Limit shapes of pattern-avoiding permutations

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talk based on joint works with
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Including additional pictures by Jacopo Borga and Carine Pivoteau

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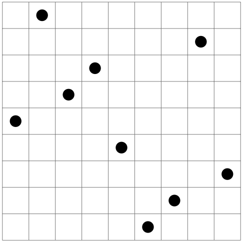
What are permutations? (in this talk)

A permutation of size n is a bijection from $\{1, 2, \ldots, n\}$ to itself.

We often write a permutation σ of size n as the word $\sigma(1)\sigma(2)\ldots\sigma(n)$.

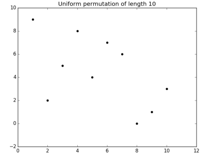
For the purpose of this talk, we represent permutations by their permutation matrices, or rather their diagram.

Example: the diagram of $\sigma = 596741283$ is



What are random permutations (and their limit shapes)?

Diagrams of permutations of various sizes picked uniformly at random:



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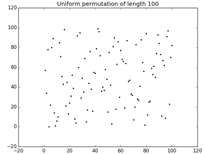

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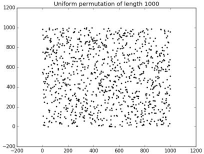

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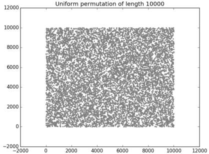

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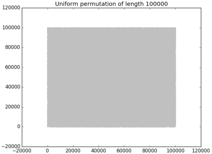

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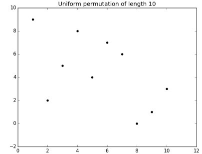

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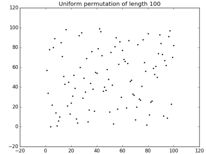

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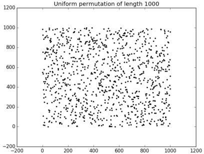

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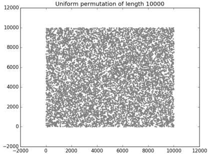

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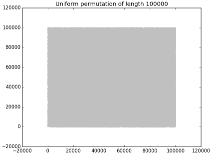

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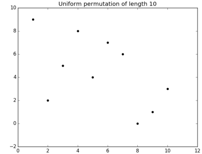

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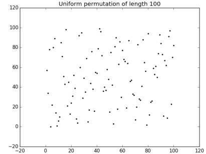

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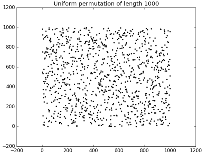

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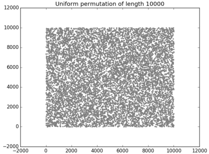

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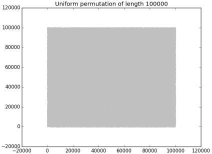

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Goal of the talk: Describe limit shapes of (the diagrams of) pattern-avoiding permutations.

Patterns in permutations

A permutation π of size k is a pattern of a permutation σ of size n if there exist $1 \leq i_1 < \ldots < i_k \leq n$ such that $\sigma(i_1)\ldots\sigma(i_k)$ is in the same relative order ($\equiv$) as π.

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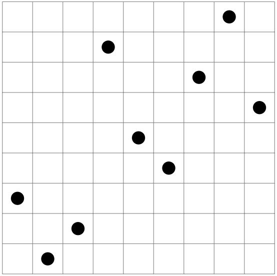

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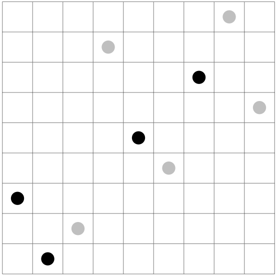

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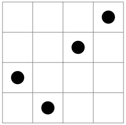

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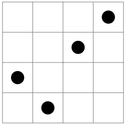

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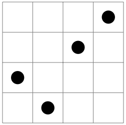

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There is a large literature on permutation classes, with main focus on enumerative aspects, but also in algorithmic and more recently logic and (discrete) probability theory.

Analoguous framework for graphs

Similarities:

- permutation patterns $\leftrightarrow$ induced subgraph
- permutation class $\leftrightarrow$ hereditary family
- similar theory for limit shapes, as we shall see

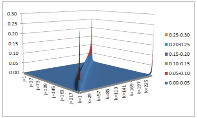
Differences:

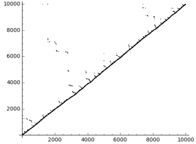
- Less counting, more algorithms
- For enumeration, additional difficulties arising from internal symmetries (automorphisms)
- However, permutations are not a special case of graphs via their permutation graphs, since this correspondence is not one-to-one.

$\Rightarrow$ Two parallel worlds, with similarities, but also with their specificities.

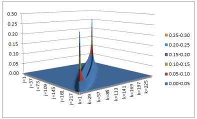
Uniform random permutations in $Av(\tau)$ for τ of size 3

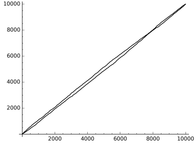
$Av(231)$





$Av(321)$





from Miner-Pak, 2013   from Hoffman-Rizzolo-Slivken, 2015

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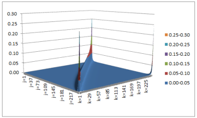

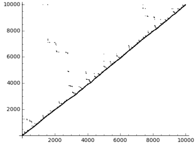

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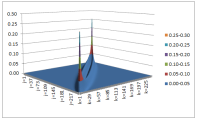
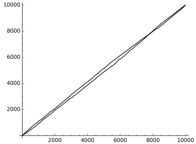
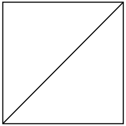




from Miner-Pak, 2013 from Hoffman-Rizzolo-Slivken, 2015

- Miner-Pak, also Madras with Atapour, Liu and Pehlivan: very precise local description of the average asymptotic shape
- Hoffman-Rizzolo-Slivken: scaling limits and link with the Brownian excursion (for the fluctuations around the main diagonal)

Diagrams of uniform random permutations in classes

At first order, the limit of the diagram of a uniform random permutation in $Av(\tau)$ for $\tau = 231$ or 321 is just  .

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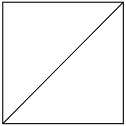
At first order, the limit of the diagram of a uniform random permutation in $Av(\tau)$ for $\tau = 231$ or 321 is just  .

