A general theory of Wilf-equivalence for Catalan structures

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joint work with Michael Albert (University of Otago)

arXiv:1407.8261

Discrete Math Seminar, Uni. Zürich.
September 23, 2014.

Enumeration sequences and Wilf-equivalence

Let $\mathcal{C}$ be any combinatorial class, *i.e.*

- $\mathcal{C}$ is equipped with a notion of size
- such that for any n there are finitely many objects of size n in $\mathcal{C}$.
- The number of objects of size n in $\mathcal{C}$ is denoted c_n.

To $\mathcal{C}$, we associate:

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Sometimes (or very often!), two classes have the same enumeration sequences (or equivalently generating function).

Such enumeration coincidences are called Wilf-equivalences (terminology from the *Permutation Patterns* literature).

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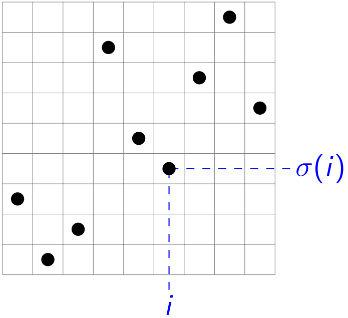

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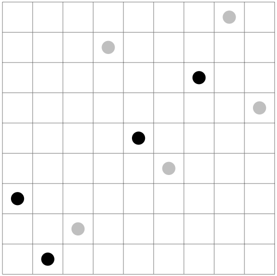

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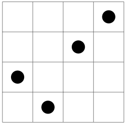

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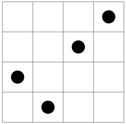

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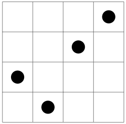

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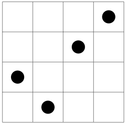

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Some Wilf-equivalences for pattern-avoiding permutations

Small excluded patterns:

- $\mathrm{Av}(123)$ and $\mathrm{Av}(231)$ are Wilf-equivalent, and enumerated by the Catalan numbers $Cat_n = \frac{1}{n+1}\binom{2n}{n}$
- There are three Wilf-equivalence classes for permutation classes $\mathrm{Av}(\pi)$ with π of size 4, the enumeration of $\mathrm{Av}(1324)$ being open.
- Check all Wilf-equivalences between $\mathrm{Av}(\pi, \tau)$ when π and τ have size 3 or 4 on Wikipedia.

Some results for arbitrary long patterns:

- $\mathrm{Av}(231 \oplus \pi)$ and $\mathrm{Av}(312 \oplus \pi)$ [West & Stankova 02]

First unbalanced Wilf-equivalences:

- $\mathrm{Av}(1324, 3416725)$ and $\mathrm{Av}(1234)$;
  $\mathrm{Av}(2143, 3142, 246135)$ and $\mathrm{Av}(2413, 3142)$ [Burstein & Pantone 14+]

Novelty of our work: a global look

Our goal: find all Wilf-equivalences between classes $\mathrm{Av}(231, \pi)$.

Harmless assumption: In $\mathrm{Av}(231, \pi)$, throughout the talk, π avoids 231.
(or we are just studying $\mathrm{Av}(231)$. . .)

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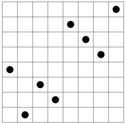
So, equivalently but somehow more generally, our goal rephrases as:

find all Wilf-equivalences between **"pattern-avoiding Catalan objects"**.

Substructures in Catalan objects

Some Catalan structures, and their substructures

- 231-avoiding permutations

$41327658 =$ 

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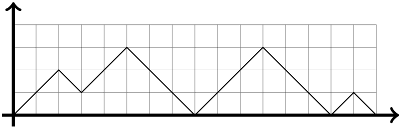

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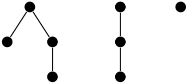
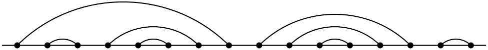


- Arch systems



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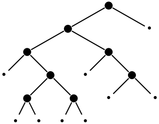

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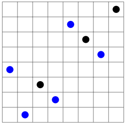
$41327658 =$ 

