Generating trees, a method for enumeration

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A bit of history

Generating trees were introduced in the nineties independently by
- J. West, for pattern-avoiding permutations;
- the Florentine combinatorics group (R. Pinzani, E. Barcucci, A. Del Lungo, ...), for a variety of combinatorial objects, including pattern-avoiding permutations.
(They use the name “ECO method”.)

Once combined with the kernel method on functional equations for generating functions (as explained by M. Bousquet-Mélou), it is a general method that can be used to enumerate some families of discrete objects.

In this talk, I present this method, illustrated by several examples.

Note: I have another talk ready, about how generating trees can be used to establish local and scaling limit results for permutations (results of J. Borga), but not the topic here.
A toy example:
312-avoiding permutations
A permutation of size $n$ is a sequence containing exactly once each symbol between 1 and $n$.

**Ex:** $5 3 7 2 4 1 6 8$ is a permutation of size 8.

**Notation:** We usually write a permutation $\sigma = \sigma_1 \sigma_2 \ldots \sigma_n$.

We often represent a permutation by its **diagram**: the $n \times n$ grid which contains a dot in each column $i$, in row $\sigma_i$.

(Rows are numbered from bottom to top.)

**Ex:** The diagram of our example is

![Diagram of permutation](image)
Patterns in permutations

For \( \sigma \) a permutation of size \( n \) and \( \pi \) a permutation of size \( k \leq n \), we say that \( \sigma \) contains \( \pi \) as a pattern when there exists \( i_1 < i_2 < \cdots < i_k \) such that \( \sigma_{i_a} < \sigma_{i_b} \) if and only if \( \pi_a < \pi_b \).

This is written \( \pi \prec \sigma \).

The subsequence \( \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k} \) is an occurrence of \( \pi \).

Ex: \( \sigma = 5 3 7 2 4 1 6 8 \) contains the pattern \( \pi = 3 1 2 4 \), an occurrence being \( \sigma_1 \sigma_4 \sigma_5 \sigma_7 = 5 2 4 6 \).

We can see patterns and occurrences on the diagrams:
Pattern-avoiding permutations

\( \sigma \) avoids \( \pi \) when \( \pi \) has no occurrence in \( \sigma \).

For \( B \) any set of patterns, we denote by \( Av(B) \) the set of permutations (of all sizes) avoiding all patterns in \( B \).

**Ex:** \( 53724168 \not\in Av(3124) \), but \( 25134768 \in Av(321) \).

In this first part, we consider the permutation class \( Av(312) \), which we will enumerate using generating trees.
Letting permutations grow on the right

One way of building all permutations of size $n + 1$:

- Start from all permutations $\sigma$ of size $n$
- For each such $\sigma$, append to $\sigma$ a new final value $a \in \{1, 2, \ldots, n + 1\}$, adding 1 to any $\sigma_i$ such that $\sigma_i \geq a$.

Ex: Appending 3 to 35124 gives 461253

On diagrams:
Restriction to $Av(312)$

To build all permutations of size $n + 1$ in $Av(312)$, we can

- Start from all permutations of size $n$ avoiding 312
- For each such $\sigma$, append a new final value as before, in all possible “places” which do not create an occurrence of 312.

Such places are called active sites ($\circ$), the others are inactive sites ($\times$).

Remark: For every family $Av(B)$ defined by the avoidance of (classical) patterns, it is possible to build permutations appending a new final value.
The combinatorial generating tree for $Av(312)$ growing on the right

It is the infinite tree

- whose root is $\bullet$ (the permutation of size 1),
- and where the children of any permutation $\sigma$ are the permutations obtained appending a new final value to $\sigma$, in all possible ways which do not create a pattern 312.

**Remark:** The nodes at level $n$ are the 312-avoiding permutations of size $n$. 
In general, a **combinatorial generating tree** for a combinatorial class \( C \) is

- an **infinite tree**,  
- whose **nodes** are the elements of \( C \), each occurring **exactly once**,  
- whose **root** is the element of size 1 in \( C \) (assumed to exist and be unique),  
- and where the **children** of any node \( c \) are obtained from \( c \) by performing **local expansions** according to some **prescribed rules**.