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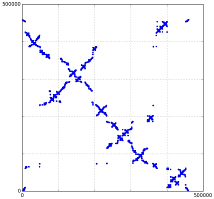
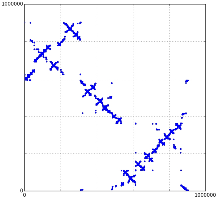
Typical large permutations in $Av(2413, 3142)$, the class of separable permutations, also described as the substitution-closed class with set of simple permutations $\emptyset$:





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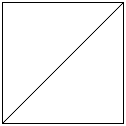
At first order, the limit of the diagram of a uniform random permutation in $Av(\tau)$ for $\tau = 231$ or 321 is just  .

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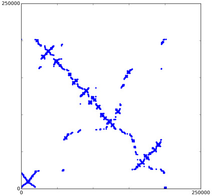
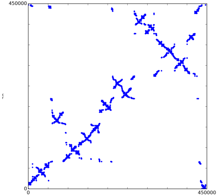
Typical large permutations in the substitution-closed class with set of simple permutations $\{2413, 3142, 24153\}$:





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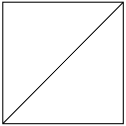
At first order, the limit of the diagram of a uniform random permutation in $Av(\tau)$ for $\tau = 231$ or 321 is just  .

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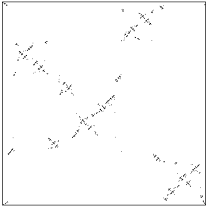
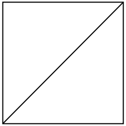
Typical large permutation in the substitution-closed class with (infinite) set of simple permutations $Av(321) \cap \{Simples\}$, *i.e.* in the substitution closure of $Av(321)$:



At first order, the limit of the diagram of a uniform random permutation in $Av(\tau)$ for $\tau = 231$ or 321 is just  .

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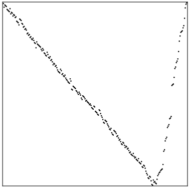
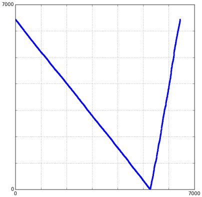
Typical (large) permutations in $Av(2413, 1243, 2341, 41352, 531642)$:





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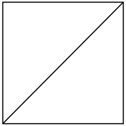
At first order, the limit of the diagram of a uniform random permutation in $Av(\tau)$ for $\tau = 231$ or 321 is just  .

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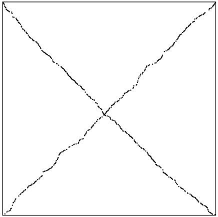
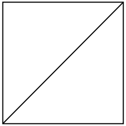
Typical (large) permutation in $Av(2413, 3142, 2143, 3412)$, called the X-class and denoted $\mathcal{X}$ later:



At first order, the limit of the diagram of a uniform random permutation in $Av(\tau)$ for $\tau = 231$ or 321 is just  .

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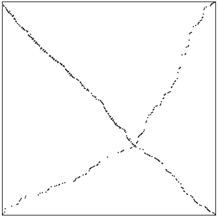
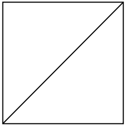


At first order, the limit of the diagram of a uniform random permutation in $Av(\tau)$ for $\tau = 231$ or 321 is just  .

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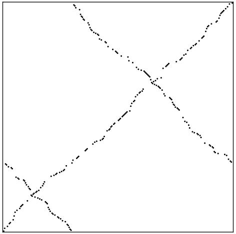
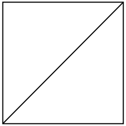


At first order, the limit of the diagram of a uniform random permutation in $Av(\tau)$ for $\tau = 231$ or 321 is just  .

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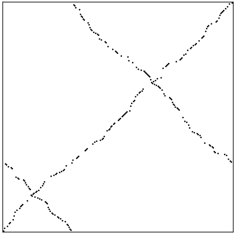

How can we explain these pictures?

Describing limits of permutations: the framework of permutons

What type of objects are the limiting diagrams?

A permuton (defined by Hoppen-Kohayakawa-Moreira-Rath-Sampaio, 2013, named by Glebov-Grzesik-Klimošová-Král, 2015) is a probability measure on the unit square with uniform projections,
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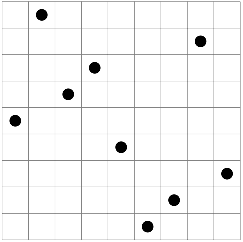

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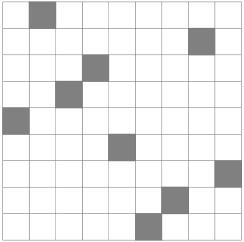

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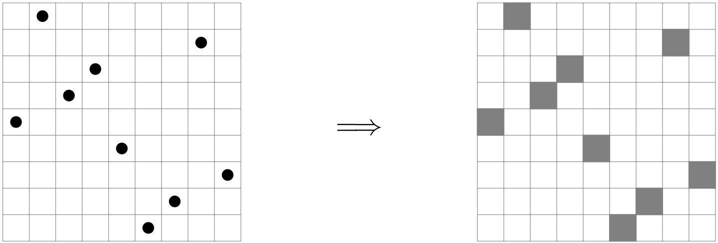

Informally, permutons can represent permutations of finite size, but also "permutations of infinite size".

Permuton convergence

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Theorem [Hoppen-Kohayakawa-Moreira-Rath-Sampaio, 2013]:
In the deterministic setting, permuton convergence is characterized by the convergence of probabilities (or densities) of all patterns.

Permuton convergence

We say that a sequence of permutations (σ_n) converges to a permuton μ when the sequence of permutons (μ_{σ_n}) converges to μ (for the weak convergence of measures).

This extends to sequences of random permutations $(\boldsymbol{\sigma}_n)$, converging to a (*a priori* random) permuton $\boldsymbol{\mu}$.

Theorem [Hoppen-Kohayakawa-Moreira-Rath-Sampaio, 2013]:
In the deterministic setting, permuton convergence is characterized by the convergence of probabilities (or densities) of all patterns.

Theorem [Bassino-Bouvel-Féray-Gerin-Maazoun-Pierrot, 2020]:
In the random setting, permuton convergence in distribution is characterized by the convergence of probabilities (or densities) of all patterns in expectation.

Characterization of permuton convergence

Let $(\sigma_n)_{n\geq 1}$ be a sequence of permutations of size tending to infinity. Let μ be a permuton. For any pattern π, writing $|\pi| = k$, define:

- $\widetilde{\text{occ}}(\pi, \sigma_n)$ = probability that k points picked uniformly at random in σ_n form an occurrence of the pattern $\pi = \frac{\text{number of occurrences of } \pi \text{ in } \sigma_n}{\binom{|\sigma_n|}{k}}$.
- $\widetilde{\text{occ}}(\pi, \mu)$ = the probability that k points of the unit square picked at random according to μ induce the pattern π.

