Bubble Sort and Permutation Classes

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Joint work with
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The Bubble Sort Operator B

B = one pass of **bubble sort**.
On sequences that are **permutations**.

Definition(s):

- Algorithmically:

$\hookrightarrow$ B processes a permutation σ from left to right, and modifies σ dynamically exchanging $\sigma(i)$ and $\sigma(i+1)$ when $\sigma(i) > \sigma(i+1)$.

- Recursively:

$$\hookrightarrow \begin{cases} B(\sigma_1 n \sigma_2) = B(\sigma_1)\sigma_2 n \text{ if } \sigma = \sigma_1 n \sigma_2 \in S_n \\ B(\varepsilon) = \varepsilon \end{cases}$$

NB *Stack-sorting operator S*
$S(\sigma_1 n \sigma_2) = S(\sigma_1)S(\sigma_2)n$

- Explicitely:

$\hookrightarrow$ If $\sigma = n_1\lambda_1 n_2\lambda_2 \cdots n_k\lambda_k$ where $n_1, \ldots, n_k$ are the left to right maxima of σ then $B(\sigma) = \lambda_1 n_1 \lambda_2 n_2 \cdots \lambda_k n_k$.

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Permutation Classes

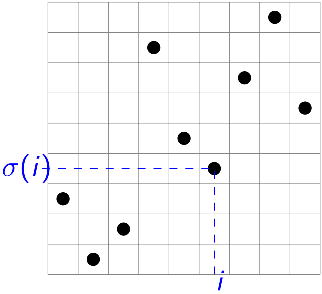
Permutations

- S_n = permutations σ of $\{1, 2, \ldots, n\}$
- Representation by a word: $\sigma(1)\sigma(2)\cdots\sigma(n)$, by its diagram, ...

Patterns

- Subpermutation of σ
- Subword or subset of points of the diagram that is normalized

Example: $2\,1\,3\,4 \preccurlyeq 3\,1\,2\,8\,5\,4\,7\,9\,6$ since $3\,2\,7\,9 \equiv 2\,1\,3\,4$



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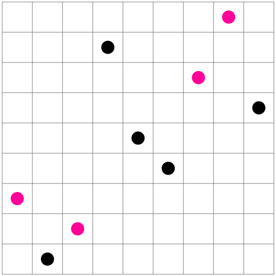

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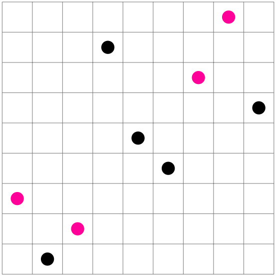

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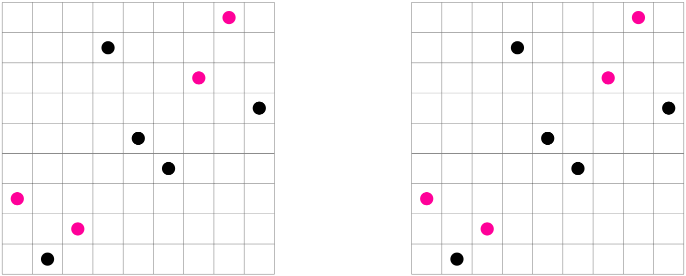

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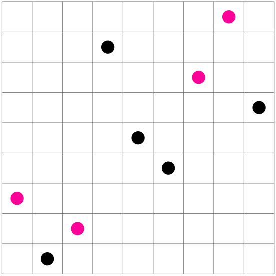

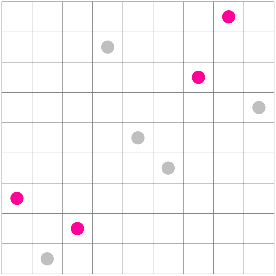

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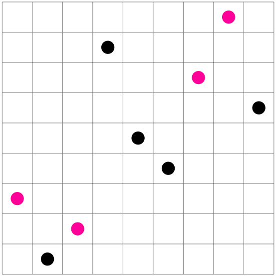

