Baxter Tree-like Tableaux

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talk based on joint work with
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Permutation Patterns, July 6, 2023, Dijon, Université de Bourgogne.

Goal of the talk

The Baxter numbers are defined by $\mathrm{Bax}_n = \frac{2}{n(n+1)^2} \sum_{k=1}^{n} \binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}$.

They are known to enumerate many families of discrete objects, including

- Baxter permutations $Av(2\underline{41}3, 3\underline{14}2)$
  [Chung, Graham, Hoggatt, Kleiman, 1978; Bousquet-Mélou, 2002]
- Twisted-Baxter permutations $Av(2\underline{41}3, 3\underline{41}2)$
  [Reading, 2005; West, 2006]
- Mosaic floorplans
  [Yao, Chen, Cheng, Graham, 2003; Ackerman, Barequet, Pinter, 2006]
- Triples of non-intersecting lattice paths
  [Dulucq, Guibert, 1998; among others]

We give bijections from Baxter tree-like tableaux (new objects) to twisted-Baxter permutations, mosaic floorplans and triples of non-intersecting lattice paths.

Tree-like Tableaux
and Baxter Tree-like Tableaux

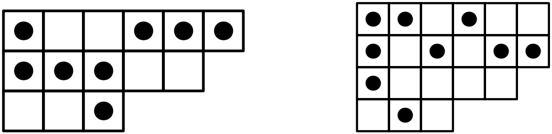
Tree-like tableaux (TLTs): definition

A tree-like tableau (TLT) is a Ferrers diagram where cells are either empty or pointed (=occupied by a point), and such that:

- every column and every row contains at least one pointed cell;
- the top leftmost cell of the diagram is occupied by a point, called the root point;
- for every non-root pointed cell c, there exists a pointed cell p either above c in the same column, or to its left in the same row, but not both; p is called the parent of c in the TLT.

The size is the number of points.

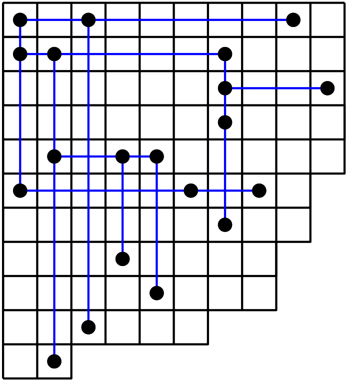
Examples:



Some facts about TLTs

- In size-preserving bijection with permutations
  (*via* a labeling of the points which we shall present shortly)

  - TLTs carry an underlying tree structure, induced by the parent/child relation.



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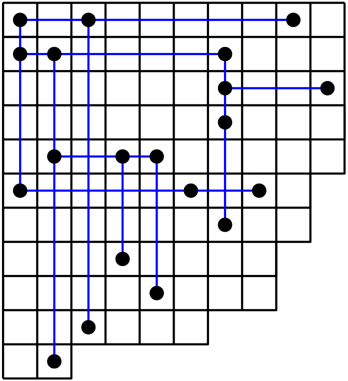
  - An empty cell is called a crossing when it has a point above it in its column and a point to its left in its row.
    These are indeed crossings of blue lines (extended to reach the boundary of the Ferrers shape).



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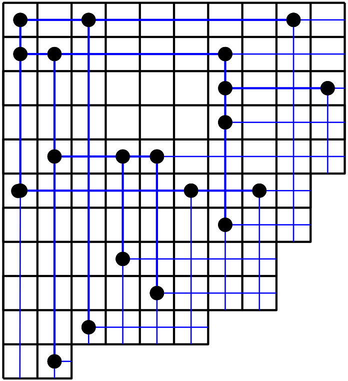
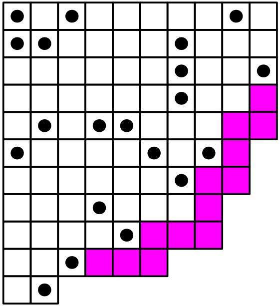


Labeling the points of a TLT

- The special point of a TLT is the rightmost among the points that are in the bottommost cell of their column.



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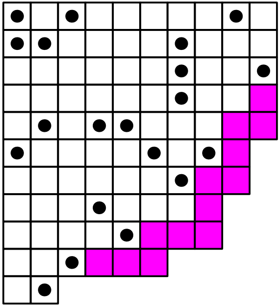
- The special point of a TLT is the rightmost among the points that are in the bottommost cell of their column.