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Theorems:

- (σ_n) converges to some permuton $\mu \Leftrightarrow$ for every pattern π, $\widetilde{\mathrm{occ}}(\pi, \sigma_n)$ converges to some value $\delta_\pi \in [0, 1]$, which characterize μ through $\widetilde{\mathrm{occ}}(\pi, \mu) = \delta_\pi$ for all π.

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- $(\boldsymbol{\sigma}_n)$ converges in distribution to some permuton $\boldsymbol{\mu} \Leftrightarrow$ for every pattern π, $\mathbb{E}[\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)]$ converges to some value $\Delta_\pi \in [0, 1]$.
- If this holds, $\mathbb{E}[\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\mu})] = \Delta_\pi$ for all π and $(\Delta_\pi)_\pi$ characterizes $\boldsymbol{\mu}$.

Before permutons were graphons

- Graphons are used to express limits of dense graphs [Lovász, 2012, based on prior works].
- Graphons are **essentially** continuous analogues of adjacency matrices of graphs (up to permuting rows and columns in all possible ways)
- The space of graphons is less nice than that of permutons, but it is possible to define graphon convergence.
- Graphon convergence is characterized by the convergence of all induces subgraph densities.
- There is a large literature on graphons. Graphon limits of hereditary graph classes is a drop in the ocean.

Summary so far, and what comes next

Goal: Prove limit shape results for the diagrams of uniform random permutations in permutation classes.

Framework: Express these limit shape results in the framework of permutons.

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Goal: Prove limit shape results for the diagrams of uniform random permutations in permutation classes.

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Key tools:

- Permuton convergence is the convergence of all pattern probabilities;
- Thanks to their substitution decomposition, permutations are trees and their patterns are subtrees;
- Limit shape results on random trees [BBFGP, 2018]

OR

- Singularity analysis of generating functions for trees [BBFGMP, 2020 and 2022].

Substitution decomposition and decomposition trees

Substitution decomposition

Ingredients:

- A way of building bigger permutations from smaller ones
  $\rightsquigarrow$ substitution or inflation;
- "Building blocks" allowing to build all permutations
  $\rightsquigarrow$ simple permutations.

Essential property:

- For every permutation σ, there exists a unique way of obtaining it recursively using inflations of simple permutations.

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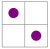
Outcome:

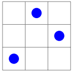
- Bijection between permutations and decomposition trees.
- (Some) permutation classes are (nice) families of trees:
  - easiest case: substitution-closed classes;
  - beyond those: classes with a finite combinatorial specification.

Inflating permutations

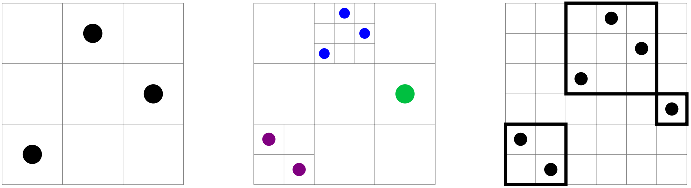
Substitution or inflation: $\pi[\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(k)}]$, for π of size k.

Example: Here, $\pi = 1\,3\,2$, and

$\alpha^{(1)} = 2\,1 =$ 

$\alpha^{(2)} = 1\,3\,2 =$ 

$\alpha^{(3)} = 1 =$  .



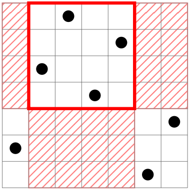
Hence $1\,3\,2[2\,1, 1\,3\,2, 1] = 2\,1\,4\,6\,5\,3$.

Simple permutations

Interval (or block) = set of elements of σ whose positions **and** values form intervals of integers
Example: 5 7 4 6 is an interval of 2 5 7 4 6 1 3

Simple permutation = permutation with no interval, except the trivial ones: $1, 2, \ldots, n$ and σ
Example: 3 1 7 4 6 2 5 is simple

Not simple:



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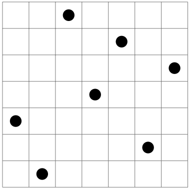

Simple permutations

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Example: $5\,7\,4\,6$ is an interval of $2\,5\,7\,4\,6\,1\,3$

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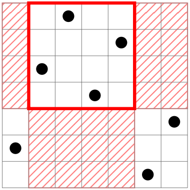
Example: $3\,1\,7\,4\,6\,2\,5$ is simple

The smallest simple permutations:
$12, 21$, $2413, 3142$, 6 of size 5, ...
Remark:
It is convenient to consider 12 and 21 **not** simple.

Not simple:



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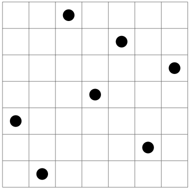

Substitution decomposition theorem for permutations

Theorem [Albert-Atkinson, 2005]:
Every σ ($\neq 1$) is uniquely decomposed as

- $12\ldots k[\alpha^{(1)},\ldots,\alpha^{(k)}] = \oplus[\alpha^{(1)},\ldots,\alpha^{(k)}]$, where the $\alpha^{(i)}$ are $\oplus$-indecomposable
- $k\ldots 21[\alpha^{(1)},\ldots,\alpha^{(k)}] = \ominus[\alpha^{(1)},\ldots,\alpha^{(k)}]$, where the $\alpha^{(i)}$ are $\ominus$-indecomposable
- $\pi[\alpha^{(1)},\ldots,\alpha^{(k)}]$, where π is simple of size $k \geq 4$

Notations:

- $\oplus$-indecomposable: that cannot be written as $\oplus[\beta^{(1)},\beta^{(2)}]$
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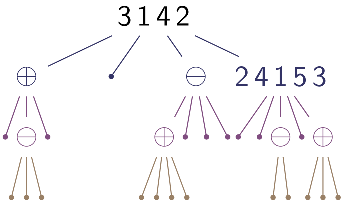
Decomposing recursively inside the $\alpha^{(i)}$ $\Rightarrow$ decomposition tree

Example: Decomposition tree of

$\sigma =$ 10 13 12 11 14 1 18 19 20 21 17 16 15 4 8 3 2 9 5 6 7



Notations and properties:

- $\oplus = 12\dots k$, $\ominus = k\dots 21$ = linear nodes.
- π simple of size ≥ 4 = prime node.
- No edge $\oplus - \oplus$ nor $\ominus - \ominus$.
- Rooted ordered trees.
- These conditions characterize decomposition trees.

