Limits of permutations and graphs avoiding substructures

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talk based on joint works with
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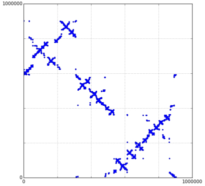
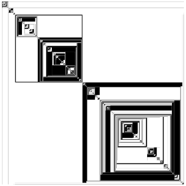
The problem:

- Consider a class $\mathcal{C}$ of permutations or graphs defined by the avoidance of substructures (patterns or induced subgraphs).
- For any n, let $\boldsymbol{\sigma}_n$ or $\boldsymbol{G}_n$ be an object of size n in $\mathcal{C}$, taken uniformly at random among objects of size n in $\mathcal{C}$.
- We would like to describe the typical global behavior of $\boldsymbol{\sigma}_n$ or $\boldsymbol{G}_n$ as n tends to ∞, through its permuton or graphon limit.



Permutation matrix of a typical large permutation avoiding 2413 and 3142



Adjacency matrix of typical large graph with no induced P_4

The proof strategy:

- Permutons and graphons describe global limits of permutations and graphs. But permuton and graphon convergence are characterized by convergence of the densities of substructures.

- Using the substitution or modular decomposition, we can represent permutations or graphs by trees (decorated on their internal nodes).
- Substructures in permutations or graphs correspond to induced subtrees in these trees (subtrees induced by a set of leaves).

- We write functional equations for the generating functions counting decomposition trees, possibly with specified induced subtrees.
- Using analytic combinatorics, we derive the limiting densities of substructures in our permutations or graphs, proving permuton or graphon convergence.

A caveat

- Only some classes of permutations or graphs are amenable to the presented strategy:
  when the substitution/modular decomposition is "nice".
- These represent very few cases in the whole landscape of permutation classes/hereditary families of graphs.
- But it still covers quite many classes compared to what was previously known (especially in the permutation case).

The talk will mainly discuss the simplest case in the graph setting: the family of cographs, avoiding an induced P_4 (the path on 4 vertices).

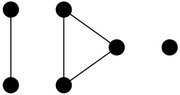
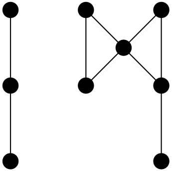
We will also discuss briefly its permutation analogue: the family of separable permutations, avoiding the patterns 2413 and 3142, as well as hint at some generalizations.

Induced subgraphs,

and a biased view of graphon convergence

Induced subgraphs and hereditary families of graphs

- g is an induced subgraph of G when

$g =$  $\preccurlyeq$ $G =$ 

In words, the subgraph of $G = (V, E)$ induced by $V' \subset V$ is the graph with vertex set V' and edge set $E \cap (V' \times V')$.

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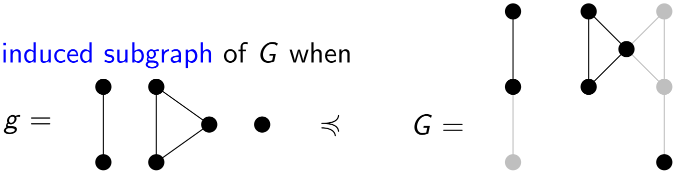

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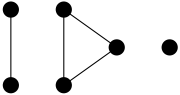
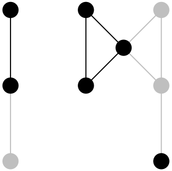
$g =$  $\preccurlyeq$ $G =$ 

In words, the subgraph of $G = (V, E)$ induced by $V' \subset V$ is the graph with vertex set V' and edge set $E \cap (V' \times V')$.

- A hereditary family of graphs is a set of graphs $\mathcal{C}$ such that for every $G \in C$, if g is an induced subgraph of G, then $g \in \mathcal{C}$.
- Examples include the families of cographs, comparability graphs, permutation graphs, circle graphs, parity graphs, ...
- Equivalently, hereditary families of graphs are characterized as sets of graphs whose induced subgraphs avoid a prescribed set (which may be finite or infinite).

Densities of induced subgraphs

- **Definition:** The density of an induced subgraph g in G is

$$\mathrm{Dens}(g, G) = \mathbb{P}(\mathrm{SubGraph}_k(G) = g)$$

where k is the number of vertices of g and $\mathrm{SubGraph}_k(G)$ is the (random) subgraph of G induced by a k-tuple of i.i.d. uniform random vertices of G.

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- **Variant:** The "injective density" is defined by

$$\mathrm{Dens}^{inj}(g, G) = \mathbb{P}(\mathrm{SubGraph}_k^{inj}(G) = g),$$

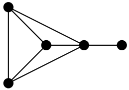
where $\mathrm{SubGraph}_k^{inj}(G)$ is the (random) subgraph of G induced by a uniform random k-tuple of distinct vertices of G.

