Limits of constrained permutations and graphs *via* decomposition trees

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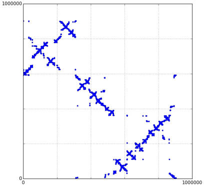
talk based on joint works with
Frédérique Bassino, Jacopo Borga, Valentin Féray,
Lucas Gerin, Michael Drmota, Mickaël Maazoun,
Adeline Pierrot and Benedikt Stufler

Permutation Patterns, June 2022, Valparaiso University.

The problem:

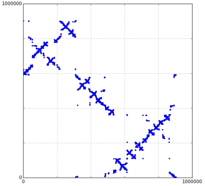
- Consider a class $\mathcal{C}$ of permutations defined by the avoidance of substructures (patterns).
- For any n, let $\boldsymbol{\sigma}_n$ be an object of size n in $\mathcal{C}$, taken uniformly at random among objects of size n in $\mathcal{C}$.
- We would like to describe the typical global behavior of $\boldsymbol{\sigma}_n$ as n tends to ∞, through its permuton limit.



Permutation diagram of a typical large permutation avoiding 2413 and 3142

The problem:

- Consider a class $\mathcal{C}$ of permutations or graphs defined by the avoidance of substructures (patterns or induced subgraphs).
- For any n, let $\boldsymbol{\sigma}_n$ or $\boldsymbol{G}_n$ be an object of size n in $\mathcal{C}$, taken uniformly at random among objects of size n in $\mathcal{C}$.
- We would like to describe the typical global behavior of $\boldsymbol{\sigma}_n$ or $\boldsymbol{G}_n$ as n tends to ∞, through its permuton or graphon limit.



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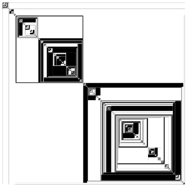


Adjacency matrix of typical large graph with no induced P_4

Plan for the talk (2/2)

The proof strategy:

- Permutons and graphons describe global limits of permutations and graphs. But permuton and graphon convergence are characterized by convergence of the densities of substructures.

- Using the substitution or modular decomposition, we can represent permutations or graphs by trees (decorated on their internal nodes).
- Substructures in permutations or graphs correspond to induced subtrees in these trees (subtrees induced by a set of leaves).

- We write functional equations for the generating functions counting decomposition trees, possibly with specified induced subtrees.
- Using analytic combinatorics, we derive the limiting densities of substructures in our permutations or graphs, proving permuton or graphon convergence.

A caveat

- Only some classes of permutations or graphs are amenable to the presented strategy:
  when the substitution/modular decomposition is "nice".
- These represent very few cases in the whole landscape of permutation classes/hereditary families of graphs.
- But it still covers quite many classes compared to what was previously known (especially in the permutation case).

The talk will mainly discuss the simplest case in the graph setting: the family of cographs, avoiding an induced P_4 (the path on 4 vertices).

We will also discuss its permutation analogue: the family of separable permutations[1], avoiding the patterns 2413 and 3142, as well as hint at some generalizations.

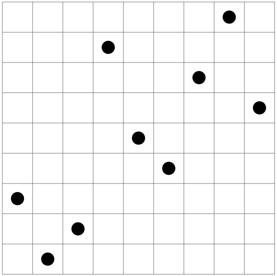
[1] see also the invited talk of Lucas Gerin at PP 2021 on-line.

Patterns in permutations,
and a biased view of permutons;

and their graph analogues:
induced subgraphs,
and a more biased view of graphons

You all know... but just to be on the same page

A permutation $\sigma = \sigma(1) \ldots \sigma(n)$ of size n is a bijection from $\{1, 2, \ldots, n\}$ to itself, which we represent by its permutation matrix, or diagram.



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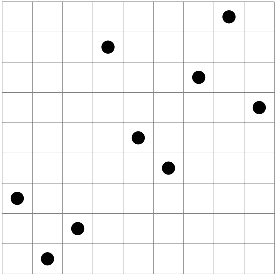

A permutation π of size k is a pattern of a permutation σ of size n if there exist $1 \leq i_1 < \ldots < i_k \leq n$ such that $\sigma(i_1) \ldots \sigma(i_k)$ is in the same relative order ($\equiv$) as π.

Example: 2 1 3 4 is a pattern of **3 1** 2 8 **5** 4 **7** 9 6 since 3 1 5 7 $\equiv$ 2 1 3 4.

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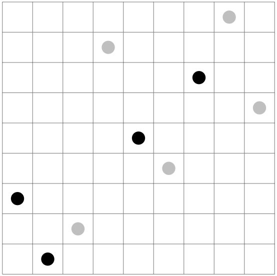

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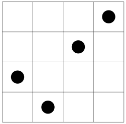

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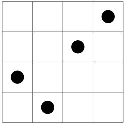

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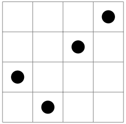

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Permutation classes are sets of permutations closed downwards for the pattern partial order relation.

They are equivalently characterized by the avoidance of patterns.

Permutons

A permuton is a probability measure on the unit square with uniform marginals (or projections),
i.e. the total mass on any vertical or horizontal strip of width x is x.