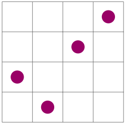

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Occurrence of a pattern

- Occurrence = subpermutation without normalization

Example : $3\,2\,7\,9 \subseteq 3\,1\,2\,8\,5\,4\,7\,9\,6$

Classes

- Subset of $S = \cup_n S_n$ downward closed for $\preccurlyeq$
- Characterization by a basis of excluded patterns: $\mathcal{C} = Av(\mathcal{B})$
- Principal classes: $\mathcal{C} = Av(\pi)$

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Example: 2 1 3 4 $\preccurlyeq$ 3 1 2 8 5 4 7 9 6 since 3 2 7 9 $\equiv$ 2 1 3 4

Occurrence of a pattern

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B-sortable permutations

Proposition

The permutations that are sorted by B are a class.
Namely: $B(\sigma) = Id$ iff $\sigma \in Av(231, 321)$.

Proof: by induction.

Decompose $\sigma = \sigma_1 n \sigma_2$ around its maximum n.

Recall that $B(\sigma) = B(\sigma_1)\sigma_2 n$.

$$\begin{aligned} & \sigma \text{ is sorted by } B \\ \Leftrightarrow\ & \sigma_1 \text{ is sorted by } B,\ \sigma_2 \text{ is increasing, and } \sigma_1 < \sigma_2 \\ \Leftrightarrow\ & \sigma_1 \in Av(231, 321),\ \sigma_2 \text{ is increasing, and } \sigma_1 < \sigma_2 \\ \Leftrightarrow\ & \sigma \in Av(231, 321) \end{aligned}$$

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σ is sorted by B

$\Leftrightarrow$ σ_1 is sorted by B, σ_2 is increasing, and $\sigma_1 < \sigma_2$

$\Leftrightarrow$ $\sigma_1 \in Av(231, 321)$, σ_2 is increasing, and $\sigma_1 < \sigma_2$

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Motivation and main result

- B-sortable permutations

$\hookrightarrow$ $B^{-1}(Av(21)) = Av(231, 321)$

- SB-sortable permutations?

$\hookrightarrow$ $(SB)^{-1}(Av(21)) = B^{-1}(Av(231))$

- B^2-sortable permutations?

$\hookrightarrow$ $(BB)^{-1}(Av(21)) = B^{-1}(Av(231, 321))$

- In general, what can we say about $B^{-1}(\mathcal{C})$?

For $\mathcal{C} = Av(\pi)$ a principal permutation class, we can determine

- when $B^{-1}(Av(\pi))$ is a class,
- and in this case give its basis.

This result is proved by considering the LtoR-maxima of π.

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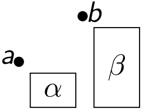
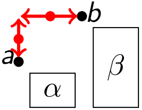
Summary of results

π	$B^{-1}(Av(\pi))$	Basis
1	is a class	1
12	is a class	$12, 21$
21	is a class	$231, 321$
$n\alpha$, $\alpha \neq \varepsilon$	is a class	$n(n+1)\alpha, (n+1)n\alpha$
$(n-1)\alpha n$, $\alpha \neq \varepsilon$	is a class	$(n-1)n\alpha, n(n-1)\alpha$
$a\alpha b\beta$, $\beta \neq \varepsilon$	is a class	$R(\pi)$
$a\alpha b\beta n$, $\beta \neq \varepsilon$	is a class	$R(a\alpha b\beta)$
$(n-2)\alpha(n-1)n$	is a class	$(n-2)(n-1)\alpha\ n, (n-1)(n-2)\alpha n,$ $(n-2)n\ \alpha(n-1), n(n-2)\alpha\ (n-1)$
$a\alpha b\beta c\gamma$, $\gamma \neq \varepsilon$	is not a class	

Remarks: n, $(n-1)$, $(n-2)$, a, b and c are LtoR-maxima.