These rules must be carefully chosen to ensure that every element of \( C \) appears, and does not appear multiple times.

**Remarks:**

- Objects of size \( n \) are at level \( n \) in the tree. Hence enumerating \( C \) amounts to **counting the number of nodes at each level**.
- There may be **several combinatorial generating trees** for \( C \), depending on the “local expansion rule” which is chosen.
Labels in the combinatorial generating tree of $Av(312)$ ... 

... or how to find a simplified or concise description of this tree.

To each 312-avoiding permutation, assign a label: its number of active sites.

Conjecture:
If a permutation has label $k$, then its $k$ children have labels $2, 3 \ldots, k + 1$. 
Proving the conjecture (active sites of $Av(312)$)

- Observe that the bottommost and topmost sites are always active.
- Observe that a site cannot become active if it was previously inactive.
- Number the active sites 1 to $k$ from bottom to top.

- When inserting in the $j$-th active site for $j \neq k$, all active sites above it become inactive, except the topmost one.

So insertion in active site $j$ produces a permutation with label $j + 1$.
- Insertion in the topmost site produces a permutation with label $k + 1$.

Prop.: If $\sigma$ has label $k$, then its children have labels $2, 3 \ldots, k + 1$.  

□
The “simplified” generating tree, and the rewriting rule

Keeping only the labels, the generating tree for $\text{Av}(312)$ becomes

\[ c_n = |\text{Av}_n(312)| \text{ is the number of nodes at level } n. \]

This tree is completely described by the rewriting rule (or succession rule)

\[ \Omega_{\text{Cat}} = \left\{ \begin{array}{c} (2) \\ (k) \end{array} \mapsto (2), \ldots, (k), (k + 1). \right\} \]
Labels and rewriting rules in general

For a generating tree to be useful in some way, we need to identify

- **labels** for the objects
- a **rewriting rule** describing the labels of the children of an object from just the label of that object.

Labels can be integers, pairs of integers, or ... essentially anything.

We say that a generating tree is **concise** when, for all vertices $v$ and $w$, the subtrees rooted at $v$ and $w$ are isomorphic if and only if $v$ and $w$ have the same label.

**Remarks:**

- The simplified generating tree for $Av(312)$ from above is concise.
- Terminology introduced by B. Testart recently.
- Most (if not all) generating trees used in the literature so far are indeed concise.
Enumerating $Av(312)$ from its generating tree:
one way using generating functions
Recall the rewriting rule for 312-avoiding permutations:

\[ \Omega_{\text{Cat}} = \begin{cases} 
(2) \\
(2), \ldots, (2), (2), (k), (k + 1).
\end{cases} \]

Let \( c_{n,k} \) be the number of 312-avoiding permutations having size \( n \) and label \( k \). Consider the bivariate generating function \( C(x; y) = \sum_{n,k} c_{n,k} x^n y^k \).

Remark: \( C(x; 1) \) is the generating function of 312-avoiding permutations.

The rewriting rule gives

\[
C(x; y) = xy^2 + \sum_{n,k} c_{n,k} x^{n+1} (y^2 + \ldots y^k + y^{k+1})
\]

\[
= xy^2 + \sum_{n,k} c_{n,k} x^{n+1} y^2 \frac{1 - y^k}{1 - y}.
\]
Putting the equation in kernel form

\[ C(x; y) = xy^2 + \sum_{n, k} c_{n,k} x^{n+1} y^2 \frac{1 - y^k}{1 - y} \]

\[ = xy^2 + \frac{xy^2}{1 - y} \left( \sum_{n, k} c_{n,k} x^n - \sum_{n, k} c_{n,k} x^n y^k \right) \]

\[ = xy^2 + \frac{xy^2}{1 - y} (C(x; 1) - C(x; y)) \]

It follows that \( (1 - y + xy^2)C(x; y) = xy^2(1 - y + C(x; 1)) \).

The coefficient of \( C(x; y) \) is the kernel of this equation.

Remark: Putting \( y = 1 \) in the equation gives no information on \( C(x; 1) \).
Solving the equation: the kernel method

Method:

- Find a formal power series $Y(x)$ which cancels the kernel.
- Substituting $y$ by $Y(x)$ gives an equation for $C(x; 1)$.

