Limit shapes of pattern-avoiding permutations

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(Institut für Mathematik, Universität Zürich)

talk based on joint works with
Frédérique Bassino, Valentin Féray, Lucas Gerin,
Mickaël Maazoun and Adeline Pierrot

Including additional pictures by Jacopo Borga and Carine Pivoteau

Winter Combinatorics Meeting, The Open University, Feb. 2019.

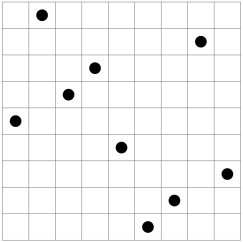
What are permutations? (in this talk)

A permutation of size n is a bijection from $\{1, 2, \ldots, n\}$ to itself.

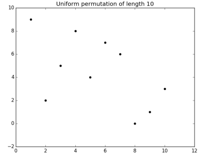
We often write a permutation σ of size n as the word $\sigma(1)\sigma(2)\ldots\sigma(n)$.

For the purpose of this talk, we represent permutations by their permutation matrices, or rather their diagram.

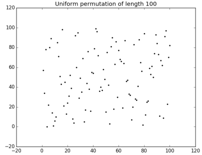
Example: the diagram of $\sigma = 596741283$ is



Diagrams of permutations of various sizes picked uniformly at random:



size 10



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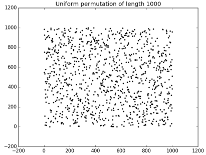

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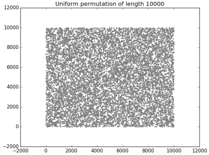

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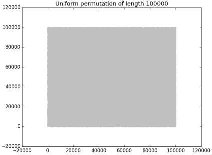

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What are random permutations (and their limit shapes)?

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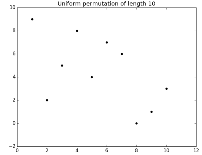

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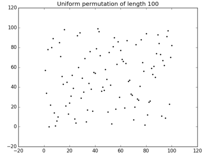

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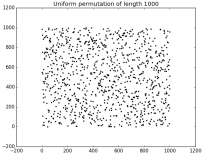

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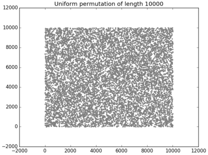

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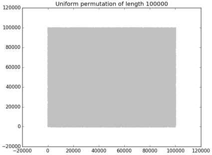

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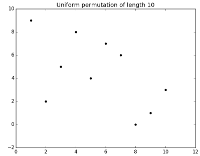

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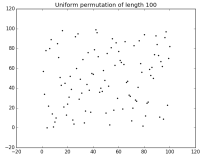

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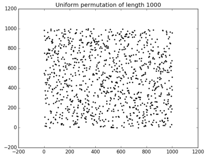

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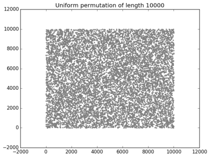

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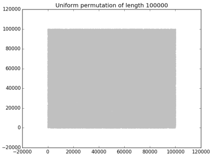

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Goal of the talk: Describe limit shapes of (the diagrams of) pattern-avoiding permutations.

Patterns in permutations

A permutation π of size k is a pattern of a permutation σ of size n if there exist $1 \leq i_1 < \ldots < i_k \leq n$ such that $\sigma(i_1) \ldots \sigma(i_k)$ is in the same relative order ($\equiv$) as π.

Example: 2 1 3 4 is a pattern of **3** **1** 2 8 **5** 4 **7** 9 6 since 3 1 5 7 $\equiv$ 2 1 3 4.

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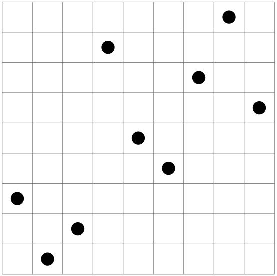
Example: $2\,1\,3\,4$ is a pattern of $\mathbf{3}\,\mathbf{1}\,2\,8\,\mathbf{5}\,4\,\mathbf{7}\,9\,6$ since $3\,1\,5\,7 \equiv 2\,1\,3\,4$.



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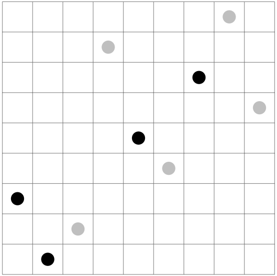

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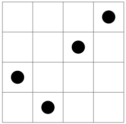

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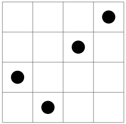
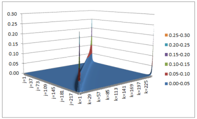
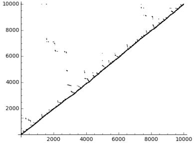


Permutation classes are sets of permutations defined by the avoidance of patterns. They are denoted $Av(B)$ for B a set of excluded patterns.

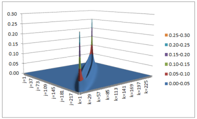
Uniform random permutations in $Av(\tau)$ for τ of size 3

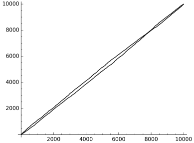
$Av(231)$





$Av(321)$





from Miner-Pak (2013) from Hoffman-Rizzolo-Slivken (2015)

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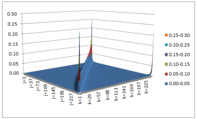

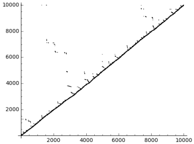

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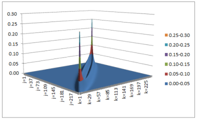
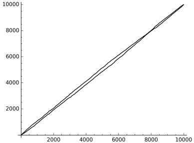
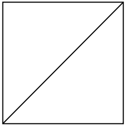




from Miner-Pak (2013) from Hoffman-Rizzolo-Slivken (2015)

- Miner-Pak, also Madras with Atapour, Liu and Pehlivan: very precise local description of the average asymptotic shape
- Hoffman-Rizzolo-Slivken: scaling limits and link with the Brownian excursion (for the fluctuations around the main diagonal)

At first order, the limit of the diagram of a uniform random permutation in $Av(\tau)$ for $\tau = 231$ or 321 is just  .

