A general and algorithmic method for computing the generating function of permutation classes and for their random generation

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Guideline for the talk

Data:

- B a finite set of permutations (the excluded patterns),
- $\mathcal{C} = Av(B)$ the class of permutations that avoid every pattern of B.

Problem:

Describe an algorithm to obtain automatically from B

- some enumerative results on $\mathcal{C}$, in terms of generating function $C(z) = \sum |Av_n(B)| z^n$,
- a random sampler of permutations in $\mathcal{C}$, that is uniform on $Av_n(B)$ for each n.

Result:

Such an algorithm ... that works when $\mathcal{C}$ contains a finite number of simple permutations. Additional algorithms for:

- testing if $\mathcal{C}$ contains a finite number of simple permutations
- computing from B the finite set of simple permutations of $\mathcal{C}$

Outline

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Representation of permutations

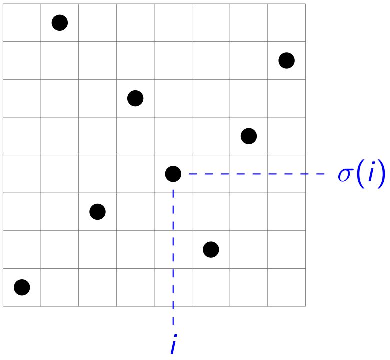
Permutation: Bijection from $[1..n]$ to itself. Set $\mathfrak{S}_n$.

- Linear representation:
  $\sigma = 1\ 8\ 3\ 6\ 4\ 2\ 5\ 7$

- Two lines representation:
  $$\sigma = \begin{pmatrix} 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8 \\ 1\ 8\ 3\ 6\ 4\ 2\ 5\ 7 \end{pmatrix}$$

- Representation as a product of cycles:
  $\sigma = (1)\ (2\ 8\ 7\ 5\ 4\ 6)\ (3)$

- Graphical representation:



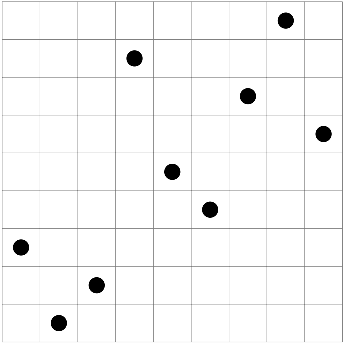
Patterns in permutations

Pattern (order) relation $\preccurlyeq$:
$\pi \in \mathfrak{S}_k$ is a pattern of $\sigma \in \mathfrak{S}_n$ if $\exists\ 1 \leq i_1 < \ldots < i_k \leq n$ such that $\sigma_{i_1} \ldots \sigma_{i_k}$ is order isomorphic ($\equiv$) to π.

Notation: $\pi \preccurlyeq \sigma$.

Equivalently:
The normalization of $\sigma_{i_1} \ldots \sigma_{i_k}$ on $[1..k]$ yields π.

Example: $2\,1\,3\,4 \preccurlyeq \mathbf{3}\,\mathbf{1}\,2\,8\,\mathbf{5}\,4\,\mathbf{7}\,9\,6$
since $3\,1\,5\,7 \equiv 2\,1\,3\,4$.



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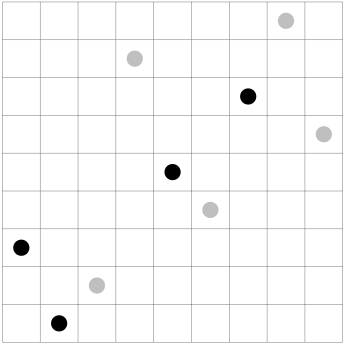

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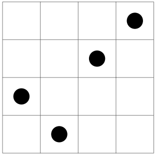

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Permutation classes

Permutation class : set of permutations downward-closed for $\preccurlyeq$.

$Av(B)$: the class of permutations that avoid every pattern of B.
If B is an antichain then B is the basis of $Av(B)$.

Conversely : Every class $\mathcal{C}$ can be characterized by its basis:

$$\mathcal{C} = Av(B) \text{ for } B = \{\sigma \notin \mathcal{C} : \forall \pi \preccurlyeq \sigma \text{ such that } \pi \neq \sigma, \pi \in \mathcal{C}\}$$

A class has a unique basis.
A basis can be either finite or infinite.

