Generating trees, a method for enumeration

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A bit of history

Generating trees were introduced in the nineties independently by

- J. West, for pattern-avoiding permutations;
- the Florentine combinatorics group (R. Pinzani, E. Barcucci, A. Del Lungo, ...), for a variety of combinatorial objects, including pattern-avoiding permutations.
  (They use the name "ECO method".)

Once combined with the kernel method on functional equations for generating functions (as explained by M. Bousquet-Mélou), it is a general method that can be used to enumerate some families of discrete objects.

In this talk, I present this method, illustrated by several examples.

Note: I have another talk ready, about how generating trees can be used to establish local and scaling limit results for permutations (results of J. Borga), but not the topic here.

A toy example:
312-avoiding permutations

Permutations

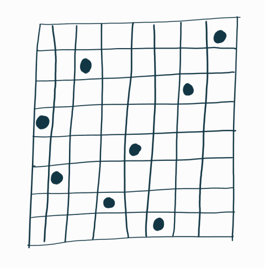
A permutation of size n is a sequence containing exactly once each symbol between 1 and n.

Ex: 5 3 7 2 4 1 6 8 is a permutation of size 8.

Notation: We usually write a permutation $\sigma = \sigma_1 \sigma_2 \dots \sigma_n$.

We often represent a permutation by its diagram: the $n \times n$ grid which contains a dot in each column i, in row σ_i.
(Rows are numbered from bottom to top.)

Ex: The diagram of our example is



Patterns in permutations

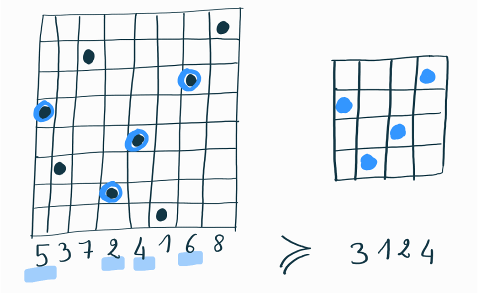
For σ a permutation of size n and π a permutation of size $k \leq n$,
we say that σ contains π as a pattern when
there exists $i_1 < i_2 < \cdots < i_k$ such that $\sigma_{i_a} < \sigma_{i_b}$ if and only if $\pi_a < \pi_b$.

This is written $\pi \preccurlyeq \sigma$.

The subsequence $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_k}$ is an occurrence of π.

Ex: $\sigma = 5\,3\,7\,2\,4\,1\,6\,8$ contains the pattern $\pi = 3\,1\,2\,4$, an occurrence being $\sigma_1\sigma_4\sigma_5\sigma_7 = 5\,2\,4\,6$.

We can see patterns and occurrences on the diagrams:

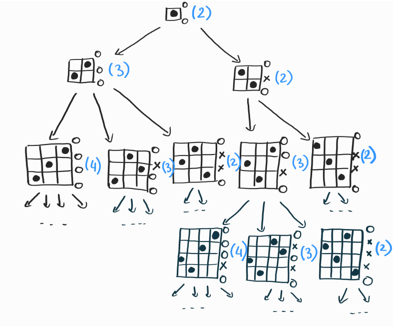


σ avoids π when π has no occurrence in σ.

For B any set of patterns, we denote by $Av(B)$ the set of permutations (of all sizes) avoiding all patterns in B.

Ex: $5\,3\,7\,2\,4\,1\,6\,8 \notin Av(3124)$, but $2\,5\,1\,3\,4\,7\,6\,8 \in Av(321)$.

In this first part, we consider the permutation class $Av(312)$, which we will enumerate using generating trees.



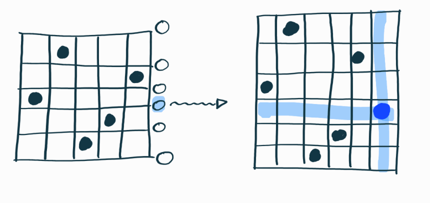
Letting permutations grow on the right

One way of building all permutations of size $n+1$:

- Start from all permutations σ of size n
- For each such σ, append to σ a new final value $a \in \{1, 2, \ldots, n+1\}$, adding 1 to any σ_i such that $\sigma_i \geq a$.