If $\pi =$ 

, then $R(\pi)$ is the set of permutations 

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Short patterns π

Proposition

There are no permutations σ of length $n \geq 1$ such that $B(\sigma)$ avoids 1. Hence $B^{-1}(Av(1)) = \{\varepsilon\} = Av(1)$.

Proposition

The only permutations σ such that $B(\sigma)$ avoids 12 are ε and 1. Hence $B^{-1}(Av(12)) = \{\varepsilon, 1\} = Av(12, 21)$.

Proof: $B(\sigma)$ always ends with its maximum.

Proposition

The permutations σ such that $B(\sigma)$ avoids 21 are the B-sortable permutations. Hence $B^{-1}(Av(21)) = Av(231, 321)$.

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There are no permutations σ of length $n \geq 1$ such that $B(\sigma)$ avoids 1*.*
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Hence $B^{-1}(Av(12)) = \{\varepsilon, 1\} = Av(12, 21)$.

Proof: $B(\sigma)$ always ends with its maximum.

Proposition

The permutations σ such that $B(\sigma)$ avoids 21 *are the B-sortable permutations.*
Hence $B^{-1}(Av(21)) = Av(231, 321)$.

Summary of results

π	$B^{-1}(Av(\pi))$	Basis	Proof
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12	is a class	$12, 21$	✓
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$(n-1)\alpha n$, $\alpha \neq \varepsilon$	is a class	$(n-1)n\alpha, n(n-1)\alpha$	
$a\alpha b\beta$, $\beta \neq \varepsilon$	is a class	$R(\pi)$	
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$(n-2)\alpha(n-1)n$	is a class	$(n-2)(n-1)\alpha\ n$, $(n-1)(n-2)\alpha n$, $(n-2)n\ \alpha(n-1)$, $n(n-2)\alpha\ (n-1)$	
$a\alpha b\beta c\gamma$, $\gamma \neq \varepsilon$	is not a class		

Patterns $\pi \in S_n$ ending with n but not with $(n-1)n$

Lemma

If $\pi \in S_n$ with $n \geq 3$ is such that $\pi(n) = n$ but $\pi(n-1) \neq n-1$, then setting $\pi' = \pi(1)\pi(2)\dots\pi(n-1)$ we have $B^{-1}(Av(\pi)) = B^{-1}(Av(\pi'))$.

Proof:

- $\sigma \in B^{-1}(Av(\pi')) \Rightarrow B(\sigma)$ avoids π'
  $\Rightarrow B(\sigma)$ avoids $\pi \Rightarrow \sigma \in B^{-1}(Av(\pi))$
- $\sigma \in B^{-1}(Av(\pi)) \Rightarrow B(\sigma)$ avoids $\pi = \pi' n$
  But $B(\sigma) = B(\sigma_1)\sigma_2 m$ ends with its maximum m.
  Hence $B(\sigma_1)\sigma_2$ avoids π'.
  But π' does not end with its maximum.
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This lemmas applies in particular for $\pi = (n-1)\alpha n$ with $\alpha \neq \varepsilon$ and $\pi = a\alpha b\beta n$ with $\beta \neq \varepsilon$.

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$a\alpha b\beta$, $\beta \neq \varepsilon$	is a class	$R(\pi)$	
$a\alpha b\beta n$, $\beta \neq \varepsilon$	is a class	$R(a\alpha b\beta)$	✓
$(n-2)\alpha(n-1)n$	is a class	$(n-2)(n-1)\alpha\ n$, $(n-1)(n-2)\alpha n$, $(n-2)n\ \alpha(n-1)$, $n(n-2)\alpha\ (n-1)$	
$a\alpha b\beta c\gamma$, $\gamma \neq \varepsilon$	is not a class		

Patterns $\pi \in S_n$ with at least three LtoR-maxima $\neq \pi(n)$

Proposition

If $\pi = a\alpha b\beta c\gamma$, with a, b and c the first three LtoR-maxima of π and $\gamma \neq \varepsilon$, then $B^{-1}(Av(\pi))$ is not a class.