- **Fact:** For $\boldsymbol{G}_n$ a sequence of random graphs of size tending to infinity, $\mathbb{E}[\mathrm{Dens}(g, \boldsymbol{G}_n)] \to \Delta_g$ iff $\mathbb{E}[\mathrm{Dens}^{inj}(g, \boldsymbol{G}_n)] \to \Delta_g$.

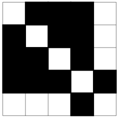
What is (informally) a graphon?

In the discrete setting:

(Unlabeled) graph G $\longrightarrow$ Adjacency matrix M_G (symmetric) $\longrightarrow$ Function $w_G : [0,1]^2 \to [0,1]$



$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



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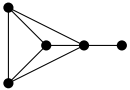

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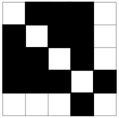

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Continuous extension:

In general, a graphon is obtained as above, from a symmetric matrix M, possibly with a continuum of rows and columns, and with values in $[0,1]$.

It is an equivalence class of symmetric functions from $[0,1]^2 \to [0,1]$ under the action of permuting rows and columns of M.

Subgraph densities in graphons

Fix g a graph with k vertices, unlabeled.

- **Reminder of the discrete case:**
  For a graph G, $\mathrm{Dens}(g, G) = \mathbb{P}(\mathrm{SubGraph}_k(G) = g)$,
  where $\mathrm{SubGraph}_k(G)$ is the (random) subgraph of G induced by a k-tuple of i.i.d. uniform random vertices of G.

- **Continuous generalization:**
  For a graphon W, $\mathrm{Dens}(g, W) = \mathbb{P}(\mathrm{Sample}_k(W) = g)$,
  where $\mathrm{Sample}_k(W)$ is the (random) graph with k vertices $v_1, \dots, v_k$ such that v_i and v_j are connected with probability $w(x_i, x_j)$, for $x_1, \dots, x_k$ i.i.d. uniform random variables in $[0,1]$ and $w : [0,1]^2 \to [0,1]$ a representative of W.

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Remark: For any graph G, $\text{Dens}(g, W_G) = \text{Dens}(g, G)$.

Characterization of (deterministic) graphon convergence

(Non-)definition:
The space of graphons is (up to technicalities) metric, for the cut-distance (and in addition is compact).

So, it makes sense to study convergence of a sequence of graphons $(W_n)_{n \geq 0}$ to a graphon W (for this cut-distance). We write $W_n \to W$.

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Typically, $W_n = W_{G_n}$, the graphon associated to a graph G_n, with the sequence of graphs (G_n) such that the size of G_n grows to infinity with n. In this case, we also write $G_n \to W$.

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Combinatorial characterization of convergence:
For (W_n) a sequence of graphons and W a graphon, $W_n \to W$ iff for any (finite) graph g, $\mathrm{Dens}(g, W_n) \to \mathrm{Dens}(g, W)$.

Characterization of graphon convergence: the random case

Reminder: $G_n \to W$ iff $\text{Dens}(g, G_n) \to \text{Dens}(g, W)$ for all g, for (G_n) a sequence of (deterministic) graphs and W a (deterministic) graphon.

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What if we take $(\boldsymbol{G_n})$ random? ($\text{Dens}(g, \boldsymbol{G_n})$ being then a real r.v.)

Theorem [Diaconis-Janson, 2008]:
The distribution of a random graphon $\boldsymbol{W}$ is characterized by all expected subgraph densities $\mathbb{E}[\text{Dens}(g, \boldsymbol{W})]$ (for all g).

Theorem [Diaconis-Janson, 2008]:
Let $(\boldsymbol{G_n})$ be a sequence of random graphs. TFAE:

- $\boldsymbol{G_n}$ tends in distribution to some random graphon, $\boldsymbol{W}$.
- For all g, $\mathbb{E}[\text{Dens}(g, \boldsymbol{G_n})]$ converges to some value $\Delta_g \in [0, 1]$.

If this holds, in addition we have:
for all g, $\mathbb{E}[\text{Dens}(g, \boldsymbol{W})] = \Delta_g$, so that $(\Delta_g)_g$ characterizes $\boldsymbol{W}$.