Our example:

- The equation is $(1 - y + xy^2)C(x; y) = xy^2(1 - y + C(x; 1))$.
- The formal power series canceling the kernel is $Y(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$.
- Substitution gives $C(x; 1) = Y(x) - 1$.
- It follows that there are $c_n = \frac{1}{n+1} \binom{2n}{n}$ 312-avoiding permutations of any size $n \geq 1$.

Remark: For “similar” generating trees with integer labels, the generating functions are always algebraic.

See the “Generating functions for generating trees” paper.
Generating trees
where labels are pairs of integers:
the case of $\text{Av}(2\ 4\ 1\ 3)$
The pattern 2413

A permutation $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$ contains the pattern 2413 if there exists indices $i < j < j+1 < k$ such that $\sigma_{j+1} < \sigma_i < \sigma_k < \sigma_j$ i.e. such that the subsequence $\sigma_i\sigma_j\sigma_{j+1}\sigma_k$ is an occurrence of 2413. Otherwise $\sigma$ avoids 2413.

$Av(2413)$ denotes the set of all permutations avoiding 2413.
Letting permutations avoiding $2413$ grow on the right

Remark: If $\sigma_1 \ldots \sigma_n \sigma_{n+1}$ avoids $2413$, then so does $\sigma_1 \ldots \sigma_n$.
(Be careful! Not true for any element removed, e.g. $25314$!)

Thus, to build all $2413$-avoiding permutations of size $n + 1$, we can

- Start from all $2413$-avoiding permutations of size $n$
- For each such $\sigma$, append a new final value in all active sites (= the sites which do not create an occurrence of $2413$).

This induces a combinatorial generating tree for $\text{Av}(2413)$.
Non-empty descents are the reason for sites to be inactive

We say that a non-empty descent of $\sigma$ is an occurrence of the pattern $2\ 3\ 1$ in $\sigma$, i.e. a subsequence $\sigma_i\sigma_j\sigma_{j+1}$ (with $i < j$) such that $\sigma_j > \sigma_i > \sigma_{j+1}$.

A site is inactive if and only if it is above the 2 of a non-empty descent.
The non-empty descents determine the active sites.

Appending a new final value affects the set of non-empty descents of \( \sigma = \sigma_1 \sigma_2 \ldots \sigma_n \) differently if we insert below or above \( \sigma_n \).

Thus, we record separately the active sites below and above \( \sigma_n \).

We take the label of \( \sigma \) of size \( n \) avoiding 2413 to be \((h, k)\)

- with \( h = \) number of active sites below \( \sigma_n \)
- and \( k = \) number of active sites above \( \sigma_n \).
Remark: The site immediately below $\sigma_n$ is always active.

For $\sigma$ of label $(h, k)$, insertion in the site immediately below $\sigma_n$ produces an empty descent. Hence, all active sites stay active.

The label of the corresponding child of $\sigma$ is $(h, k+1)$. 
Labels of the children (2/3)

Insertion in an active site above \(\sigma_n\) produces an ascent. Hence, all active sites stay active.

For insertion in the \(i\)-th such active site from the top, the label of the corresponding child of \(\sigma\) is \((h + k - i + 1, i)\).
Insertion in an active site below $\sigma_n$ (and not immediately below) produces a non-empty descent. Hence, all active sites between $\sigma_n$ and the new final element $\sigma_{n+1}$ become inactive (except the site immediately above $\sigma_{n+1}$).

For insertion in the $i$-th such active site from the bottom, the label of the corresponding child of $\sigma$ is $(i, k+1)$. 
Rewriting rule

The generating tree for permutations avoiding 2413 growing on the right is described by the following rewriting rule:

\[ \Omega_{semi} = \begin{cases} 
(1, 1) \\
(h, k) \implies (1, k + 1), \ldots, (h, k + 1) \\
(h + k, 1), \ldots, (h + 1, k).
\end{cases} \]