Proof:

By the previous lemma, we may assume that $\pi = a\alpha b\beta c\gamma n$.

Set $\theta_1 = ba\alpha n\beta c\gamma$ and $\theta_2 = (n+1)\theta_1$. Notice that $\theta_1 \preccurlyeq \theta_2$.

- Clearly, $B(\theta_1) = \pi$ and $\theta_1 \notin B^{-1}(Av(\pi))$.
- $B(\theta_2) = ba\alpha n\beta c\gamma(n+1)$
  Since $B(\theta_2)$ is only one term longer than π, we easily check that $B(\theta_2)$ avoids π. Hence $\theta_2 \in B^{-1}(Av(\pi))$.

We have $B^{-1}(Av(\pi)) \not\ni \theta_1 \preccurlyeq \theta_2 \in B^{-1}(Av(\pi))$. Consequently, $B^{-1}(Av(\pi))$ is not a class.

Patterns $\pi \in S_n$ with at least three LtoR-maxima $\neq \pi(n)$

Proposition

If $\pi = a\alpha b\beta c\gamma$, with a, b and c the first three LtoR-maxima of π and $\gamma \neq \varepsilon$, then $B^{-1}(Av(\pi))$ is not a class.

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Summary of results

π	$B^{-1}(Av(\pi))$	Basis	Proof
1	is a class	1	✓
12	is a class	$12, 21$	✓
21	is a class	$231, 321$	✓
$n\alpha$, $\alpha \neq \varepsilon$	is a class	$n(n+1)\alpha, (n+1)n\alpha$	
$(n-1)\alpha n$, $\alpha \neq \varepsilon$	is a class	$(n-1)n\alpha, n(n-1)\alpha$	✓
$a\alpha b\beta$, $\beta \neq \varepsilon$	is a class	$R(\pi)$	
$a\alpha b\beta n$, $\beta \neq \varepsilon$	is a class	$R(a\alpha b\beta)$	✓
$(n-2)\alpha(n-1)n$	is a class	$(n-2)(n-1)\alpha\ n$, $(n-1)(n-2)\alpha n$, $(n-2)n\ \alpha(n-1)$, $n(n-2)\alpha\ (n-1)$	
$a\alpha b\beta c\gamma$, $\gamma \neq \varepsilon$	is not a class		✓

Common framework for the remaining cases

Lemma

For any pattern π, if there exists a set $\mathcal{R}$ of permutations such that $\forall \sigma, \pi \preccurlyeq B(\sigma) \Leftrightarrow \rho \preccurlyeq \sigma$ for some $\rho \in \mathcal{R}$, then $B^{-1}(Av(\pi))$ is a class. Furthermore, if $\mathcal{R}$ is minimal, it is the basis of $B^{-1}(Av(\pi))$.

Proof: Show that $B^{-1}(Av(\pi))$ is downward closed for $\preccurlyeq$.

$$\begin{aligned} & \sigma \notin B^{-1}(Av(\pi)) \\ \Leftrightarrow\ & B(\sigma) \notin Av(\pi) \\ \Leftrightarrow\ & \pi \preccurlyeq B(\sigma) \\ \Leftrightarrow\ & \exists \rho \in \mathcal{R}, \rho \preccurlyeq \sigma \end{aligned}$$

so that $\sigma \in B^{-1}(Av(\pi)) \Leftrightarrow \forall \rho \in \mathcal{R}, \rho \not\preccurlyeq \sigma$.

This show that $B^{-1}(Av(\pi))$ is a downset, hence a class.

This also shows that the minimal $\mathcal{R}$ is its basis.

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Patterns $\pi \in S_n$ starting with n

Proposition

If $\pi \in S_n$ is such that $\pi = n\alpha$ for $\alpha \neq \varepsilon$, then $B^{-1}(Av(\pi))$ is a class whose basis is $\{n(n+1)\alpha, (n+1)n\alpha\}$.