Ex: Appending 3 to 35124 gives 461253

On diagrams:



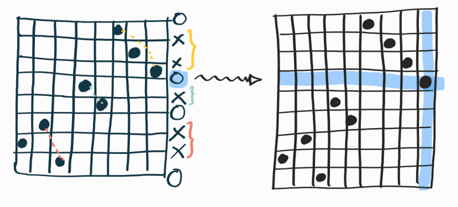
Restriction to $Av(312)$

To build all permutations of size $n+1$ in $Av(312)$, we can

- Start from all permutations of size n avoiding 312
- For each such σ, append a new final value as before, in all possible "places" which do not create an occurrence of 312.

Such places are called active sites ($\circ$), the others are inactive sites ($\times$).

Ex:

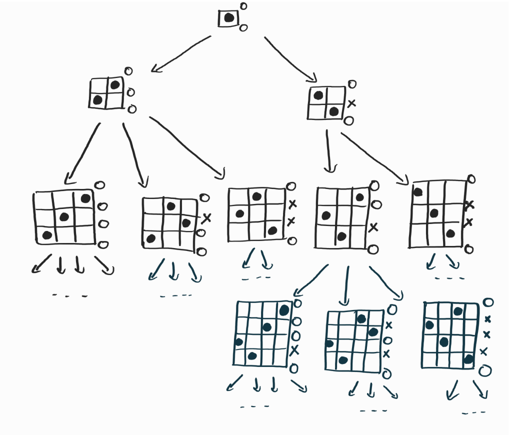


Remark: For every family $Av(B)$ defined by the avoidance of (classical) patterns, it is possible to build permutations appending a new final value.

The combinatorial generating tree for $Av(312)$ growing on the right

It is the infinite tree

- whose root is $\boxed{\bullet}$ (the permutation of size 1),
- and where the children of any permutation σ are the permutations obtained appending a new final value to σ, in all possible ways which do not create a pattern 312.



Remark: The nodes at level n are the 312-avoiding permutations of size n.

Combinatorial generating tree for a class of discrete objects

In general, a combinatorial generating tree for a combinatorial class $\mathcal{C}$ is

- an infinite tree,
- whose nodes are the elements of $\mathcal{C}$, each occurring exactly once,
- whose root is the element of size 1 in $\mathcal{C}$ (assumed to exist and be unique),
- and where the children of any node c are obtained from c by performing local expansions according to some prescribed rules.

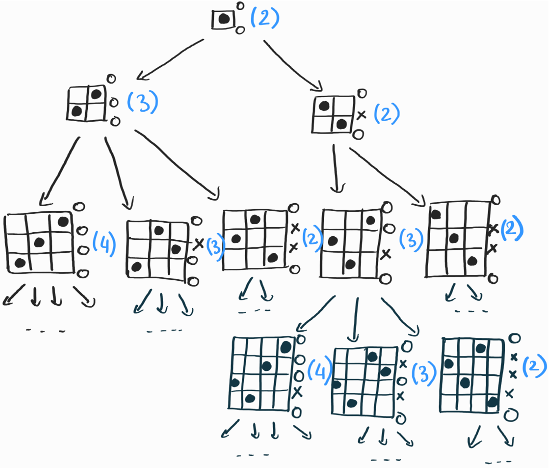
These rules must be carefully chosen to ensure that every element of $\mathcal{C}$ appears, and does not appear multiple times.

Remarks:

- Objects of size n are at level n in the tree. Hence enumerating $\mathcal{C}$ amounts to counting the number of nodes at each level.
- There may be several combinatorial generating trees for $\mathcal{C}$, depending on the "local expansion rule" which is chosen.

. . . or how to find a simplified or concise description of this tree.