Origin : [Knuth 73] with stack-sortable permutations $= Av(231)$

Enumeration[Stanley & Wilf 92][Marcus & Tardos 04] : $|\mathcal{C} \cap \mathfrak{S}_n| \leq c^n$

Problematics

- **Combinatorics**: study of classes defined by their basis.

$\hookrightarrow$ Enumeration.

$\hookrightarrow$ Exhaustive generation.

- **Algorithmics**: problematics from text algorithmics.

$\hookrightarrow$ Pattern matching, longest common pattern.

$\hookrightarrow$ Linked with testing the membership of σ to a class.

- **Combinatorics (and algorithms)**: study families of classes.

$\hookrightarrow$ A class is not always described by its basis.

$\hookrightarrow$ Obtain general results on the structure of a class...

$\hookrightarrow$ ...and do it automatically (with algorithms).

Outline

Substitution decomposition: main ideas

Analogous to the decomposition of integers as products of primes.

- [Möhring & Radermacher 84]: general framework.
- Specialization: Modular decomposition of graphs.

Relies on:

- a principle for building objects (permutations, graphs) from smaller objects: the substitution.
- some "basic objects" for this construction: **simple** permutations, **prime** graphs.

Required properties:

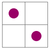
- every object can be decomposed using only "basic objects".
- this decomposition is unique.

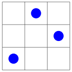
Substitution for permutations

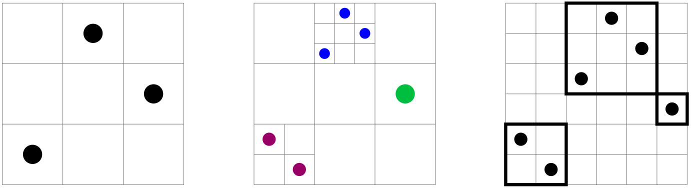
Substitution or inflation : $\sigma = \pi[\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}]$.

Example : Here, $\pi = 1\,3\,2$, and

$$\begin{cases} \alpha^{(1)} = 2\,1 = \\ \alpha^{(2)} = 1\,3\,2 = \\ \alpha^{(3)} = 1 = \end{cases}$$



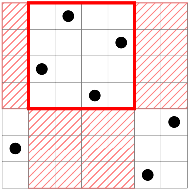




Hence $\sigma = 1\,3\,2[2\,1, 1\,3\,2, 1] = 2\,1\,4\,6\,5\,3$.

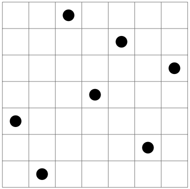
Simple permutations

Interval (or block) = set of elements of σ whose positions **and** values form intervals of integers
Example: $5\,7\,4\,6$ is an interval of $2\,5\,7\,4\,6\,1\,3$



Simple permutation = permutation that has no interval, except the trivial intervals: $1, 2, \ldots, n$ and σ
Example: $3\,1\,7\,4\,6\,2\,5$ is simple.

The smallest simple: $1\,2, 2\,1, 2\,4\,1\,3, 3\,1\,4\,2$



Substitution decomposition of permutations

Theorem: Every σ ($\neq 1$) is uniquely decomposed as

- $12[\alpha^{(1)}, \alpha^{(2)}]$, where $\alpha^{(1)}$ is $\oplus$-indecomposable
- $21[\alpha^{(1)}, \alpha^{(2)}]$, where $\alpha^{(1)}$ is $\ominus$-indecomposable
- $\pi[\alpha^{(1)}, \dots, \alpha^{(k)}]$, where π is simple of size $k \geq 4$

Remarks:

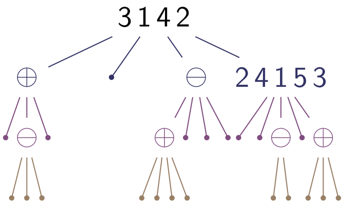
- $\oplus$-indecomposable : that cannot be written as $12[\alpha^{(1)}, \alpha^{(2)}]$
- Result stated as in [Albert & Atkinson 05]
- Can be rephrased changing the first two items into:
  - $12 \dots k[\alpha^{(1)}, \dots, \alpha^{(k)}]$, where the $\alpha^{(i)}$ are $\oplus$-indecomposable
  - $k \dots 21[\alpha^{(1)}, \dots, \alpha^{(k)}]$, where the $\alpha^{(i)}$ are $\ominus$-indecomposable

Decomposing recursively inside the $\alpha^{(i)}$ $\Rightarrow$ **decomposition tree**

Decomposition tree: witness of this decomposition

Example: Decomposition tree of $\sigma =$

10 13 12 11 14 1 18 19 20 21 17 16 15 4 8 3 2 9 5 6 7



Notations and properties:

- $\oplus = 12\dots k$ and $\ominus = k\dots 21$ = linear nodes.
- π simple of size ≥ 4 = prime node.
- No edge $\oplus - \oplus$ nor $\ominus - \ominus$.
- Ordered trees.