Next steps:
- Consider the trivariate generating function

\[ S(y, z) = S(x; y, z) = \sum_{n,h,k} s_{n,h,k} x^n y^h z^k, \]

where \( s_{n,h,k} \) is the number of 2413-avoiding permutations having size \( n \) and label \((h, k)\).
- Translate \( \Omega_{semi} \) into a functional equation for \( S(y, z) \).
- Apply the (obstinate) kernel method to solve this equation.
Recall the rewriting rule $\Omega_{\text{semi}} = \left\{ \begin{array}{ll} (1, 1) \\ (h, k) \mapsto (1, k + 1), \ldots, (h, k + 1) \\ (h + k, 1), \ldots, (h + 1, k) \end{array} \right.$

Therefore,

$$S(y, z) = xyz + \sum_{n, h, k \geq 1} s_{n, h, k} x^{n+1} \left( (y + y^2 + \cdots + y^h) z^{k+1} ight. \\
+ \left. (y^{h+k} z + y^{h+k-1} z^2 + \cdots + y^{h+1} z^k) \right)$$

$$= xyz + \sum_{n, h, k \geq 1} s_{n, h, k} x^{n+1} \left( \frac{1 - y^h}{1 - y} y z^{k+1} + \frac{1 - \left( \frac{y}{z} \right)^k}{1 - \frac{y}{z}} y^{h+1} z^k \right)$$

$$= xyz + \frac{xyz}{1 - y} (S(1, z) - S(y, z)) + \frac{xyz}{z - y} (S(y, z) - S(y, y)) .$$
Kernel form of the equation

We obtained

\[ S(y, z) = xyz + \frac{xyz}{1 - y} (S(1, z) - S(y, z)) + \frac{xyz}{z - y} (S(y, z) - S(y, y)) \].

In kernel form, and substituting \( y \) with \( 1 + a \), this is

\[ K(a, z)S(1 + a, z) = xz(1 + a) - \frac{xz(1 + a)}{a} S(1, z) \]

\[ - \frac{xz(1 + a)}{z - 1 - a} S(1 + a, 1 + a), \]

where the kernel is \( K(a, z) = 1 - \frac{xz(1 + a)}{a} - \frac{xz(1 + a)}{z - 1 - a} \).

**Notation:** We denote with \( R(x, a, z, S(1, z), S(1 + a, 1 + a)) \) the right-hand side of the equation in kernel form.
Canceling the kernel

Recall that the kernel is \( K(a, z) = 1 - \frac{xz(1 + a)}{a} - \frac{xz(1 + a)}{z - 1 - a} \).

Solving for \( z \) the (quadratic) equation \( K(a, z) = 0 \) gives two solutions:

\[
Z_+(a) = \frac{1}{2} \frac{a + x + ax - Q}{x(1 + a)} = (1 + a) + (1 + a)^2x + O(x^2),
\]

\[
Z_-(a) = \frac{1}{2} \frac{a + x + ax + Q}{x(1 + a)} = \frac{a}{(1 + a)x} - a - (1 + a)^2x + O(x^2),
\]

where \( Q = \sqrt{a^2 - 2ax - 6a^2x + x^2 + 2ax^2 + a^2x^2 - 4a^3x} \).

Substituting \( z \) for \( Z_+ \), we obtain an equation relating the formal power series \( Z_+, S(1, Z_+) \) and \( S(1 + a, 1 + a) \), namely:

\[
R(x, a, Z_+, S(1, Z_+), S(1 + a, 1 + a)) = 0.
\]

We would like to eliminate \( S(1, Z_+) \) in order to find \( S(1 + a, 1 + a) \).
Being obstinate in canceling the kernel

- Look for the transformations leaving the kernel unchanged.
  - Here observe that 
    \[ K(a, z) = K\left(\frac{z-1-a}{1+a}, z\right) \] 
    and 
    \[ K(a, z) = K\left(a, \frac{z+za-1-a}{z-1-a}\right). \]
  - Therefore, define the involutions
    \[ \Phi : (a, z) \rightarrow \left(\frac{z-1-a}{1+a}, z\right) \]
    \[ \text{and} \]
    \[ \Psi : (a, z) \rightarrow \left(a, \frac{z+za-1-a}{z-1-a}\right). \]
- Examine the group generated by \( \Phi \) and \( \Psi \).
  - Here, they generate a group of order 10.
Substituting $z$ for $Z_+$, each element $(f_1(a, z), f_2(a, z))$ in this group cancels the kernel.