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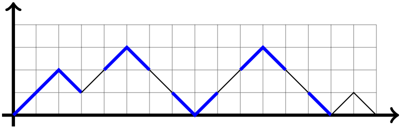

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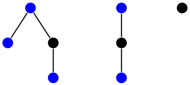

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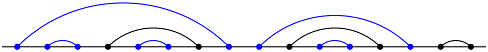

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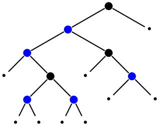

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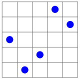
$31254 =$ 

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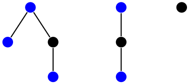

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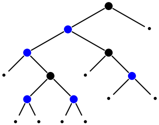

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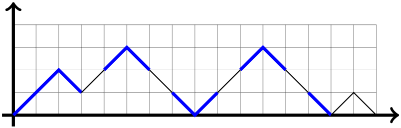

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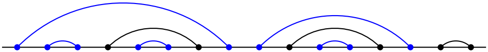

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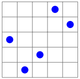
$31254 =$ 

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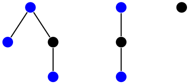

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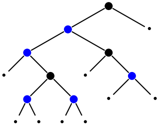

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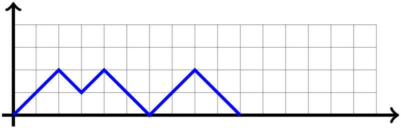

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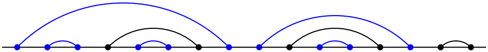

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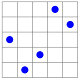
$31254 =$ 

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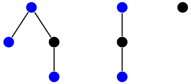

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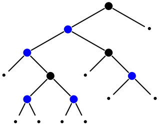

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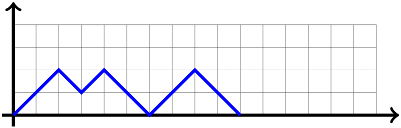

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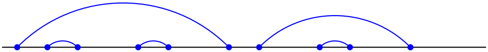

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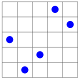
$31254 =$ 

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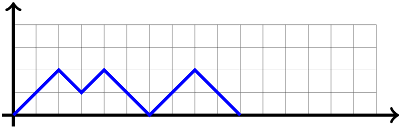

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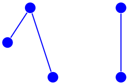

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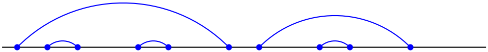

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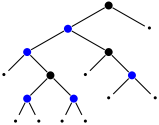

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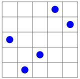
$31254 =$ 

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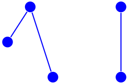

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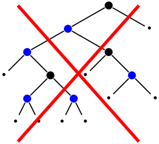

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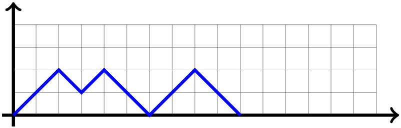

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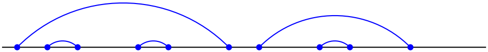

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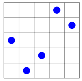
$31254 =$ 

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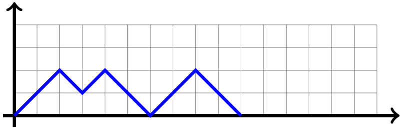

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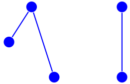

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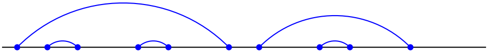

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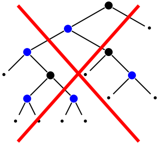

Essential fact: The usual bijections relating our quartet of Catalan structures preserve the substructure relation.

Outline for (the rest of) the talk

For any Catalan family in our quartet, we are interested in classes defined by the avoidance of one Catalan object.

- Motivation: permutation classes $\mathrm{Av}(231, \pi)$
- In practice: classes $\mathrm{Av}(A)$ of arch systems avoiding some subsystem A

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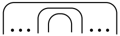
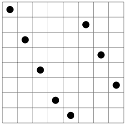
But all four contexts are equivalent!