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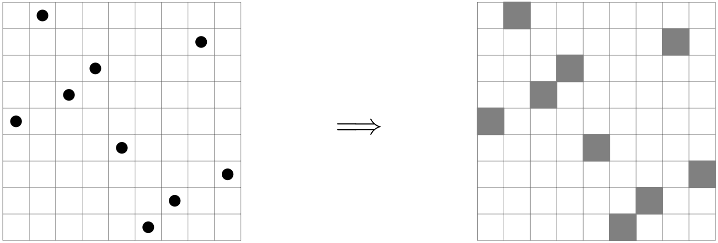

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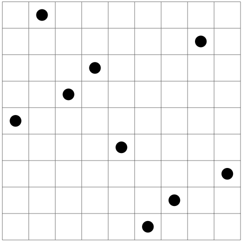
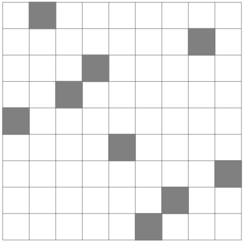


$\Longrightarrow$



Informally, permuton can represent permutations of finite size, but also "permutations of infinite size".

Defining permuton convergence

- Consider (σ_n) a sequence of permutations of size growing with n (typically $|\sigma_n| = n$).

  How can we describe the limit of σ_n as $n \to \infty$ (if it exists)?

- Consider the permutons (μ_{σ_n}) associated with this sequence.

  These are measures on the unit square.

  So, using the weak convergence of measures, it makes sense to speak about the limit of (μ_{σ_n}).

- We define that (σ_n) converges to the permuton μ when (μ_{σ_n}) converges to μ.

This extends to sequences of random permutations $(\boldsymbol{\sigma}_n)$, converging (in distribution) to a (*a priori* random) permuton $\boldsymbol{\mu}$.

Characterizing permuton convergence

Theorem: Permuton convergence is characterized by the convergence of frequencies of all patterns π (of all sizes).

- $\widetilde{\text{occ}}(\pi, \sigma_n)$ = frequency of occurrence of the pattern π in σ_n
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- These are defined as follows, with $|\pi| = k$:
  - $\widetilde{\text{occ}}(\pi, \sigma_n)$ = probability that k points picked uniformly at random in σ_n form an occurrence of the pattern $\pi = \frac{\text{number of occurrences of } \pi \text{ in } \sigma_n}{\binom{|\sigma_n|}{k}}$.
  - $\widetilde{\text{occ}}(\pi, \mu)$ = the probability that k points of the unit square picked at random according to μ induce the pattern π.

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  - $\widetilde{\text{occ}}(\pi, \mu)$ = the probability that k points of the unit square picked at random according to μ induce the pattern π.

Extension to the random setting:

- $(\boldsymbol{\sigma}_n)$ converges in distribution to some permuton $\boldsymbol{\mu} \Leftrightarrow$ for every pattern π, $\mathbb{E}[\widetilde{\text{occ}}(\pi, \boldsymbol{\sigma}_n)]$ converges to some value $\Delta_\pi \in [0, 1]$.
- If this holds, $\mathbb{E}[\widetilde{\text{occ}}(\pi, \boldsymbol{\mu})] = \Delta_\pi$ for all π and $(\Delta_\pi)_\pi$ characterizes $\boldsymbol{\mu}$.

Summary so far

Notions about permutations which we have seen:

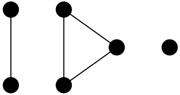
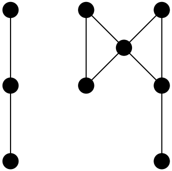
- Definition of patterns and of permutation classes
- Notion of permuton as a “rescaled diagram”
- Combinatorial characterization of permuton convergence: by the convergence of the frequencies of patterns (in expectation in the random case)

Graph analogues which we now discuss:

- Definition of induced subgraphs and of hereditary classes of graphs
- Notion of graphon as a “rescaled adjacency matrix”
- Combinatorial characterization of graphon convergence: by the convergence of the densities of induced subgraphs (in expectation in the random case)

Induced subgraphs and hereditary families of graphs

- g is an induced subgraph of G when

$g =$  $\preccurlyeq$ $G =$ 

In words, the subgraph of $G = (V, E)$ induced by $V' \subset V$ is the graph with vertex set V' and edge set $E \cap (V' \times V')$.

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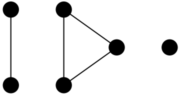
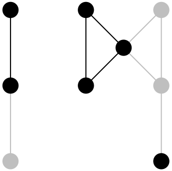
$g =$  $\preccurlyeq$ $G =$ 

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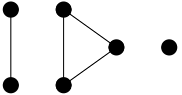
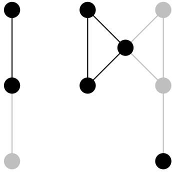
$g =$  $\preccurlyeq$ $G =$ 

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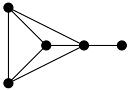
- A hereditary family of graphs is a set of graphs $\mathcal{C}$ such that for every $G \in C$, if g is an induced subgraph of G, then $g \in \mathcal{C}$.
- Equivalently, hereditary families of graphs are characterized as sets of graphs whose induced subgraphs avoid a prescribed set (which may be finite or infinite).
- Examples include the families of cographs, comparability graphs, permutation graphs, circle graphs, parity graphs, ...

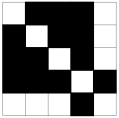
What is (informally) a graphon?

In the discrete setting:

(Unlabeled) graph G $\longrightarrow$ Adjacency matrix M_G (symmetric) $\longrightarrow$ Function $w_G : [0,1]^2 \to [0,1]$



$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



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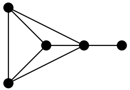
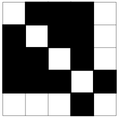
		$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$		

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Continuous extension:

In general, a graphon is obtained as above, from a symmetric matrix M, possibly with a continuum of rows and columns, and with values in $[0,1]$.