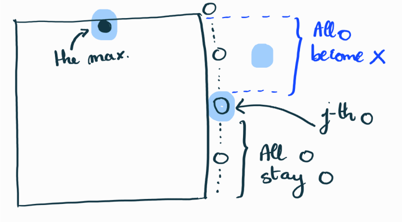
To each 312-avoiding permutation, assign a label: its number of active sites.



Conjecture:
If a permutation has label k,
then its k children have labels $2, 3 \ldots, k+1$.

Proving the conjecture (active sites of $Av(312)$)

- Observe that the bottommost and topmost sites are always active.
- Observe that a site cannot become active it if was previously inactive.
- Number the active sites 1 to k from bottom to top.
- When inserting in the j-th active site for $j \neq k$, all active sites above it become inactive, except the topmost one.



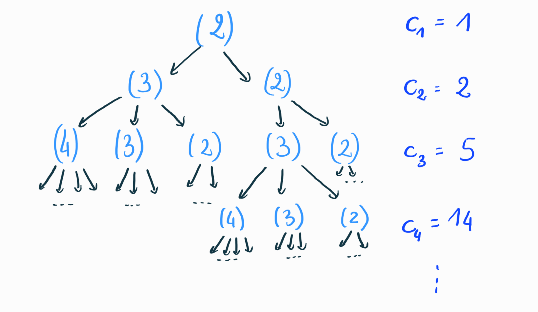
So insertion in active site j produces a permutation with label $j+1$.

- Insertion in the topmost site produces a permutation with label $k+1$.

Prop.: If σ has label k, then its children have labels $2, 3 \ldots, k+1$. □

The "simplified" generating tree, and the rewriting rule

Keeping only the labels, the generating tree for $Av(312)$ becomes



$c_n = |Av_n(312)|$ is the number of nodes at level n.

This tree is completely described by the rewriting rule (or succession rule)

$$\Omega_{Cat} = \begin{cases} (2) & \\ (k) & \rightsquigarrow (2), \dots, (k), (k+1). \end{cases}$$

Labels and rewriting rules in general

For a generating tree to be useful in some way, we need to identify

- labels for the objects
- a rewriting rule describing the labels of the children of an object from just the label of that object.

Labels can be integers, pairs of integers, or … essentially anything.

We say that a generating tree is concise when, for all vertices v and w, the subtrees rooted at v and w are isomorphic if and only if v and w have the same label.

Remarks:

- The simplified generating tree for $Av(312)$ from above is concise.
- Terminology introduced by B. Testart recently.
- Most (if not all) generating trees used in the literature so far are indeed concise.

Enumerating $Av(312)$
from its generating tree:
one way using generating functions

From the rewriting rule to a functional equation

Recall the rewriting rule for 312-avoiding permutations:

$$\Omega_{Cat} = \left\{ \begin{array}{ll} (2) & \\ (k) & \rightsquigarrow (2), \ldots, (k), (k+1). \end{array} \right.$$

Let $c_{n,k}$ be the number of 312-avoiding permutations having size n and label k. Consider the bivariate generating function $C(x;y) = \sum_{n,k} c_{n,k} x^n y^k$.

Remark: $C(x;1)$ is the generating function of 312-avoiding permutations.

The rewriting rule gives

$$\begin{aligned} C(x;y) &= xy^2 + \sum_{n,k} c_{n,k} x^{n+1} (y^2 + \ldots y^k + y^{k+1}) \\ &= xy^2 + \sum_{n,k} c_{n,k} x^{n+1} y^2 \frac{1-y^k}{1-y}. \end{aligned}$$

Putting the equation in kernel form

$$
\begin{aligned}
C(x;y) &= xy^2 + \sum_{n,k} c_{n,k} x^{n+1} y^2 \frac{1-y^k}{1-y} \\
&= xy^2 + \frac{xy^2}{1-y} \left(\sum_{n,k} c_{n,k} x^n - \sum_{n,k} c_{n,k} x^n y^k \right) \\
&= xy^2 + \frac{xy^2}{1-y} \big(C(x;1) - C(x;y) \big)
\end{aligned}
$$

It follows that $(1 - y + xy^2) C(x;y) = xy^2 (1 - y + C(x;1))$.