- Which arch systems A are Wilf-equivalent?
  i.e. which classes $\mathrm{Av}(A)$ have the same enumeration?
- Bijections between $\mathrm{Av}(A)$ and $\mathrm{Av}(B)$ for Wilf-equivalent arch systems A and B?
- How many Wilf-equivalence classes of arch systems are there?
- The special case of the Wilf-equivalence class of $N_n =$ .
- Comparison between the enumeration sequences of $\mathrm{Av}(A)$ and $\mathrm{Av}(B)$ for some A and B that are not equivalent.

Quick detour: What about other Catalan structures?

Other Catalan objects having a natural notion of substructure:

- 123-avoiding permutations

$86421753 =$ 

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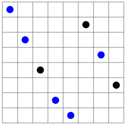
$86421753 =$ 

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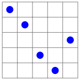
$54213 =$ 

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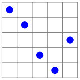
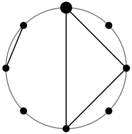
$54213 =$ 

- Non-crossing partitions



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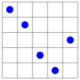
$54213 =$ 

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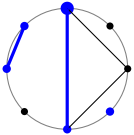

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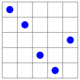
  54213 = 

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  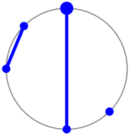

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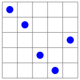
$54213 =$ 

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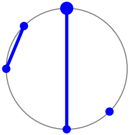

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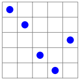
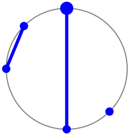
$54213 =$ 



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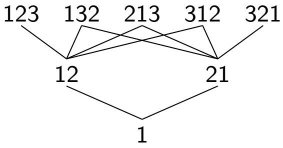
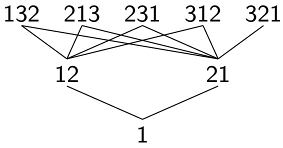
$\mathrm{Av}(231)$ is 

but $\mathrm{Av}(123)$ is 

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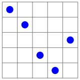
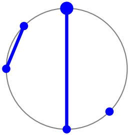
$54213 =$ 



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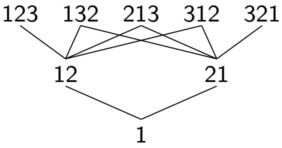
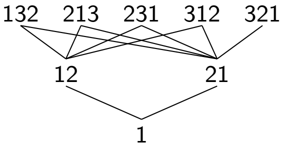
$\mathrm{Av}(231)$ is 

but $\mathrm{Av}(123)$ is 

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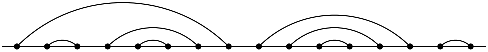
$\Rightarrow$ These Catalan objects are **not** part of our study. (Future work maybe?)

An equivalence relation strongly related to Wilf-equivalence

An equivalence relation on arch systems

Observation and terminology:

An arch system is a concatenation of atoms, i.e. (non-empty) arch systems having a single outermost arch.



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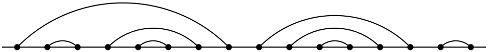

The binary relation, $\sim$, is the finest equivalence relation that satisfies:

(0)	$A \sim A$
(1)	$A \sim B \implies \overgroup{A} \sim \overgroup{B}$
(2)	$a \sim b \implies PaQ \sim PbQ$
(3)	$PabQ \sim PbaQ$
(4)	$a\overgroup{bc} \sim \overgroup{ab}c$

where A, B, P and Q denote arbitrary arch systems and a, b and c denote atoms or empty arch systems.

$\sim$ is (a refinement of?) Wilf-equivalence

Main theorem: If A and B are arch systems such that $A \sim B$ then $\mathrm{Av}(A)$ and $\mathrm{Av}(B)$ have the same enumeration, *i.e.* are Wilf-equivalent.

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In other words, $\sim$ refines Wilf-equivalence.

Conjecture: $\sim$ coincides with Wilf-equivalence.