- If the special point has a (necessarily empty) neighboring cell on its right, then a ribbon is associated to it.
- The ribbon of such a special point is the maximal set of cells along the southeast border that is connected, does not contain any 2×2 square, and consists only of empty cells.



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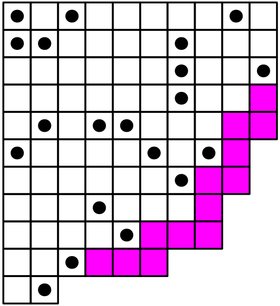
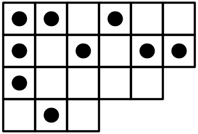
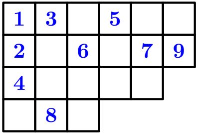
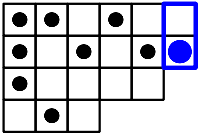
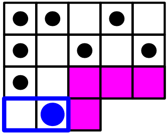
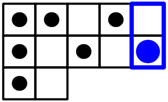
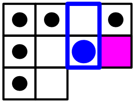
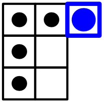
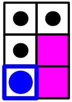
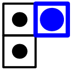
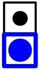


- Inductive labeling of the points:
  In a TLT of size n, the special point receives the label n, and the other points are labeled as in the smaller TLT obtained removing the special point, its empty row or column, and its ribbon (when there is one).

$T =$  $=$ 

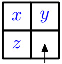
$T = T_9 =$ , $T_8 =$ , $T_7 =$ ,

$T_6 =$ , $T_5 =$ , $T_4 =$ , $T_3 =$ , $T_2 =$ ,

$T_1 =$ .

Extending the labeling and bijection with permutations

We can propagate the labeling of the points of a TLT to its empty cells according to local rules. For a cell c as in



- if there is a point above c and a point to its left (*i.e.* if c is a crossing), then c receives the label x;
- if there is a point above c but none to its left, then c receives the label y;
- if there is a point to the left of c but none above it, then c receives the label z;
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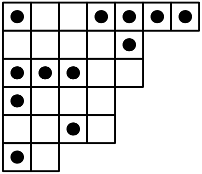
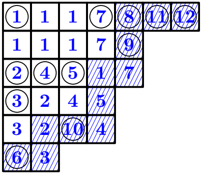


is labeled



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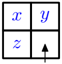
We can propagate the labeling of the points of a TLT to its empty cells according to local rules. For a cell c as in 

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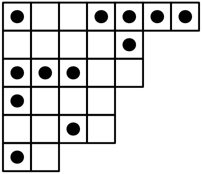

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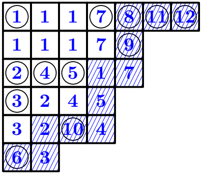


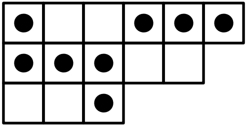
The permutation $\varphi_{perm}(T)$ is read along the southeast border of the TLT T. This is a bijection. [Aval, Boussicault, Nadeau, 2013]

and gives $\varphi_{perm}(T) =$ 6 3 2 10 4 5 1 7 9 8 11 12.

Avoiding patterns: Baxter TLTs

A Baxter TLT is a TLT which avoids the patterns  and 
(where · can be either an empty or a pointed cell).

Equivalently, a Baxter TLT is a TLT with no point below or to the right of a crossing.

Examples: 

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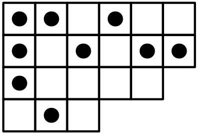

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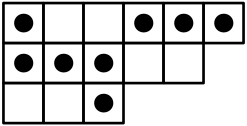

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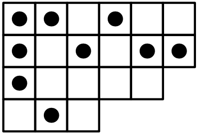

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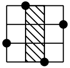
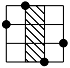
Next: bijections from Baxter TLTs to

- twisted-Baxter permutations
- mosaic floorplans
- triples of non-intersecting lattice paths

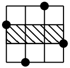
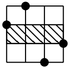
Bijection to twisted-Baxter permutations

Baxter family of permutations in bijection with Baxter TLT

- Twisted-Baxter permutations: defined by the avoidance of

  $2\,\underbrace{41}\,3 =$  and $3\,\underbrace{41}\,2 =$  .

- Their inverses are defined by the avoidance of

  $2^{+}132 =$  and $2^{+}312 =$  .