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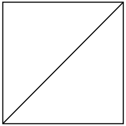
At first order, the limit of the diagram of a uniform random permutation in $Av(\tau)$ for $\tau = 231$ or 321 is just  .

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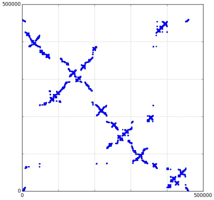
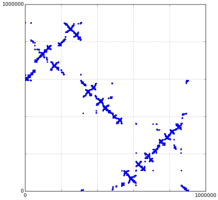
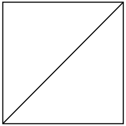
Typical large permutations in $Av(2413, 3142)$, the class of separable permutations, also described as the substitution-closed class with set of simple permutations $\emptyset$:





At first order, the limit of the diagram of a uniform random permutation in $Av(\tau)$ for $\tau = 231$ or 321 is just  .

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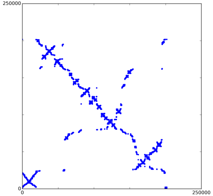
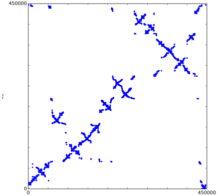
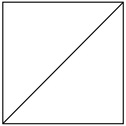
Typical large permutations in the substitution-closed class with set of simple permutations $\{2413, 3142, 24153\}$:





Diagrams of uniform random permutations in classes

At first order, the limit of the diagram of a uniform random permutation in $Av(\tau)$ for $\tau = 231$ or 321 is just  .

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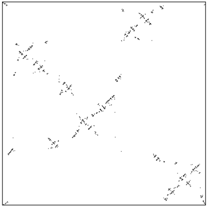
Typical large permutation in the substitution-closed class with (infinite) set of simple permutations $Av(321) \cap \{Simples\}$, *i.e.* in the substitution closure of $Av(321)$:



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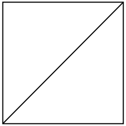
At first order, the limit of the diagram of a uniform random permutation in $Av(\tau)$ for $\tau = 231$ or 321 is just  .

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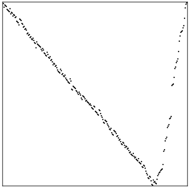
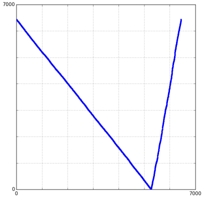
Typical (large) permutations in $Av(2413, 1243, 2341, 41352, 531642)$:





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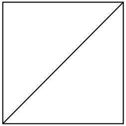
At first order, the limit of the diagram of a uniform random permutation in $Av(\tau)$ for $\tau = 231$ or 321 is just  .

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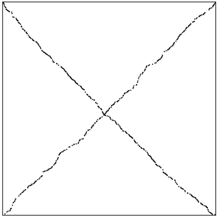
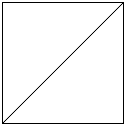


At first order, the limit of the diagram of a uniform random permutation in $Av(\tau)$ for $\tau = 231$ or 321 is just  .

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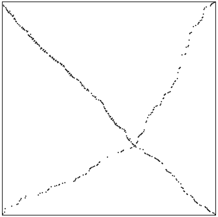

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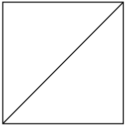
At first order, the limit of the diagram of a uniform random permutation in $Av(\tau)$ for $\tau = 231$ or 321 is just  .

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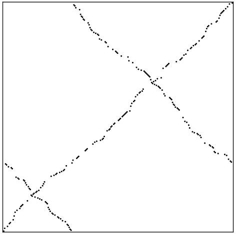

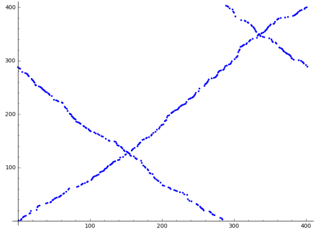

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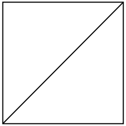
At first order, the limit of the diagram of a uniform random permutation in $Av(\tau)$ for $\tau = 231$ or 321 is just  .

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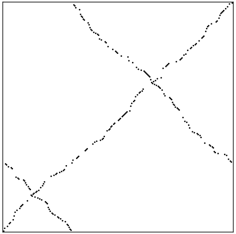

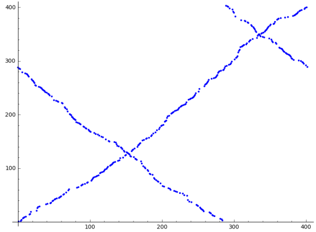

How can we explain these pictures?