Summary so far

Graphs:

- Definition of induced subgraphs
- Definition of hereditary classes of graphs
- Notion of graphon as a "rescaled adjacency matrix"
- Combinatorial characterization of graphon convergence: by the convergence of the densities of induced subgraphs (in expectation in the random case)

Analogue notions for permutations:

- Induced subgraphs correspond to patterns
- Hereditary families are called permutation classes
- Notion of permuton as a "rescaled permutation matrix"
- Combinatorial characterization of permuton convergence: by the convergence of the frequencies of patterns (in expectation in the random case)

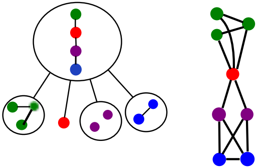
Decomposition trees

Modular decomposition of graphs

- A module in a graph $G = (V, E)$ is a set $S \subseteq V$ of vertices which cannot be distinguished by vertices outside of S:

$$\textit{for every } v \in V \setminus S, \textit{ either } \{v, s\} \in E \textit{ for all } s \in S$$
$$\textit{or } \{v, s\} \notin E \textit{ for all } s \in S$$

- Given a partition of V into modules, G can be described
  - the subgraph induced keeping exactly one vertex in each module (sometimes called quotient)
  - the subgraph induced by each module (sometimes called factors)

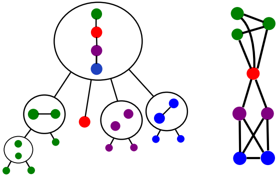


- Repeating this construction inside the modules, we obtain a modular decomposition tree of G (which is rooted, non planar, with internal vertices labeled by quotient graphs, and leaves corresponding to V).

Modular decomposition trees

- The trivial modules of G are $\emptyset$, V, and $\{v\}$ for any $v \in V$.
- A graph G is prime if it contains no non-trivial module.

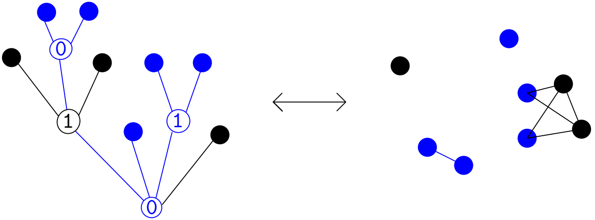
Theorem: Every graph has a unique modular decomposition tree whose vertices are either cliques (denoted 1), or independent sets (denoted 0), or prime graphs (denoted P), and with no $0-0$ nor $1-1$ edges.
We call it canonical and denote it $T(G)$.



$T(G)$ is obtained considering recursively the quotients resulting from the partition of V into maximal modules different from V (in the prime case, with special cases for cliques and independent sets).

Induced subgraphs in decomposition trees

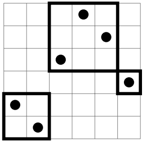
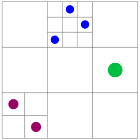
- Let G be a graph. Let S be a subset of its vertices.
- Consider the subgraph of G induced by S.
- Let $T(G)$ be its canonical modular decomposition tree.



Fact: A decomposition tree for the induced subgraph of G corresponding to S is obtained considering the subtree of $T(G)$ induced by the set of leaves corresponding to S.

Remark: The induced tree is not necessarily the canonical tree of the induced subgraph (*e.g.* it may contain $0-0$ edges).

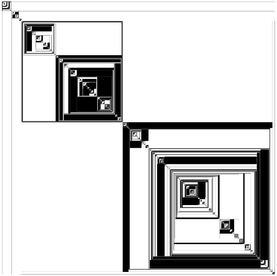
- A module in a graph corresponds to an interval in a permutation.
- A permutation can be decomposed into quotients and factors *via* the substitution decomposition.

$2\,1\,4\,6\,5\,3 =$  $=$  $= 1\,3\,2[2\,1, 1\,3\,2, 1]$

- The trees recording these decompositions are called (substitution) decomposition trees.
- There exists a unique canonical decomposition tree.
- Patterns correspond to subtrees induced by leaves.

Remark: For a permutation σ, consider its inversion graph G. Up to planarity and adapting the decorations, it holds that $T(\sigma) = T(G)$.

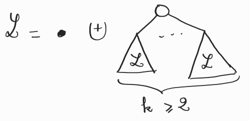
Graphon limit of (labeled) cographs



Cographs and their modular decomposition trees

- Cographs are defined by the avoidance of P_4 (induced path on 4 vertices, which is the smallest prime graph).
- Equivalently, cographs are all graphs whose modular decomposition trees involve only 0 (indep. set) and 1 (clique) nodes (no prime node). We call cotrees their modular decomposition trees.
- Therefore, labeled[1] cographs can be described from the combinatorial specification:

$$\mathcal{L} = \bullet \uplus \text{Set}_{\geq 2}(\mathcal{L}) \qquad i.e.,$$



Indeed, *via* its canonical modular decomposition tree, a cograph correspond to a tree of $\mathcal{L}$ with a label 0 or 1 on the root (propagating labels alternating between 0 and 1 along each edge).

[1] meaning vertices are labeled by the integers from 1 to n; in the unlabeled case, we need to consider Multiset, and hence later Polya operators for the GF

Expressing $\mathbb{E}[\text{Dens}^{inj}(g, \boldsymbol{G}_n)]$

Notation:
Let $\boldsymbol{G}_n$ be a uniform random labeled cograph with n vertices.