The coefficient of $C(x;y)$ is the kernel of this equation.

Remark: Putting $y = 1$ in the equation gives no information on $C(x;1)$.

Solving the equation: the kernel method

Method:

- Find a formal power series $Y(x)$ which cancels the kernel.
- Substituting y by $Y(x)$ gives an equation for $C(x;1)$.

Our example:

- The equation is $(1-y+xy^2)C(x;y) = xy^2(1-y+C(x;1))$.
- The formal power series canceling the kernel is $Y(x) = \frac{1-\sqrt{1-4x}}{2x}$.
- Substitution gives $C(x;1) = Y(x) - 1$.
- It follows that there are $c_n = \frac{1}{n+1}\binom{2n}{n}$ 312-avoiding permutations of any size $n \geq 1$.

Remark: For “similar” generating trees with integer labels, the generating functions are always algebraic.
See the “Generating functions for generating trees” paper.

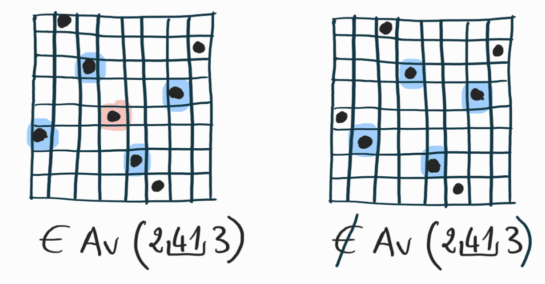
Generating trees where labels are pairs of integers: the case of $Av(2\,\underline{41}\,3)$

The pattern $2\underline{41}3$

A permutation $\sigma = \sigma_1\sigma_2\dots\sigma_n$ contains the pattern $2\underline{41}3$ if there exists indices $i < j < j+1 < k$ such that $\sigma_{j+1} < \sigma_i < \sigma_k < \sigma_j$
i.e. such that the subsequence $\sigma_i\sigma_j\sigma_{j+1}\sigma_k$ is an occurrence of 2413.

Otherwise σ avoids $2\underline{41}3$.

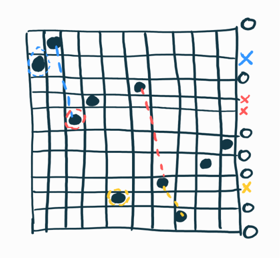
$Av(2\underline{41}3)$ denotes the set of all permutations avoiding $2\underline{41}3$.



Remark: If $\sigma_1 \dots \sigma_n \sigma_{n+1}$ avoids $2\,\underline{41}\,3$, then so does $\sigma_1 \dots \sigma_n$.
(Be careful! Not true for any element removed, *e.g.* 25314!)

Thus, to build all $2\,\underline{41}\,3$-avoiding permutations of size $n+1$, we can

- Start from all $2\,\underline{41}\,3$-avoiding permutations of size n
- For each such σ, append a new final value in all active sites ($=$ the sites which do not create an occurrence of $2\,\underline{41}\,3$).

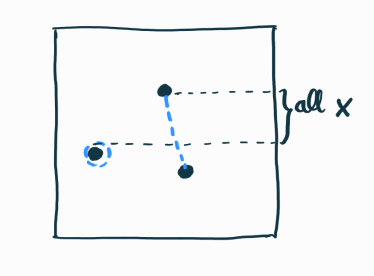


This induces a combinatorial generating tree for $Av(2\,\underline{41}\,3)$.

Non-empty descents are the reason for sites to be inactive

We say that a non-empty descent of σ is an occurrence of the pattern $2\underline{31}$ in σ, *i.e.* a subsequence $\sigma_i\sigma_j\sigma_{j+1}$ (with $i < j$) such that $\sigma_j > \sigma_i > \sigma_{j+1}$.

A site is inactive if and only if it is above the 2 of a non-empty descent.