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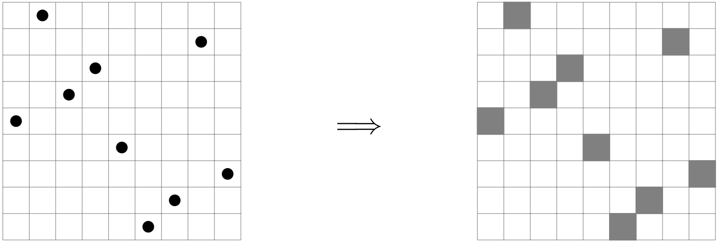

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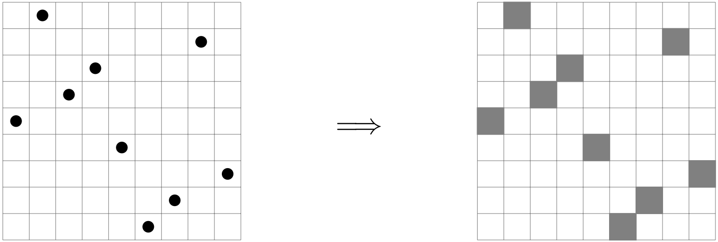

Informally, permuton can represent permutations of finite size,
but also "permutations of infinite size".

Permuton convergence

We say that a sequence of permutations (σ_n) converges to a permuton μ when the sequence of permutons (μ_{σ_n}) converges to μ (for the weak convergence of measures).

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More precisely, with $\pi = k$

- $\widetilde{\text{occ}}(\pi, \sigma) =$ probability that k points picked uniformly at random in σ form an occurrence of the pattern $\pi = \frac{\text{number of occurrences of } \pi \text{ in } \sigma}{\binom{|\sigma|}{k}}$.
- $\widetilde{\text{occ}}(\pi, \mu) =$ the probability that k points of the unit square picked at random according to μ induce the pattern π.

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This extends to sequences of random permutations $(\boldsymbol{\sigma}_n)$, converging to a (*a priori* random) permuton $\boldsymbol{\mu}$.

Permuton convergence is characterized by the convergence of probabilities of all patterns.

- $\widetilde{\text{occ}}(\pi, \sigma)$ = probability of occurrence of the pattern π in σ
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- If $(\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n))_\pi$ converges (jointly) to some $(\Lambda_\pi)_\pi$ in distribution, then there exists a permuton $\boldsymbol{\mu}$ such that $(\boldsymbol{\sigma}_n)$ converges to $\boldsymbol{\mu}$ and $(\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\mu}))_\pi \stackrel{(d)}{=} (\Lambda_\pi)_\pi$.

Summary so far, and what comes next

Goal: Prove limit shape results for the diagrams of uniform random permutations in permutation classes.

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Key tools:

- Permuton convergence is the convergence of all pattern probabilities;
- Thanks to their substitution decomposition, permutations are trees and their patterns are subtrees;
- Limit shape results on random trees

OR

- Singularity analysis of generating functions for trees.

Substitution decomposition

Ingredients:

- A way of building bigger permutations from smaller ones
  $\rightsquigarrow$ substitution or inflation;
- "Building blocks" allowing to build all permutations
  $\rightsquigarrow$ simple permutations.

Essential property:

- For every permutation σ, there exists a unique way of obtaining it recursively using inflations of simple permutations.

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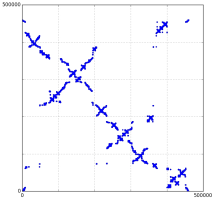
Outcome:

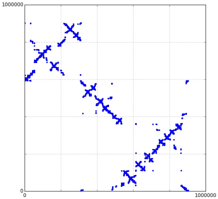
- Bijection between permutations and decomposition trees.
- (Some) permutation classes are (nice) families of trees:
  - easiest case: substitution-closed classes;
  - beyond those: classes with a finite combinatorial specification.

Separable permutations
and the Brownian separable permuton

They are equivalently described as

- $Av(2413, 3142)$;
- the substitution-closed class with set of simple permutations $\emptyset$.

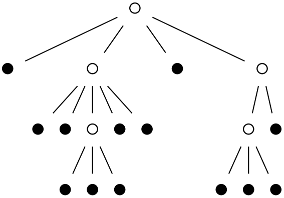




Theorem:
Uniform random separable permutations converge to a genuinely random permuton: the Brownian separable permuton.

Decomposition trees of separable permutations

- A Schröder tree of size n is a rooted plane tree with n leaves whose internal vertices have at least two children.



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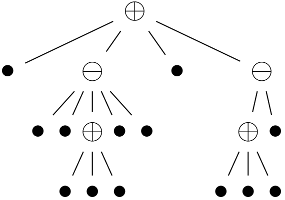
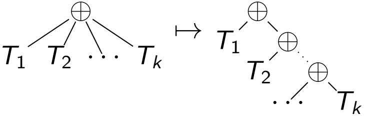


- Decomposition trees of separable permutations are signed Schröder trees, as above with additional signs $\oplus$ and $\ominus$ on the internal vertices, which alternate on any path from the root to a leaf (*i.e.* the sign of the root determines all others).

Up to the "right binarization"



(and same with $\ominus$),
the decomposition trees of separable permutations are generated by the following combinatorial specification:

$$\begin{cases} \mathcal{T}_{\text{sep}} = \{\bullet\} \uplus \overset{\oplus}{\mathcal{T}^{\text{not}\oplus}_{\text{sep}} \quad \mathcal{T}_{\text{sep}}} \uplus \overset{\ominus}{\mathcal{T}^{\text{not}\ominus}_{\text{sep}} \quad \mathcal{T}_{\text{sep}}} \; ; \\ \mathcal{T}^{\text{not}\oplus}_{\text{sep}} = \{\bullet\} \uplus \overset{\ominus}{\mathcal{T}^{\text{not}\ominus}_{\text{sep}} \quad \mathcal{T}_{\text{sep}}} \; ; \\ \mathcal{T}^{\text{not}\ominus}_{\text{sep}} = \{\bullet\} \uplus \overset{\oplus}{\mathcal{T}^{\text{not}\oplus}_{\text{sep}} \quad \mathcal{T}_{\text{sep}}} \; . \end{cases}$$

Starting point of the "analytic combinatorics" proof, discussed later.