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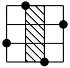
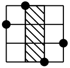
  $2\,\underline{41}\,3 =$  and $3\,\underline{41}\,2 =$  .

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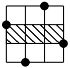
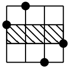
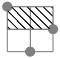
  $2^{+}132 =$  and $2^{+}312 =$  .

Theorem: The bijection φ_{perm} bijectively sends Baxter TLTs to inverses of twisted-Baxter permutations.

Key lemma for this proof: The crossings of a TLT T correspond to occurrences of $2^{+}12=$  in $\varphi_{perm}(T)$.

Hence, a point below or to the right of a crossing corresponds to an occurrence of $2^{+}132$ or $2^{+}312$.

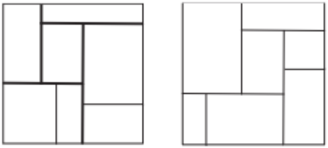
Bijection to mosaic floorplans

Mosaic floorplans

- A floorplan is a partition of a rectangle into rectangles, such that any two intersecting segments form a $\perp$, $\top$, $\vdash$ or $\dashv$ (but never $+$).
- Two floorplans are R-equivalent if one can pass from one to the other by sliding the segments to adjust the sizes of the rectangles.
- A mosaic floorplan is an equivalence class of floorplans under R.

Example, from [Ackerman, Barequet, Pinter, 2006]:

Two R-equivalent floorplans:



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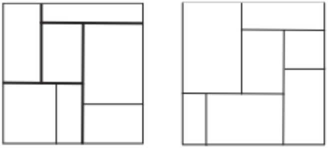
Two R-equivalent floorplans:  .

They represent the same mosaic floorplan.

Mosaic floorplans are counted by Baxter numbers.

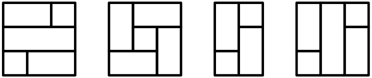
[Yao, Chen, Cheng, Graham, 2003; Ackerman, Barequet, Pinter, 2006]

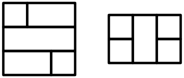
Representatives as packed floorplans (PFP)

A packed floorplan (PFP) is a floorplan

- whose rectangular bounding box has integer coordinates
- every line of integer coordinate inside this bounding box is the support of exactly one segment,
- the pattern  is avoided.

Examples:

Some packed floorplans: 

These are not packed floorplans: 

Proposition: Every mosaic floorplan has exactly one representative as a packed floorplan.

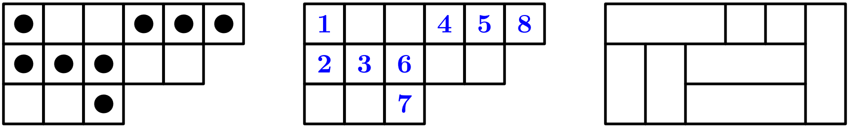
The bijection, from Baxter TLTs to PFPs

Let T be a Baxter TLT of size n.
Consider the inductive labeling of its points explained before.

We build a PFP $\varphi_{PFP}(T)$ as follows.

- Use as bounding box the smallest rectangle containing T, called R.
- For each i from n to 1, draw a rectangle inside R, whose top-left corner is the point of T labeled by i, and which is the largest possible (without stepping on the rectangles already placed).

Example:



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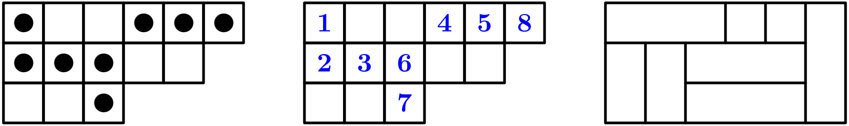


Theorem: φ_{PFP} is a size-preserving bijection between TLTs and PFPs (where the size of a PFP is its number of rectangles).

Bijection to
triples of non-intersecting lattice paths

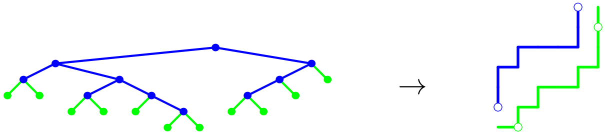
Binary trees and pairs of non-intersecting lattice paths

A pair of non-intersecting lattice paths (NILPs) is a pair of lattice paths with unitary N and E steps, which never meet, starting at $(1,0)$ and $(0,1)$ and ending at $(n-i,i)$ and $(n-i-1,i+1)$ for some $i \in [0..(n-1)]$.