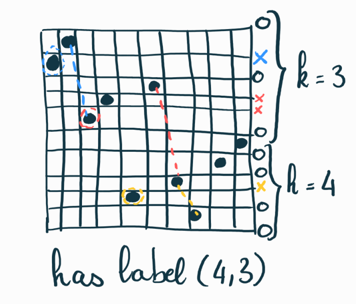
Labels for $2\,\underline{41}\,3$-avoiding permutations

- The non-empty descents determine the active sites.
- Appending a new final value affects the set of non-empty descents of $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ differently if we insert below or above σ_n.

Thus, we record separately the active sites below and above σ_n.

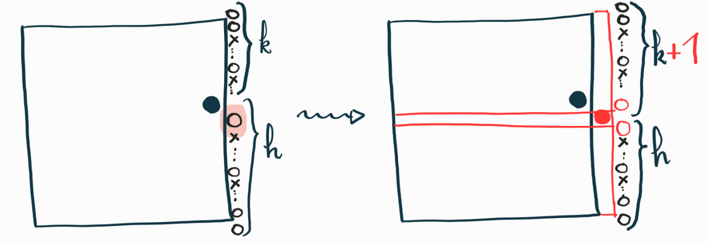
We take the label of σ of size n avoiding $2\,\underline{41}\,3$ to be (h, k)

- with h = number of active sites below σ_n
- and k = number of active sites above σ_n.



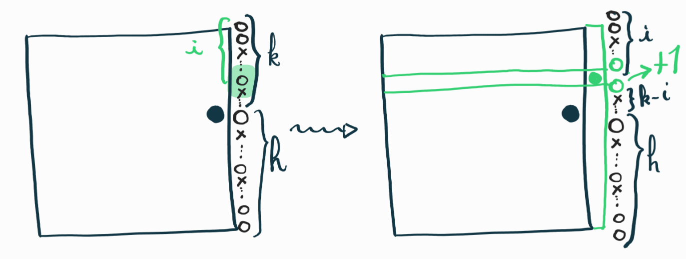
Remark: The site immediately below σ_n is always active.

For σ of label (h, k), insertion in the site immediately below σ_n produces an empty descent. Hence, all active sites stay active.



The label of the corresponding child of σ is $(h, k+1)$.

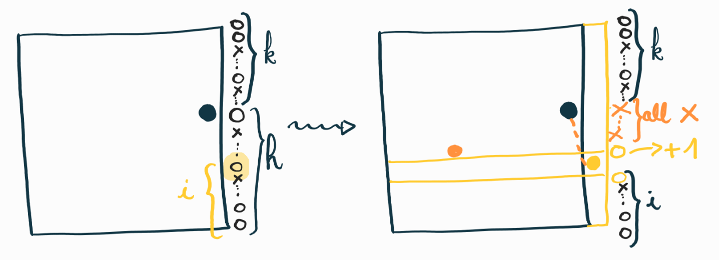
Insertion in an active site above σ_n produces an ascent.
Hence, all active sites stay active.



For insertion in the i-th such active site from the top, the label of the corresponding child of σ is $(h + k - i + 1, i)$.

Labels of the children (3/3)

Insertion in an active site below σ_n (and not immediately below) produces a non-empty descent. Hence, all active sites between σ_n and the new final element σ_{n+1} become inactive (except the site immediately above σ_{n+1}).



For insertion in the i-th such active site from the bottom, the label of the corresponding child of σ is $(i, k+1)$.

Rewriting rule

The generating tree for permutations avoiding $2\underline{41}3$ growing on the right is described by the following rewriting rule:

$$\Omega_{semi} = \begin{cases} (1,1) & \\ (h,k) \rightsquigarrow & (1,k+1),\ldots,(h,k+1) \\ & (h+k,1),\ldots,(h+1,k). \end{cases}$$

Next steps:

- Consider the trivariate generating function

$$S(y,z) = S(x;y,z) = \sum_{n,h,k} s_{n,h,k} x^n y^h z^k,$$

  where $s_{n,h,k}$ is the number of $2\underline{41}3$-avoiding permutations having size n and label (h,k).
- Translate Ω_{semi} into a functional equation for $S(y,z)$.
- Apply the (obstinate) kernel method to solve this equation.