Find the pairs $(f_1(a, z), f_2(a, z))$ such that $f_1(a, Z_+)$ and $f_2(a, Z_+)$ are formal power series in $x$.

Here, we obtain the following pairs:

$$[a, z] \overset{\Phi}{\leftrightarrow} \left[ \frac{z-1-a}{1+a}, Z \right] \overset{\Psi}{\leftrightarrow} \left[ \frac{z-1-a}{1+a}, \frac{z-1}{a} \right] \overset{\Phi}{\leftrightarrow} \left[ \frac{z-1-a}{az}, \frac{z-1}{a} \right] \overset{\Psi}{\leftrightarrow} \left[ \frac{z-1-a}{az}, \frac{1+a}{a} \right].$$

Each such pair $(f_1(a, Z_+), f_2(a, Z_+))$ can be substituted in the kernel equation $K(a, z)S(1 + a, z) = R(x, a, z, S(1, z), S(1 + a, 1 + a))$. It results in an equation

- involving only formal power series,
- and where the kernel is 0.

Therefore, each pair satisfies

$$R(x, f_1(a, Z_+), f_2(a, Z_+), S(1, f_2(a, Z_+)), S(1 + f_1(a, Z_+), 1 + f_1(a, Z_+))) = 0.$$
We obtain the following system, with 5 equations and 6 unknowns:

\[
\begin{cases}
0 = R(x, a, Z_+, S(1, Z_+), S(1 + a, 1 + a)) \\
0 = R \left( x, \frac{Z_+-1-a}{1+a}, Z_+, S(1, Z_+), S(1 + \frac{Z_+-1-a}{1+a}, 1 + \frac{Z_+-1-a}{1+a}) \right) \\
0 = R \left( x, \frac{Z_+-1-a}{aZ_+}, \frac{Z_+-1}{a}, S(1, \frac{Z_+-1}{a}), S(1 + \frac{Z_+-1-a}{1+a}, 1 + \frac{Z_+-1-a}{1+a}) \right) \\
0 = R \left( x, \frac{Z_+-1-a}{aZ_+}, \frac{1+a}{a}, S(1, \frac{1+a}{a}), S(1 + \frac{Z_+-1-a}{aZ_+}, 1 + \frac{Z_+-1-a}{aZ_+}) \right) \\
0 = R \left( x, \frac{Z_+-1-a}{aZ_+}, \frac{1+a}{a}, S(1, \frac{1+a}{a}), S(1 + \frac{Z_+-1-a}{aZ_+}, 1 + \frac{Z_+-1-a}{aZ_+}) \right) \\
\end{cases}
\]

We eliminate all unknowns except \(S(1 + a, 1 + a)\) and \(S(1, \frac{1+a}{a})\).
A single resulting equation

We usually write $\bar{a} = a^{-1}$. Observe that $\frac{1+a}{a} = 1 + \bar{a}$.

Elimination from the previous system yields

$$S(1 + a, 1 + a) + \frac{(1 + a)^2 x}{a^4} S(1, 1 + \bar{a}) = P(a, Z_+),$$

where $P(a, z) = (z - 1 - a)(-za^4 + z^2 a^4 - za^3 + z^2 a^3 - z^3 a^2 - 2a^2 + z^2 a^2 + za^2 - 4a + 5az - 3az^2 + z^3 a + 3z - z^2 - 2)/(za^4(z - 1))$.

In this equation, we can separate powers of $a$:

- $S(1 + a, 1 + a)$ involves only powers of $a$ that are $\geq 0$.
- $\frac{(1+a)^2 x}{a^4} S(1, 1 + \bar{a})$ involves only powers of $a$ that are $\leq -2$.

Therefore, $S(1 + a, 1 + a) = \Omega_{\geq}[P(a, Z_+)]$, where for

$$G(x; a) = \sum_{n \geq 0} \sum_{i \in \mathbb{Z}} g_{n,i} x^n a^i,$$

we define $\Omega_{\geq}[G(x; a)] = \sum_{n \geq 0} \sum_{i \geq 0} g_{n,i} x^n a^i$. 
Who is $P(a, Z_+)$?