It is an equivalence class of symmetric functions from $[0,1]^2 \to [0,1]$ under the action of permuting rows and columns of M.

Subgraph densities in graphs and graphons

Fix g a graph with k vertices, unlabeled.

- **Definition in the discrete case:**
  For a graph G, $\mathrm{Dens}(g, G) = \mathbb{P}(\mathrm{SubGraph}_k(G) = g)$,
  where $\mathrm{SubGraph}_k(G)$ is the (random) subgraph of G
  induced by a k-tuple of i.i.d. uniform random vertices of G
  (or a uniform random k-tuple of distinct vertices).

- **Continuous generalization:**
  For a graphon W, $\mathrm{Dens}(g, W) = \mathbb{P}(\mathrm{Sample}_k(W) = g)$,
  where $\mathrm{Sample}_k(W)$ is the (random) graph with k vertices $v_1, \dots, v_k$
  such that v_i and v_j are connected with probability $w(x_i, x_j)$,
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Remark: For any graph G, $\mathrm{Dens}(g, W_G) = \mathrm{Dens}(g, G)$.

Characterization of graphon convergence

- The space of graphons is (up to technicalities) metric, for the cut-distance (and in addition is compact).
- So, we can define convergence of a sequence of graphons $(W_n)_{n\geq 0}$ to a graphon W, written $W_n \to W$.
- Typically, $W_n = W_{G_n}$ for a sequence of graphs (G_n) with $|G_n| \to \infty$ with n and we write $G_n \to W$.

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Combinatorial characterization of convergence:
$G_n \to W$ iff for any (finite) graph g, $\text{Dens}(g, G_n) \to \text{Dens}(g, W)$.

In the random setting, with $(\boldsymbol{G_n})$ a sequence of random graphs:

- $\boldsymbol{G_n}$ tends in distribution to some random graphon $\boldsymbol{W}$ iff for all g, $\mathbb{E}[\text{Dens}(g, \boldsymbol{G_n})]$ converges to some value $\Delta_g \in [0, 1]$.
- If this holds, in addition we have:
for all g, $\mathbb{E}[\text{Dens}(g, \boldsymbol{W})] = \Delta_g$, so that $(\Delta_g)_g$ characterizes $\boldsymbol{W}$.

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Graph analogues which we have seen next:

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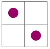
Decomposition trees

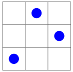
Substitution for permutations

Substitution is an operation building a permutation from smaller ones.

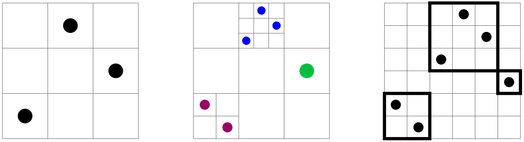
Notation for **substitution** (or inflation): $\sigma = \pi[\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(k)}]$ with k = size of π.

Example: Here, $\pi = 1\,3\,2$, and

$\alpha^{(1)} = 2\,1 =$ 

$\alpha^{(2)} = 1\,3\,2 =$ 

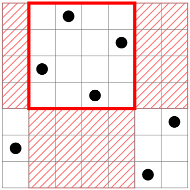
$\alpha^{(3)} = 1 =$  .



Hence $\sigma = 1\,3\,2[2\,1, 1\,3\,2, 1] = 2\,1\,4\,6\,5\,3$.

Many substitutions give σ, but there is a canonical one.

Substitution decomposition



Definitions:

- An interval of σ is a factor $\sigma_i \dots \sigma_j$ whose value form an interval of integers.
- A simple permutation is a permutation containing only trivial intervals (which are $\emptyset$, σ_i, $\sigma_1 \dots \sigma_n$).
- π-indecomposable = which cannot be written as $\pi[\dots]$.
- $\oplus$ (resp. $\ominus$) represents $12 \dots k$ (resp. $k \dots 21$) for some/all k.

Theorem: Every σ ($\neq 1$) is uniquely decomposed as

- $\oplus[\alpha^{(1)}, \dots, \alpha^{(k)}]$, where the $\alpha^{(i)}$ are $\oplus$-indecomposable, or
- $\ominus[\alpha^{(1)}, \dots, \alpha^{(k)}]$, where the $\alpha^{(i)}$ are $\ominus$-indecomposable, or
- $\pi[\alpha^{(1)}, \dots, \alpha^{(k)}]$, where π is simple of size $k \geq 4$

Proof idea: The $\alpha^{(i)}$ represent the maximal proper intervals of σ.

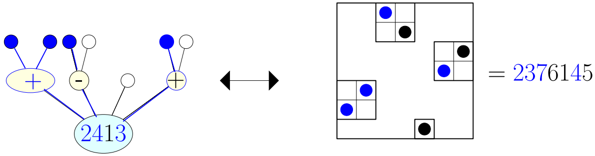
Substitution decomposition trees, and patterns

Decomposing recursively inside the α_i, we obtain:

Consequence: Every permutation admits a unique (substitution) decomposition tree whose internal vertices are labeled by $\oplus$, $\ominus$ and simple permutations, with no $\oplus - \oplus$ nor $\ominus - \ominus$ edges. We call it canonical and denote it $T(\sigma)$.

