Limits of permutations and graphs avoiding substructures

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talk based on joint works with
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The problem:

- Consider a class $\mathcal{C}$ of permutations or graphs defined by the avoidance of substructures (patterns or induced subgraphs).
- For any $n$, let $\sigma_n$ or $G_n$ be an object of size $n$ in $\mathcal{C}$, taken uniformly at random among objects of size $n$ in $\mathcal{C}$.
- We would like to describe the typical global behavior of $\sigma_n$ or $G_n$ as $n$ tends to $\infty$, through its permutohedral or graphon limit.

Permutation matrix of a typical large permutation avoiding 2413 and 3142  

Adjacency matrix of typical large graph with no induced $P_4$
The proof strategy:

- Permutons and graphons describe **global limits** of permutations and graphs. But **permuton and graphon** convergence are characterized by convergence of the **densities of substructures**.

- Using the **substitution or modular decomposition**, we can represent permutations or graphs by **trees** (decorated on their internal nodes).

- **Substructures** in permutations or graphs correspond to induced **subtrees** in these trees (subtrees induced by a set of leaves).

- We write functional equations for the **generating functions** counting decomposition trees, possibly with specified induced subtrees.

- Using **analytic combinatorics**, we derive the **limiting densities** of substructures in our permutations or graphs, proving permuton or graphon convergence.
A caveat

- Only some classes of permutations or graphs are amenable to the presented strategy: when the substitution/modular decomposition is “nice”.
- These represent very few cases in the whole landscape of permutation classes/hereditary families of graphs.
- But it still covers quite many classes compared to what was previously known (especially in the permutation case).

The talk will mainly discuss the simplest case in the graph setting: the family of cographs, avoiding an induced $P_4$ (the path on 4 vertices).

We will also discuss briefly its permutation analogue: the family of separable permutations, avoiding the patterns 2413 and 3142, as well as hint at some generalizations.
Induced subgraphs,
and a biased view of
graphon convergence
Induced subgraphs and hereditary families of graphs

- $g$ is an **induced subgraph** of $G$ when

$$
g = \begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array} \quad \subseteq \quad \begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array} \quad \Rightarrow \quad G = \begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array} \quad \cup \quad \begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array}
$$

In words, the **subgraph** of $G = (V, E)$ induced by $V' \subset V$ is the graph with vertex set $V'$ and edge set $E \cap (V' \times V')$. 
Induced subgraphs and hereditary families of graphs

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\[ g = \quad \Leftrightarrow \quad G = \]

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Induced subgraphs and hereditary families of graphs

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In words, the subgraph of $G = (V, E)$ induced by $V' \subset V$ is the graph with vertex set $V'$ and edge set $E \cap (V' \times V')$.

- A hereditary family of graphs is a set of graphs $C$ such that for every $G \in C$, if $g$ is an induced subgraph of $G$, then $g \in C$.
- Examples include the families of cographs, comparability graphs, permutation graphs, circle graphs, parity graphs, . . .
- Equivalently, hereditary families of graphs are characterized as sets of graphs whose induced subgraphs avoid a prescribed set (which may be finite or infinite).
**Definition:** The density of an induced subgraph $g$ in $G$ is

$$\text{Dens}(g, G) = \mathbb{P}(\text{SubGraph}_k(G) = g)$$

where $k$ is the number of vertices of $g$ and \text{SubGraph}_k(G) is the (random) subgraph of $G$ induced by a $k$-tuple of i.i.d. uniform random vertices of $G$. 
Densities of induced subgraphs

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**Variant:** The “injective density” is defined by

$$Dens^{inj}(g, G) = \mathbb{P}(\text{SubGraph}^{inj}_k(G) = g),$$

where $\text{SubGraph}^{inj}_k(G)$ is the (random) subgraph of $G$ induced by a uniform random $k$-tuple of distinct vertices of $G$.

**Fact:** For $G_n$ a sequence of random graphs of size tending to infinity, $\mathbb{E}[Dens(g, G_n)] \to \Delta_g$ iff $\mathbb{E}[Dens^{inj}(g, G_n)] \to \Delta_g$. 
What is (informally) a graphon?

In the discrete setting:

(Unlabeled) Adjacency matrix Function
graph $G$ $\rightarrow$ $M_G$ (symmetric) $\rightarrow$ $w_G : [0, 1]^2 \rightarrow [0, 1]$

The graphon $W_G$ associated with $G$ is the equivalence class of $w_G$ under the action of permuting rows and columns of $M_G$. 
What is (informally) a graphon?

**In the discrete setting:**

(Unlabeled) graph $G$ $\rightarrow$ Adjacency matrix $M_G$ (symmetric) $\rightarrow$ Function $w_G : [0, 1]^2 \rightarrow [0, 1]$

![Adjacency matrix example]

The graphon $W_G$ associated with $G$ is the equivalence class of $w_G$ under the action of permuting rows and columns of $M_G$.