The combinatorial specification of separable permutations

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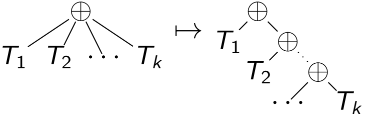

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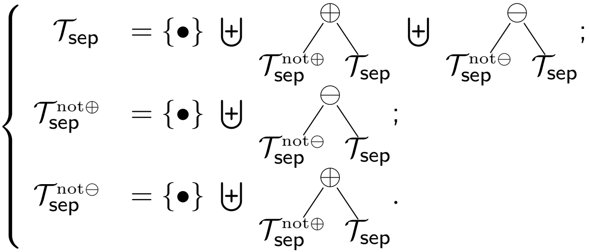

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For now, we present the "random trees" proof.

Contours and their limits

The contour of a uniform random Schröder tree converges to the Brownian excursion.

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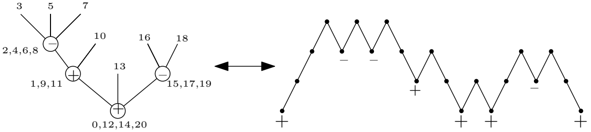

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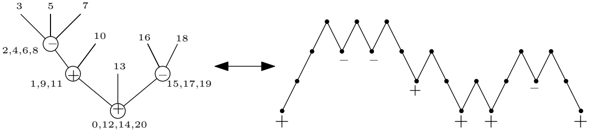

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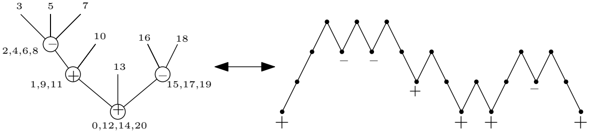

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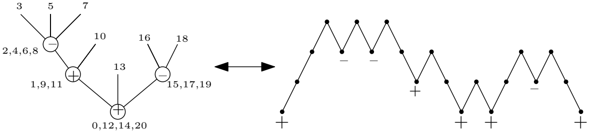

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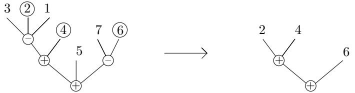
On finite objects:

Extracting a pattern π of size k from a separable permutation σ

$$\sigma = 3214576 \longmapsto \pi = 123$$

$\equiv$ Extracting a signed subtree (induced by k leaves) in a signed Schröder tree of σ



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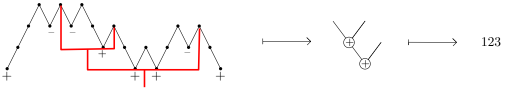

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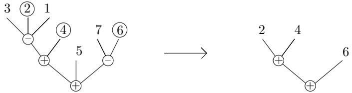

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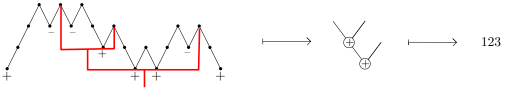

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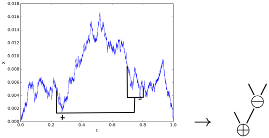

Extracting a signed tree from a set of k uniformly chosen points in the signed Brownian excursion $\mathbf{e}_{\pm}$

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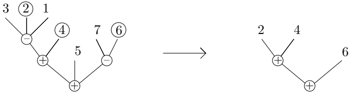

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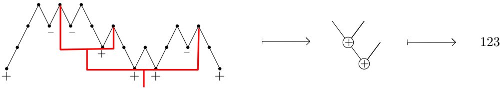

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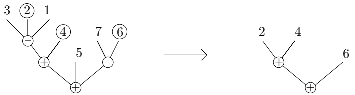

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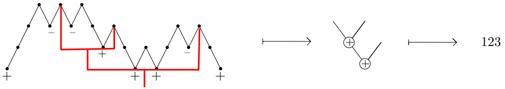

In the limit:

This characterizes the probabilities of all subtrees extracted from $\mathbf{e}_\pm$, hence of all patterns extracted from the Brownian separable permuton (so defined).

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Extracting a signed tree from a set of k uniformly chosen points in the signed Brownian excursion $\mathbf{e}_\pm$

Building the Brownian separable permuton from $\mathbf{e}_{\pm}$ 1/2

This was described by Mickaël Maazoun.

Idea: Imitate the discrete construction "σ as a word $\mapsto$ permuton of σ using trees and contours" on the continuous objects.

