\ \mathcal{T}_{\text{not}\oplus} = \{\bullet\} \uplus \begin{array}{c} \ominus \\ \swarrow \quad \searrow \\ \mathcal{T}_{\text{not}\ominus} \quad \mathcal{T} \end{array} \uplus \begin{array}{c} \pi \\ \swarrow \quad \searrow \\ \mathcal{T} \quad \mathcal{T} \end{array} \uplus \dots \uplus \begin{array}{c} \pi \\ \swarrow \quad \searrow \\ \mathcal{T} \quad \mathcal{T} \end{array} ; \\ \mathcal{T}_{\text{not}\ominus} = \{\bullet\} \uplus \begin{array}{c} \oplus \\ \swarrow \quad \searrow \\ \mathcal{T}_{\text{not}\oplus} \quad \mathcal{T} \end{array} \uplus \begin{array}{c} \pi \\ \swarrow \quad \searrow \\ \mathcal{T} \quad \mathcal{T} \end{array} . \end{array} \right.$$

where \mathcal{S} is the set of simple permutations in the class $\mathcal{C}_{\mathcal{S}}$ considered.

Theorem:

Under an analytic condition on the generating function $S(z)$ of \mathcal{S} , uniform random permutations in the substitution-closed class $\mathcal{C}_{\mathcal{S}}$ converge to a **biased Brownian separable permutation**.

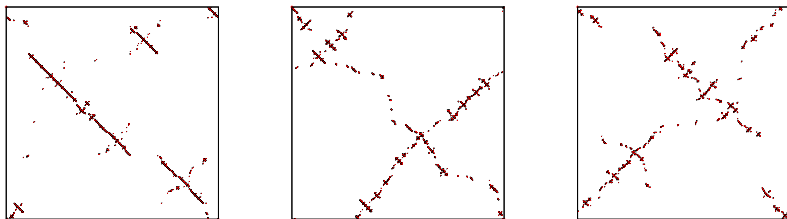
The biased Brownian separable permuton of parameter p

- Biased version of the signed Brownian excursion (for $p \in [0, 1]$): in $\mathbf{e}_{\pm, p}$, local minima carry independent signs, but not balanced; instead, $+$ with probability p , $-$ with probability $1 - p$.
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The higher p is, the more drift there is towards the direction of the main diagonal in μ_p .



simulations of μ_p for $p = 0.2, 0.45, 0.5$.

Statement and examples

Theorem:

Let $\mathcal{C}_{\mathcal{S}}$ be the **substitution-closed class** with set of simple permutations \mathcal{S} . Let $S(z)$ be the generating function of \mathcal{S} , and let $R_{\mathcal{S}}$ be its (positive) radius of convergence.

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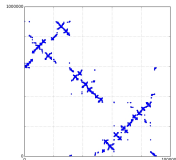
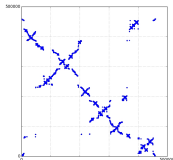
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Example 1: Separable permutations, i.e. \mathcal{C}_{\emptyset} , $\Rightarrow p = 0.5$



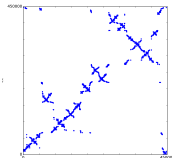
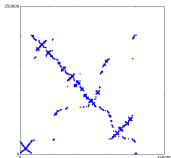
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Example 2: $\mathcal{C}_{\mathcal{S}}$ with $\mathcal{S} = \{2413, 3142, 24153\}$, $\Rightarrow p = 0.5$



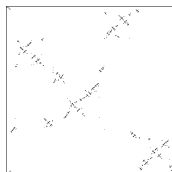
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Example 3: $\mathcal{C}_{\mathcal{S}}$ with $\mathcal{S} = Av(321) \cap \{\text{Simples}\}$, $\Rightarrow p \in [0.577, 0.622]$



Statement and examples

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Let \mathcal{C}_S be the **substitution-closed class** with set of simple permutations \mathcal{S} . Let $S(z)$ be the generating function of \mathcal{S} , and let R_S be its (positive) radius of convergence.

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Further examples: All substitution-closed classes

- with finitely many simple permutations,
- or more generally such that $R_S = 1$,
- or such that S' diverges at R_S (rational, square root singularities, ...).

This covers all substitution-closed classes whose simple permutations have been enumerated.

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Non-example: $Av(2413)$

Limit not known.

We believe it is “degenerate”: typical permutations in $Av(2413)$ look like typical **simple** permutations in $Av(2413)$.

$\sigma_n =$ uniform random permutation of size n in \mathcal{C}_S .