Expansion of $\oplus[T_1, T_2, T_3, \dots]$ into $\oplus[T_1, \oplus[T_2, T_3, \dots]]$ and recursively, for the version of the trees of [AA05]

$$\sigma = 3\,1\,4\,2[\oplus[1, \ominus[1, 1, 1], 1], 1, \ominus[\oplus[1, 1, 1, 1], 1, 1, 1], 2\,4\,1\,5\,3[1, 1, \ominus[1, 1], 1, \oplus[1, 1, 1]]]$$

Bijection between permutations and their decomposition trees.

Computation and examples of application

Computation: in linear time. [Uno & Yagiura 00] [Bui Xuan, Habib & Paul 05] [Bergeron, Chauve, Montgolfier & Raffinot 08]

In algorithms:

- Pattern matching [Bose, Buss & Lubiw 98] [Ibarra 97]
- Algorithms for bio-informatics [Bérard, Bergeron, Chauve & Paul 07] [Bérard, Chateau, Chauve, Paul & Tannier 08]

In combinatorics:

- Simple permutations [Albert, Atkinson & Klazar 03]
- Classes closed by substitution product [Atkinson & Stitt 02] [Brignall 07] [Atkinson, Ruškuc & Smith 09]
- Exhibit the structure of classes [Albert & Atkinson 05] [Brignall, Huczynska & Vatter 08a,08b] [Brignall, Ruškuc & Vatter 08]

Outline

Combinatorial classes and generating functions

Notations:

- $\mathcal{C} = \cup_{n\geq 0}\mathcal{C}_n$ with finite number $c_n = |\mathcal{C}_n|$ of objects of size n
- Generating function $C(z) = \sum c_n z^n$

Recursive description with constructors $\Rightarrow$ Equation on the g.f.:

Constructor	Notation	$C(z)$
Atom	$\mathcal{Z}$	z
Disjoint Union	$\mathcal{A} + \mathcal{B}$	$A(z) + B(z)$
Cartesian Product	$\mathcal{A} \times \mathcal{B}$	$A(z)B(z)$
Sequence	$\textsc{Seq}(\mathcal{A})$	$\dfrac{1}{1 - A(z)}$
Restricted Seq.	$\textsc{Seq}_{=k}(\mathcal{A})$	$A(z)^k$

[Flajolet & Sedgewick 09]

Combinatorial classes and random samplers

Uniform sampling: objects of size n have the same probability

Two methods based on the recursive description of objects:

- Recursive method [Flajolet, Zimmerman & Van Cutsem 94]: size n chosen in advance. Requires to know the c_k for $k \leq n$.
- **Boltzmann method** [Duchon, Flajolet, Louchard & Schaeffer 04]: size n not fixed. Needs the evaluation of $C(z)$ at one point x.

$\mathcal{Z}$	return an atom
$\mathcal{A}+\mathcal{B}$	call $\Gamma A(x)$ with proba. $\frac{A(x)}{A(x)+B(x)}$, else $\Gamma B(x)$
$\mathcal{A}\times\mathcal{B}$	call $\Gamma A(x)$ and $\Gamma B(x)$
$\textsc{Seq}(\mathcal{A})$	choose k according to a geometric law of parameter $A(x)$ and call $\Gamma A(x)$ k times
$\textsc{Seq}_{=k}(\mathcal{A})$	call the sampler $\Gamma A(x)$ k times

Example: binary trees

$\mathcal{B} = \cup_{n\geq 1}\mathcal{B}_n$
where $\mathcal{B}_n$ denotes the set of binary trees with n leaves.

Recursive description (also called specification): $\mathcal{B} = \bullet \quad + \quad \mathcal{B} \quad \mathcal{B}$

Equation for the g.f.: $B(z) = z + B(z)^2$, hence $B(z) = \frac{1-\sqrt{1-4z}}{2}$.