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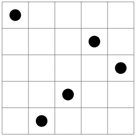

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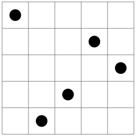

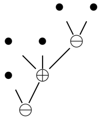

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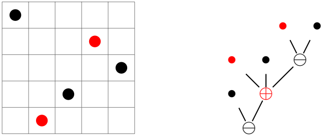

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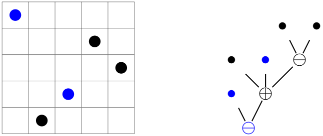

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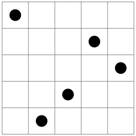

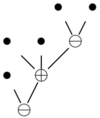

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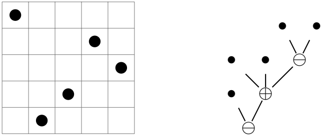

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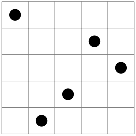

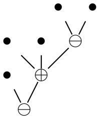

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In the continuous:

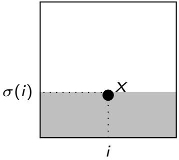
Consider two orders $<_h, <_v$ on $[0, 1]$:

- $<_h$ is the natural order $<$
- $x <_v y$ when
$$\begin{cases} x <_h y \text{ and } sign(m) = \oplus \\ y <_h x \text{ and } sign(m) = \ominus \end{cases}$$
where m is the (a.s. unique) local minimum of $\mathbf{e}_{\pm}$ between x and y.

Building the Brownian separable permuton from $\mathbf{e}_\pm$ 2/2

In the discrete case:

- Denoting i the rank of x for $<_h$, $\sigma(i)$ is $\sigma(i) = \#\{y \text{ s.t. } y \leq_v x\}$.



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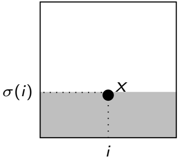

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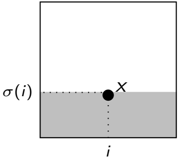

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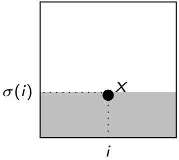

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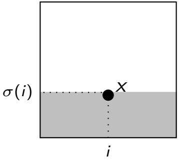

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- Define the function $\phi : [0,1] \to [0,1]$ by $\phi(t) = \lambda(\{u \in [0,1] \text{ s.t. } u \leq_v t\})$, where λ is the Lebesgue measure on $[0,1]$.

- The Brownian separable permuton is the pushforward of the Lebesgue measure on $[0,1]$ by the function $x \mapsto (x, \phi(x))$.

Consequences: genuinely random permuton, fractal behavior, Hausdorff dimension 1, ...

Substitution-closed classes and universality of the Brownian separable permuton

The class of separable permutations is the one whose decomposition trees are described by the combinatorial specification

$$
\begin{cases}
\mathcal{T}_{\mathsf{sep}} = \{\bullet\} \uplus \overset{\oplus}{\mathcal{T}^{\mathrm{not}\oplus}_{\mathsf{sep}} \quad \mathcal{T}_{\mathsf{sep}}} \uplus \overset{\ominus}{\mathcal{T}^{\mathrm{not}\ominus}_{\mathsf{sep}} \quad \mathcal{T}_{\mathsf{sep}}} \quad ; \\
\mathcal{T}^{\mathrm{not}\oplus}_{\mathsf{sep}} = \{\bullet\} \uplus \overset{\ominus}{\mathcal{T}^{\mathrm{not}\ominus}_{\mathsf{sep}} \quad \mathcal{T}_{\mathsf{sep}}} \quad ; \\
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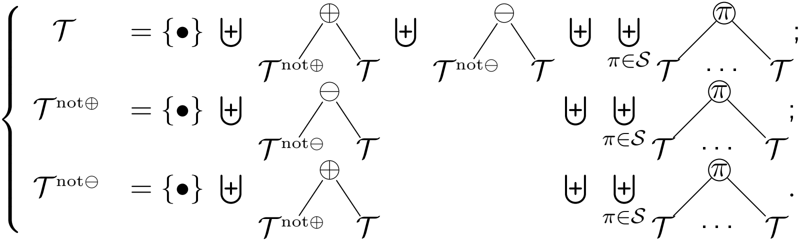
\end{cases}
$$

Substitution-closed classes are those whose decomposition trees described by a combinatorial specification of the form



where $\mathcal{S}$ is the set of simple permutations in the class $\mathcal{C}_{\mathcal{S}}$ considered.

Separable permutations and substitution-closed classes

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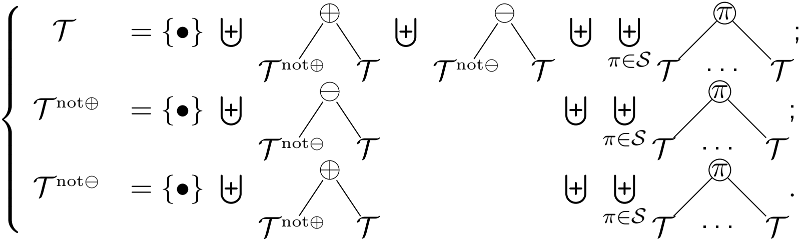

where $\mathcal{S}$ is the set of simple permutations in the class $\mathcal{C}_{\mathcal{S}}$ considered.

Theorem:
Under an analytic condition on the generating function $S(z)$ of $\mathcal{S}$, uniform random permutations in the substitution-closed class $\mathcal{C}_{\mathcal{S}}$ converge to a biased Brownian separable permuton.

The biased Brownian separable permuton of parameter p

- Biased version of the signed Brownian excursion (for $p \in [0,1]$): in $\mathbf{e}_{\pm,p}$, local minima carry independent signs, but not balanced; instead, $+$ with probability p, $-$ with probability $1-p$.
- The biased Brownian separable permuton $\boldsymbol{\mu}_p$ of parameter p is characterized as above but starting from $\mathbf{e}_{\pm,p}$ instead of $\mathbf{e}_{\pm}$.