Reminder:
Knowing $\mathbb{E}[\text{Dens}^{inj}(g, \boldsymbol{G}_n)]$ for all g characterizes the graphon limit of $\boldsymbol{G}_n$.

Notation:
for all n, and all $k \leq n$,
$\boldsymbol{t}^{(n)}$ is a uniform random labeled canonical cotree of size n, and
$\boldsymbol{t}_k^{(n)}$ is the subtree of $\boldsymbol{t}^{(n)}$ induced by a uniform k-tuple of distinct leaves.

Observation:
For any cograph g, we have:

$$\mathbb{E}[\text{Dens}^{inj}(g, \boldsymbol{G}_n)] = \mathbb{P}(\text{SubGraph}_k^{inj}(\boldsymbol{G}_n) = g) = \sum \mathbb{P}(\boldsymbol{t}_k^{(n)} = t_0),$$

where the sum runs over all cotrees t_0 corresponding to g.

Expressing $\mathbb{P}(\boldsymbol{t}_k^{(n)} = t_0)$

Observation: $\mathbb{P}(\boldsymbol{t}_k^{(n)} = t_0) = \frac{n![z^n]M_{t_0}(z)}{n![z^n]M(z)\times n(n-1)\ldots(n-k+1)}$, where

- $\mathcal{M}$ is the set of labeled canonical cotrees
- for any cotree t_0 with k leaves, $\mathcal{M}_{t_0}$ is the set of labeled canonical cotrees with a marked k-tuple of distinct leaves, which induce t_0,
- $M(z)$ and $M_{t_0}(z)$ are the corresponding exponential generating functions
- as usual, $[z^n]F(z)$ denotes the coefficient of z^n in the generating function $F(z)$

Next: Use symbolic and analytic combinatorics to compute the asymptotic behavior of the numerator and the denominator in

$$\frac{n![z^n]M_{t_0}(z)}{n![z^n]M(z) \times n(n-1)\ldots(n-k+1)}.$$

Estimating the denominator

- Recall that $\mathcal{L} = \bullet \uplus \text{Set}_{\geq 2}(\mathcal{L})$.
- Let $L(z)$ be the exponential generating function of $\mathcal{L}$.
- From [Flajolet-Sedgewick] (rather a variant on trees counted by leaves), $L(z)$ satisfies $L(z) = z + \exp(L(z)) - 1 - L(z)$.
- The generating function of cographs is $M(z) = 2L(z) - z$.
- $L(z)$ and $M(z)$ have the same radius of convergence $\rho = 2\log(2) - 1$ and are Δ-analytic.
- Near $z = \rho$, $L(z) = \log(2) - \sqrt{\rho}\sqrt{1 - z/\rho} + \mathcal{O}(1 - z/\rho)$ and $M(z) = 1 - 2\sqrt{\rho}\sqrt{1 - z/\rho} + \mathcal{O}(1 - z/\rho)$.
- From the transfer theorem,

$$n(n-1)\dots(n-k+1)[z^n]M(z) \underset{n\to+\infty}{\sim} \frac{n^{k-3/2}}{\rho^{n-1/2}\sqrt{\pi}}.$$

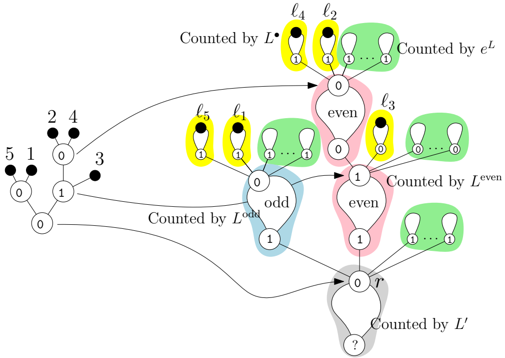
Estimating the numerator

Prop.: If t_0 with k leaves has n_v internal vertices, $n_=$ edges of the form $0-0$ or $1-1$, and $n_{\neq}$ edges of the form $0-1$ or $1-0$, then

$$M_{t_0} = (L')(\exp(L))^{n_v}(L^\bullet)^k(L^{\text{odd}})^{n_=}(L^{\text{even}})^{n_{\neq}},$$

these series being variations on $L(z)$ whose singular behavior results from that of $L(z)$.

Proof:



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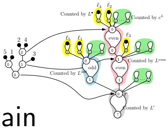

Corollary: Like before, we obtain

- the behavior at ρ of $M_{t_0}(z)$,
- and the asymptotic estimate of $[z^n]M_{t_0}(z)$.

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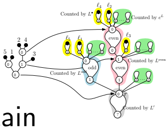

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More precisely, we have

$$[z^n]M_{t_0}(z) \underset{n\to+\infty}{\sim} \frac{(k-1)!}{(2k-2)!}\frac{n^{k-3/2}}{\rho^{n-1/2}\sqrt{\pi}},$$

if t_0 is binary (which implies $n_v = k-1$ and $n_= + n_{\neq} = k-2$).