**Continuous extension:**

In general, a graphon is obtained as above, from a symmetric matrix $M$, possibly with a continuum of rows and columns, and with values in $[0, 1]$. It is an equivalence class of symmetric functions from $[0, 1]^2 \rightarrow [0, 1]$ under the action of permuting rows and columns of $M$. 
Subgraph densities in graphons

Fix $g$ a graph with $k$ vertices, unlabeled.

- **Reminder of the discrete case:**
  For a graph $G$, $\text{Dens}(g, G) = \mathbb{P}(\text{SubGraph}_k(G) = g)$, where $\text{SubGraph}_k(G)$ is the (random) subgraph of $G$ induced by a $k$-tuple of i.i.d. uniform random vertices of $G$.

- **Continuous generalization:**
  For a graphon $W$, $\text{Dens}(g, W) = \mathbb{P}(\text{Sample}_k(W) = g)$, where $\text{Sample}_k(W)$ is the (random) graph with $k$ vertices $v_1, \ldots, v_k$ such that $v_i$ and $v_j$ are connected with probability $w(x_i, x_j)$, for $x_1, \ldots, x_k$ i.i.d. uniform random variables in $[0, 1]$ and $w : [0, 1]^2 \rightarrow [0, 1]$ a representative of $W$. 
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**Remark:** For any graph $G$, $\text{Dens}(g, W_G) = \text{Dens}(g, G)$. 
(Non-)definition:
The space of graphons is (up to technicalities) metric, for the cut-distance (and in addition is compact).

So, it makes sense to study convergence of a sequence of graphons \((W_n)_{n \geq 0}\) to a graphon \(W\) (for this cut-distance). We write \(W_n \rightarrow W\).
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Typically, \(W_n = W_{G_n}\), the graphon associated to a graph \(G_n\), with the sequence of graphs \((G_n)\) such that the size of \(G_n\) grows to infinity with \(n\). In this case, we also write \(G_n \rightarrow W\).
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Combinatorial characterization of convergence:
For \((W_n)\) a sequence of graphons and \(W\) a graphon, \(W_n \to W\) iff for any (finite) graph \(g\), \(\text{Dens}(g, W_n) \to \text{Dens}(g, W)\).
Characterization of graphon convergence: the random case

Reminder: $G_n \rightarrow W$ iff $\text{Dens}(g, G_n) \rightarrow \text{Dens}(g, W)$ for all $g$, for $(G_n)$ a sequence of (deterministic) graphs and $W$ a (deterministic) graphon.
Reminder: $G_n \rightarrow W$ iff $\text{Dens}(g, G_n) \rightarrow \text{Dens}(g, W)$ for all $g$, for $(G_n)$ a sequence of (deterministic) graphs and $W$ a (deterministic) graphon.

What if we take $(G_n)$ random? $(\text{Dens}(g, G_n)$ being then a real r.v.)
Reminder: $G_n \to W$ iff $\text{Dens}(g, G_n) \to \text{Dens}(g, W)$ for all $g$, for $(G_n)$ a sequence of (deterministic) graphs and $W$ a (deterministic) graphon.

What if we take $(G_n)$ random? ($\text{Dens}(g, G_n)$ being then a real r.v.)

**Theorem** [Diaconis-Janson, 2008]:
The distribution of a random graphon $W$ is characterized by all expected subgraph densities $\mathbb{E}[\text{Dens}(g, W)]$ (for all $g$).

**Theorem** [Diaconis-Janson, 2008]:
Let $(G_n)$ be a sequence of random graphs. TFAE:
- $G_n$ tends in distribution to some random graphon, $W$.
- For all $g$, $\mathbb{E}[\text{Dens}(g, G_n)]$ converges to some value $\Delta_g \in [0, 1]$.

If this holds, in addition we have:
for all $g$, $\mathbb{E}[\text{Dens}(g, W)] = \Delta_g$, so that $(\Delta_g)_g$ characterizes $W$. 
Summary so far

Graphs:
- Definition of induced subgraphs
- Definition of hereditary classes of graphs
- Notion of graphon as a “rescaled adjacency matrix”
- Combinatorial characterization of graphon convergence: by the convergence of the densities of induced subgraphs (in expectation in the random case)

Analogue notions for permutations:
- Induced subgraphs correspond to patterns
- Hereditary families are called permutation classes
- Notion of permuton as a “rescaled permutation matrix”
- Combinatorial characterization of permuton convergence: by the convergence of the frequencies of patterns (in expectation in the random case)
Decomposition trees
A module in a graph $G = (V, E)$ is a set $S \subseteq V$ of vertices which cannot be distinguished by vertices outside of $S$: 

for every $v \in V \setminus S$, either $\{v, s\} \in E$ for all $s \in S$

or $\{v, s\} \notin E$ for all $s \in S$

Given a partition of $V$ into modules, $G$ can be described

- the subgraph induced keeping exactly one vertex in each module (sometimes called quotient)
- the subgraph induced by each module (sometimes called factors)

Repeating this construction inside the modules, we obtain a modular decomposition tree of $G$ (which is rooted, non-planar, with internal vertices labeled by quotient graphs, and leaves corresponding to $V$).
The trivial modules of $G$ are $\emptyset$, $V$, and $\{v\}$ for any $v \in V$.