Data, obtained with PermLab:
The conjecture holds for arch systems of size up to 15 (where $\sim$ has 16,709 equivalence classes on the $Cat_{15} = 9{,}694{,}845$ arch systems).

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To any arch system A, we can associate:

- its $\sim$-equivalence class, *i.e.*, its cohort;
- its avoidance class $\mathrm{Av}(A)$;
- the enumeration sequence, or generating function F_A, of $\mathrm{Av}(A)$.

Overview of the proof

Main theorem: If A and B are arch systems such that $A \sim B$ then $\mathrm{Av}(A)$ and $\mathrm{Av}(B)$ have the same enumeration, *i.e.* are Wilf-equivalent.

Overview of the proof. . . by induction!

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Base case: If $A = B$ then $\mathrm{Av}(A)$ and $\mathrm{Av}(B)$ are Wilf-equivalent. . .

Inductive case: One case for each rule defining $\sim$.

Rule	bijective proof	analytic proof
(1) $A \sim B \implies A \sim B$	yes	_
(2) $a \sim b \implies PaQ \sim PbQ$	yes	_
(3) $PabQ \sim PbaQ$	yes	_
(4) $a\,bc \sim ab\,c$	no	yes

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(4 weak) $a\,\overparen{b} \sim \overparen{ba}$	yes	_

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Having only bijective proofs would allow to "unfold" the induction into a bijective proof that $\mathrm{Av}(A)$ and $\mathrm{Av}(B)$ are Wilf-equivalent, for all $A \sim B$.

Bijective proof in case (2)

$$(2) \qquad a \sim b \implies PaQ \sim PbQ$$

Take $a \sim b$ and suppose that $\mathrm{Av}(a)$ and $\mathrm{Av}(b)$ are Wilf-equivalent.

Take a size-preserving bijection $\sigma : X \mapsto X^\sigma$ from $\mathrm{Av}(a)$ to $\mathrm{Av}(b)$.

Build a size-preserving bijection τ from $\mathrm{Av}(PaQ)$ to $\mathrm{Av}(PbQ)$ as follows:

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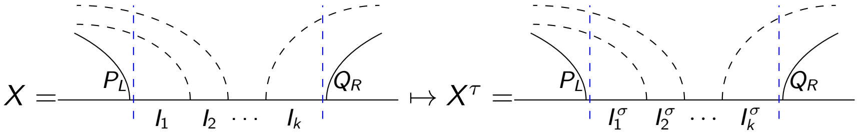
Build a size-preserving bijection τ from $\mathrm{Av}(PaQ)$ to $\mathrm{Av}(PbQ)$ as follows:

- If X avoids PQ, then take $X^\tau = X$.
- Otherwise, apply σ to all intervals determined by the arches having one extremity between the leftmost P and the rightmost Q:



- X^τ avoids PbQ if and only if X avoids PaQ.

Analytic proof in case (4)

$$(4) \quad a\overparen{bc} \sim \overparen{ab}c$$

Notations: $a = \overparen{A}$, $b = \overparen{B}$ and $c = \overparen{C}$.
$F_X =$ the generating function of $\mathrm{Av}(X)$.

We want that $F_{a\overparen{bc}} = F_{\overparen{ab}c}$.

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- Compute a system for $F_{a\lceil bc\rceil}$:

$$F_{a\lceil bc\rceil} = 1 + tF_A F_{a\lceil bc\rceil} + t(F_{a\lceil bc\rceil} - F_A)F_{\lceil bc\rceil}$$

$$\mathrm{Av}(a\lceil bc\rceil) = \varepsilon + \underset{X \text{ avoids } A}{\lceil X\rceil Y} + \underset{Z \text{ contains } A}{\lceil Z\rceil T}$$

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F_c &= 1 + tF_C F_c
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- Consequently, $F_{a\overparen{bc}} = F_{c\overparen{ab}} = F_{\overparen{ab}c}$.
- Using $F_{\overparen{X}} = 1/(1 - tF_X)$, we can write:

$$F_{a\overparen{bc}} = \frac{1 - t(F_aF_b + F_bF_c + F_cF_a - F_aF_bF_c)}{1 - t(F_a + F_b + F_c - F_aF_bF_c)}$$

How many cohorts?