Conclusion of the proof

Notation (reminder):

- $\boldsymbol{t}^{(n)}$: uniform random labeled canonical cotree of size n
- $\boldsymbol{t}_k^{(n)}$: subtree of $\boldsymbol{t}^{(n)}$ induced by a uniform k-tuple of distinct leaves
- t_0: cotree with k leaves

What we proved: If t_0 is binary, then $\lim_{n\to\infty} \mathbb{P}(\boldsymbol{t}_k^{(n)} = t_0) = \frac{(k-1)!}{(2k-2)!}$.

Conclusion of the proof

Notation (reminder):

- $\boldsymbol{t}^{(n)}$: uniform random labeled canonical cotree of size n
- $\boldsymbol{t}_k^{(n)}$: subtree of $\boldsymbol{t}^{(n)}$ induced by a uniform k-tuple of distinct leaves
- t_0: cotree with k leaves

What we proved: If t_0 is binary, then $\lim_{n\to\infty} \mathbb{P}(\boldsymbol{t}_k^{(n)} = t_0) = \frac{(k-1)!}{(2k-2)!}$.

Remark: $\frac{(k-1)!}{(2k-2)!} = \frac{1}{\text{number of binary cotrees with } k \text{ leaves}}$.

Consequence: If t_0 is not binary, then $\lim_{n\to\infty} \mathbb{P}(\boldsymbol{t}_k^{(n)} = t_0) = 0$.

Conclusion of the proof

Notation (reminder):

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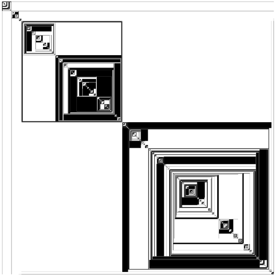
Remark/reminder:
Summing over all t_0 encoding a cograph g, this gives $\lim_{n\to\infty} \mathbb{E}[\mathrm{Dens}(g, \boldsymbol{G}_n)]$.
These quantities characterize the graphon limit of cographs.

The Brownian cographon

- Starting from a Brownian excursion, whose local minima receive unbiased decorations by 0 and 1, we can build the Brownian cographon of parameter 1/2, denoted $\boldsymbol{W}^{1/2}$.
- We can compute $\Delta_g = \mathbb{E}[\mathrm{Dens}(g, \boldsymbol{W}^{1/2})]$ for all cographs g.

But this is a story for another time...

- We observe that $\lim_{n\to\infty} \mathbb{E}[\mathrm{Dens}(g, \boldsymbol{G}_n)] = \Delta_g$ for all g.
- Therefore, not only do we prove existence of a graphon limit for uniform random cographs, but we also provide a construction of this limit.
- The limiting graphon is a genuinely random and fractal object.



From cographs to separable permutations

- Separable permutations are those avoiding the patterns 2413 and 3142 (the two smallest simple permutations).
- Equivalently, it is the family of all permutations whose decomposition trees involve only $\oplus$ and $\ominus$ nodes (no simple permutations).
- Separable permutations are therefore the permutation analogue of cographs.

- From a combinatorial specification for the decomposition trees of separable permutations, and using analytic combinatorics as before, we obtain the limiting behavior of $\mathbb{E}[\widetilde{\text{occ}}(\pi, \boldsymbol{\sigma}_n)]$ for π any pattern, and $\boldsymbol{\sigma}_n$ a uniform random separable permutation of size n. These quantities characterize the permuton limit of separable permutations.

- Again, we have an explicit construction of the limiting object $\boldsymbol{\mu}^{1/2}$ (called the Brownian separable permuton of parameter $1/2$) from a Brownian excursion with decorations.

From cographs to separable permutations

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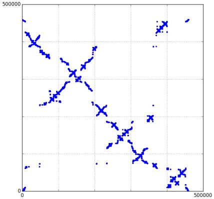
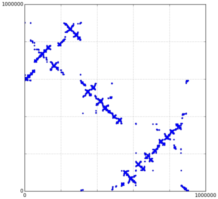




More classes of permutations

Transposing the proof strategy to a more general setting

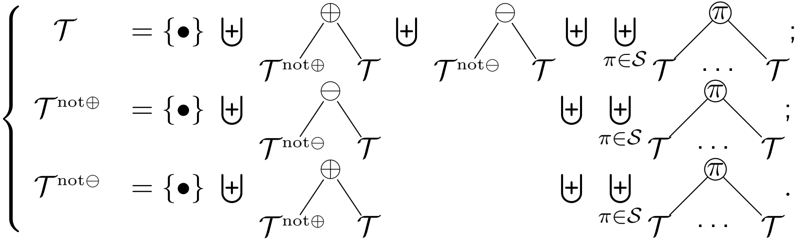
Idea of the method:

- Assume that you know a combinatorial specification for the decomposition trees of permutations in some class $\mathcal{C}$.
- It translates into a system of equations for the GF of $\mathcal{C}$.
- We can in addition "track patterns" in these equations.
- IF the method of analytic combinatorics goes through, we obtain convergence to a certain permuton, as for separable permutations.