Lemma

If $\pi \preccurlyeq B(\sigma)$, consider an occurrence $p\lambda \subseteq B(\sigma)$.
Then there exists $q > p > \lambda$ such that $pq\lambda \subseteq \sigma$ or $qp\lambda \subseteq \sigma$.
Hence $n(n+1)\alpha$ or $(n+1)n\alpha \preccurlyeq \sigma$.

Lemma

If $n(n+1)\alpha$ or $(n+1)n\alpha \preccurlyeq \sigma$, consider an occurrence $pq\lambda$ or $qp\lambda \subseteq \sigma$.
Then $p\lambda \subseteq B(\sigma)$.
Hence $\pi \preccurlyeq B(\sigma)$.

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If $\pi \in S_n$ is such that $\pi = n\alpha$ for $\alpha \neq \varepsilon$, then $B^{-1}(Av(\pi))$ is a class whose basis is $\{n(n+1)\alpha, (n+1)n\alpha\}$.

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Proof of the first lemma for $\pi = n\alpha$ with $\alpha \neq \varepsilon$

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Proof: by induction on $|\sigma|$.

- If $|\sigma| \leq 2$, result vacuously true (since $B(\sigma)$ ends with its maximum).
- If $\sigma = \sigma_1 m \sigma_2$ with $m = |\sigma| > 2$, then $p\lambda \subseteq B(\sigma_1)\sigma_2 m$.
  Because $p\lambda$ does not end with its maximum, $p\lambda \subseteq B(\sigma_1)\sigma_2$.

⋆ If $\lambda = \lambda_1\lambda_2$ with $\lambda_1 \neq \varepsilon$, $p\lambda_1 \subseteq B(\sigma_1)$ and $\lambda_2 \subseteq \sigma_2$, then by induction $p\lambda_1 \subseteq B(\sigma_1)$ implies that $\exists q > p$ such that $pq\lambda_1 \subseteq \sigma_1$ or $qp\lambda_1 \subseteq \sigma_1$.
Hence $\sigma = \sigma_1 m \sigma_2$ contains an occurrence of $pq\lambda_1\lambda_2$ or of $qp\lambda_1\lambda_2$.

⋆ If $p \subseteq B(\sigma_1)$ and $\lambda \subseteq \sigma_2$, then $p \subseteq \sigma_1$ and $pm\lambda \subseteq \sigma_1 m \sigma_2 = \sigma$.

⋆ If $p\lambda \subseteq \sigma_2$, then $mp\lambda \subseteq m\sigma_2 \subseteq \sigma$.

Proof of the second lemma for $\pi = n\alpha$ with $\alpha \neq \varepsilon$

Lemma

If $n(n+1)\alpha$ *or* $(n+1)n\alpha \preccurlyeq \sigma$*, consider an occurrence* $pq\lambda$ *or* $qp\lambda \subseteq \sigma$*.*
Then $p\lambda \subseteq B(\sigma)$*.*
Hence $\pi \preccurlyeq B(\sigma)$*.*

Proof:

Recall that if $\sigma = n_1\lambda_1 n_2\lambda_2 \cdots n_k\lambda_k$ where $n_1, \ldots, n_k$ are the left to right maxima of σ then $B(\sigma) = \lambda_1 n_1 \lambda_2 n_2 \cdots \lambda_k n_k$.
Hence, the order of the elements not LtoR-maxima is preserved by B.

- If $qp\lambda \subseteq \sigma$, $p\lambda$ are not LtoR-maxima. Hence $p\lambda \subseteq B(\sigma)$.
- This also holds when $pq\lambda \subseteq \sigma$ and p is not a LtoR-maximum.
- If $pq\lambda \subseteq \sigma$ and p is a LtoR-maximum, then there exists some r between p and q (possibly $r = q$) in σ that is a LtoR-maximum. This implies that p still precedes λ in $B(\sigma)$, hence $p\lambda \subseteq B(\sigma)$.