- The law of $(\widetilde{\text{occ}}(\pi, \sigma_n))_\pi$ is **determined by its moments**.
 - These moments are all determined by $(\mathbb{E}[\widetilde{\text{occ}}(\pi, \sigma_n)])_\pi$.
- \Rightarrow It is enough to prove the convergence of $\left(\mathbb{E}[\widetilde{\text{occ}}(\pi, \sigma_n)]\right)_n$ for all π .

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- Numerator and denominator are expressed as **coefficients of generating series** of trees (possibly with marked leaves).
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- Numerator and denominator are expressed as **coefficients of generating series** of trees (possibly with marked leaves).
- The combinatorial specification for \mathcal{C}_S yields equations for these tree series.
- Estimate their coefficients with **analytic combinatorics**.

- Determine the **singular behavior** of the trees series.

They all depend on the one of $T_{\text{not}\oplus}$ satisfying

$$T_{\text{not}\oplus}(z) = z + \Lambda(T_{\text{not}\oplus}(z))$$

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Rk: The limit of $\mathbb{E}[\widetilde{\text{occ}}(\pi, \sigma_n)]$ is non-zero if and only if π is separable.

**Finitely generated classes,
not necessarily substitution-closed**

Combinatorial specifications of classes

- For substitution-closed classes: there is **always** a combinatorial specification for the associated decomposition trees.
- For other classes, there is **sometimes** a combinatorial specification for the associated decomposition trees.
- It is always the case when the **number of simple permutations** in the class is **finite**.
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Moreover, in this case, the specification is automatically produced.

Example: $Av(132)$

$$\left\{ \begin{array}{l} \mathcal{T} = \{\bullet\} \uplus \mathcal{T}^{\text{not}\oplus} \uplus \mathcal{T}_{\langle 21 \rangle} \\ \mathcal{T}^{\text{not}\oplus} = \{\bullet\} \uplus \mathcal{T}^{\text{not}\ominus} \uplus \mathcal{T} \\ \mathcal{T}^{\text{not}\ominus} = \{\bullet\} \uplus \mathcal{T}^{\text{not}\oplus} \uplus \mathcal{T}_{\langle 21 \rangle} \\ \mathcal{T}_{\langle 21 \rangle} = \{\bullet\} \uplus \mathcal{T}_{\langle 21 \rangle}^{\text{not}\oplus} \uplus \mathcal{T}_{\langle 21 \rangle} \\ \mathcal{T}_{\langle 21 \rangle}^{\text{not}\oplus} = \{\bullet\}. \end{array} \right.$$

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Consider a specification for the decomposition trees of permutations in a class $\mathcal{C} = \mathcal{T}_0$ where the families $(\mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_k)$ appear.

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Consider a specification for the decomposition trees of permutations in a class $\mathcal{C} = \mathcal{T}_0$ where the families $(\mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_k)$ appear.

Define that \mathcal{T}_i is **critical** when its generating function has minimal radius of convergence among all those of the $(\mathcal{T}_j)_j$ (which is that of \mathcal{T}_0).

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The limiting behavior of uniform permutations in \mathcal{C} is determined by:

- the **strongly connected components** of the specification restricted to critical families;
- whether the restriction of the specification to each strongly connected component is **linear** or **branching**.

Essentially branching specifications

The restriction of the specification to critical families is strongly connected, and contains a **product**.

⇒ The limiting permuton of the class is a biased **Brownian separable permuton** (of parameter ρ possibly 0 or 1).

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Example 1: $Av(132)$, with critical families in **blue**.

$$\left\{ \begin{array}{l} \mathcal{T} = \{\bullet\} \uplus \mathcal{T}_{\text{not}\oplus}^{\oplus} \mathcal{T}_{\langle 21 \rangle} \uplus \mathcal{T}_{\text{not}\ominus}^{\ominus} \mathcal{T} \\ \mathcal{T}_{\text{not}\oplus} = \{\bullet\} \uplus \mathcal{T}_{\text{not}\ominus}^{\ominus} \mathcal{T} \\ \mathcal{T}_{\text{not}\ominus} = \{\bullet\} \uplus \mathcal{T}_{\text{not}\oplus}^{\oplus} \mathcal{T}_{\langle 21 \rangle} \\ \mathcal{T}_{\langle 21 \rangle} = \{\bullet\} \uplus \mathcal{T}_{\text{not}\oplus}^{\oplus} \mathcal{T}_{\langle 21 \rangle} \\ \mathcal{T}_{\langle 21 \rangle}^{\text{not}\oplus} = \{\bullet\}. \end{array} \right.$$

The limit is the Brownian separable permuton of parameter $p = 0$.

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The restriction of the specification to critical families is strongly connected, and contains a **product**.

⇒ The limiting permuton of the class is a biased **Brownian separable permuton** (of parameter ρ possibly 0 or 1).