Remarks:

- The leaves of $T(\sigma)$ correspond to the elements of σ.
- Patterns in σ correspond to subtrees of $T(\sigma)$ induced by leaves (although not giving canonical trees in general).

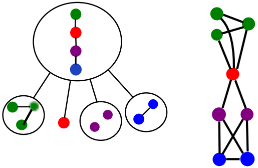


Modular decomposition of graphs

- A module in a graph $G = (V, E)$ is a set $S \subseteq V$ of vertices which cannot be distinguished by vertices outside of S:

$$\textit{for every } v \in V \setminus S, \textit{ either } \{v, s\} \in E \textit{ for all } s \in S$$
$$\textit{or } \{v, s\} \notin E \textit{ for all } s \in S$$

- Given a partition of V into modules, G can be described
  - the subgraph induced keeping exactly one vertex in each module (sometimes called quotient)
  - the subgraph induced by each module (sometimes called factors)

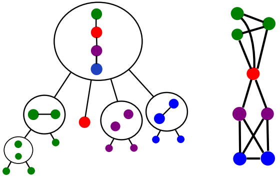


- Repeating this construction inside the modules, we obtain a modular decomposition tree of G (which is rooted, non planar, with internal vertices labeled by quotient graphs, and leaves corresponding to V).

Modular decomposition trees

- The trivial modules of G are $\emptyset$, V, and $\{v\}$ for any $v \in V$.
- A graph G is prime if it contains no non-trivial module.

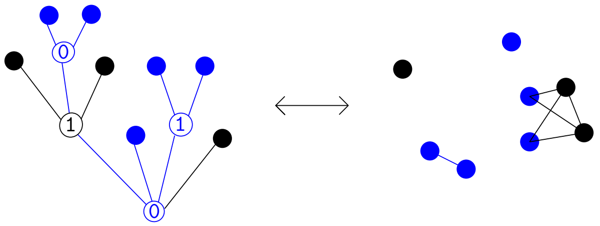
Theorem: Every graph has a unique modular decomposition tree whose vertices are either cliques (denoted 1), or independent sets (denoted 0), or prime graphs (denoted P), and with no $0-0$ nor $1-1$ edges.
We call it canonical and denote it $T(G)$.



Proof idea: $T(G)$ is obtained considering recursively the quotients resulting from the partition of V into maximal modules different from V (in the prime case, with special cases for cliques and independent sets).

Induced subgraphs in decomposition trees

- Let G be a graph. Let S be a subset of its vertices.
- Consider the subgraph of G induced by S.
- Let $T(G)$ be its canonical modular decomposition tree.



Fact: A decomposition tree for the induced subgraph of G corresponding to S is obtained considering the subtree of $T(G)$ induced by the set of leaves corresponding to S.

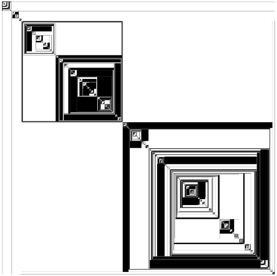
Remark: The induced tree is not necessarily the canonical tree of the induced subgraph (*e.g.* it may contain $0-0$ edges).

A summary highlighting similarities and differences

- Using substitution/modular decomposition, permutations and graphs can be encoded by "decorated" trees.
- However, the trees associated with permutations are planar, while those associated with graphs are non-planar.
- Patterns/induced subgraphs correspond to subtrees induced by leaves.
- A permutation/graph is encoded by possibly many decomposition trees, but exactly one canonical decomposition tree.

Next: How to use these trees to prove permuton/graphon convergence for uniform permutations/graphs avoiding substructures.

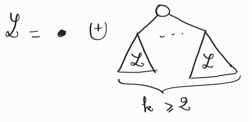
Graphon limit of (labeled) cographs



Cographs and their modular decomposition trees

- Cographs are defined by the avoidance of P_4 (induced path on 4 vertices, which is the smallest prime graph).
- Equivalently, cographs are all graphs whose modular decomposition trees involve only 0 (indep. set) and 1 (clique) nodes (no prime node). We call cotrees their modular decomposition trees.
- Therefore, labeled[2] cographs can be described from the combinatorial specification:

$$\mathcal{L} = \bullet \uplus \text{Set}_{\geq 2}(\mathcal{L}) \qquad i.e.,$$



Indeed, *via* its canonical modular decomposition tree, a cograph correspond to a tree of $\mathcal{L}$ with a label 0 or 1 on the root (propagating labels alternating between 0 and 1 along each edge).

[2] meaning vertices are labeled by the integers from 1 to n; in the unlabeled case, we need to consider Multiset, and hence later Polya operators for the GF

Expressing $\mathbb{E}[\mathrm{Dens}(g, \boldsymbol{G}_n)]$

Notation:
Let $\boldsymbol{G}_n$ be a uniform random labeled cograph with n vertices.

Reminder:
Knowing $\mathbb{E}[\mathrm{Dens}(g, \boldsymbol{G}_n)]$ for all g characterizes the graphon limit of $\boldsymbol{G}_n$.

Notation:
for all n, and all $k \leq n$,
$\boldsymbol{t}^{(n)}$ is a uniform random labeled canonical cotree of size n, and
$\boldsymbol{t}_k^{(n)}$ is the subtree of $\boldsymbol{t}^{(n)}$ induced by a uniform k-tuple of distinct leaves.