How many Wilf-equivalence classes ?

Number of Wilf-equivalence classes: upper bounds

Up to size 15, there are as many Wilf-equivalence as cohorts:

1, 1, 2, 4, 8, 16, 32, 67, 142, 307, 669, 1 478, 3 290, 7 390, 16 709 . . .

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For any size n, an upper bound on the number of Wilf-equivalence classes of classes $\mathrm{Av}(A)$, where A is an arch system with n arches is:

- $Cat_n =$ number of arch systems with n arches
  $=$ number of plane forests of size n: $\sim \frac{1}{\sqrt{\pi}} \cdot 4^n \cdot n^{-3/2}$

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- Number of cohorts of arch systems of size n: $\sim 0.455 \cdot 2.4975^n \cdot n^{-3/2}$

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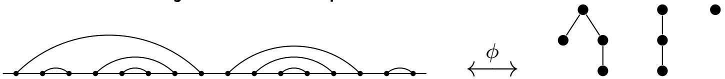
Moral of the story:
Many Wilf-equivalences between classes $\mathrm{Av}(A)$ avoiding an arch system A!

Fewer cohorts than non-plane forests

Arch systems are in bijection with plane forests:



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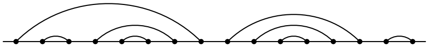
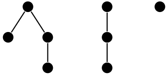
 $\xleftrightarrow{\phi}$ 

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Proposition: If $\phi(A) = \phi(B)$ as non-plane forests, then $A \sim B$.

Sketch of proof:

- (3) $PabQ \sim PbaQ$: The order of the trees does not affect the cohort.
- (1) $A \sim B \implies \overset{\frown}{A} \sim \overset{\frown}{B}$ and (2) $a \sim b \implies PaQ \sim PbQ$:
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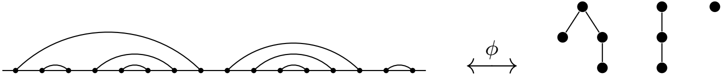

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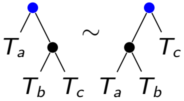
Corollary: There are fewer cohorts than non-plane forests,
hence fewer Wilf-equivalence classes than non-plane forests.

Asymptotic estimate of the number of cohorts

Interpretation of (4) $a\overset{\frown}{bc} \sim \overset{\frown}{ab}c$ on forests:

Asymptotic estimate of the number of cohorts

Interpretation of (4') $\overline{a\,\overline{bc}} \sim \overline{\overline{ab}\,c}$ on trees:



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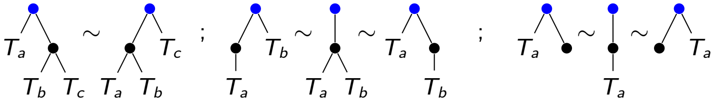

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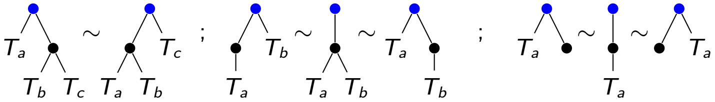

Proposition: The generating function of cohorts is $A(t)/t$ where

$$A = t + tA + \frac{1}{t} MSet_{\geq 2}(t^2 MSet_{\geq 3}(A)) + tMSet_{\geq 3}(A)$$

$$\text{where } MSet(Z) = \exp\left(\frac{Z(t)}{1} + \frac{Z(t^2)}{2} + \frac{Z(t^3)}{3} + \frac{Z(t^4)}{4} + \ldots\right)$$

$$MSet_{\geq 2}(Z) = MSet(Z) - 1 - Z(t)$$

$$MSet_{\geq 3}(Z) = MSet(Z) - 1 - Z(t) - \frac{1}{2}\left(Z(t)^2 + Z(t^2)\right)$$

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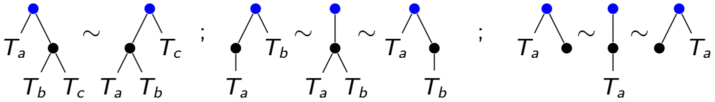

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Proposition: The number of cohorts is asymptotically equivalent to $c \cdot \gamma^n \cdot n^{-3/2}$ where $c \approx 0.455$ and $\gamma \approx 2.4975$.