Example 2: $Av(2413, 31452, 41253, 41352, 531246)$, with critical families in **blue**.

$$\left\{ \begin{array}{l} \mathcal{T}_0 = \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_0] \uplus \ominus[\mathcal{T}_2, \mathcal{T}_0] \uplus 3142[\mathcal{T}_0, \mathcal{T}_3, \mathcal{T}_3, \mathcal{T}_0] \\ \mathcal{T}_1 = \{\bullet\} \uplus \ominus[\mathcal{T}_2, \mathcal{T}_0] \uplus 3142[\mathcal{T}_0, \mathcal{T}_3, \mathcal{T}_3, \mathcal{T}_0] \\ \mathcal{T}_2 = \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_0] \uplus 3142[\mathcal{T}_0, \mathcal{T}_3, \mathcal{T}_3, \mathcal{T}_0] \\ \mathcal{T}_3 = \{\bullet\} \uplus \ominus[\mathcal{T}_4, \mathcal{T}_3] \\ \mathcal{T}_4 = \{\bullet\} \end{array} \right.$$

The limit is the Brownian separable permuton of parameter $\rho \approx 0.4748692376\dots$ (only real root of a certain polynomial of degree 9).

Essentially linear specifications

The restriction of the specification to critical families is strongly connected, and contains **no product**.

⇒ The limiting permuton of the class is the **X-permuton** of parameter $\mathbf{p} = (p_1, p_2, p_3, p_4)$ (not necessarily centered, possibly degenerate).

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Example 1: $Av(2413, 3142, 2143, 3412)$, a.k.a. the X-class, with critical families in **red** and **blue** (for two strongly connected components).

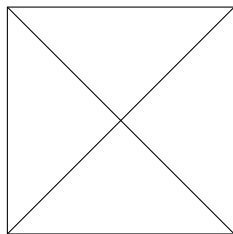
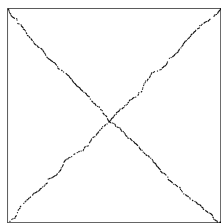
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Essentially linear specifications

The restriction of the specification to critical families is strongly connected, and contains **no product**.

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The limit is the centered X-permuton (of parameter $(1/4, 1/4, 1/4, 1/4)$).

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Example 2: $Av(2413, 3142, 2143, 34512)$, with critical families in **red** and **blue** (for two strongly connected components).

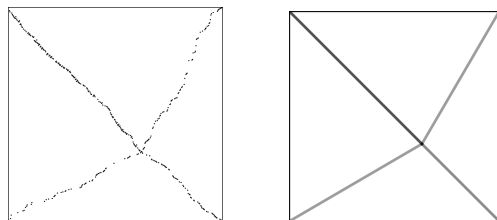
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The restriction of the specification to critical families is strongly connected, and contains **no product**.

⇒ The limiting permuton of the class is the **X-permuton** of parameter $\mathbf{p} = (p_1, p_2, p_3, p_4)$ (not necessarily centered, possibly degenerate).

Example 2: $Av(2413, 3142, 2143, 34512)$, with critical families in **red** and **blue** (for two strongly connected components).



The limit is the X-permuton of parameter $\approx (0.2003, 0.2003, 0.4313, 0.1681)$.

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Example 3: $Av(2413, 1243, 2341, 531642, 41352)$, with critical families in **red** and **blue** (for two strongly connected components).

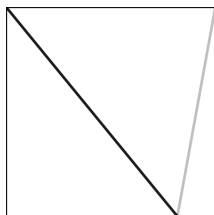
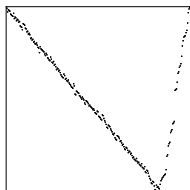
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Example 3: $Av(2413, 1243, 2341, 531642, 41352)$, with critical families in **red** and **blue** (for two strongly connected components).



The limit is the X-permuton of parameter $(0, 0, 1 - p, p)$ with $p \approx 0.81863$.

Several strongly connected components

Assume the specification for \mathcal{C} restricted to critical families has **several strongly connected components**.

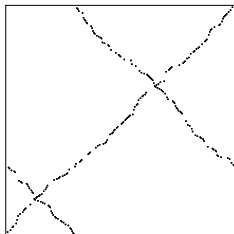
- Limit shape **in each strongly connected component**: as above.
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Example: the downward closure of $\oplus[\mathcal{X}, \mathcal{X}]$:



Summary of the described limit shapes

(Biased) Brownian separable permuton:

- Separable permutations;
- Substitution-closed class under an analytic condition on $S(z)$;
- Classes with a specification that is essentially strongly connected and essentially branching.

(Parametrized) X -permuton:

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A mix of the above:

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Thank you!