Some results:

- Convergence to Brownian separable permutons of parameters $p \in [0, 1]$ for substitution-closed classes, under some analytic condition on the GF of the simple permutations in the class.
- Dichotomy for classes for which a specification is known (in particular: whenever they contain finitely many simple permutations): (random) Brownian permutons VS (deterministic) X-permutons.

- Their specification adds some simple permutations to that of separable permutations. We denote by $\mathcal{S}$ the set of allowed simple permutations.



Substitution-closed classes

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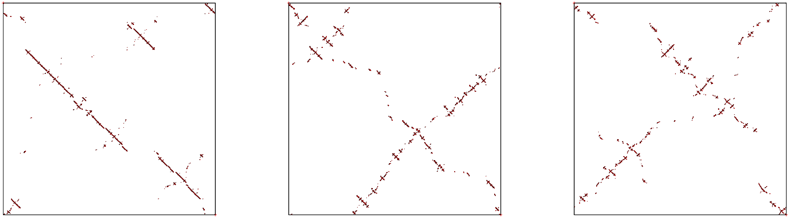
- Limit permutons are (biased) Brownian separable permutons.



simulations of $\boldsymbol{\mu}_p$ for $p = 0.2, 0.45, 0.5$.

Substitution-closed classes

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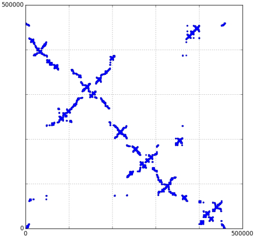
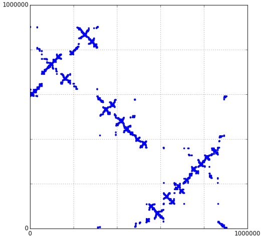
- Limit permutons are (biased) Brownian separable permutons.

Example 1: Separable permutations, *i.e.* $\mathcal{S} = \emptyset$, $\Rightarrow p = 0.5$

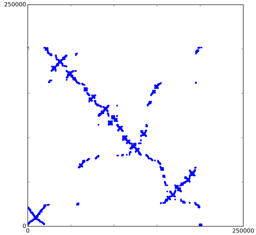


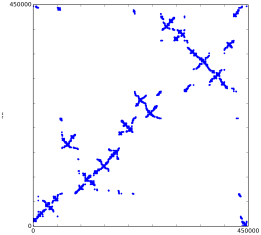


Substitution-closed classes

- Their specification adds some simple permutations to that of separable permutations. We denote by $\mathcal{S}$ the set of allowed simple permutations.
- Limit permutons are (biased) Brownian separable permutons.

Example 2: $\mathcal{S} = \{2413, 3142, 24153\}$, $\Rightarrow p = 0.5$

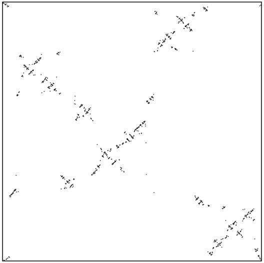




Substitution-closed classes

- Their specification adds some simple permutations to that of separable permutations. We denote by $\mathcal{S}$ the set of allowed simple permutations.
- Limit permutons are (biased) Brownian separable permutons.

Example 3: $\mathcal{S} = Av(321) \cap \{Simples\}$ (infinite), $\Rightarrow p \in [0.577, 0.622]$



Brownian case of the dichotomy

When the specification contains a product of critical families.

$\Rightarrow$ The limiting permuton of the class is a biased Brownian separable permuton (of parameter p possibly 0 or 1).

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Example 1: $Av(132)$, with critical families in blue.

$$\begin{cases} \mathcal{T} = \{\bullet\} \uplus \oplus\left[\mathcal{T}^{\text{not}\oplus}, \mathcal{T}_{\langle 21\rangle}\right] \uplus \ominus\left[\mathcal{T}^{\text{not}\ominus}, \mathcal{T}\right] \\ \mathcal{T}^{\text{not}\oplus} = \{\bullet\} \uplus \ominus\left[\mathcal{T}^{\text{not}\ominus}, \mathcal{T}\right] \\ \mathcal{T}^{\text{not}\ominus} = \{\bullet\} \uplus \oplus\left[\mathcal{T}^{\text{not}\oplus}, \mathcal{T}_{\langle 21\rangle}\right] \\ \mathcal{T}_{\langle 21\rangle} = \{\bullet\} \uplus \oplus\left[\mathcal{T}_{\langle 21\rangle}^{\text{not}\oplus}, \mathcal{T}_{\langle 21\rangle}\right] \\ \mathcal{T}_{\langle 21\rangle}^{\text{not}\oplus} = \{\bullet\}. \end{cases}$$

The limit is the Brownian separable permuton of parameter $p = 0$.