A graph $G$ is prime if it contains no non-trivial module.

**Theorem:** Every graph has a unique modular decomposition tree whose vertices are either cliques (denoted 1), or independent sets (denoted 0), or prime graphs (denoted $P$), and with no 0–0 nor 1–1 edges. We call it canonical and denote it $T(G)$.

$T(G)$ is obtained considering recursively the quotients resulting from the partition of $V$ into maximal modules different from $V$ (in the prime case, with special cases for cliques and independent sets).
Induced subgraphs in decomposition trees

- Let $G$ be a graph. Let $S$ be a subset of its vertices.
- Consider the subgraph of $G$ induced by $S$.
- Let $T(G)$ be its canonical modular decomposition tree.

**Fact:** A decomposition tree for the induced subgraph of $G$ corresponding to $S$ is obtained considering the subtree of $T(G)$ induced by the set of leaves corresponding to $S$.

**Remark:** The induced tree is not necessarily the canonical tree of the induced subgraph (e.g. it may contain $0 - 0$ edges).
Permutation analogues

- A module in a graph corresponds to an interval in a permutation.
- A permutation can be decomposed into quotients and factors via the substitution decomposition.

\[ 2 1 4 6 5 3 = \] 

The trees recording these decompositions are called (substitution) decomposition trees.

- There exists a unique canonical decomposition tree.
- Patterns correspond to subtrees induced by leaves.

Remark: For a permutation \( \sigma \), consider its inversion graph \( G \). Up to planarity and adapting the decorations, it holds that \( T(\sigma) = T(G) \).
Graphon limit of (labeled) cographs
Cographs and their modular decomposition trees

- Cographs are defined by the avoidance of $P_4$ (induced path on 4 vertices, which is the smallest prime graph).
- Equivalently, cographs are all graphs whose modular decomposition trees involve only 0 (indep. set) and 1 (clique) nodes (no prime node). We call cotrees their modular decomposition trees.
- Therefore, labeled\(^1\) cographs can be described from the combinatorial specification:

$$\mathcal{L} = \bullet \uplus \text{Set}_{\geq 2}(\mathcal{L}) \quad \text{i.e.,}$$

Indeed, via its canonical modular decomposition tree, a cograph correspond to a tree of $\mathcal{L}$ with a label 0 or 1 on the root (propagating labels alternating between 0 and 1 along each edge).

\(^1\)meaning vertices are labeled by the integers from 1 to $n$; in the unlabeled case, we need to consider Multiset, and hence later Polya operators for the GF
Expressing $\mathbb{E}[\text{Dens}^{\text{inj}}(g, G_n)]$

**Notation:**
Let $G_n$ be a uniform random labeled cograph with $n$ vertices.

**Reminder:**
Knowing $\mathbb{E}[\text{Dens}^{\text{inj}}(g, G_n)]$ for all $g$ characterizes the graphon limit of $G_n$.

**Notation:**
for all $n$, and all $k \leq n$,
- $t^{(n)}$ is a uniform random labeled canonical cotree of size $n$, and
- $t_k^{(n)}$ is the subtree of $t^{(n)}$ induced by a uniform $k$-tuple of distinct leaves.

**Observation:**
For any cograph $g$, we have:
$$\mathbb{E}[\text{Dens}^{\text{inj}}(g, G_n)] = \mathbb{P}(\text{SubGraph}^{\text{inj}}_k(G_n) = g) = \sum \mathbb{P}(t_k^{(n)} = t_0),$$
where the sum runs over all cotrees $t_0$ corresponding to $g$. 
Expressing $\mathbb{P}(t_k^{(n)} = t_0)$

**Observation:** $\mathbb{P}(t_k^{(n)} = t_0) = \frac{n![z^n]M_{t_0}(z)}{n![z^n]M(z) \times n(n-1) \ldots (n-k+1)}$, where

- $\mathcal{M}$ is the set of labeled canonical cotrees
- for any cotree $t_0$ with $k$ leaves, $\mathcal{M}_{t_0}$ is the set of labeled canonical cotrees with a marked $k$-tuple of distinct leaves, which induce $t_0$,
- $M(z)$ and $M_{t_0}(z)$ are the corresponding exponential generating functions
- as usual, $[z^n]F(z)$ denotes the coefficient of $z^n$ in the generating function $F(z)$

**Next:** Use symbolic and analytic combinatorics to compute the asymptotic behavior of the numerator and the denominator in

$$
\frac{n![z^n]M_{t_0}(z)}{n![z^n]M(z) \times n(n-1) \ldots (n-k+1)}.
$$
Recall that $\mathcal{L} = \bullet \uplus \text{Set}_{\geq 2}(\mathcal{L})$.