Observation:
For any cograph g, we have:

$$\mathbb{E}[\mathrm{Dens}(g, \boldsymbol{G}_n)] = \mathbb{P}(\mathrm{SubGraph}_k(\boldsymbol{G}_n) = g) = \sum \mathbb{P}(\boldsymbol{t}_k^{(n)} = t_0),$$ where

the sum runs over all cotrees t_0 corresponding to g.

Expressing $\mathbb{P}(\boldsymbol{t}_k^{(n)} = t_0)$

Observation: $\mathbb{P}(\boldsymbol{t}_k^{(n)} = t_0) = \frac{n![z^n]M_{t_0}(z)}{n![z^n]M(z) \times n(n-1)\dots(n-k+1)}$, where

- $\mathcal{M}$ is the set of labeled canonical cotrees
- for any cotree t_0 with k leaves, $\mathcal{M}_{t_0}$ is the set of labeled canonical cotrees with a marked k-tuple of distinct leaves, which induce t_0,
- $M(z)$ and $M_{t_0}(z)$ are the corresponding exponential generating functions
- as usual, $[z^n]F(z)$ denotes the coefficient of z^n in the generating function $F(z)$

Next: Use symbolic and analytic combinatorics to compute the asymptotic behavior of the numerator and the denominator in

$$\frac{n![z^n]M_{t_0}(z)}{n![z^n]M(z) \times n(n-1)\dots(n-k+1)}.$$

Estimating the denominator

- Recall that $\mathcal{L} = \bullet \uplus \text{Set}_{\geq 2}(\mathcal{L})$.
- Let $L(z)$ be the exponential generating function of $\mathcal{L}$.
- From [Flajolet-Sedgewick] (rather a variant on trees counted by leaves), $L(z)$ satisfies $L(z) = z + \exp(L(z)) - 1 - L(z)$.
- The generating function of cographs is $M(z) = 2L(z) - z$.
- $L(z)$ and $M(z)$ have the same radius of convergence $\rho = 2\log(2) - 1$ and are Δ-analytic.
- Near $z = \rho$, $L(z) = \log(2) - \sqrt{\rho}\sqrt{1 - z/\rho} + \mathcal{O}(1 - z/\rho)$ and $M(z) = 1 - 2\sqrt{\rho}\sqrt{1 - z/\rho} + \mathcal{O}(1 - z/\rho)$.
- From the transfer theorem,

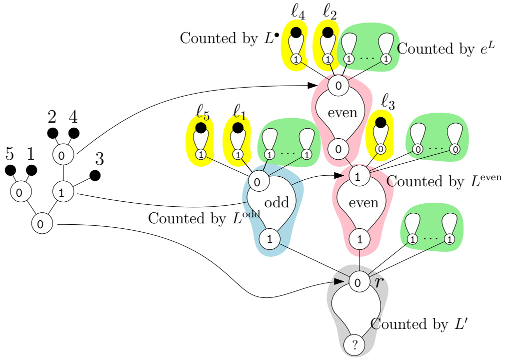
$$n(n-1)\dots(n-k+1)[z^n]M(z) \underset{n\to+\infty}{\sim} \frac{n^{k-3/2}}{\rho^{n-1/2}\sqrt{\pi}}.$$

Prop.: If t_0 with k leaves has n_v internal vertices, $n_=$ edges of the form $0-0$ or $1-1$, and $n_{\neq}$ edges of the form $0-1$ or $1-0$, then

$$M_{t_0} = (L')(\exp(L))^{n_v}(L^{\bullet})^k(L^{\text{odd}})^{n_=}(L^{\text{even}})^{n_{\neq}},$$

these series being variations on $L(z)$ whose singular behavior results from that of $L(z)$.

Proof:



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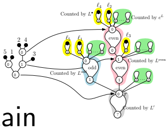

Corollary: Like before, we obtain

- the behavior at ρ of $M_{t_0}(z)$,
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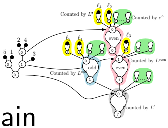

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More precisely, we have

$$[z^n]M_{t_0}(z) \underset{n\to+\infty}{\sim} \frac{(k-1)!}{(2k-2)!} \frac{n^{k-3/2}}{\rho^{n-1/2}\sqrt{\pi}},$$

if t_0 is binary (which implies $n_v = k-1$ and $n_= + n_{\neq} = k-2$).

Conclusion of the proof

Notation (reminder):

- $\boldsymbol{t}^{(n)}$: uniform random labeled canonical cotree of size n
- $\boldsymbol{t}_k^{(n)}$: subtree of $\boldsymbol{t}^{(n)}$ induced by a uniform k-tuple of distinct leaves
- t_0: cotree with k leaves

What we proved: If t_0 is binary, then $\lim_{n\to\infty} \mathbb{P}(\boldsymbol{t}_k^{(n)} = t_0) = \frac{(k-1)!}{(2k-2)!}$.

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Remark: $\frac{(k-1)!}{(2k-2)!} = \frac{1}{\text{number of binary cotrees with } k \text{ leaves}}$.

Consequence: If t_0 is not binary, then $\lim_{n\to\infty} \mathbb{P}(\boldsymbol{t}_k^{(n)} = t_0) = 0$.

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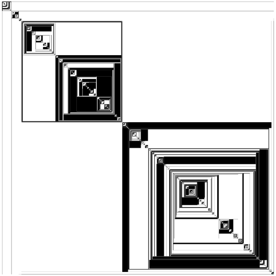
Remark/reminder:
Summing over all t_0 encoding a cograph g, this gives $\lim_{n\to\infty} \mathbb{E}[\mathrm{Dens}(g, \boldsymbol{G}_n)]$.
These quantities characterize the graphon limit of cographs.