$\sigma = 3\,1\,4\,2[\oplus[1, \ominus[1,1,1], 1], 1, \ominus[\oplus[1,1,1,1],1,1,1], 2\,4\,1\,5\,3[1,1,\ominus[1,1],1,\oplus[1,1,1]]]$

Decomposition tree: witness of this decomposition

Example: Decomposition tree of

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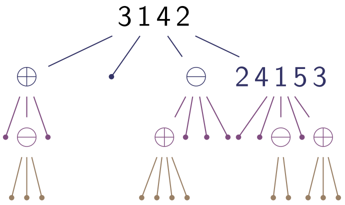

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Bijection between permutations and their decomposition trees.

Decomposition tree: witness of this decomposition

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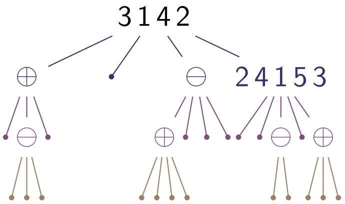

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$\boxed{\text{Bijection between permutations and their decomposition trees.}}$

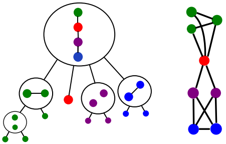
A substitution-closed class of permutations is characterized by a (pattern-closed) set of simple permutations.

Several graph analogues

There are several ways of encoding graphs with "decomposition trees", where leaves correspond to vertices of the graph.

Modular decomposition

- intervals ↔ modules
- simple permutations ↔ prime graphs
- the resulting tree is rooted but not fully ordered



Split decomposition

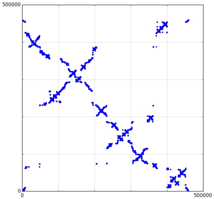
- tree-edge ↔ cut which is a complete bipartite graph
- the resulting tree is neither rooted nor ordered

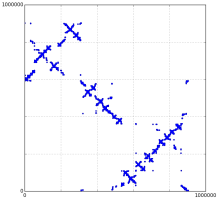
Separable permutations
and the Brownian separable permuton

Separable permutations

They are equivalently described as

- $Av(2413, 3142)$;
- the substitution-closed class with set of simple permutations $\emptyset$.

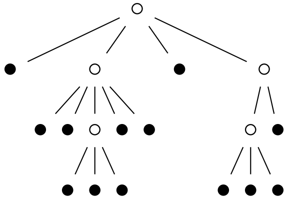




Theorem:
Uniform random separable permutations converge to a genuinely random permuton: the Brownian separable permuton.

Decomposition trees of separable permutations

- A Schröder tree of size n is a rooted plane tree with n leaves whose internal vertices have at least two children.



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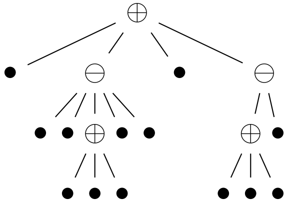
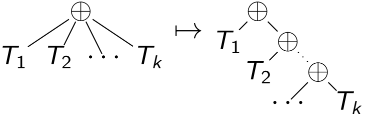


- Decomposition trees of separable permutations are signed Schröder trees, as above with additional signs $\oplus$ and $\ominus$ on the internal vertices, which alternate on any path from the root to a leaf (*i.e.* the sign of the root determines all others).

Up to the "right binarization"



(and same with $\ominus$),
the decomposition trees of separable permutations are generated by the following combinatorial specification:

$$
\begin{cases}
\mathcal{T}_{\text{sep}} = \{\bullet\} \uplus \begin{array}{c}\oplus\\ \mathcal{T}^{\text{not}\oplus}_{\text{sep}} \quad \mathcal{T}_{\text{sep}}\end{array} \uplus \begin{array}{c}\ominus\\ \mathcal{T}^{\text{not}\ominus}_{\text{sep}} \quad \mathcal{T}_{\text{sep}}\end{array} ;\\
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Starting point of the "analytic combinatorics" proof, discussed later.

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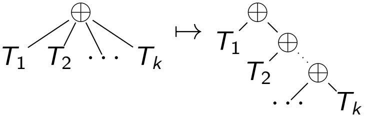

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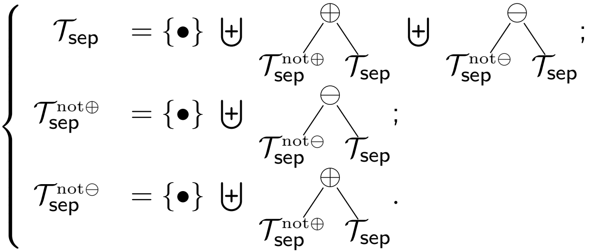

Starting point of the "analytic combinatorics" proof, discussed later.
For now, we present the "random trees" proof.

Contours and their limits

The contour of a uniform random Schröder tree converges to the Brownian excursion.

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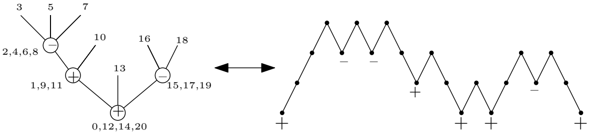
We can define signed contours of signed Schröder trees:

- Peaks $\leftrightarrow$ leaves.
- Local minima with signs $\leftrightarrow$ signed internal nodes.



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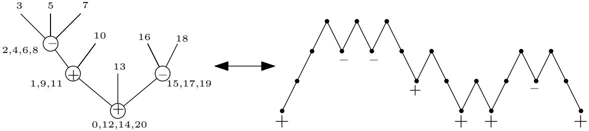

We can define a signed version of the Brownian excursion:

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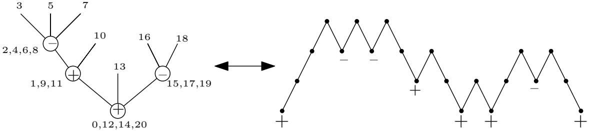

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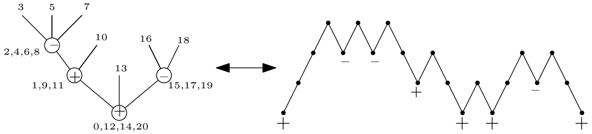

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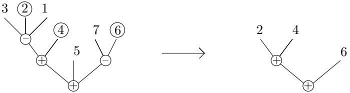
Convergence of the extracted patterns/subtrees

On finite objects:

Extracting a pattern π of size k from a separable permutation σ

$$\sigma = 3214576 \longmapsto \pi = 123$$

$\equiv$ Extracting a signed subtree (induced by k leaves) in a signed Schröder tree of σ



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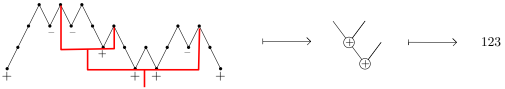

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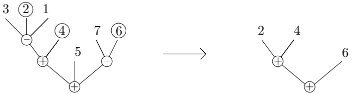