Boltzmann random sampler $\Gamma\mathcal{B}(x)$ for $\mathcal{B}$:

- Data: x, $B(x)$
- Result: a random binary tree
- Procedure:
  - Choose r uniformly at random on $[0,1]$
  - If $r < \frac{x}{B(x)}$ then return $\bullet$
  - Else return $\Gamma\mathcal{B}(x)\,\Gamma\mathcal{B}(x)$

Specifications for permutation classes

For all permutations, with $\mathcal{S}$ the set of all simple permutations:

$$\begin{cases} \mathfrak{S} = \bullet + \overset{\oplus}{\mathfrak{S}^+ \;\; \mathfrak{S}} + \overset{\ominus}{\mathfrak{S}^- \;\; \mathfrak{S}} + \sum_{\pi \in \mathcal{S}} \overset{\pi}{\mathfrak{S} \;\; \mathfrak{S} \;\; \cdots \;\; \mathfrak{S}} \\ \mathfrak{S}^+ = \bullet + \overset{\ominus}{\mathfrak{S}^- \;\; \mathfrak{S}} + \sum_{\pi \in \mathcal{S}} \overset{\pi}{\mathfrak{S} \;\; \mathfrak{S} \;\; \cdots \;\; \mathfrak{S}} \\ \mathfrak{S}^- = \bullet + \overset{\oplus}{\mathfrak{S}^+ \;\; \mathfrak{S}} + \sum_{\pi \in \mathcal{S}} \overset{\pi}{\mathfrak{S} \;\; \mathfrak{S} \;\; \cdots \;\; \mathfrak{S}} \end{cases}$$

$\Rightarrow$The generating functions of $\mathfrak{S}$ and $\mathcal{S}$ are related [Albert, Atkinson & Klazar 03].

This can be adapted to (substitution-closed and arbitrary) permutation classes [Albert & Atkinson 05].

The simpler case of substitution-closed classes

A permutation class $\mathcal{C}$ is **substitution-closed** when $\pi[\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(k)}] \in \mathcal{C}$ for all $\pi, \alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(k)} \in \mathcal{C}$.

Hence, with $\mathcal{S}_\mathcal{C} = \mathcal{C} \cap \mathcal{S}$ the set of simple permutations in $\mathcal{C}$:

$$\begin{cases} \mathcal{C} = \bullet + \overset{\oplus}{\mathcal{C}^+ \quad \mathcal{C}} + \overset{\ominus}{\mathcal{C}^- \quad \mathcal{C}} + \sum_{\pi \in \mathcal{S}_\mathcal{C}} \overset{\pi}{\mathcal{C} \quad \mathcal{C} \quad \dots \quad \mathcal{C}} \\ \dots \end{cases}$$

When $\mathcal{S}_\mathcal{C}$ is finite, this is a simple family of trees in the sense of [Flajolet & Sedgewick 09].

$\Rightarrow$Enumerative results and random samplers can be obtained by efficient algorithms.

For general permutation classes

For non substitution-closed classes, we have only a strict inclusion:

$$\mathcal{C} \subsetneq \bullet + \overset{\oplus}{\mathcal{C}^+ \quad \mathcal{C}} + \overset{\ominus}{\mathcal{C}^- \quad \mathcal{C}} + \sum_{\pi \in \mathcal{S}_{\mathcal{C}}} \overset{\pi}{\mathcal{C} \quad \mathcal{C} \quad \cdots \quad \mathcal{C}}$$

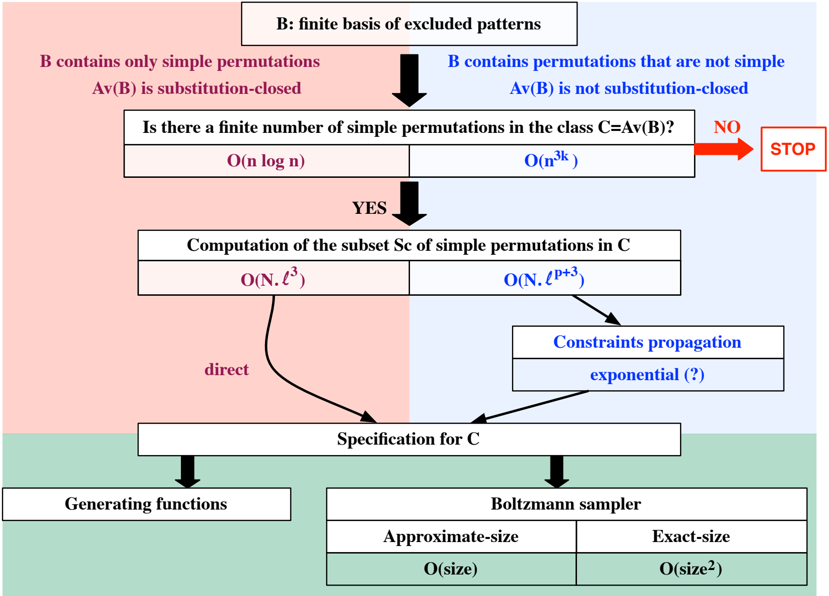
Example: $\ominus[12, 1] \notin Av(231)$ whereas $12, 1 \in Av(231)$.