The Brownian cographon

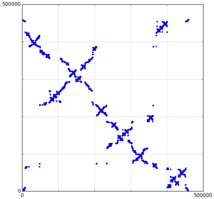
- Starting from a Brownian excursion, whose local minima receive unbiased decorations by 0 and 1, we can build the Brownian cographon of parameter 1/2, denoted $\boldsymbol{W}^{1/2}$.
- We can compute $\Delta_g = \mathbb{E}[\mathrm{Dens}(g, \boldsymbol{W}^{1/2})]$ for all cographs g.

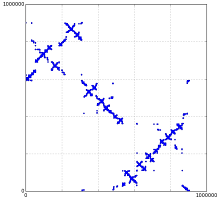
But this is a story for another time...

- We observe that $\lim_{n\to\infty} \mathbb{E}[\mathrm{Dens}(g, \boldsymbol{G}_n)] = \Delta_g$ for all g.
- Therefore, not only do we prove existence of a graphon limit for uniform random cographs, but we also provide a construction of this limit.
- The limiting graphon is a genuinely random and fractal object.



Permuton limit of separable permutations





Obtaining similarly the limit of separable permutations

- Separable permutations are those avoiding the patterns 2413 and 3142 (the two smallest simple permutations).
- Equivalently, it is the family of all permutations whose decomposition trees involve only $\oplus$ and $\ominus$ nodes (no simple permutations).
- Separable permutations are therefore the permutation analogue of cographs.

- From a combinatorial specification for the decomposition trees of separable permutations, and using analytic combinatorics as before, we obtain the limiting behavior of $\mathbb{E}[\widetilde{\mathrm{occ}}(\pi, \boldsymbol{\sigma}_n)]$ for π any pattern, and $\boldsymbol{\sigma}_n$ a uniform random separable permutation of size n. These quantities characterize the permuton limit of separable permutations.

- Again, we have an explicit construction of the limiting object $\boldsymbol{\mu}^{1/2}$ (called the Brownian separable permuton of parameter $1/2$) from a Brownian excursion with decorations.

More classes of permutations

and universality of
the Brownian separable permuton

Transposing the proof strategy to a more general setting

Idea of the method:

- Assume that you know a combinatorial specification for the decomposition trees of permutations in some class $\mathcal{C}$.
- It translates into a system of equations for the GF of $\mathcal{C}$.
- We can in addition "track patterns" in these equations.
- IF the method of analytic combinatorics goes through, we obtain convergence to a certain permuton, as for separable permutations.

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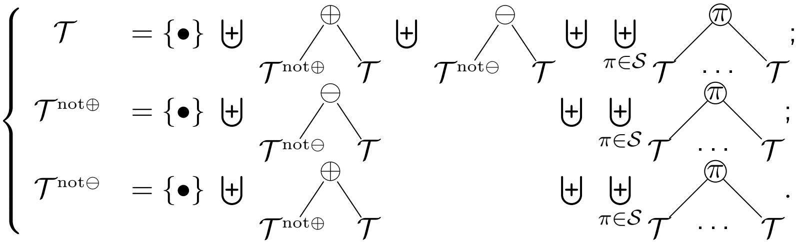
Some results, indicating a universality phenomenon:

- Convergence to Brownian separable permutons of parameters $p \in [0, 1]$ for substitution-closed classes, under some analytic condition on the GF of the simple permutations in the class.
- Dichotomy for classes for which a specification is known (in particular: whenever they contain finitely many simple permutations): (random) Brownian permutons VS (deterministic) X-permutons.

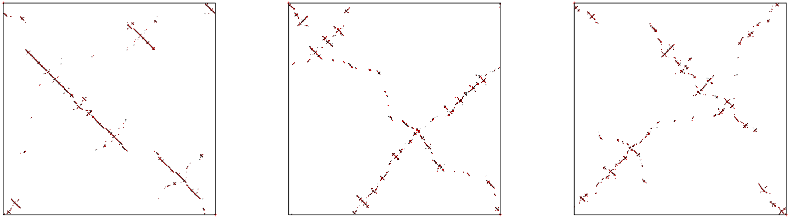
Substitution-closed classes

- Their specification adds some simple permutations to that of separable permutations. We denote by $\mathcal{S}$ the set of allowed simple permutations.



Substitution-closed classes

- Their specification adds some simple permutations to that of separable permutations. We denote by $\mathcal{S}$ the set of allowed simple permutations.
- Limit permutons are (biased) Brownian separable permutons.



simulations of $\boldsymbol{\mu}_p$ for $p = 0.2, 0.45, 0.5$.

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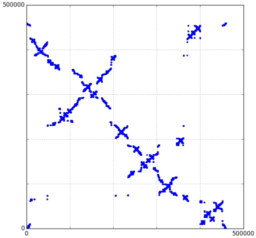
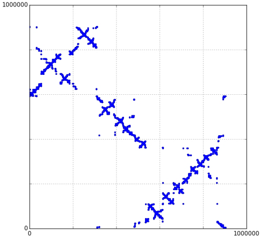
Example 1: Separable permutations, *i.e.* $\mathcal{S} = \emptyset$, $\Rightarrow p = 0.5$

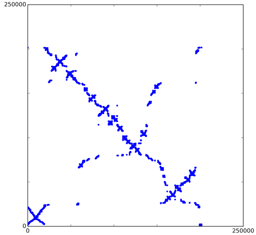


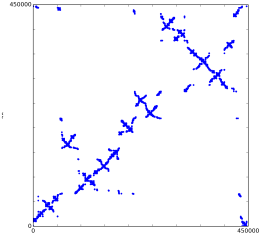


Substitution-closed classes

- Their specification adds some simple permutations to that of separable permutations. We denote by $\mathcal{S}$ the set of allowed simple permutations.
- Limit permutons are (biased) Brownian separable permutons.