Brownian case of the dichotomy

When the specification contains a product of critical families.

$\Rightarrow$ The limiting permuton of the class is a biased Brownian separable permuton (of parameter p possibly 0 or 1).

Example 2: $Av(2413, 31452, 41253, 41352, 531246)$, with critical families in blue.

$$\begin{cases} \mathcal{T}_0 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_0] \uplus \ominus[\mathcal{T}_2, \mathcal{T}_0] \uplus 3142[\mathcal{T}_0, \mathcal{T}_3, \mathcal{T}_3, \mathcal{T}_0] \\ \mathcal{T}_1 = & \{\bullet\} \uplus \ominus[\mathcal{T}_2, \mathcal{T}_0] \uplus 3142[\mathcal{T}_0, \mathcal{T}_3, \mathcal{T}_3, \mathcal{T}_0] \\ \mathcal{T}_2 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_0] \uplus 3142[\mathcal{T}_0, \mathcal{T}_3, \mathcal{T}_3, \mathcal{T}_0] \\ \mathcal{T}_3 = & \{\bullet\} \uplus \ominus[\mathcal{T}_4, \mathcal{T}_3] \\ \mathcal{T}_4 = & \{\bullet\} \end{cases}$$

The limit is the Brownian separable permuton of parameter $p \approx 0.4748692376...$ (only real root of a certain polynomial of degree 9).

X case of the dichotomy

When the specification contains no product of critical families.

$\Rightarrow$ The limiting permuton of the class has a deterministic X shape (not necessarily centered, possibly missing some of the 4 branches).

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Example 1: $Av(2413, 3142, 2143, 3412)$, a.k.a. the X-class, with critical families in blue.

$$\begin{cases} \mathcal{T}_0 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_2] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_5] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_7, \mathcal{T}_5] \\ \mathcal{T}_1 = & \{\bullet\} \\ \mathcal{T}_2 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_2] \\ \mathcal{T}_3 = & \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_5] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_7, \mathcal{T}_5] \\ \mathcal{T}_4 = & \ominus[\mathcal{T}_1, \mathcal{T}_5] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_7, \mathcal{T}_5] \\ \mathcal{T}_5 = & \{\bullet\} \uplus \ominus[\mathcal{T}_1, \mathcal{T}_5] \\ \mathcal{T}_6 = & \oplus[\mathcal{T}_1, \mathcal{T}_2] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_7, \mathcal{T}_5] \\ \mathcal{T}_7 = & \oplus[\mathcal{T}_1, \mathcal{T}_2] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2]. \end{cases}$$

X case of the dichotomy

When the specification contains no product of critical families.

$\Rightarrow$ The limiting permuton of the class has a deterministic X shape (not necessarily centered, possibly missing some of the 4 branches).

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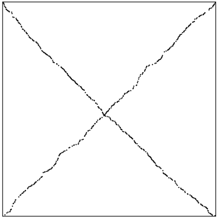

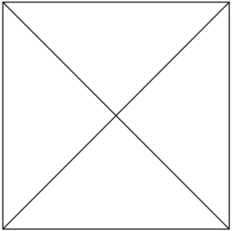

The limit is the centered X-permuton.

X case of the dichotomy

When the specification contains no product of critical families.

$\Rightarrow$ The limiting permuton of the class has a deterministic X shape (not necessarily centered, possibly missing some of the 4 branches).

Example 2: $Av(2413, 3142, 2143, 34512)$, with critical families in blue.