The system describing $\mathcal{C}$ has to be refined with new equations for these constraints. The system can be computed by an algorithm.

$\Rightarrow$Enumerative results and random samplers can be obtained algorithmically, but this is less efficient.

Outline

Summary of the overall procedure



From the simple permutations to the specification for $\mathcal{C}$

From the set $\mathcal{S}_\mathcal{C}$ of simple permutations in $\mathcal{C}$, the specification for the substitution closure $\hat{\mathcal{C}}$ of $\mathcal{C}$ is obtained immediately:

$$\begin{cases} \hat{\mathcal{C}} = \bullet + \oplus[\hat{\mathcal{C}}^+, \hat{\mathcal{C}}] + \ominus[\hat{\mathcal{C}}^-, \hat{\mathcal{C}}] + \sum_{\pi\in\mathcal{S}_\mathcal{C}} \pi[\hat{\mathcal{C}}, \hat{\mathcal{C}}, \dots, \hat{\mathcal{C}}] \\ \hat{\mathcal{C}}^+ = \bullet + \ominus[\hat{\mathcal{C}}^-, \hat{\mathcal{C}}] + \sum_{\pi\in\mathcal{S}_\mathcal{C}} \pi[\hat{\mathcal{C}}, \hat{\mathcal{C}}, \dots, \hat{\mathcal{C}}] \\ \hat{\mathcal{C}}^- = \bullet + \oplus[\hat{\mathcal{C}}^+, \hat{\mathcal{C}}] + \sum_{\pi\in\mathcal{S}_\mathcal{C}} \pi[\hat{\mathcal{C}}, \hat{\mathcal{C}}, \dots, \hat{\mathcal{C}}] \end{cases}$$

- If $\mathcal{C}$ is substitution-closed, $\mathcal{C} = \hat{\mathcal{C}}$ and we are done.
- Otherwise, $\mathcal{C} = \hat{\mathcal{C}}\langle B^\star\rangle$ and propagate the constraints from $B^\star = \{\beta \in B \ : \ \beta \text{ is not simple } \}$ into the subtrees.

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Pushing constraints into the subtrees

Embeddings of $\beta \in B^{\star}$ into $\pi \in \mathcal{S}_{\mathcal{C}} \cup \{12, 21\}$

- Example:
  for $\ominus[\mathcal{C}^{-}, \mathcal{C}]\langle 3412 \rangle$, there are 3 embeddings of 3412 into 21, $(34, 12) \hookrightarrow (2, 1)$ and the trivial ones $(3412, \emptyset) \hookrightarrow (2, 1)$ and $(\emptyset, 3412) \hookrightarrow (2, 1)$.
- additional restrictions α in $B^{?}$ that are blocks of $\beta \in B^{\star}$
- and do it inductively while new constraints α appear
- this terminates since each $\alpha \preccurlyeq \beta$ for some $\beta \in B^{\star}$

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Result: A system describing $\mathcal{C}$, that may be ambiguous

An ambiguous system

Example: At the first step for $\ominus[\mathcal{C}^-, \mathcal{C}]\langle 3412\rangle$, we get:

$$\ominus[\hat{\mathcal{C}}^-, \hat{\mathcal{C}}\langle 3412\rangle] = \ominus[\hat{\mathcal{C}}^-\langle 3412\rangle, \hat{\mathcal{C}}] \cap \ominus[\hat{\mathcal{C}}^-, \hat{\mathcal{C}}\langle 3412\rangle] \cap \left(\ominus[\hat{\mathcal{C}}^-\langle 12\rangle, \hat{\mathcal{C}}] \cup \ominus[\hat{\mathcal{C}}^-, \hat{\mathcal{C}}\langle 12\rangle]\right)$$

$$= \ominus[\hat{\mathcal{C}}^-\langle 12\rangle, \hat{\mathcal{C}}\langle 3412\rangle] \cup \ominus[\hat{\mathcal{C}}^-\langle 3412\rangle, \hat{\mathcal{C}}\langle 12\rangle]$$