Proof: Use the "twenty steps" of [Harary, Robinson & Schwenk 75].

This is an upper bound (conjecturally tight) on the number of Wilf-equivalence classes of classes $\mathrm{Av}(A)$ defined by the avoidance of an arch system A of size n.

Further results:
the “main” cohort,
and comparison between cohorts

Original motivation for our work

Define the sequence $(C^{(n)})$ of generating functions by

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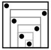
- $\pi = k \dots 21 \cdot n \dots (k+2)(k+1)$ for any $1 \leq k \leq n$ [Mansour & Vainshtein 01]
- π is a "wedge permutation" of size n [Mansour & Vainshtein 02]
- $\pi = \lambda_k \oplus \lambda_{n-k}$ for any $1 \leq k \leq n$, with *e.g.* $\lambda_6 =$  [A. & B. 13]

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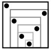
- $\pi = \lambda_k \oplus \lambda_{n-k}$ for any $1 \leq k \leq n$, with *e.g.* $\lambda_6 =$  [A. & B. 13]

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Remark:
$C^{(n)}$ is also the generating function of Dyck path of height at most n.

New results:
We can explain these statements (and more) studying the "main" cohort.

The main cohort

Definition: $N_n =$  is the nested arch system with n arches.
The main cohort (of size n) $\mathcal{M}_n$ is the cohort of N_n.

Theorem: The arch systems A such that the generating function F_A of $\mathrm{Av}(A)$ is $C^{(n)}$ are exactly those of $\mathcal{M}_n$.

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- It also generalizes them to more excluded patterns.

$\hookrightarrow$ There are $Motz_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} Cat_k$ objects in the main cohort.

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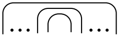
- It provides a bijective explanation of all these Wilf-equivalences.

$\hookrightarrow$ Because rule (4) defining $\sim$ is useless to explain $\sim$-equivalences inside the main cohort, the proof of our main theorem gives bijections between $\mathrm{Av}(A)$ and $\mathrm{Av}(B)$ for $A, B \in \mathcal{M}_n$.

The main cohort

Definition: $N_n =$  is the nested arch system with n arches.
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- Why not for other A? For A of size n, if $A \notin \mathcal{M}_n$ then $C^{(n)}$ dominates F_A term by term (and eventually strictly).

$\hookrightarrow$ This follows from results on the comparison of cohorts.

Comparison of cohorts

A, B arch systems. Generating functions F_A and F_B for $\mathrm{Av}(A)$ and $\mathrm{Av}(B)$.

Definition: Write $A \leq B$ when F_B dominates F_A term by term,
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Proof: Recall the bijective proof for case (1) of main theorem:
from a bijection $\mathrm{Av}(A) \to \mathrm{Av}(B)$, build a bijection $\mathrm{Av}(\lceil A \rceil) \to \mathrm{Av}(\lceil B \rceil)$.

The same construction applies to injections instead of bijections
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Similar results and proofs for rules (2) and (4 weak).

Corollary: For A of size n, either $F_A = C^{(n)}$ or $C^{(n)}$ dominates F_A term by term (and eventually strictly).

Summary of results and open questions (1/2)

- Main theorem: $\sim$ refines Wilf-equivalence between classes of Catalan objects with one excluded substructure.
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- Conjecture: $\sim$ and Wilf-equivalence coincide.
- Stronger conjecture: Given two arch systems A and B both with n arches, either $A \sim B$ or $|\mathrm{Av}_{2n-2}(A)| \neq |\mathrm{Av}_{2n-2}(B)|$.

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- What about other Catalan posets?