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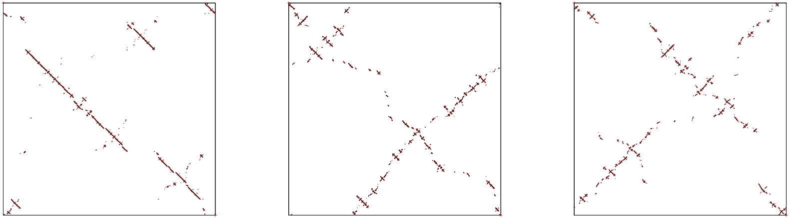
The higher p is, the more drift there is towards the direction of the main diagonal in $\boldsymbol{\mu}_p$.



simulations of $\boldsymbol{\mu}_p$ for $p = 0.2, 0.45, 0.5$.

Statement and examples

Theorem:
Let $\mathcal{C}_{\mathcal{S}}$ be the substitution-closed class with set of simple permutations $\mathcal{S}$. Let $S(z)$ be the generating function of $\mathcal{S}$, and let R_S be its (positive) radius of convergence.

Statement and examples

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Assuming that $\lim_{\substack{r \to R_S \\ r < R_S}} S'(r) > \frac{2}{(1+R_S)^2} - 1$, uniform random permutations in $\mathcal{C}_\mathcal{S}$ converge to the biased Brownian separable permuton $\boldsymbol{\mu}_p$,

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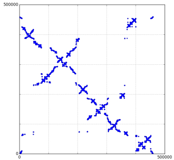
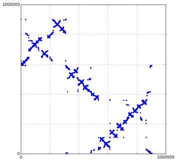
Example 1: Separable permutations, *i.e.* $\mathcal{C}_{\emptyset}$, $\Rightarrow p = 0.5$





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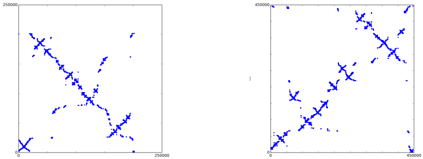
Example 2: $\mathcal{C}_{\mathcal{S}}$ with $\mathcal{S} = \{2413, 3142, 24153\}$, $\Rightarrow p = 0.5$



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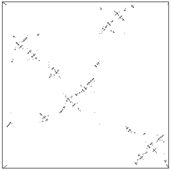
Example 3: $\mathcal{C}_{\mathcal{S}}$ with $\mathcal{S} = Av(321) \cap \{Simples\}$, $\Rightarrow p \in [0.577, 0.622]$



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Further examples: All substitution-closed classes

- with finitely many simple permutations,
- or more generally such that $R_S = 1$,
- or such that S' diverges at R_S (in particular for S rational, or in case of square root singularity for S, ...).

This covers all substitution-closed classes whose simple permutations have been enumerated.

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Non-example: $Av(2413)$

Limit not known.

It is "degenerate": typical permutations in $Av(2413)$ look like typical simple permutations in $Av(2413)$.

But we don't know the limit of simple permutations in $Av(2413)$.

Proof schema 1/2

$\boldsymbol{\sigma}_n =$ uniform random permutation of size n in $\mathcal{C}_{\mathcal{S}}$.

- The law of $(\widetilde{\text{occ}}(\pi, \boldsymbol{\sigma}_n))_\pi$ is determined by its moments.
- These moments are all determined by $(\mathbb{E}\left[\widetilde{\text{occ}}(\pi, \boldsymbol{\sigma}_n)\right])_\pi$.

$\Rightarrow$ It is enough to prove the convergence of $\left(\mathbb{E}\left[\widetilde{\text{occ}}(\pi, \boldsymbol{\sigma}_n)\right]\right)_n$ for all π.

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- The combinatorial specification for $\mathcal{C}_\mathcal{S}$ yields equations for these tree series.
- Estimate their coefficients with analytic combinatorics.

Proof schema 2/2

- Determine the singular behavior of the trees series.

  They all depend on the one of $T_{\text{not}\oplus}$ satisfying

  $$T_{\text{not}\oplus}(z) = z + \Lambda(T_{\text{not}\oplus}(z))$$

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Rk: The limit of $\mathbb{E}\left[\widetilde{\text{occ}}(\pi, \boldsymbol{\sigma}_n)\right]$ is non-zero if and only if π is separable.

Finitely generated classes,
not necessarily substitution-closed

Combinatorial specifications of classes

- For substitution-closed classes: there is always a combinatorial specification for the associated decomposition trees.
- For other classes, there is sometimes a combinatorial specification for the associated decomposition trees.
- It is always the case when the number of simple permutations in the class is finite.

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  Moreover, in this case, the specification is automatically produced.

Example: $Av(132)$

$$\begin{cases} \mathcal{T} = \{\bullet\} \uplus \oplus\left(\mathcal{T}^{\mathrm{not}\oplus}, \mathcal{T}_{\langle 21 \rangle}\right) \uplus \ominus\left(\mathcal{T}^{\mathrm{not}\ominus}, \mathcal{T}\right) \\ \mathcal{T}^{\mathrm{not}\oplus} = \{\bullet\} \uplus \ominus\left(\mathcal{T}^{\mathrm{not}\ominus}, \mathcal{T}\right) \\ \mathcal{T}^{\mathrm{not}\ominus} = \{\bullet\} \uplus \oplus\left(\mathcal{T}^{\mathrm{not}\oplus}, \mathcal{T}_{\langle 21 \rangle}\right) \\ \mathcal{T}_{\langle 21 \rangle} = \{\bullet\} \uplus \oplus\left(\mathcal{T}^{\mathrm{not}\oplus}_{\langle 21 \rangle}, \mathcal{T}_{\langle 21 \rangle}\right) \\ \mathcal{T}^{\mathrm{not}\oplus}_{\langle 21 \rangle} = \{\bullet\}. \end{cases}$$

Relevant properties of the specifications

Consider a specification for the decomposition trees of permutations in a class $\mathcal{C} = \mathcal{T}_0$ where the families $(\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_k)$ appear.