Let $L(z)$ be the exponential generating function of $\mathcal{L}$.

From [Flajolet-Sedgewick] (rather a variant on trees counted by leaves), $L(z)$ satisfies $L(z) = z + \exp(L(z)) - 1 - L(z)$.

The generating function of cographs is $M(z) = 2L(z) - z$.

$L(z)$ and $M(z)$ have the same radius of convergence $\rho = 2 \log(2) - 1$ and are $\Delta$-analytic.

Near $z = \rho$, $L(z) = \log(2) - \sqrt{\rho} \sqrt{1 - z/\rho} + O(1 - z/\rho)$ and $M(z) = 1 - 2\sqrt{\rho} \sqrt{1 - z/\rho} + O(1 - z/\rho)$.

From the transfer theorem,

$$n(n - 1) \ldots (n - k + 1)[z^n]M(z) \sim_{n \to +\infty} \frac{n^{k-3/2}}{\rho^{n-1/2} \sqrt{\pi}}.$$
**Prop.** If \( t_0 \) with \( k \) leaves has \( n_v \) internal vertices, \( n_\) edges of the form \( 0 \to 0 \) or \( 1 \to 1 \), and \( n_\neq \) edges of the form \( 0 \to 1 \) or \( 1 \to 0 \), then

\[
M_{t_0} = (L')(\exp(L))^{n_v}(L^\bullet)^k(L^{\text{odd}})^n_\)(L^{\text{even}})^{n_\neq},
\]

these series being variations on \( L(z) \) whose singular behavior results from that of \( L(z) \).

**Proof:**
Estimating the numerator

**Prop.:** If \( t_0 \) with \( k \) leaves has \( n_v \) internal vertices, \( n_\equiv \) edges of the form \( 0 \rightarrow 0 \) or \( 1 \rightarrow 1 \), and \( n_\neq \) edges of the form \( 0 \rightarrow 1 \) or \( 1 \rightarrow 0 \), then

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these series being variations on \( L(z) \) whose singular behavior results from that of \( L(z) \).

**Proof:**

**Corollary:** Like before, we obtain

- the behavior at \( \rho \) of \( M_{t_0}(z) \),
- and the asymptotic estimate of \([z^n]M_{t_0}(z)\).
Estimating the numerator

**Prop.** If $t_0$ with $k$ leaves has $n_v$ internal vertices, $n=$ edges of the form $0\rightarrow 0$ or $1\rightarrow 1$, and $n\neq$ edges of the form $0\rightarrow 1$ or $1\rightarrow 0$, then

$$M_{t_0} = (L')(\exp(L))^{n_v}(L^\bullet)^k(L^{\text{odd}})^n=L^{\text{even}})^{n\neq},$$

these series being variations on $L(z)$ whose singular behavior results from that of $L(z)$.

**Proof:**

**Corollary:** Like before, we obtain

- the behavior at $\rho$ of $M_{t_0}(z)$,
- and the asymptotic estimate of $[z^n]M_{t_0}(z)$.

More precisely, we have

$$[z^n]M_{t_0}(z) \sim_{n \to +\infty} \frac{(k-1)!}{(2k-2)!} \frac{n^{k-3/2}}{\rho^{n-1/2}\sqrt{\pi}}.$$  

if $t_0$ is binary (which implies $n_v = k - 1$ and $n= + n\neq = k - 2$).
Conclusion of the proof

Notation (reminder):

- $t^{(n)}$: uniform random labeled canonical cotree of size $n$
- $t^{(n)}_k$: subtree of $t^{(n)}$ induced by a uniform $k$-tuple of distinct leaves
- $t_0$: cotree with $k$ leaves

What we proved: If $t_0$ is binary, then $\lim_{n \to \infty} \mathbb{P}(t^{(n)}_k = t_0) = \frac{(k-1)!}{(2k-2)!}$. 
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Remark: $\frac{(k-1)!}{(2k-2)!} = \frac{1}{\text{number of binary cotrees with } k \text{ leaves}}$.

Consequence: If $t_0$ is not binary, then $\lim_{n \to \infty} \mathbb{P}(t^{(n)}_k = t_0) = 0$. 
Conclusion of the proof

Notation (reminder):
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Consequence: If $t_0$ is not binary, then $\lim_{n \to \infty} P(t_k^{(n)} = t_0) = 0$.

Remark/reminder:
Summing over all $t_0$ encoding a cograph $g$, this gives $\lim_{n \to \infty} E[Dens(g, G_n)]$. These quantities characterize the graphon limit of cographs.
The Brownian cographon

- Starting from a Brownian excursion, whose local minima receive unbiased decorations by 0 and 1, we can build the Brownian cographon of parameter $1/2$, denoted $W^{1/2}$.

- We can compute $\Delta_g = \mathbb{E}[\text{Dens}(g, W^{1/2})]$ for all cographs $g$.

  But this is a story for another time...