From a (complete) binary tree, we can build two words w_1 and w_2 by performing a depth-first traversal and writing:

- an N (resp. E) in w_1 for each internal left (resp. right) edge;
- an E (resp. N) in w_2 for each left (resp. right) leaf
  (and then forgetting the initial E and the final N is w_2).

Example:



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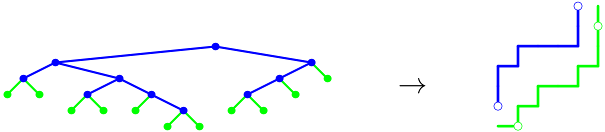
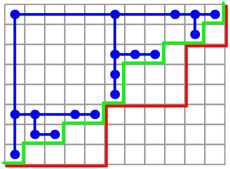
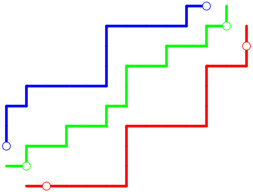
Example: 

Proposition: The above construction is a bijection between (complete) binary trees and pairs of NILPs. [Delest, Viennot, 1984; Dulucq, Guibert, 1998]

Extension to Baxter TLTs

With a Baxter TLT T of size n, we associate 3 words w_1, w_2 and w_3, each in $\{N, E\}^{n-1}$, as follows:

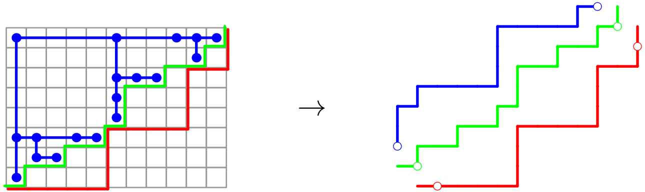
- w_1 and w_2 as before, from the (completed) binary tree underlying T;
- w_3 is the word describing the southeast border of T (up to forgetting the initial E and the final N).

Example:  $\rightarrow$ 

Extension to Baxter TLTs

With a Baxter TLT T of size n, we associate 3 words w_1, w_2 and w_3, each in $\{N, E\}^{n-1}$, as follows:

- w_1 and w_2 as before, from the (completed) binary tree underlying T;
- w_3 is the word describing the southeast border of T (up to forgetting the initial E and the final N).

Example: 

Theorem: The above construction is a bijection between Baxter TLTs and triples of NILPs.

Key lemma: In a Baxter TLT T, w_2 also describes the southeast border of the thinnest Ferrers shape containing the points of T.

Final remarks

We can do a little more

- With NILPs, we can use the Lindström-Gessel-Viennot lemma to obtained enumeration of our objects according to some parameters.

Example: The number of twisted-Baxter permutations of size n, with k ascents and r left-to-right minima is $\sum_{p,q,s} LGV(n,k,r,p,s,q)$, with

$$LGV(n,k,r,p,s,q) = \begin{vmatrix} \binom{n-1-r-p}{k-p} & \binom{n-1-p}{k-p} & \binom{n-1-s-p}{k-s-p} \\ \binom{n-1-r}{k} & \binom{n-1}{k} & \binom{n-1-s}{k-s} \\ \binom{n-1-r-q}{k} & \binom{n-1-q}{k} & \binom{n-1-s-q}{k-s} \end{vmatrix}.$$

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- There is an interesting restriction of our bijections where
  - Baxter TLTs are "almost complete",
  - permutations are alternating starting with an ascent,
  - PFPs have all their rectangles touching the main diagonal,
  - and triples NILPs are such that $w_2 = ENENENE \dots ENE$.

An intriguing enumerative coincidence

Proposition: Denoting (C_n) the Catalan numbers, it holds that for any n, there are C_n^2 (resp. $C_n \cdot C_{n+1}$) permutations of size $2n$ (resp. $2n+1$) which avoid the patterns 2^+132 and 2^+312 and are alternating starting with an ascent.

This follows from the previous restriction, through the chain of bijections: permutations $\leftrightarrow$ Baxter TLTs $\leftrightarrow$ NILPs.

Question: Can we provide a direct proof of this proposition?

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Observation: For σ avoiding 2^+132 and 2^+312 which is alternating starting with an ascent, the permutation σ_{odd} (resp. σ_{even}) read on the odd (resp. even) positions avoids 312 (resp. 231).

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Thank you!