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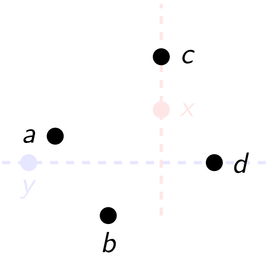
Introducing ambiguity in diagram representations

- A set of points in the plane that are pairwise neither horizontally nor vertically aligned represents a permutation.
- When some points are horizontally or vertically aligned, **sets of permutations** are represented (considering all possible disambiguations).

Example:



- $\{a, b, c, d\}$ represents 3142.
- $\{a, b, c, d, x, y\}$ represents the set $\{241563, 241653, 341562, 341652\}$.

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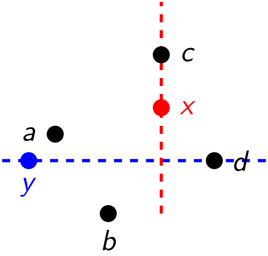

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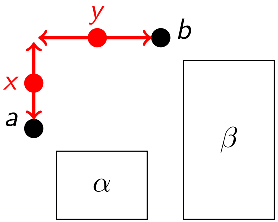
Definition of $R(\pi)$ for $\pi = a\alpha b\beta$ with $\beta \neq \varepsilon$

$R(\pi)$ is the set of minimal permutations in the set



When x is above β and y is to the left of α, x and y coalesce into a unique point.

Remark

$R(\pi)$ contains exactly

- 4 *one-point extensions of* π
- $4|\alpha|(n-a-1)$ *two-points extensions of* π

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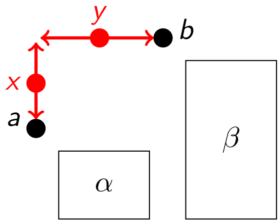

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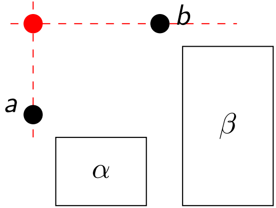

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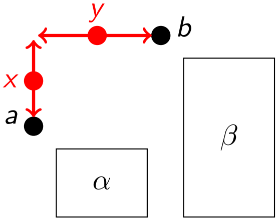

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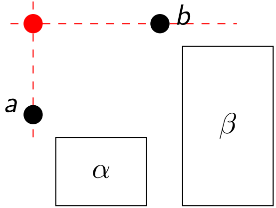

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Patterns $\pi \in S_n$ with two LtoR-maxima $\neq \pi(n)$

Proposition

If $\pi \in S_n$ is such that $\pi = a\alpha b\beta$ for $\beta \neq \varepsilon$, then $B^{-1}(Av(\pi))$ is a class whose basis is $R(\pi)$.

Lemma

If $\pi \preccurlyeq B(\sigma)$, consider an occurrence $p\lambda q\mu \subseteq B(\sigma)$. Then there exists a subsequence of σ which is an occurrence of some pattern in $R(\pi)$.

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Proof of the first lemma

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Proof: We prove that $px\lambda_1 y\lambda_2 z\mu$ or $xp\lambda_1 y\lambda_2 z\mu \subseteq \sigma$ with

$$\begin{cases} \lambda = \lambda_1\lambda_2, \quad p < x \\ y \text{ and } z \text{ are the two largest terms of this sequence} \\ \text{if } \lambda_1 = \varepsilon \text{ and } x > \mu, \text{ then } x \text{ and } y \text{ coalesce} \end{cases}$$

Such a sequence is a permutation in $R(\pi)$.

The proof follows by induction on $|\sigma|$.

- If $|\sigma| \leq 3$, result vacuously true (since $B(\sigma)$ ends with its maximum).
- If $\sigma = \sigma_1 m \sigma_2$ with $m = |\sigma| > 3$, then $p\lambda q\mu \subseteq B(\sigma_1)\sigma_2 m$. Because $p\lambda q\mu$ does not end with its maximum, $p\lambda q\mu \subseteq B(\sigma_1)\sigma_2$.

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  Because $p\lambda q\mu$ does not end with its maximum, $p\lambda q\mu \subseteq B(\sigma_1)\sigma_2$.