- We observe that
  \[ \lim_{n \to \infty} \mathbb{E}[\text{Dens}(g, G_n)] = \Delta_g \text{ for all } g. \]

- Therefore, not only do we prove existence of a graphon limit for uniform random cographs, but we also provide a construction of this limit.

- The limiting graphon is a genuinely random and fractal object.
Separable permutations are those avoiding the patterns 2413 and 3142 (the two smallest simple permutations).

Equivalently, it is the family of all permutations whose decomposition trees involve only $\oplus$ and $\ominus$ nodes (no simple permutations).

Separable permutations are therefore the permutation analogue of cographs.

From a combinatorial specification for the decomposition trees of separable permutations, and using analytic combinatorics as before, we obtain the limiting behavior of $\mathbb{E}[\widetilde{\text{occ}}(\pi, \sigma_n)]$ for $\pi$ any pattern, and $\sigma_n$ a uniform random separable permutation of size $n$. These quantities characterize the permuton limit of separable permutations.

Again, we have an explicit construction of the limiting object $\mu^{1/2}$ (called the Brownian separable permuton of parameter 1/2) from a Brownian excursion with decorations.
Separable permutations are those avoiding the patterns 2413 and 3142 (the two smallest simple permutations).

Equivalently, it is the family of all permutations whose decomposition trees involve only $\oplus$ and $\ominus$ nodes (no simple permutations).

Separable permutations are therefore the permutation analogue of cographs.
More classes of permutations
Transposing the proof strategy to a more general setting

**Idea of the method:**
- Assume that you know a combinatorial specification for the decomposition trees of permutations in some class \( C \).
- It translates into a system of equations for the GF of \( C \).
- We can in addition “track patterns” in these equations.
- IF the method of analytic combinatorics goes through, we obtain convergence to a certain permuton, as for separable permutations.

**Some results:**
- Convergence to Brownian separable permutons of parameters \( p \in [0, 1] \) for substitution-closed classes, under some analytic condition on the GF of the simple permutations in the class.
- Dichotomy for classes for which a specification is known (in particular: whenever they contain finitely many simple permutations): (random) Brownian permutons VS (deterministic) \( X \)-permutons.
Substitution-closed classes

- Their specification adds some simple permutations to that of separable permutations. We denote by $S$ the set of allowed simple permutations.

\[
\begin{align*}
\mathcal{T} &= \{\bullet\} \cup \mathcal{T}_{\text{not} \oplus} \cup \mathcal{T}_{\text{not} \ominus} \cup \mathcal{T}_{\text{not} \ominus} \cup \mathcal{T}_{\text{not} \ominus} \\
\mathcal{T}_{\text{not} \oplus} &= \{\bullet\} \cup \mathcal{T}_{\text{not} \ominus} \\
\mathcal{T}_{\text{not} \ominus} &= \{\bullet\} \cup \mathcal{T}_{\text{not} \ominus}
\end{align*}
\]
Substitution-closed classes

- Their specification adds some simple permutations to that of separable permutations. We denote by $\mathcal{S}$ the set of allowed simple permutations.
- Limit permutons are (biased) Brownian separable permutons.

Simulations of $\mu_p$ for $p = 0.2, 0.45, 0.5$. 
Substitution-closed classes

- Their specification adds some simple permutations to that of separable permutations. We denote by $\mathcal{S}$ the set of allowed simple permutations.
- Limit permutons are (biased) Brownian separable permutons.

Example 1: Separable permutations, i.e. $\mathcal{S} = \emptyset$, $\Rightarrow p = 0.5$
Substitution-closed classes

- Their specification adds some simple permutations to that of separable permutations. We denote by $S$ the set of allowed simple permutations.
- Limit permutons are (biased) Brownian separable permutons.

Example 2: $S = \{2413, 3142, 24153\}$, $\Rightarrow p = 0.5$
Substitution-closed classes

- Their specification adds some simple permutations to that of separable permutations. We denote by $S$ the set of allowed simple permutations.
- Limit permutons are (biased) Brownian separable permutons.

Example 3: $S = Av(321) \cap \{\text{Simples}\}$ (infinite), $\Rightarrow p \in [0.577, 0.622]$
Brownian case of the dichotomy

When the specification contains a product of critical families.

⇒ The limiting permuton of the class is a biased Brownian separable permuton (of parameter $p$ possibly 0 or 1).
Brownian case of the dichotomy

When the specification contains a \textbf{product} of \textbf{critical families}.

⇒ The limiting permuton of the class is a biased \textbf{Brownian separable permuton} (of parameter $p$ possibly 0 or 1).