Recall that $Z_+$ is the unique formal power series canceling the kernel

$$K(a, z) = 1 - \frac{xz(1+a)}{a} - \frac{xz(1+a)}{z-1-a}.$$

Therefore $W = Z_+ - (1 + a)$ is the unique formal power series solution of

$$W = x\bar{\alpha}(1 + a)(W + 1 + a)(W + a).$$

Then, $P(a, Z_+)$ can be expressed from $W$ as $P(a, Z_+) = F(a, W)$ for

$$F(a, W) = (1 + a)^2 x + \left(\frac{1}{a^5} + \frac{1}{a^4} + 2 + 2a\right) x W + \left(-\frac{1}{a^5} - \frac{1}{a^4} + \frac{1}{a^3} - \frac{1}{a^2} - \frac{1}{a} + 1\right) x W^2 + \left(\frac{1}{a^4} - \frac{1}{a^2}\right) x W^3.$$

This gives a more direct definition of $\Omega_{\geq}[P(a, Z_+)] = S(1 + a, 1 + a)$. 
What about the number of 2413-avoiding permutations?

- There are $a_n = [x^n]S(1, 1)$ 2413-avoiding permutations of size $n$.
- It holds that $[x^n]S(1, 1) = [x^n a^0]S(1 + a, 1 + a)$.
- Since $S(1 + a, 1 + a) = \Omega_{\geq}[F(a, W)]$, it follows that
  \[ [x^n]S(1, 1) = [x^n a^0]S(1 + a, 1 + a) = [x^n a^0]F(a, W). \]
- From the equation for $W$ and the expression of $F(a, W)$, Lagrange inversion gives an (ugly) summation formula for $[x^n]S(1, 1)$.
- The method of creative telescoping of Zeilberger produces a nice recursive formula for $a_n$:
  \[ a_n = \frac{11n^2 + 11n - 6}{(n + 4)(n + 3)}a_{n-1} + \frac{(n - 3)(n - 2)}{(n + 4)(n + 3)}a_{n-2}. \]
- Nicer (previously conjectured) summation formulas for $a_n$ then follow.
Here are the nicer formulas

The number $a_n$ of 2413-avoiding permutations of size $n \geq 2$ is

$$a_n = \frac{24}{(n - 1)n^2(n + 1)(n + 2)} \sum_{j=0}^{n} \left( \begin{array}{c} n \\ j+2 \end{array} \right) \left( \begin{array}{c} n+2 \\ j \end{array} \right) \left( \begin{array}{c} n+j+2 \\ j+1 \end{array} \right)$$

$$= \frac{24}{(n - 1)n^2(n + 1)(n + 2)} \sum_{j=0}^{n} \left( \begin{array}{c} n+1 \\ j+3 \end{array} \right) \left( \begin{array}{c} n+2 \\ j+1 \end{array} \right) \left( \begin{array}{c} n+j+3 \\ j \end{array} \right)$$

$$= \frac{24}{(n - 1)n^2(n + 1)(n + 2)} \sum_{j=0}^{n} \left( \begin{array}{c} n \\ j+2 \end{array} \right) \left( \begin{array}{c} n+1 \\ j \end{array} \right) \left( \begin{array}{c} n+j+2 \\ j+3 \end{array} \right)$$

$$= \frac{24}{(n - 1)n(n+1)^2(n+2)} \sum_{j=0}^{n} \left( \begin{array}{c} n+1 \\ j \end{array} \right) \left( \begin{array}{c} n+1 \\ j+3 \end{array} \right) \left( \begin{array}{c} n+j+2 \\ j+2 \end{array} \right)$$
Summary and perspectives

- What is a combinatorial generating tree, and its (hopefully concise) encoding by a rewriting rule.
- How to turn a rewriting rule into a functional equation for the multivariate generating function.
- How to solve it with the (obstinate) kernel method.
- Presented applications on pattern-avoiding permutations.

- Using generating trees to obtain local and scaling limit results (works of J. Borga, partly joint with M. Maazoun).
- Extensions to other objets.
- Generalizations of the generating trees, to widen their scope of application.
- Next talk by Benjamin Testart in two weeks!
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Merci de votre présence!