Proof of the first lemma, continued

As before, distinguish how $p\lambda q\mu$ can lie across $B(\sigma_1)\sigma_2$.

- ⋆ If $\mu = \mu_1\mu_2$ with $\mu_1 \neq \varepsilon$, $p\lambda q\mu_1 \subseteq B(\sigma_1)$ and $\mu_2 \subseteq \sigma_2$ then by induction σ_1 contains a subsequence of the form $px\lambda_1 y\lambda_2 z\mu_1$ or $xp\lambda_1 y\lambda_2 z\mu_1$ to which μ_2 can be appended.
- ⋆ If $p\lambda q \subseteq B(\sigma_1)$ and $\mu \subseteq \sigma_2$, then by a previous lemma $\exists t > p$ such that $tp\lambda$ or $pt\lambda \subseteq \sigma_1$. If q is to the left of λ in σ, then $pq\lambda m\mu$ or $qp\lambda m\mu \subseteq \sigma$ is of the required form. Otherwise, q and t can play the rôle of y and x, and appending $m\mu$ gives the desired subsequence.
- ⋆ If $\lambda = \lambda_1\lambda_2$ with $\lambda_1 \neq \varepsilon$, $p\lambda_1 \subseteq B(\sigma_1)$ and $\lambda_2 q\mu \subseteq \sigma_2$, then as before $\exists x > p$ such that $xp\lambda_1$ or $px\lambda_1 \subseteq \sigma_1$. Appending $m\lambda_2 q\mu$ gives the desired subsequence.
- ⋆ If $p \subseteq B(\sigma_1)$ and $\lambda q\mu \subseteq \sigma_2$, then $pm\lambda q\mu \subseteq \sigma_1 m\sigma_2 = \sigma$ is of the desired form, with x and y coalesing in m.
- ⋆ If $p\lambda q\mu \subseteq \sigma_2$, then $mp\lambda q\mu \subseteq \sigma$. Again, x and y coalese in m.

Proof of the second lemma

Lemma

If σ contains an occurrence $px\lambda_1 y\lambda_2 q\mu$ or $xp\lambda_1 y\lambda_2 q\mu$ of some pattern in $R(\pi)$, then there exists a subsequence of $B(\sigma)$ which is an occurrence of π.

Proof: Recall that if $\sigma = n_1\lambda_1 n_2\lambda_2 \cdots n_k\lambda_k$ where $n_1, \ldots, n_k$ are the left to right maxima of σ then $B(\sigma) = \lambda_1 n_1 \lambda_2 n_2 \cdots \lambda_k n_k$.

Hence $\lambda\mu \subseteq B(\sigma)$. Notice also that $p\lambda q\mu$ is an occurrence of π in σ.

1. We show that p is to the left of λ in $B(\sigma)$.
   - If p is not a LtoR-maximum, this is true.
   - If p is a LtoR-maximum, then $px\lambda_1 y\lambda_2 q\mu \subseteq \sigma$ and there exists some t between p and x (possibly $t = x$) in σ that is a LtoR-maximum. This implies that p still precedes λ in $B(\sigma)$.
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1. We show that p is to the left of λ in $B(\sigma)$. ✓
2. We show that there exists r in $B(\sigma)$ between λ and μ with $r > p\lambda\mu$.
   - If q is not a LtoR-maximum, choose $r = q$.
   - If q is a LtoR-maximum, choose $r =$ the LtoR-maximum of σ immediately to the left of q. Then $p\lambda r\mu \subseteq B(\sigma)$.
     By contradiction, assume that $r < y$ then in σ we have
     - ⋆ either $\cdots y \cdots r \cdots q \cdots$, and r is not a LtoR-maximum,
     - ⋆ or $\cdots r \cdots y \cdots q \cdots$, and there is a LtoR-maximum between r and q.

   Hence $r \geq y$, and $r > p\lambda\mu$ as desired.