\textbf{Example 1:} $Av(132)$, with critical families in \textbf{blue}.

$$
\begin{align*}
\mathcal{T} &= \{\bullet\} \bigcup \mathcal{T}^{\not\oplus}_{\langle 21 \rangle} \bigcup \mathcal{T}^{\not\ominus}_{\langle 21 \rangle} \\
\mathcal{T}^{\not\oplus} &= \{\bullet\} \bigcup \mathcal{T}^{\not\ominus} \\
\mathcal{T}^{\not\ominus} &= \{\bullet\} \bigcup \mathcal{T}^{\not\oplus}_{\langle 21 \rangle} \\
\mathcal{T}_{\langle 21 \rangle} &= \{\bullet\} \bigcup \mathcal{T}^{\not\oplus}_{\langle 21 \rangle} \\
\mathcal{T}^{\not\oplus}_{\langle 21 \rangle} &= \{\bullet\}.
\end{align*}
$$

The limit is the Brownian separable permuton of parameter $p = 0$. 
Brownian case of the dichotomy

When the specification contains a product of critical families.

⇒ The limiting permuton of the class is a biased Brownian separable permuton (of parameter $p$ possibly 0 or 1).

Example 2: $Av(2413, 31452, 41253, 41352, 531246)$, with critical families in blue.

\[
\begin{align*}
\mathcal{T}_0 &= \{\bullet\} \uplus \ominus[\mathcal{T}_1, \mathcal{T}_0] \uplus \ominus[\mathcal{T}_2, \mathcal{T}_0] \uplus 3142[\mathcal{T}_0, \mathcal{T}_3, \mathcal{T}_3, \mathcal{T}_0] \\
\mathcal{T}_1 &= \{\bullet\} \uplus \ominus[\mathcal{T}_2, \mathcal{T}_0] \uplus 3142[\mathcal{T}_0, \mathcal{T}_3, \mathcal{T}_3, \mathcal{T}_0] \\
\mathcal{T}_2 &= \{\bullet\} \uplus \ominus[\mathcal{T}_1, \mathcal{T}_0] \uplus 3142[\mathcal{T}_0, \mathcal{T}_3, \mathcal{T}_3, \mathcal{T}_0] \\
\mathcal{T}_3 &= \{\bullet\} \uplus \ominus[\mathcal{T}_4, \mathcal{T}_3] \\
\mathcal{T}_4 &= \{\bullet\}
\end{align*}
\]

The limit is the Brownian separable permuton of parameter $p \approx 0.4748692376...$ (only real root of a certain polynomial of degree 9).
X case of the dichotomy

When the specification contains no product of critical families.

⇒ The limiting permuton of the class has a deterministic X shape (not necessarily centered, possibly missing some of the 4 branches).
**X case of the dichotomy**

When the specification contains no product of critical families.

⇒ The limiting permuton of the class has a **deterministic X shape** (not necessarily centered, possibly missing some of the 4 branches).

**Example 1:** $Av(2413, 3142, 2143, 3412)$, a.k.a. the X-class, with critical families in **blue**.

\[
\begin{align*}
\mathcal{T}_0 & = \{•\} \uplus \mathcal{T}_1 \uplus \mathcal{T}_2 \uplus \mathcal{T}_3 \uplus \mathcal{T}_4 \uplus \mathcal{T}_5 \uplus \mathcal{T}_6 \uplus \mathcal{T}_7 \uplus \mathcal{T}_5 \\
\mathcal{T}_1 & = \{•\} \\
\mathcal{T}_2 & = \{•\} \uplus \mathcal{T}_1 \uplus \mathcal{T}_2 \\
\mathcal{T}_3 & = \mathcal{T}_1 \uplus \mathcal{T}_3 \uplus \mathcal{T}_4 \uplus \mathcal{T}_5 \uplus \mathcal{T}_1 \uplus \mathcal{T}_6 \uplus \mathcal{T}_7 \uplus \mathcal{T}_5 \\
\mathcal{T}_4 & = \mathcal{T}_1 \uplus \mathcal{T}_5 \uplus \mathcal{T}_1 \uplus \mathcal{T}_6 \uplus \mathcal{T}_7 \uplus \mathcal{T}_5 \\
\mathcal{T}_5 & = \{•\} \uplus \mathcal{T}_1 \uplus \mathcal{T}_5 \\
\mathcal{T}_6 & = \mathcal{T}_1 \uplus \mathcal{T}_2 \uplus \mathcal{T}_3 \uplus \mathcal{T}_4 \uplus \mathcal{T}_2 \uplus \mathcal{T}_1 \uplus \mathcal{T}_6 \uplus \mathcal{T}_7 \uplus \mathcal{T}_5 \\
\mathcal{T}_7 & = \mathcal{T}_1 \uplus \mathcal{T}_2 \uplus \mathcal{T}_3 \uplus \mathcal{T}_4 \uplus \mathcal{T}_2.
\end{align*}
\]
X case of the dichotomy

When the specification contains no product of critical families.

⇒ The limiting permuton of the class has a deterministic X shape (not necessarily centered, possibly missing some of the 4 branches).

Example 1: $\text{Av}(2413, 3142, 2143, 3412)$, a.k.a. the X-class, with critical families in blue.

The limit is the centered X-permuton.
X case of the dichotomy

When the specification contains **no product of critical families**.