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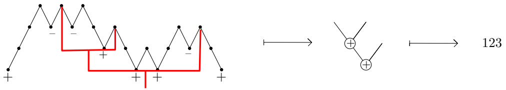

In the limit:

Extracting a signed tree from a set of k uniformly chosen points in the signed Brownian excursion $\mathbf{e}_{\pm}$

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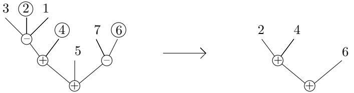

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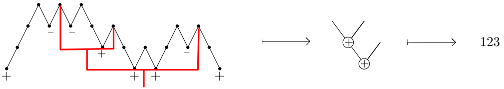

In the limit:

This characterizes the probabilities of all subtrees extracted from $\mathbf{e}_\pm$,

$\Uparrow$

Extracting a signed tree from a set of k uniformly chosen points in the signed Brownian excursion $\mathbf{e}_\pm$

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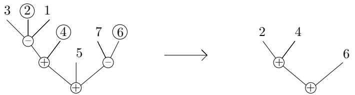

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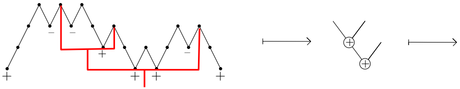

In the limit:

This characterizes the probabilities of all subtrees extracted from $\mathbf{e}_\pm$, hence of all patterns extracted from the Brownian separable permuton (so defined).

$\Uparrow$

Extracting a signed tree from a set of k uniformly chosen points in the signed Brownian excursion $\mathbf{e}_\pm$

Substitution-closed classes and universality of the Brownian separable permuton

Separable permutations and substitution-closed classes

The class of separable permutations is the one whose decomposition trees are described by the combinatorial specification

$$\begin{cases} \mathcal{T}_{\mathsf{sep}} = \{\bullet\} \uplus \overset{\oplus}{\overbrace{\mathcal{T}^{\mathrm{not}\oplus}_{\mathsf{sep}} \quad \mathcal{T}_{\mathsf{sep}}}} \uplus \overset{\ominus}{\overbrace{\mathcal{T}^{\mathrm{not}\ominus}_{\mathsf{sep}} \quad \mathcal{T}_{\mathsf{sep}}}} & ; \\ \mathcal{T}^{\mathrm{not}\oplus}_{\mathsf{sep}} = \{\bullet\} \uplus \overset{\ominus}{\overbrace{\mathcal{T}^{\mathrm{not}\ominus}_{\mathsf{sep}} \quad \mathcal{T}_{\mathsf{sep}}}} & ; \\ \mathcal{T}^{\mathrm{not}\ominus}_{\mathsf{sep}} = \{\bullet\} \uplus \overset{\oplus}{\overbrace{\mathcal{T}^{\mathrm{not}\oplus}_{\mathsf{sep}} \quad \mathcal{T}_{\mathsf{sep}}}} & . \end{cases}$$

Substitution-closed classes are those whose decomposition trees described by a combinatorial specification of the form

$$\begin{cases} \mathcal{T} = \{\bullet\} \uplus \underset{\mathcal{T}^{\text{not}\oplus}\quad \mathcal{T}}{\oplus} \uplus \underset{\mathcal{T}^{\text{not}\ominus}\quad \mathcal{T}}{\ominus} \uplus \biguplus_{\pi\in\mathcal{S}} \underset{\mathcal{T}\quad \dots \quad \mathcal{T}}{\pi} ; \\ \mathcal{T}^{\text{not}\oplus} = \{\bullet\} \uplus \underset{\mathcal{T}^{\text{not}\ominus}\quad \mathcal{T}}{\ominus} \uplus \biguplus_{\pi\in\mathcal{S}} \underset{\mathcal{T}\quad \dots \quad \mathcal{T}}{\pi} ; \\ \mathcal{T}^{\text{not}\ominus} = \{\bullet\} \uplus \underset{\mathcal{T}^{\text{not}\oplus}\quad \mathcal{T}}{\oplus} \uplus \biguplus_{\pi\in\mathcal{S}} \underset{\mathcal{T}\quad \dots \quad \mathcal{T}}{\pi} . \end{cases}$$

where $\mathcal{S}$ is the set of simple permutations in the class $\mathcal{C}_{\mathcal{S}}$ considered.

Separable permutations and substitution-closed classes

Substitution-closed classes are those whose decomposition trees described by a combinatorial specification of the form

$$
\begin{cases}
\mathcal{T} = \{\bullet\} \uplus \overset{\oplus}{\mathcal{T}^{\text{not}\oplus} \quad \mathcal{T}} \uplus \overset{\ominus}{\mathcal{T}^{\text{not}\ominus} \quad \mathcal{T}} \uplus \biguplus_{\pi \in \mathcal{S}} \overset{\pi}{\mathcal{T} \quad \ldots \quad \mathcal{T}} ; \\
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\mathcal{T}^{\text{not}\ominus} = \{\bullet\} \uplus \overset{\oplus}{\mathcal{T}^{\text{not}\oplus} \quad \mathcal{T}} \uplus \biguplus_{\pi \in \mathcal{S}} \overset{\pi}{\mathcal{T} \quad \ldots \quad \mathcal{T}} .
\end{cases}
$$

where $\mathcal{S}$ is the set of simple permutations in the class $\mathcal{C}_{\mathcal{S}}$ considered.

Theorem:
Under an analytic condition on the generating function $S(z)$ of $\mathcal{S}$, uniform random permutations in the substitution-closed class $\mathcal{C}_{\mathcal{S}}$ converge to a biased Brownian separable permuton.

The biased Brownian separable permuton of parameter p

- Biased version of the signed Brownian excursion (for $p \in [0,1]$): in $\mathbf{e}_{\pm,p}$, local minima carry independent signs, but not balanced; instead, $+$ with probability p, $-$ with probability $1-p$.
- The biased Brownian separable permuton $\boldsymbol{\mu}_p$ of parameter p is characterized as above but starting from $\mathbf{e}_{\pm,p}$ instead of $\mathbf{e}_{\pm}$.

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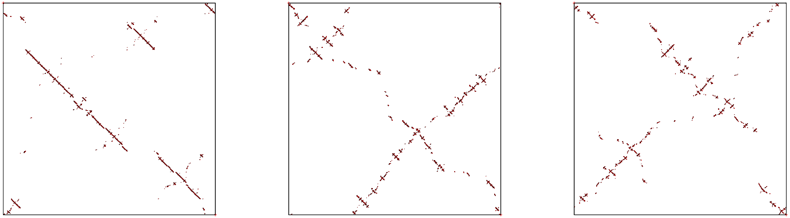
- Biased version of the signed Brownian excursion (for $p \in [0, 1]$): in $\mathbf{e}_{\pm,p}$, local minima carry independent signs, but not balanced; instead, $+$ with probability p, $-$ with probability $1 - p$.
- The biased Brownian separable permuton $\boldsymbol{\mu}_p$ of parameter p is characterized as above but starting from $\mathbf{e}_{\pm,p}$ instead of $\mathbf{e}_{\pm}$.