$$\begin{cases} \mathcal{T}_0 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_2] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_5, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_5, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_8, \mathcal{T}_6] \\ \mathcal{T}_1 = & \{\bullet\} \\ \mathcal{T}_2 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_2] \\ \mathcal{T}_3 = & \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_5, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_5, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_8, \mathcal{T}_6] \\ \mathcal{T}_4 = & \ominus[\mathcal{T}_5, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_5, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_8, \mathcal{T}_6] \\ \mathcal{T}_5 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_1] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_9] \uplus \oplus[\mathcal{T}_9, \mathcal{T}_1] \\ \mathcal{T}_6 = & \{\bullet\} \uplus \ominus[\mathcal{T}_1, \mathcal{T}_6] \\ \mathcal{T}_7 = & \oplus[\mathcal{T}_1, \mathcal{T}_2] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_{10}, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_{10}, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_8, \mathcal{T}_6] \\ \mathcal{T}_8 = & \oplus[\mathcal{T}_1, \mathcal{T}_{11}] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_{12}] \uplus \oplus[\mathcal{T}_{13}, \mathcal{T}_{11}] \uplus \oplus[\mathcal{T}_9, \mathcal{T}_{11}] \uplus \oplus[\mathcal{T}_{13}, \mathcal{T}_1] \\ \mathcal{T}_9 = & \ominus[\mathcal{T}_1, \mathcal{T}_6] \\ \mathcal{T}_{10} = & \oplus[\mathcal{T}_1, \mathcal{T}_1] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_9] \uplus \oplus[\mathcal{T}_9, \mathcal{T}_1] \\ \mathcal{T}_{11} = & \oplus[\mathcal{T}_1, \mathcal{T}_2] \\ \mathcal{T}_{12} = & \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_{10}, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_{10}, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_8, \mathcal{T}_6] \\ \mathcal{T}_{13} = & \ominus[\mathcal{T}_{10}, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_{10}, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_8, \mathcal{T}_6]. \end{cases}$$

X case of the dichotomy

When the specification contains no product of critical families.

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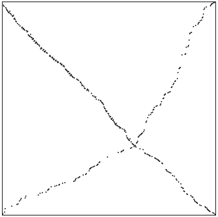

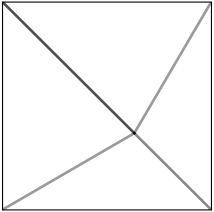

The limit is a non-centered X-permuton.

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When the specification contains no product of critical families.

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Example 3: $Av(2413, 1243, 2341, 531642, 41352)$, with critical families in blue.

$$
\begin{cases}
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\mathcal{T}_2 = & \{\bullet\} \uplus \oplus[\mathcal{T}_7, \mathcal{T}_2] \\
\mathcal{T}_3 = & \oplus[\mathcal{T}_8, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_9, \mathcal{T}_6] \\
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$$

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$\Rightarrow$ The limiting permuton of the class has a deterministic X shape (not necessarily centered, possibly missing some of the 4 branches).

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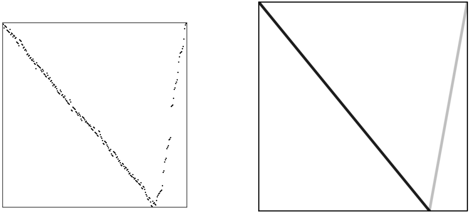

The limit is a degenerate X-permuton.

Concluding remarks

Towards more classes of graphs

- As in the permutation case, we can extend the study of cographs to families of graphs whose modular decomposition trees are described by a combinatorial specification.
- Our analytic approach can only work with GF having positive radius of convergence. This was necessary in permutation classes, but is an additional requirement for hereditary graph classes.

- This is the PhD topic of Théo Lenoir, who started working in September 2021, supervised by Frédérique Bassino and Lucas Gerin.
- The classes that he studied are the P_4-blah graphs, where blah $\in$ {reducible, sparse, lite, extensible, tidy}. All converge to the Brownian cographon.
- Recall that cographs are P_4-free graphs.

An alternative proof strategy

Consider a class of graphs/permutations $\mathcal{C}$, seen as trees.
For any n, let $\boldsymbol{G}_n$ be a uniform random object of size n in $\mathcal{C}$.
Goal: Describe the graphon/permuton limit of $\boldsymbol{G}_n$ as $n \to \infty$.

The strategy I presented (in part):

- Build the limiting graphon/permuton from a Brownian excursion
- Compute the densities $(\Delta_g)_g$ of substructures in it.
- $(\Delta_g)_g$ characterizes the graphon/permuton limit
- Use a combinatorial specification for $\mathcal{C}$ and analytic combinatorics to compute, for any g, the limiting behavior for $n \to \infty$ of the density of g in $\boldsymbol{G}_n$. Observe that it coincides with Δ_g.

The "random trees" strategy

- Compute densities of induced subtrees directly on the random tree of $\boldsymbol{G}_n$, using techniques from the random trees literature.

A nice consequence of permuton/graphon limits

Results:

- The size of the largest independent set of a uniform random cograph is sublinear.
  (hence P_4 does not have the asymptotic linear Erdős-Hajnal property, answering a question of Kang, McDiarmid, Reed and Scott in 2014)
- The length of the longest increasing subsequence of a uniform random separable permutations is sublinear.

Main proof ingredients:

- Convergence to the Brownian cographon
- The independence number of the Brownian cographon $\boldsymbol{W}^{1/2}$ is 0

Bonus: The sublinearity result applies to all classes with graphon/permuton limit $\boldsymbol{W}^p$ or a Brownian separable permuton.

A nice consequence of permuton/graphon limits

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Thank you for being there!