$\Rightarrow$ The limiting permuton of the class has a **deterministic X shape** (not necessarily centered, possibly missing some of the 4 branches).

**Example 2**: $Av(2413, 3142, 2143, 34512)$, with critical families in **blue**.

$$
\begin{align*}
\mathcal{T}_0 &= \{\bullet\} \uplus [\mathcal{T}_1, \mathcal{T}_2] \uplus [\mathcal{T}_1, \mathcal{T}_3] \uplus [\mathcal{T}_4, \mathcal{T}_2] \uplus [\mathcal{T}_5, \mathcal{T}_6] \uplus [\mathcal{T}_5, \mathcal{T}_7] \uplus [\mathcal{T}_8, \mathcal{T}_6] \\
\mathcal{T}_1 &= \{\bullet\} \\
\mathcal{T}_2 &= \{\bullet\} \uplus [\mathcal{T}_1, \mathcal{T}_2] \\
\mathcal{T}_3 &= [\mathcal{T}_1, \mathcal{T}_3] \uplus [\mathcal{T}_4, \mathcal{T}_2] \uplus [\mathcal{T}_5, \mathcal{T}_6] \uplus [\mathcal{T}_5, \mathcal{T}_7] \uplus [\mathcal{T}_8, \mathcal{T}_6] \\
\mathcal{T}_4 &= [\mathcal{T}_5, \mathcal{T}_6] \uplus [\mathcal{T}_5, \mathcal{T}_7] \uplus [\mathcal{T}_8, \mathcal{T}_6] \\
\mathcal{T}_5 &= \{\bullet\} \uplus [\mathcal{T}_1, \mathcal{T}_1] \uplus [\mathcal{T}_1, \mathcal{T}_9] \uplus [\mathcal{T}_9, \mathcal{T}_1] \\
\mathcal{T}_6 &= \{\bullet\} \uplus [\mathcal{T}_1, \mathcal{T}_6] \\
\mathcal{T}_7 &= [\mathcal{T}_1, \mathcal{T}_2] \uplus [\mathcal{T}_1, \mathcal{T}_3] \uplus [\mathcal{T}_4, \mathcal{T}_2] \uplus [\mathcal{T}_6, \mathcal{T}_6] \uplus [\mathcal{T}_10, \mathcal{T}_6] \uplus [\mathcal{T}_10, \mathcal{T}_7] \uplus [\mathcal{T}_8, \mathcal{T}_6] \\
\mathcal{T}_8 &= [\mathcal{T}_1, \mathcal{T}_11] \uplus [\mathcal{T}_1, \mathcal{T}_12] \uplus [\mathcal{T}_13, \mathcal{T}_11] \uplus [\mathcal{T}_9, \mathcal{T}_11] \uplus [\mathcal{T}_{13}, \mathcal{T}_1] \\
\mathcal{T}_9 &= [\mathcal{T}_1, \mathcal{T}_6] \\
\mathcal{T}_{10} &= [\mathcal{T}_1, \mathcal{T}_1] \uplus [\mathcal{T}_1, \mathcal{T}_9] \uplus [\mathcal{T}_9, \mathcal{T}_1] \\
\mathcal{T}_{11} &= [\mathcal{T}_1, \mathcal{T}_2] \\
\mathcal{T}_{12} &= [\mathcal{T}_1, \mathcal{T}_3] \uplus [\mathcal{T}_4, \mathcal{T}_2] \uplus [\mathcal{T}_{10}, \mathcal{T}_6] \uplus [\mathcal{T}_{10}, \mathcal{T}_7] \uplus [\mathcal{T}_1, \mathcal{T}_7] \uplus [\mathcal{T}_8, \mathcal{T}_6] \\
\mathcal{T}_{13} &= [\mathcal{T}_{10}, \mathcal{T}_6] \uplus [\mathcal{T}_{10}, \mathcal{T}_7] \uplus [\mathcal{T}_1, \mathcal{T}_7] \uplus [\mathcal{T}_8, \mathcal{T}_6].
\end{align*}
$$
X case of the dichotomy

When the specification contains **no product of critical families**.

⇒ The limiting permutohn of the class has a **deterministic X shape** (not necessarily centered, possibly missing some of the 4 branches).

**Example 2:** $Av(2413, 3142, 2143, 34512)$, with critical families in **blue**.

![Example 2](image)

The limit is a non-centered X-permutohn.
X case of the dichotomy

When the specification contains no product of critical families.

⇒ The limiting permuton of the class has a deterministic X shape (not necessarily centered, possibly missing some of the 4 branches).