The higher p is, the more drift there is towards the direction of the main diagonal in $\boldsymbol{\mu}_p$.



simulations of $\boldsymbol{\mu}_p$ for $p = 0.2, 0.45, 0.5$.

Theorem:
Let $\mathcal{C}_{\mathcal{S}}$ be the substitution-closed class with set of simple permutations $\mathcal{S}$. Let $S(z)$ be the generating function of $\mathcal{S}$, and let R_S be its (positive) radius of convergence.

Statement and examples

Theorem:
Let $\mathcal{C}_\mathcal{S}$ be the substitution-closed class with set of simple permutations $\mathcal{S}$. Let $S(z)$ be the generating function of $\mathcal{S}$, and let R_S be its (positive) radius of convergence.
Assuming that $\lim_{\substack{r \to R_S \\ r < R_S}} S'(r) > \frac{2}{(1+R_S)^2} - 1$, uniform random permutations in $\mathcal{C}_\mathcal{S}$ converge to the biased Brownian separable permuton $\boldsymbol{\mu}_p$,

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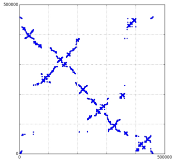
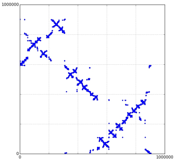
Example 1: Separable permutations, *i.e.* $\mathcal{C}_\emptyset$, $\Rightarrow p = 0.5$





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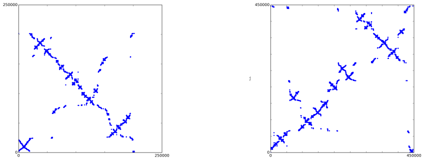
Example 2: $\mathcal{C}_{\mathcal{S}}$ with $\mathcal{S} = \{2413, 3142, 24153\}$, $\Rightarrow p = 0.5$



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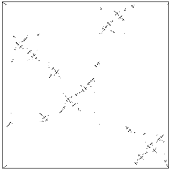
Example 3: $\mathcal{C}_{\mathcal{S}}$ with $\mathcal{S} = Av(321) \cap \{Simples\}$, $\Rightarrow p \in [0.577, 0.622]$



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Further examples: All substitution-closed classes

- with finitely many simple permutations,
- or more generally such that $R_S = 1$,
- or such that S' diverges at R_S (rational, square root singularities, ...).

This covers all substitution-closed classes whose simple permutations have been enumerated.

Statement and examples

Theorem:
Let $\mathcal{C}_{\mathcal{S}}$ be the substitution-closed class with set of simple permutations $\mathcal{S}$. Let $S(z)$ be the generating function of $\mathcal{S}$, and let R_S be its (positive) radius of convergence.
Assuming that $\lim_{\substack{r \to R_S \\ r < R_S}} S'(r) > \frac{2}{(1+R_S)^2} - 1$, uniform random permutations in $\mathcal{C}_{\mathcal{S}}$ converge to the biased Brownian separable permuton $\boldsymbol{\mu}_p$, where p is explicit in terms of (the generating function of) the number of occurrences of patterns 12 and 21 in permutations of $\mathcal{S}$.

Non-example: $Av(2413)$

Limit not known.

We believe it is "degenerate": typical permutations in $Av(2413)$ look like typical simple permutations in $Av(2413)$.

Proof schema

$\boldsymbol{\sigma}_n =$ uniform random permutation of size n in $\mathcal{C}_{\mathcal{S}}$.

- It is enough to prove the convergence of $\left(\mathbb{E}\left[\widetilde{\text{occ}}(\pi, \boldsymbol{\sigma}_n)\right]\right)_n$ for all π.

Proof schema

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- It is enough to prove the convergence of $\left(\mathbb{E}\left[\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)\right]\right)_n$ for all π.
- By definition, $\mathbb{E}\left[\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)\right] = \frac{\text{total number of occ. of } \pi \text{ in } \sigma \text{ of size } n \text{ in } \mathcal{C}_{\mathcal{S}}}{\binom{n}{|\pi|} \times \text{ number of } \sigma \text{ of size } n \text{ in } \mathcal{C}_{\mathcal{S}}}$.

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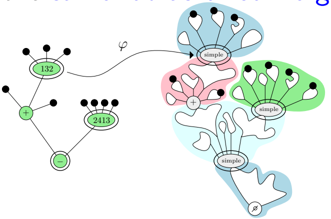
- Numerator and denominator are expressed as coefficients of generating series of trees (possibly with marked leaves).
- The combinatorial specification for $\mathcal{C}_\mathcal{S}$ yields equations for these tree series, which all have the same radius of convergence.



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- Estimate their coefficients with analytic combinatorics, *via* the singular behavior of the trees series and the transfer theorem.

  This gives the limit of $\mathbb{E}\left[\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)\right]$.

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Rk: The limit of $\mathbb{E}\left[\widetilde{\text{occ}}(\pi, \boldsymbol{\sigma}_n)\right]$ is non-zero if and only if π is separable.

Extension:
Finitely generated classes,
not necessarily substitution-closed

Combinatorial specifications of classes

- For substitution-closed classes: there is always a combinatorial specification for the associated decomposition trees.
- For other classes, there is sometimes a combinatorial specification for the associated decomposition trees.
- It is always the case when the number of simple permutations in the class is finite. Moreover, in this case, the specification is automatically produced [Bassino-Bouvel-Pierrot-Pivoteau-Rossin, 2017].

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Example: $Av(132)$

$$\left\{ \begin{array}{lllllll}
\mathcal{T} & = \{\bullet\} & \biguplus & \mathcal{T}^{\text{not}\oplus} \overset{\oplus}{\diagup\;\diagdown} \mathcal{T}_{\langle 21\rangle} & \biguplus & \mathcal{T}^{\text{not}\ominus} \overset{\ominus}{\diagup\;\diagdown} \mathcal{T} \\
\mathcal{T}^{\text{not}\oplus} & = \{\bullet\} & \biguplus & \mathcal{T}^{\text{not}\ominus} \overset{\ominus}{\diagup\;\diagdown} \mathcal{T} & & \\
\mathcal{T}^{\text{not}\ominus} & = \{\bullet\} & \biguplus & \mathcal{T}^{\text{not}\oplus} \overset{\oplus}{\diagup\;\diagdown} \mathcal{T}_{\langle 21\rangle} & & \\
\mathcal{T}_{\langle 21\rangle} & = \{\bullet\} & \biguplus & \mathcal{T}^{\text{not}\oplus}_{\langle 21\rangle} \overset{\oplus}{\diagup\;\diagdown} \mathcal{T}_{\langle 21\rangle} & & \\
\mathcal{T}^{\text{not}\oplus}_{\langle 21\rangle} & = \{\bullet\}. & & & &
\end{array} \right.$$

Relevant properties of the specifications

Consider a specification for the decomposition trees of permutations in a class $\mathcal{C} = \mathcal{T}_0$ where the families $(\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_k)$ appear.