Example 2: $\mathcal{S} = \{2413, 3142, 24153\}, \qquad \Rightarrow p = 0.5$

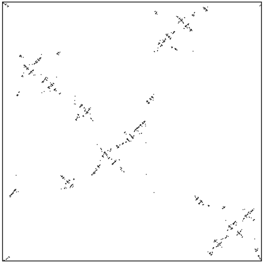




Substitution-closed classes

- Their specification adds some simple permutations to that of separable permutations. We denote by $\mathcal{S}$ the set of allowed simple permutations.
- Limit permutons are (biased) Brownian separable permutons.

Example 3: $\mathcal{S} = Av(321) \cap \{Simples\}$ (infinite), $\Rightarrow p \in [0.577, 0.622]$



Brownian case of the dichotomy

When the specification contains a product of critical families.

⇒ The limiting permuton of the class is a biased Brownian separable permuton (of parameter p possibly 0 or 1).

Brownian case of the dichotomy

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$\Rightarrow$ The limiting permuton of the class is a biased Brownian separable permuton (of parameter p possibly 0 or 1).

Example 1: $Av(132)$, with critical families in blue.

$$\begin{cases} \mathcal{T} = \{\bullet\} \uplus \oplus(\mathcal{T}^{\text{not}\oplus}, \mathcal{T}_{\langle 21\rangle}) \uplus \ominus(\mathcal{T}^{\text{not}\ominus}, \mathcal{T}) \\ \mathcal{T}^{\text{not}\oplus} = \{\bullet\} \uplus \ominus(\mathcal{T}^{\text{not}\ominus}, \mathcal{T}) \\ \mathcal{T}^{\text{not}\ominus} = \{\bullet\} \uplus \oplus(\mathcal{T}^{\text{not}\oplus}, \mathcal{T}_{\langle 21\rangle}) \\ \mathcal{T}_{\langle 21\rangle} = \{\bullet\} \uplus \oplus(\mathcal{T}^{\text{not}\oplus}_{\langle 21\rangle}, \mathcal{T}_{\langle 21\rangle}) \\ \mathcal{T}^{\text{not}\oplus}_{\langle 21\rangle} = \{\bullet\}. \end{cases}$$

The limit is the Brownian separable permuton of parameter $p = 0$.

Brownian case of the dichotomy

When the specification contains a product of critical families.

$\Rightarrow$ The limiting permuton of the class is a biased Brownian separable permuton (of parameter p possibly 0 or 1).

Example 2: $Av(2413, 31452, 41253, 41352, 531246)$, with critical families in blue.

$$\begin{cases} \mathcal{T}_0 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_0] \uplus \ominus[\mathcal{T}_2, \mathcal{T}_0] \uplus 3142[\mathcal{T}_0, \mathcal{T}_3, \mathcal{T}_3, \mathcal{T}_0] \\ \mathcal{T}_1 = & \{\bullet\} \uplus \ominus[\mathcal{T}_2, \mathcal{T}_0] \uplus 3142[\mathcal{T}_0, \mathcal{T}_3, \mathcal{T}_3, \mathcal{T}_0] \\ \mathcal{T}_2 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_0] \uplus 3142[\mathcal{T}_0, \mathcal{T}_3, \mathcal{T}_3, \mathcal{T}_0] \\ \mathcal{T}_3 = & \{\bullet\} \uplus \ominus[\mathcal{T}_4, \mathcal{T}_3] \\ \mathcal{T}_4 = & \{\bullet\} \end{cases}$$

The limit is the Brownian separable permuton of parameter $p \approx 0.4748692376...$ (only real root of a certain polynomial of degree 9).

X case of the dichotomy

When the specification contains no product of critical families.

$\Rightarrow$ The limiting permuton of the class has a deterministic X shape (not necessarily centered, possibly missing some of the 4 branches).

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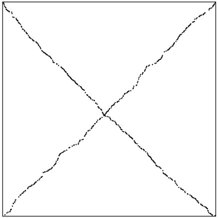

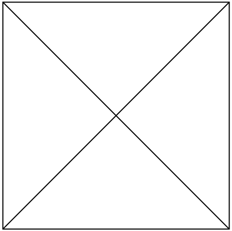

The limit is the centered X-permuton.