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$$= \ominus[\hat{\mathcal{C}}^-\langle 12\rangle, \hat{\mathcal{C}}\langle 3412\rangle] \cup \ominus[\hat{\mathcal{C}}^-\langle 3412\rangle, \hat{\mathcal{C}}\langle 12\rangle]$$

Rem. 1 The new excluded pattern 12 appears, and this new constraint should be further pushed into the substrees.

Rem. 2 The two terms of the union have a non-empty intersection $\Rightarrow$ Need of disambiguation.

Disambiguation of the system

- Use formulas of the type $A \cup B = A \cap B \;\uplus\; \overline{A} \cap B \;\uplus\; A \cap \overline{B}$
- In complement set, excluded patterns become mandatory patterns: $\mathcal{C}_{\gamma}$ for $\gamma \preccurlyeq \beta \in B^{\star}$
- Propagate also mandatory restrictions

Example: From $\ominus\big(\hat{\mathcal{C}}^{-},\ \hat{\mathcal{C}}\langle 3412\rangle\big) = \ominus\big(\hat{\mathcal{C}}^{-}\langle 12\rangle,\ \hat{\mathcal{C}}\langle 3412\rangle\big) \cup \ominus\big(\hat{\mathcal{C}}^{-}\langle 3412\rangle,\ \hat{\mathcal{C}}\langle 12\rangle\big)$,
we obtain:

$$\ominus\big(\hat{\mathcal{C}}^{-},\ \hat{\mathcal{C}}\langle 3412\rangle\big) = \ominus\big(\hat{\mathcal{C}}^{-}\langle 12\rangle,\ \hat{\mathcal{C}}\langle 12\rangle\big) \uplus \ominus\big(\hat{\mathcal{C}}^{-}_{12}\langle 3412\rangle,\ \hat{\mathcal{C}}\langle 12\rangle\big) \uplus \ominus\big(\hat{\mathcal{C}}^{-}\langle 12\rangle,\ \hat{\mathcal{C}}_{12}\langle 3412\rangle\big).$$

Notice that the terms $\ominus\big(\hat{\mathcal{C}}^{-}\langle 3412\rangle,\ \hat{\mathcal{C}}_{3412}\langle 12\rangle\big)$ and $\ominus\big(\hat{\mathcal{C}}^{-}_{3412}\langle 12\rangle,\ \hat{\mathcal{C}}\langle 3412\rangle\big)$ are empty, and have been deleted.

Disambiguation of the system

Result: An unambiguous system (i.e. a combinatorial specification) describing $\mathcal{C}$, where the left-hand-sides are $\mathcal{C}^{\varepsilon}_{\gamma_1,\ldots,\gamma_p}\langle\alpha_1,\ldots,\alpha_k\rangle$ with $\varepsilon \in \{\ \ ,+,-\}$.

Termination: all α_i and γ_j are patterns of some $\beta \in B^\star$

Theorem: The propagation of the constraints to obtain a specification for $\mathcal{C}$ is algorithmic, but there is an explosion of the number of equations in the system.

Open question: provide bounds on the number of equations of the system produced.

Outline

What next?

About the algorithm:

- Implementation in progress
- Complexity analysis of step 3 (explosion of the system)
- Dependency of the complexity of Boltzmann random samplers w.r.t. the size of the specification

With the algorithm:

- From the specifications, estimate growth rates of classes
- Are random permutations in $\mathcal{C}$ “like” in $\mathfrak{S}$?
- Compare statistics on $\mathcal{C}$ and $\mathfrak{S}$, or on $\mathcal{C}_1$ and $\mathcal{C}_2$

Related questions:

- How general is our algorithm?
- Classes with infinite set of simples, but finitely described?
- Use specification of a class to decide membership efficiently?

Almost 30 000 permutations of size 500 in $Av(2413, 1243, 2341, 531642, 41352)$