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Define that $\mathcal{T}_i$ is critical when its generating function has minimal radius of convergence among all those of the $(\mathcal{T}_j)_j$ (which is that of $\mathcal{T}_0$).

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Define that $\mathcal{T}_i$ is critical when its generating function has minimal radius of convergence among all those of the $(\mathcal{T}_j)_j$ (which is that of $\mathcal{T}_0$).

The limiting behavior of uniform permutations in $\mathcal{C}$ is determined by:

- the strongly connected components of the specification restricted to critical families;
- whether the restriction of the specification to each strongly connected component is linear or branching.

Essentially branching specifications

The restriction of the specification to critical families is strongly connected, and contains a product.

$\Rightarrow$ The limiting permuton of the class is a biased Brownian separable permuton (of parameter p possibly 0 or 1).

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Example 1: $Av(132)$, with critical families in blue.

$$
\begin{cases}
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\mathcal{T}_{\langle 21\rangle} & = \{\bullet\} \quad \uplus \quad \mathcal{T}_{\langle 21\rangle}^{\text{not}\oplus} \overset{\oplus}{\diagup\diagdown} \mathcal{T}_{\langle 21\rangle} \\
\mathcal{T}_{\langle 21\rangle}^{\text{not}\oplus} & = \{\bullet\}.
\end{cases}
$$

The limit is the Brownian separable permuton of parameter $p = 0$.

Essentially branching specifications

The restriction of the specification to critical families is strongly connected, and contains a product.

$\Rightarrow$ The limiting permuton of the class is a biased Brownian separable permuton (of parameter p possibly 0 or 1).

Example 2: $Av(2413, 31452, 41253, 41352, 531246)$, with critical families in blue.

$$\begin{cases} \mathcal{T}_0 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_0] \uplus \ominus[\mathcal{T}_2, \mathcal{T}_0] \uplus 3142[\mathcal{T}_0, \mathcal{T}_3, \mathcal{T}_3, \mathcal{T}_0] \\ \mathcal{T}_1 = & \{\bullet\} \uplus \ominus[\mathcal{T}_2, \mathcal{T}_0] \uplus 3142[\mathcal{T}_0, \mathcal{T}_3, \mathcal{T}_3, \mathcal{T}_0] \\ \mathcal{T}_2 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_0] \uplus 3142[\mathcal{T}_0, \mathcal{T}_3, \mathcal{T}_3, \mathcal{T}_0] \\ \mathcal{T}_3 = & \{\bullet\} \uplus \ominus[\mathcal{T}_4, \mathcal{T}_3] \\ \mathcal{T}_4 = & \{\bullet\} \end{cases}$$

The limit is the Brownian separable permuton of parameter $p \approx 0.4748692376...$ (only real root of a certain polynomial of degree 9).

Essentially linear specifications

The restriction of the specification to critical families is strongly connected, and contains no product.

$\Rightarrow$ The limiting permuton of the class is the X-permuton of parameter $\mathbf{p} = (p_1, p_2, p_3, p_4)$ (not necessarily centered, possibly degenerate).

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Example 1: $Av(2413, 3142, 2143, 3412)$, a.k.a. the X-class, with critical families in red and blue (for two strongly connected components).

$$\begin{cases} \mathcal{T}_0 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_2] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_5] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_7, \mathcal{T}_5] \\ \mathcal{T}_1 = & \{\bullet\} \\ \mathcal{T}_2 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_2] \\ \mathcal{T}_3 = & \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_5] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_7, \mathcal{T}_5] \\ \mathcal{T}_4 = & \ominus[\mathcal{T}_1, \mathcal{T}_5] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_7, \mathcal{T}_5] \\ \mathcal{T}_5 = & \{\bullet\} \uplus \ominus[\mathcal{T}_1, \mathcal{T}_5] \\ \mathcal{T}_6 = & \oplus[\mathcal{T}_1, \mathcal{T}_2] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_7, \mathcal{T}_5] \\ \mathcal{T}_7 = & \oplus[\mathcal{T}_1, \mathcal{T}_2] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2]. \end{cases}$$

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The restriction of the specification to critical families is strongly connected, and contains no product.

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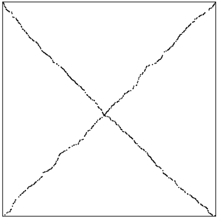

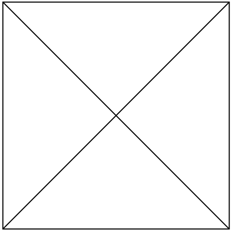

The limit is the centered X-permuton (of parameter $(1/4, 1/4, 1/4, 1/4)$).

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Example 2: $Av(2413, 3142, 2143, 34512)$, with critical families in red and blue (for two strongly connected components).

$$\begin{cases} \mathcal{T}_0 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_2] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_5, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_5, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_8, \mathcal{T}_6] \\ \mathcal{T}_1 = & \{\bullet\} \\ \mathcal{T}_2 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_2] \\ \mathcal{T}_3 = & \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_5, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_5, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_8, \mathcal{T}_6] \\ \mathcal{T}_4 = & \ominus[\mathcal{T}_5, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_5, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_8, \mathcal{T}_6] \\ \mathcal{T}_5 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_1] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_9] \uplus \oplus[\mathcal{T}_9, \mathcal{T}_1] \\ \mathcal{T}_6 = & \{\bullet\} \uplus \ominus[\mathcal{T}_1, \mathcal{T}_6] \\ \mathcal{T}_7 = & \oplus[\mathcal{T}_1, \mathcal{T}_2] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_{10}, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_{10}, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_8, \mathcal{T}_6] \\ \mathcal{T}_8 = & \oplus[\mathcal{T}_1, \mathcal{T}_{11}] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_{12}] \uplus \oplus[\mathcal{T}_{13}, \mathcal{T}_{11}] \uplus \oplus[\mathcal{T}_9, \mathcal{T}_{11}] \uplus \oplus[\mathcal{T}_{13}, \mathcal{T}_1] \\ \mathcal{T}_9 = & \ominus[\mathcal{T}_1, \mathcal{T}_6] \\ \mathcal{T}_{10} = & \oplus[\mathcal{T}_1, \mathcal{T}_1] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_9] \uplus \oplus[\mathcal{T}_9, \mathcal{T}_1] \\ \mathcal{T}_{11} = & \oplus[\mathcal{T}_1, \mathcal{T}_2] \\ \mathcal{T}_{12} = & \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_{10}, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_{10}, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_8, \mathcal{T}_6] \\ \mathcal{T}_{13} = & \ominus[\mathcal{T}_{10}, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_{10}, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_8, \mathcal{T}_6]. \end{cases}$$

Essentially linear specifications

The restriction of the specification to critical families is strongly connected, and contains no product.