X case of the dichotomy

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Example 2: $Av(2413, 3142, 2143, 34512)$, with critical families in blue.

$$\begin{cases} \mathcal{T}_0 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_2] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_5, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_5, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_8, \mathcal{T}_6] \\ \mathcal{T}_1 = & \{\bullet\} \\ \mathcal{T}_2 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_2] \\ \mathcal{T}_3 = & \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_5, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_5, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_8, \mathcal{T}_6] \\ \mathcal{T}_4 = & \ominus[\mathcal{T}_5, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_5, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_8, \mathcal{T}_6] \\ \mathcal{T}_5 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_1] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_9] \uplus \oplus[\mathcal{T}_9, \mathcal{T}_1] \\ \mathcal{T}_6 = & \{\bullet\} \uplus \ominus[\mathcal{T}_1, \mathcal{T}_6] \\ \mathcal{T}_7 = & \oplus[\mathcal{T}_1, \mathcal{T}_2] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_{10}, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_{10}, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_8, \mathcal{T}_6] \\ \mathcal{T}_8 = & \oplus[\mathcal{T}_1, \mathcal{T}_{11}] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_{12}] \uplus \oplus[\mathcal{T}_{13}, \mathcal{T}_{11}] \uplus \oplus[\mathcal{T}_9, \mathcal{T}_{11}] \uplus \oplus[\mathcal{T}_{13}, \mathcal{T}_1] \\ \mathcal{T}_9 = & \ominus[\mathcal{T}_1, \mathcal{T}_6] \\ \mathcal{T}_{10} = & \oplus[\mathcal{T}_1, \mathcal{T}_1] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_9] \uplus \oplus[\mathcal{T}_9, \mathcal{T}_1] \\ \mathcal{T}_{11} = & \oplus[\mathcal{T}_1, \mathcal{T}_2] \\ \mathcal{T}_{12} = & \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_{10}, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_{10}, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_8, \mathcal{T}_6] \\ \mathcal{T}_{13} = & \ominus[\mathcal{T}_{10}, \mathcal{T}_6] \uplus \ominus[\mathcal{T}_{10}, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_1, \mathcal{T}_7] \uplus \ominus[\mathcal{T}_8, \mathcal{T}_6]. \end{cases}$$

X case of the dichotomy

When the specification contains no product of critical families.

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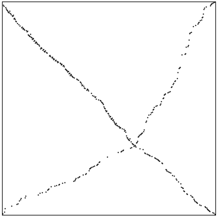

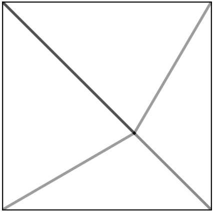

The limit is a non-centered X-permuton.

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Example 3: $Av(2413, 1243, 2341, 531642, 41352)$, with critical families in blue.

$$\begin{cases}
\mathcal{T}_0 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_2] \uplus \oplus[\mathcal{T}_1, \mathcal{T}_3] \uplus \oplus[\mathcal{T}_4, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_5, \mathcal{T}_0] \uplus 3142[\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_6] \\
\mathcal{T}_1 = & \{\bullet\} \uplus \ominus[\mathcal{T}_7, \mathcal{T}_1] \\
\mathcal{T}_2 = & \{\bullet\} \uplus \oplus[\mathcal{T}_7, \mathcal{T}_2] \\
\mathcal{T}_3 = & \oplus[\mathcal{T}_8, \mathcal{T}_2] \uplus \ominus[\mathcal{T}_9, \mathcal{T}_6] \\
\mathcal{T}_4 = & \ominus[\mathcal{T}_{10}, \mathcal{T}_{11}] \uplus \ominus[\mathcal{T}_{10}, \mathcal{T}_1] \uplus \ominus[\mathcal{T}_7, \mathcal{T}_{11}] \uplus 3142[\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_6] \\
\mathcal{T}_5 = & \{\bullet\} \uplus \oplus[\mathcal{T}_1, \mathcal{T}_1] \uplus 3142[\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1] \\
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\mathcal{T}_8 = & \ominus[\mathcal{T}_9, \mathcal{T}_6] \\
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\mathcal{T}_{10} = & \oplus[\mathcal{T}_1, \mathcal{T}_1] \uplus 3142[\mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1, \mathcal{T}_1] \\
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\end{cases}$$

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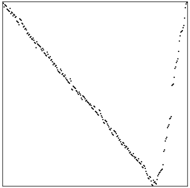

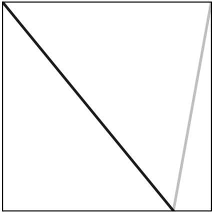

The limit is a degenerate X-permuton.

Concluding remarks

Towards more classes of graphs

- As in the permutation case, we can extend the study of cographs to families of graphs whose modular decomposition trees are described by a combinatorial specification.
- Our analytic approach can only work with GF having positive radius of convergence. This was necessary in permutation classes, but is an additional requirement for hereditary graph classes.

- This is the PhD topic of Théo Lenoir, who started working in September 2021, supervised by Frédérique Bassino and Lucas Gerin.
- The classes that he studied are the P_4-`blah` graphs, where `blah` $\in$ {reducible, sparse, lite, extensible, tidy}. All converge to the Brownian cographon.
- Recall that cographs are P_4-free graphs.

A nice consequence of permuton/graphon limits

Results:

- The size of the largest independent set of a uniform random cograph is sublinear.
  (hence P_4 does not have the asymptotic linear Erdős-Hajnal property, answering a question of Kang, McDiarmid, Reed and Scott in 2014)
- The length of the longest increasing subsequence of a uniform random separable permutations is sublinear.

Main proof ingredients:

- Convergence to the Brownian cographon
- The independence number of the Brownian cographon $\boldsymbol{W}^{1/2}$ is 0

Bonus: The sublinearity result applies to all classes with graphon/permuton limit $\boldsymbol{W}^{p}$ or a Brownian separable permuton.

A nice consequence of permuton/graphon limits

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Bonus: The sublinearity result applies to all classes with graphon/permuton limit $\boldsymbol{W}^p$ or a Brownian separable permuton.

Thank you for being there!