Example 3: $Av(2413, 1243, 2341, 531642, 41352)$, with critical families in blue.

$$
\begin{align*}
\mathcal{T}_0 &= \{\bullet\} \uplus \mathcal{T}_1 \uplus \mathcal{T}_2 \uplus \mathcal{T}_3 \uplus \mathcal{T}_4 \uplus \mathcal{T}_5 \uplus \mathcal{T}_6 \\
\mathcal{T}_1 &= \{\bullet\} \ominus \mathcal{T}_7 \times \mathcal{T}_8 \\
\mathcal{T}_2 &= \{\bullet\} \uplus \mathcal{T}_7 \times \mathcal{T}_8 \\
\mathcal{T}_3 &= \ominus \mathcal{T}_9 \times \mathcal{T}_7 \times \mathcal{T}_8 \\
\mathcal{T}_4 &= \ominus \mathcal{T}_{10} \times \mathcal{T}_{11} \ominus \mathcal{T}_{10} \times \mathcal{T}_7 \times \mathcal{T}_{11} \ominus 3142[\mathcal{T}_1 \times \mathcal{T}_1 \times \mathcal{T}_1 \times \mathcal{T}_6] \\
\mathcal{T}_5 &= \{\bullet\} \uplus \mathcal{T}_9 \times \mathcal{T}_1 \ominus 3142[\mathcal{T}_1 \times \mathcal{T}_1 \times \mathcal{T}_1 \times \mathcal{T}_1] \\
\mathcal{T}_6 &= \{\bullet\} \uplus \mathcal{T}_{10} \times \mathcal{T}_1 \ominus 3142[\mathcal{T}_1 \times \mathcal{T}_1 \times \mathcal{T}_1 \times \mathcal{T}_1] \\
\mathcal{T}_7 &= \{\bullet\} \\
\mathcal{T}_8 &= \ominus \mathcal{T}_9 \times \mathcal{T}_6 \\
\mathcal{T}_9 &= \{\bullet\} \uplus \mathcal{T}_1 \times \mathcal{T}_7 \\
\mathcal{T}_{10} &= \ominus \mathcal{T}_1 \times \mathcal{T}_1 \times \mathcal{T}_1 \times \mathcal{T}_1 \\
\mathcal{T}_{11} &= \ominus \mathcal{T}_1 \times \mathcal{T}_2 \times \mathcal{T}_3 \times \mathcal{T}_4 \times \mathcal{T}_5 \times \mathcal{T}_6 \\
\mathcal{T}_{12} &= \{\bullet\} \ominus \mathcal{T}_9 \times \mathcal{T}_6
\end{align*}$$
X case of the dichotomy

When the specification contains **no product of critical families**.

⇒ The limiting permutohedral of the class has a **deterministic X shape** (not necessarily centered, possibly missing some of the 4 branches).

**Example 3:** $\text{Av}(2413, 1243, 2341, 531642, 41352)$, with critical families in blue.

![Diagram of a degenerate X-permutohedral]

The limit is a degenerate X-permutohedral.
Concluding remarks
Towards more classes of graphs

- As in the permutation case, we can extend the study of cographs to families of graphs whose modular decomposition trees are described by a combinatorial specification.

- Our analytic approach can only work with GF having positive radius of convergence. This was necessary in permutation classes, but is an additional requirement for hereditary graph classes.

- This is the PhD topic of Théo Lenoir, who started working in September 2021, supervised by Frédérique Bassino and Lucas Gerin.

- The classes that he studied are the $P_4$-blah graphs, where blah $\in \{\text{reducible, sparse, lite, extensible, tidy}\}$. All converge to the Brownian cographon.

- Recall that cographs are $P_4$-free graphs.
An alternative proof strategy

Consider a class of graphs/permutations $\mathcal{C}$, seen as trees. For any $n$, let $G_n$ be a uniform random object of size $n$ in $\mathcal{C}$. 

**Goal:** Describe the graphon/permuton limit of $G_n$ as $n \to \infty$.

**The strategy I presented (in part):**
- Build the limiting graphon/permuton from a Brownian excursion
- Compute the densities $(\Delta_g)_g$ of substructures in it.
- $(\Delta_g)_g$ characterizes the graphon/permuton limit
- Use a combinatorial specification for $\mathcal{C}$ and analytic combinatorics to compute, for any $g$, the limiting behavior for $n \to \infty$ of the density of $g$ in $G_n$. Observe that it coincides with $\Delta_g$.

**The “random trees” strategy**
- Compute densities of induced subtrees directly on the random tree of $G_n$, using techniques from the random trees literature.
A nice consequence of permuton/graphon limits

Results:

- The size of the largest independent set of a uniform random cograph is sublinear.
  (hence $P_4$ does not have the asymptotic linear Erdős-Hajnal property, answering a question of Kang, McDiarmid, Reed and Scott in 2014)
- The length of the longest increasing subsequence of a uniform random separable permutations is sublinear.

Main proof ingredients:

- Convergence to the Brownian cographon
- The independence number of the Brownian cographon $W^{1/2}$ is 0

Bonus: The sublinearity result applies to all classes with graphon/permuton limit $W^p$ or a Brownian separable permuton.
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Bonus: The sublinearity result applies to all classes with graphon/permuton limit $W^p$ or a Brownian separable permuton.

Thank you for being there!