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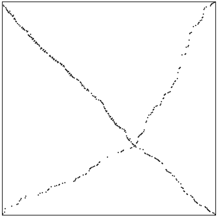

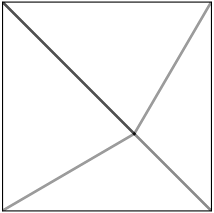

The limit is the X-permuton of parameter $\approx (0.2003, 0.2003, 0.4313, 0.1681)$.

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Example 3: $Av(2413, 1243, 2341, 531642, 41352)$, with critical families in red and blue (for two strongly connected components).

$$
\begin{cases}
\mathcal{T}_0 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_2] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_5, \mathcal{T}_0] \uplus 3142[\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_6] \\
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\mathcal{T}_7 = & \{\bullet\} \\
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\end{cases}
$$

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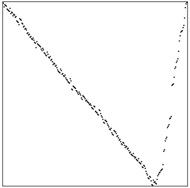

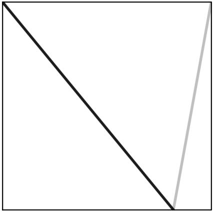

The limit is the X-permuton of parameter $(0, 0, 1-p, p)$ with $p \approx 0.81863$.

Several strongly connected components

Assume the specification for $\mathcal{C}$ restricted to critical families has several strongly connected components.

- Limit shape in each strongly connected component: as above.
- Sometimes, the limit shape of $\mathcal{C}$ is a combination of those in each component.

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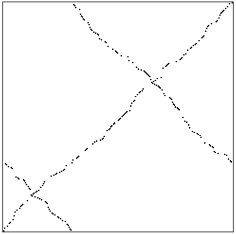
Example: the downward closure of $\oplus[\mathcal{X}, \mathcal{X}]$:



Summary of the described limit shapes

(Biased) Brownian separable permuton:

- Separable permutations;
- Substitution-closed class under an analytic condition on $S(z)$;
- Classes with a specification that is essentially strongly connected and essentially branching.

(Parametrized) X-permuton:

- Classes with a specification that is essentially strongly connected and essentially linear.

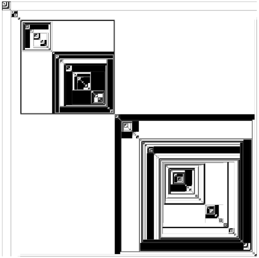
A mix of the above:

- Classes with a specification that is not essentially strongly connected.

Further extensions,
or related results

Cographs and the Brownian cographon

Cographs are the graph analogues of separable permutations.
With the same general approach, we can prove convergence of uniform random cographs to the Brownian cographon.



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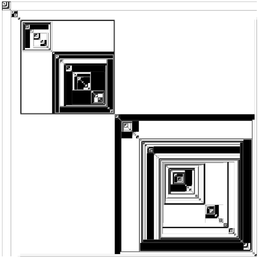
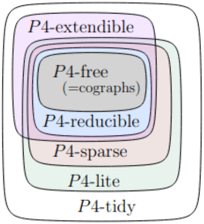


In addition, we can deduce that the size of the largest independent set of a uniform random cograph is sublinear.
(hence P_4 does not have the asymptotic linear Erdős-Hajnal property, answering a question of Kang, McDiarmid, Reed and Scott in 2014)

More families of graphs

- Théo Lenoir's PhD thesis: Convergence to the Brownian cographon of
  - various families of graphs with few P_4,
  - families of graphs with "nice" modular decompositions.



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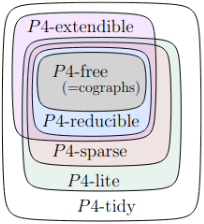

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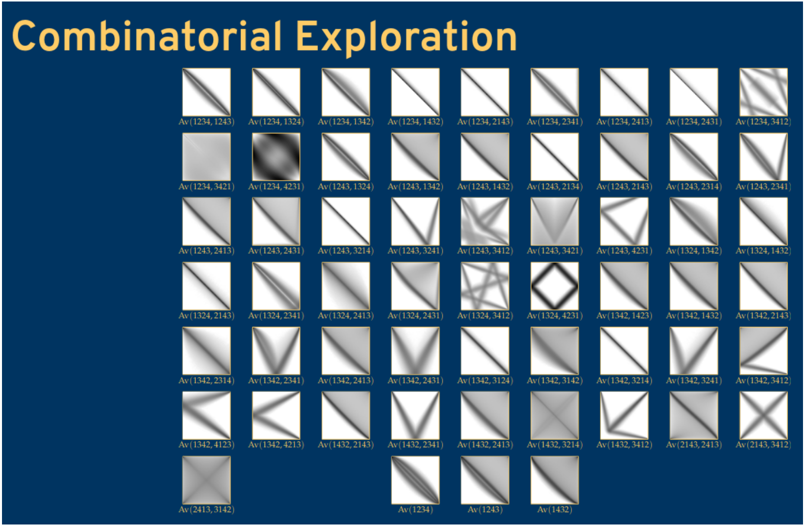
- Convergence to Aldous' Brownian CRT, with respect to the Gromov–Prokhorov topology, for distance-hereditary graphs (and a few relevant subclasses), using the split-decomposition [BBFGP, 2025$^+$].

Other tree-encodings of permutations

- The software `PermPal` developed by Albert-Bean-Claesson-Nadeau-Pantone-Ulfarsson automatically produces combinatorial specifications of permutation classes, where permutations are encoded by proof-trees.
- It includes random samplers whenever a specification is automatically derived, allowing observation of candidate limit permuton.
- Our current project is to adapt our method proving permuton convergence to these proof-trees, explaining their observations...
- ...and perhaps to automatize the derivation of the permuton limit in `PermPal`.

Thank you!



©Jay Pantone, talk at *Permutation Patterns 2023*