The functional equation

Recall the rewriting rule $\Omega_{semi} = \begin{cases} (1,1) & \\ (h,k) \rightsquigarrow & (1,k+1),\dots,(h,k+1) \\ & (h+k,1),\dots,(h+1,k). \end{cases}$

Therefore,

$$\begin{aligned} S(y,z) &= xyz + \sum_{n,h,k\geq 1} s_{n,h,k} x^{n+1} \Big((y+y^2+\cdots+y^h) z^{k+1} \\ &\qquad\qquad + (y^{h+k}z + y^{h+k-1}z^2 + \cdots + y^{h+1}z^k) \Big) \\ &= xyz + \sum_{n,h,k\geq 1} s_{n,h,k} x^{n+1} \left(\frac{1-y^h}{1-y} y\, z^{k+1} + \frac{1-\left(\frac{y}{z}\right)^k}{1-\frac{y}{z}} y^{h+1} z^k \right) \\ &= xyz + \frac{xyz}{1-y}\left(S(1,z) - S(y,z)\right) + \frac{xyz}{z-y}\left(S(y,z) - S(y,y)\right). \end{aligned}$$

Kernel form of the equation

We obtained

$$S(y,z) = xyz + \frac{xyz}{1-y}\left(S(1,z) - S(y,z)\right) + \frac{xyz}{z-y}\left(S(y,z) - S(y,y)\right) .$$

In kernel form, and substituting y with $1+a$, this is

$$\begin{aligned} K(a,z)S(1+a,z) = xz(1+a) - \frac{xz(1+a)}{a}S(1,z) \\ - \frac{xz(1+a)}{z-1-a}S(1+a,1+a), \end{aligned}$$

where the kernel is $K(a,z) = 1 - \frac{xz(1+a)}{a} - \frac{xz(1+a)}{z-1-a}$.

Notation: We denote with $R(x,a,z,S(1,z),S(1+a,1+a))$ the right-hand side of the equation in kernel form.

Canceling the kernel

Recall that the kernel is $K(a,z) = 1 - \frac{xz(1+a)}{a} - \frac{xz(1+a)}{z-1-a}$.

Solving for z the (quadratic) equation $K(a,z) = 0$ gives two solutions:

$$Z_+(a) = \frac{1}{2}\frac{a+x+ax-Q}{x(1+a)} = (1+a) + (1+a)^2 x + O(x^2),$$
$$Z_-(a) = \frac{1}{2}\frac{a+x+ax+Q}{x(1+a)} = \frac{a}{(1+a)x} - a - (1+a)^2 x + O(x^2),$$

where $Q = \sqrt{a^2 - 2ax - 6a^2x + x^2 + 2ax^2 + a^2x^2 - 4a^3x}$.

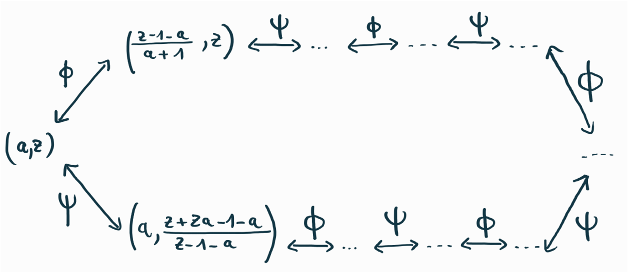
Substituting z for Z_+, we obtain an equation relating the formal power series $Z_+, S(1, Z_+)$ and $S(1+a, 1+a)$, namely:

$$R(x, a, Z_+, S(1, Z_+), S(1+a, 1+a)) = 0.$$

We would like to eliminate $S(1, Z_+)$ in order to find $S(1+a, 1+a)$.

Being obstinate in canceling the kernel

- Look for the transformations leaving the kernel unchanged.
  - Here observe that $K(a,z) = K(\frac{z-1-a}{1+a}, z)$ and $K(a,z) = K(a, \frac{z+za-1-a}{z-1-a})$.
  - Therefore, define the involutions $\Phi : (a,z) \to \left(\frac{z-1-a}{1+a}, z\right)$ and $\Psi : (a,z) \to \left(a, \frac{z+za-1-a}{z-1-a}\right)$.
- Examine the group generated by Φ and Ψ.
  - Here, they generate a group of order 10.