Summary of results

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Patterns $\pi \in S_n$ with 3 LtoR-max. $\pi(1)$, $\pi(n-1)$ and $\pi(n)$

Proposition

If $\pi \in S_n$ is such that $\pi = (n-2)\alpha(n-1)n$, then $B^{-1}(Av(\pi))$ is a class whose basis is
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Lemma

If $\pi \preccurlyeq B(\sigma)$, consider an occurrence $p\lambda qr \subseteq B(\sigma)$. Then there exists a subsequence of σ which is an occurrence of some pattern among the four above.

Lemma

If σ contains an occurrence of some pattern among the four above, then there exists a subsequence of $B(\sigma)$ which is an occurrence of π.

Patterns $\pi \in S_n$ with 3 LtoR-max. $\pi(1)$, $\pi(n-1)$ and $\pi(n)$

Proposition

If $\pi \in S_n$ is such that $\pi = (n-2)\alpha(n-1)n$, then $B^{-1}(Av(\pi))$ is a class whose basis is
$\{(n-2)(n-1)\alpha n, (n-1)(n-2)\alpha n, (n-2)n\alpha(n-1), n(n-2)\alpha(n-1)\}$.

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Summary of results

π	$B^{-1}(Av(\pi))$	Basis	Proof
1	is a class	1	✓
12	is a class	$12, 21$	✓
21	is a class	$231, 321$	✓
$n\alpha$, $\alpha \neq \varepsilon$	is a class	$n(n+1)\alpha, (n+1)n\alpha$	✓
$(n-1)\alpha n$, $\alpha \neq \varepsilon$	is a class	$(n-1)n\alpha, n(n-1)\alpha$	✓
$a\alpha b\beta$, $\beta \neq \varepsilon$	is a class	$R(\pi)$	✓
$a\alpha b\beta n$, $\beta \neq \varepsilon$	is a class	$R(a\alpha b\beta)$	✓
$(n-2)\alpha(n-1)n$	is a class	$(n-2)(n-1)\alpha\ n$, $(n-1)(n-2)\alpha n$, $(n-2)n\ \alpha(n-1)$, $n(n-2)\alpha\ (n-1)$	✓
$a\alpha b\beta c\gamma$, $\gamma \neq \varepsilon$	is not a class		✓

Some open questions

Q1. When is $B^{-1}(Av(\mathcal{B}))$ a class, for $|\mathcal{B}| \geq 2$?

Partial answer: $B^{-1}(Av(\mathcal{B}))$ is a class when $B^{-1}(Av(\pi))$ is a class for every $\pi \in \mathcal{B}$, but not only.

- $B^{-1}(Av(\mathcal{B})) = \cap_{\pi \in \mathcal{B}} B^{-1}(Av(\pi))$.
- An example is $\Gamma_2 =$ the set of permutations of length 4 ending with 1: $B^{-1}(Av(\Gamma_2))$ is a class, although Γ_2 contains 2341 and $B^{-1}(Av(2341))$ is not a class.

Q2. Are the growth rates of $\mathcal{C}$ and $B^{-1}(\mathcal{C})$ related?

Growth rate of a permutation class $\mathcal{C} = \limsup_{n\to\infty} \sqrt[n]{c_n}$ where c_n is the number of permutations of length n in $\mathcal{C}$

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Composing sorting operators

- SB-sortable permutations:

$\hookrightarrow$ $(SB)^{-1}(Av(21)) = B^{-1}(Av(231)) = Av(3241, 2341, 4231, 2431)$

- B^2-sortable permutations:

$\hookrightarrow$ $(BB)^{-1}(Av(21)) = B^{-1}(Av(231, 321)) = Av(\Gamma_2)$

- B^k-sortable permutations:

$\hookrightarrow$ $(B^k)^{-1}(Av(21)) = Av(\Gamma_{k+2})$ with
Γ_{k+2} = the set of permutations of length $k+2$ ending with 1.

Other sorting operators:

- built from B, S, ... and symmetries of the permutations (i, r, c)
- with a queue
- definition of *abstract* sorting operator

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