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Define that $\mathcal{T}_i$ is critical when its generating function has minimal radius of convergence among all those of the $(\mathcal{T}_j)_j$ (which is that of $\mathcal{T}_0$).

The limiting behavior of uniform permutations in $\mathcal{C}$ is determined by:

- the strongly connected components of the specification restricted to critical families;
- whether the restriction of the specification to each strongly connected component is linear or branching.

Essentially branching specifications

The restriction of the specification to critical families is strongly connected, and contains a product.

$\Rightarrow$ The limiting permuton of the class is a biased Brownian separable permuton (of parameter p possibly 0 or 1).

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Example 1: $Av(132)$, with critical families in blue.

$$
\begin{cases}
\mathcal{T} & = \{\bullet\} \uplus \mathcal{T}^{\text{not}\oplus} \overset{\oplus}{\wedge} \mathcal{T}_{\langle 21\rangle} \uplus \mathcal{T}^{\text{not}\ominus} \overset{\ominus}{\wedge} \mathcal{T} \\
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\mathcal{T}_{\langle 21\rangle} & = \{\bullet\} \uplus \mathcal{T}^{\text{not}\oplus}_{\langle 21\rangle} \overset{\oplus}{\wedge} \mathcal{T}_{\langle 21\rangle} \\
\mathcal{T}^{\text{not}\oplus}_{\langle 21\rangle} & = \{\bullet\}.
\end{cases}
$$

The limit is the Brownian separable permuton of parameter $p = 0$.

Essentially branching specifications

The restriction of the specification to critical families is strongly connected, and contains a product.

⇒ The limiting permuton of the class is a biased Brownian separable permuton (of parameter p possibly 0 or 1).

Example 2: $Av(2413, 31452, 41253, 41352, 531246)$, with critical families in blue.

$$\begin{cases} \mathcal{T}_0 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_0] \uplus \ominus[\mathcal{T}_2, \mathcal{T}_0] \uplus 3142[\mathcal{T}_0, \mathcal{T}_3, \mathcal{T}_3, \mathcal{T}_0] \\ \mathcal{T}_1 = & \{\bullet\} \uplus \ominus[\mathcal{T}_2, \mathcal{T}_0] \uplus 3142[\mathcal{T}_0, \mathcal{T}_3, \mathcal{T}_3, \mathcal{T}_0] \\ \mathcal{T}_2 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_0] \uplus 3142[\mathcal{T}_0, \mathcal{T}_3, \mathcal{T}_3, \mathcal{T}_0] \\ \mathcal{T}_3 = & \{\bullet\} \uplus \ominus[\mathcal{T}_4, \mathcal{T}_3] \\ \mathcal{T}_4 = & \{\bullet\} \end{cases}$$

The limit is the Brownian separable permuton of parameter $p \approx 0.4748692376...$ (only real root of a certain polynomial of degree 9).

Essentially linear specifications

The restriction of the specification to critical families is strongly connected, and contains no product.

$\Rightarrow$ The limiting permuton of the class is the X-permuton of parameter $\mathbf{p} = (p_1, p_2, p_3, p_4)$ (not necessarily centered, possibly degenerate).

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Example 1: $Av(2413, 3142, 2143, 3412)$, a.k.a. the X-class, with critical families in red and blue (for two strongly connected components).

$$\begin{cases} \mathcal{T}_0 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_2] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_5] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_7, \mathcal{T}_5] \\ \mathcal{T}_1 = & \{\bullet\} \\ \mathcal{T}_2 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_2] \\ \mathcal{T}_3 = & \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_5] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_7, \mathcal{T}_5] \\ \mathcal{T}_4 = & \ominus[\mathcal{T}_1, \mathcal{T}_5] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_7, \mathcal{T}_5] \\ \mathcal{T}_5 = & \{\bullet\} \uplus \ominus[\mathcal{T}_1, \mathcal{T}_5] \\ \mathcal{T}_6 = & \oplus[\mathcal{T}_1, \mathcal{T}_2] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_7, \mathcal{T}_5] \\ \mathcal{T}_7 = & \oplus[\mathcal{T}_1, \mathcal{T}_2] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2]. \end{cases}$$

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The restriction of the specification to critical families is strongly connected, and contains no product.

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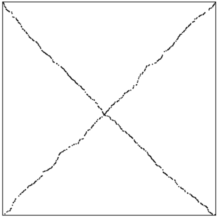

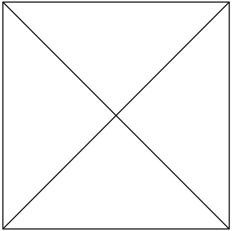

The limit is the centered X-permuton (of parameter $(1/4, 1/4, 1/4, 1/4)$).

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The restriction of the specification to critical families is strongly connected, and contains no product.

$\Rightarrow$ The limiting permuton of the class is the X-permuton of parameter $\mathbf{p} = (p_1, p_2, p_3, p_4)$ (not necessarily centered, possibly degenerate).