Being obstinate in canceling the kernel, continued

- Substituting z for Z_+, each element $(f_1(a,z), f_2(a,z))$ in this group cancels the kernel.

- Find the pairs $(f_1(a,z), f_2(a,z))$ such that $f_1(a, Z_+)$ and $f_2(a, Z_+)$ are formal power series in x.

◦ Here, we obtain the following pairs:
$[a,z] \overset{\Phi}{\leftrightarrow} \left[\frac{z-1-a}{1+a}, z\right] \overset{\Psi}{\leftrightarrow} \left[\frac{z-1-a}{1+a}, \frac{z-1}{a}\right] \overset{\Phi}{\leftrightarrow} \left[\frac{z-1-a}{az}, \frac{z-1}{a}\right] \overset{\Psi}{\leftrightarrow} \left[\frac{z-1-a}{az}, \frac{1+a}{a}\right].$

- Each such pair $(f_1(a, Z_+), f_2(a, Z_+))$ can be substituted in the kernel equation $K(a,z)S(1+a,z) = R(x,a,z,S(1,z),S(1+a,1+a))$.
  It results in an equation
  - involving only formal power series,
  - and where the kernel is 0.

- Therefore, each pair satisfies
  $R(x, f_1(a, Z_+), f_2(a, Z_+), S(1, f_2(a, Z_+)), S(1 + f_1(a, Z_+), 1 + f_1(a, Z_+))) = 0.$

Combining kernel equations

We obtain the following system, with 5 equations and 6 unknowns:

$$\begin{cases} 0 = R(x, a, Z_+, S(1, Z_+), S(1+a, 1+a)) \\ 0 = R\left(x, \frac{Z_+-1-a}{1+a}, Z_+, S(1, Z_+), S(1+\frac{Z_+-1-a}{1+a}, 1+\frac{Z_+-1-a}{1+a})\right) \\ 0 = R\left(x, \frac{Z_+-1-a}{1+a}, \frac{Z_+-1}{a}, S(1, \frac{Z_+-1}{a}), S(1+\frac{Z_+-1-a}{1+a}, 1+\frac{Z_+-1-a}{1+a})\right) \\ 0 = R\left(x, \frac{Z_+-1-a}{aZ_+}, \frac{Z_+-1}{a}, S(1, \frac{Z_+-1}{a}), S(1+\frac{Z_+-1-a}{aZ_+}, 1+\frac{Z_+-1-a}{aZ_+})\right) \\ 0 = R\left(x, \frac{Z_+-1-a}{aZ_+}, \frac{1+a}{a}, S(1, \frac{1+a}{a}), S(1+\frac{Z_+-1-a}{aZ_+}, 1+\frac{Z_+-1-a}{aZ_+})\right). \end{cases}$$

We eliminate all unknowns except $S(1+a, 1+a)$ and $S(1, \frac{1+a}{a})$.

A single resulting equation

We usually write $\bar{a} = a^{-1}$. Observe that $\frac{1+a}{a} = 1 + \bar{a}$.

Elimination from the previous system yields

$$S(1+a, 1+a) + \frac{(1+a)^2 x}{a^4} S\left(1, 1+\bar{a}\right) = P(a, Z_+),$$

where $P(a,z) = (z-1-a)(-za^4 + z^2a^4 - za^3 + z^2a^3 - z^3a^2 - 2a^2 + z^2a^2 + za^2 - 4a + 5az - 3az^2 + z^3a + 3z - z^2 - 2)/(za^4(z-1))$.

In this equation, we can separate powers of a:

- $S(1+a, 1+a)$ involves only powers of a that are ≥ 0.
- $\frac{(1+a)^2 x}{a^4} S\left(1, 1+\bar{a}\right)$ involves only powers of a that are ≤ -2.

Therefore, $S(1+a, 1+a) = \Omega_{\geq}[P(a, Z_+)]$, where for

$G(x;a) = \sum_{n \geq 0} \sum_{i \in \mathbb{Z}} g_{n,i} x^n a^i$, we define $\Omega_{\geq}[G(x;a)] = \sum_{n \geq 0} \sum_{i \geq 0} g_{n,i} x^n a^i$.

Who is $P(a, Z_+)$?

Recall that Z_+ is the unique formal power series canceling the kernel $K(a,z) = 1 - \frac{xz(1+a)}{a} - \frac{xz(1+a)}{z-1-a}$.

Therefore $W = Z_+ - (1+a)$ is the unique formal power series solution of

$$W = x\bar{a}(1+a)(W+1+a)(W+a).$$

Then, $P(a, Z_+)$ can be expressed from W as $P(a, Z_+) = F(a, W)$ for

$$F(a,W) = (1+a)^2\, x + \left(\frac{1}{a^5} + \frac{1}{a^4} + 2 + 2a\right)\, x\, W + \left(-\frac{1}{a^5} - \frac{1}{a^4} + \frac{1}{a^3} - \frac{1}{a^2} - \frac{1}{a} + 1\right)\, x\, W^2 + \left(\frac{1}{a^4} - \frac{1}{a^2}\right)\, x\, W^3.$$

This gives a more direct definition of $\Omega_{\geq}[P(a, Z_+)] = S(1+a, 1+a)$.

What about the number of $2\underline{41}3$-avoiding permutations?

- There are $a_n = [x^n]S(1,1)$ $2\underline{41}3$-avoiding permutations of size n.
- It holds that $[x^n]S(1,1) = [x^n a^0]S(1+a, 1+a)$.
- Since $S(1+a,1+a) = \Omega_{\geq}[F(a,W)]$, it follows that

$$[x^n]S(1,1) = [x^n a^0]S(1+a,1+a) = [x^n a^0]F(a,W).$$

- From the equation for W and the expression of $F(a,W)$, Lagrange inversion gives an (ugly) summation formula for $[x^n]S(1,1)$.
- The method of creative telescoping of Zeilberger produces a nice recursive formula for a_n:

$$a_n = \frac{11n^2+11n-6}{(n+4)(n+3)}a_{n-1} + \frac{(n-3)(n-2)}{(n+4)(n+3)}a_{n-2}.$$

- Nicer (previously conjectured) summation formulas for a_n then follow.

Here are the nicer formulas

The number a_n of $2\underbracket{41}3$-avoiding permutations of size $n \geq 2$ is

$$a_n = \frac{24}{(n-1)n^2(n+1)(n+2)} \sum_{j=0}^{n} \binom{n}{j+2}\binom{n+2}{j}\binom{n+j+2}{j+1}$$

$$= \frac{24}{(n-1)n^2(n+1)(n+2)} \sum_{j=0}^{n} \binom{n+1}{j+3}\binom{n+2}{j+1}\binom{n+j+3}{j}$$

$$= \frac{24}{(n-1)n^2(n+1)(n+2)} \sum_{j=0}^{n} \binom{n}{j+2}\binom{n+1}{j}\binom{n+j+2}{j+3}$$

$$= \frac{24}{(n-1)n(n+1)^2(n+2)} \sum_{j=0}^{n} \binom{n+1}{j}\binom{n+1}{j+3}\binom{n+j+2}{j+2}$$

Summary and perspectives

- What is a combinatorial generating tree, and its (hopefully concise) encoding by a rewriting rule.
- How to turn a rewriting rule into a functional equation for the multivariate generating function.
- How to solve it with the (obstinate) kernel method.
- Presented applications on pattern-avoiding permutations

- Using generating trees to obtain local and scaling limit results (works of J. Borga, partly joint with M. Maazoun).
- Extensions to other objets.
- Generalizations of the generating trees, to widen their scope of application.
- Next talk by Benjamin Testart in two weeks!

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Merci de votre présence !