Example 2: $Av(2413, 3142, 2143, 34512)$, with critical families in red and blue (for two strongly connected components).

$$\begin{cases}
\mathcal{T}_0 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_2] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_5, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_5, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_8, \mathcal{T}_6] \\
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\mathcal{T}_5 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_1] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_9] \uplus \oplus[\mathcal{T}_9, \mathcal{T}_1] \\
\mathcal{T}_6 = & \{\bullet\} \uplus \ominus[\mathcal{T}_1, \mathcal{T}_6] \\
\mathcal{T}_7 = & \oplus[\mathcal{T}_1, \mathcal{T}_2] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_{10}, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_{10}, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_8, \mathcal{T}_6] \\
\mathcal{T}_8 = & \oplus[\mathcal{T}_1, \mathcal{T}_{11}] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_{12}] \uplus \oplus[\mathcal{T}_{13}, \mathcal{T}_{11}] \uplus \oplus[\mathcal{T}_9, \mathcal{T}_{11}] \uplus \oplus[\mathcal{T}_{13}, \mathcal{T}_1] \\
\mathcal{T}_9 = & \ominus[\mathcal{T}_1, \mathcal{T}_6] \\
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\mathcal{T}_{11} = & \oplus[\mathcal{T}_1, \mathcal{T}_2] \\
\mathcal{T}_{12} = & \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_{10}, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_{10}, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_8, \mathcal{T}_6] \\
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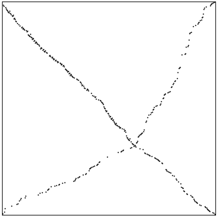
\end{cases}$$

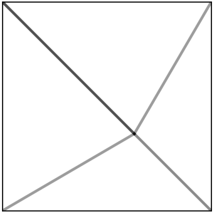
Essentially linear specifications

The restriction of the specification to critical families is strongly connected, and contains no product.

$\Rightarrow$ The limiting permuton of the class is the X-permuton of parameter $\mathbf{p} = (p_1, p_2, p_3, p_4)$ (not necessarily centered, possibly degenerate).

Example 2: $Av(2413, 3142, 2143, 34512)$, with critical families in red and blue (for two strongly connected components).





The limit is the X-permuton of parameter $\approx (0.2003, 0.2003, 0.4313, 0.1681)$.

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Example 3: $Av(2413, 1243, 2341, 531642, 41352)$, with critical families in red and blue (for two strongly connected components).

$$\begin{cases} \mathcal{T}_0 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_2] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_5, \mathcal{T}_0] \uplus 3142[\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_6] \\ \mathcal{T}_1 = & \{\bullet\} \uplus \ominus[\mathcal{T}_7, \mathcal{T}_1] \\ \mathcal{T}_2 = & \{\bullet\} \uplus \oplus[\mathcal{T}_7, \mathcal{T}_2] \\ \mathcal{T}_3 = & \oplus[\mathcal{T}_8, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_9, \mathcal{T}_6] \\ \mathcal{T}_4 = & \ominus[\mathcal{T}_{10}, \mathcal{T}_{11}] \uplus \ominus[\mathcal{T}_{10}, \mathcal{T}_1] \uplus \ominus[\mathcal{T}_7, \mathcal{T}_{11}] \uplus 3142[\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_6] \\ \mathcal{T}_5 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_1] \uplus 3142[\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1] \\ \mathcal{T}_6 = & \{\bullet\} \uplus \oplus[\mathcal{T}_{12}, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_9, \mathcal{T}_6] \\ \mathcal{T}_7 = & \{\bullet\} \\ \mathcal{T}_8 = & \ominus[\mathcal{T}_9, \mathcal{T}_6] \\ \mathcal{T}_9 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_7] \\ \mathcal{T}_{10} = & \oplus[\mathcal{T}_1, \mathcal{T}_1] \uplus 3142[\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1] \\ \mathcal{T}_{11} = & \oplus[\mathcal{T}_1, \mathcal{T}_2] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_{10}, \mathcal{T}_{11}] \uplus \ominus[\mathcal{T}_{10}, \mathcal{T}_1] \uplus \ominus[\mathcal{T}_7, \mathcal{T}_{11}] \uplus 3142[\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_6] \\ \mathcal{T}_{12} = & \{\bullet\} \uplus \ominus[\mathcal{T}_9, \mathcal{T}_6] \end{cases}$$

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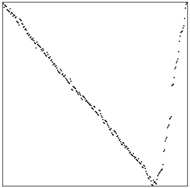

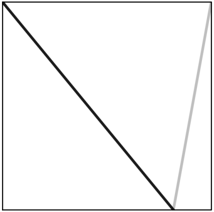

The limit is the X-permuton of parameter $(0, 0, 1-p, p)$ with $p \approx 0.81863$.

Several strongly connected components

Assume the specification for $\mathcal{C}$ restricted to critical families has several strongly connected components.

- Limit shape in each strongly connected component: as above.
- Sometimes, the limit shape of $\mathcal{C}$ is a combination of those in each component.

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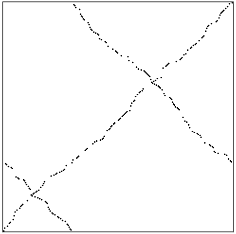
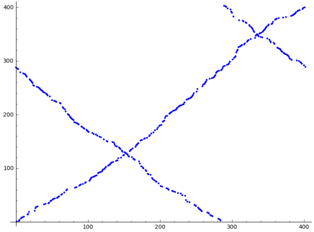
Example: the downward closure of $\oplus[\mathcal{X}, \mathcal{X}]$:





Summary of the described limit shapes

(Biased) Brownian separable permuton:

- Separable permutations;
- Substitution-closed class under an analytic condition on $S(z)$;
- Classes with a specification that is essentially strongly connected and essentially branching.

(Parametrized) X-permuton:

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A mix of the above:

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Thank you!