First-order logic for permutations

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talk based on joint work with M. Albert and V. Féray

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What is a permutation (of size n)?

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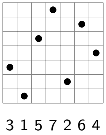
Ex.: $\sigma = (1,3,5,2)(4,7)(6) = 3\ 1\ 5\ 7\ 2\ 6\ 4 =$ 

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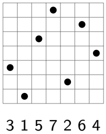
Ex.: $\sigma = (1,3,5,2)(4,7)(6) = 3\ 1\ 5\ 7\ 2\ 6\ 4 =$ 

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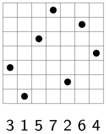

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Goal: Give a "proof" that the two points of view are hardly reconciled.

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To prove that the two points of view are essentially different, we study the expressivity of the theories:

- describe properties expressible in each theory,
- show that the properties expressible in both theories are trivial.

Two logics for permutations

TOOB: models

TOOB: the Theory Of One Bijection *(already appeared in the literature)*

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- Surjectivity: $\forall x \, \exists y \; yRx$
- Injectivity: $\neg \exists x, y, z \, (x \neq y \wedge xRz \wedge yRz)$

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Permutations are models, and every model is a permutation.
(Possibly, up to a conjugating by a bijection between X and $\{1, 2, \ldots, n\}$.)

The relation R_σ associated to σ of size n is given by:

$$i \ R_\sigma \ \sigma(i) \text{ for all } i \leq n$$

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A model of a sentence ψ is a model which in addition satisfies ψ.

Ex.: The models of $\exists x\, xRx$ are the permutations having a fixed point.

TOOB: expressivity

A property of permutations is expressible in a theory (here, TOOB) if it can be described by a sentence, *i.e.*, there is a sentence whose models are exactly the permutations for which this property holds.

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But not all such! For instance, being a full cycle is not expressible.

Thm.: If $\sigma \models \psi$, then for any τ in the conjugacy class of σ, $\tau \models \psi$.

In other words, TOOB does not distinguish between conjugate permutations.

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- Models: permutations as pairs of total orders on a finite set:
  - $<_P$ represents the position order between the elements;
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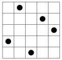
  - Ex.: $\sigma =$  25143 is represented for instance by $(\{a, b, c, d, e\}, \lhd, \blacktriangleleft)$

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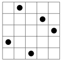
  - Ex.: $\sigma =$  25143 is represented for instance by $(\{a, b, c, d, e\}, \lhd, \blacktriangleleft)$

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Summary of differences:

- TOOB speaks about the cycle structure but the total order on $\{1, 2, \ldots, n\}$ is lost.
- TOTO speaks about the relative order of the elements, but the cycle structure is lost.

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Some concepts expressible in TOTO:

- Containment/avoidance of a classical pattern;

  Ex.: Avoidance of 231 is expressed by the sentence

$$\phi_{Av(231)} := \neg\, \exists x\, \exists y\, \exists z\, (x <_P y <_P z) \;\wedge\; (z <_V x <_V y)$$

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- Being simple;
- Being West-k-stack sortable, for any k
  (+ construction of the corresponding sentences)

Inexpressibility results in TOTO

Inexpressibility of fixed points

Thm.: There is no sentence ψ in TOTO such that $\sigma \models \psi$ if and only if σ has a fixed point.

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Intermezzo: Expressing properties of elements of permutations.

- A formula $\phi(x)$ with one (or several) free variable(s) expresses properties of one (or several) element(s) of a permutation.
- Ex: xRx expresses the property that a given element is a fixed point: For π a permutation and a an element of π, we write $(\pi, a) \models \phi(x)$ when a is a fixed point of π.

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Cor.: There is no formula with one free variable in TOTO expressing the property that a given element is a fixed point.

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Proof strategy:

- Assume such a sentence ψ exists.
  Call k its quantifier depth (=max. number of nested quantifiers in ψ).
- Exhibit two permutations σ and σ' such that
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    (Actually, σ and σ' satisfy the same sentences of quantifier depth at most k)

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To show that two permutations satisfy the same sentences, use the Ehrenfeucht-Fraïssé Theorem:

> Two permutations σ and σ' satisfy the same sentences of quantifier depth at most k if and only if Duplicator wins the EF-game with k rounds on σ and σ'.

EF-games (a.k.a. Duplicator-Spoiler games)

The setting:

- Two players: Duplicator (D) and Spoiler (S).
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Winner of the EF-game with k rounds:

- D if $\mathbf{s} = (s_1, \ldots, s_k)$ and $\mathbf{s}' = (s'_1, \ldots, s'_k)$ are isomorphic, *i.e.*, if the position- and value-orders on $\mathbf{s}$ and $\mathbf{s}'$ are identical;
- S otherwise.

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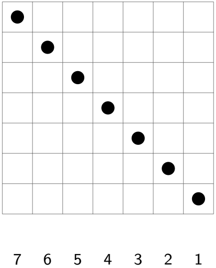

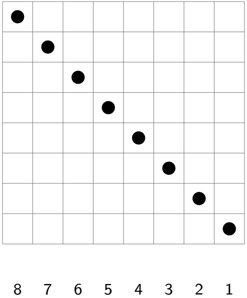

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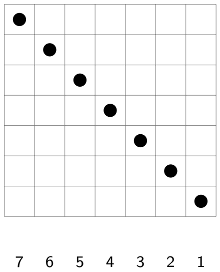

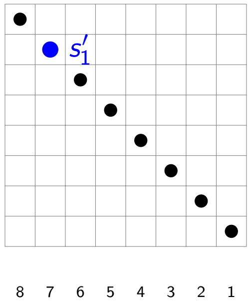

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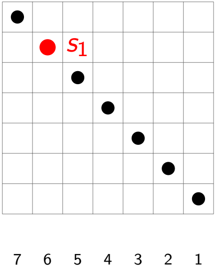

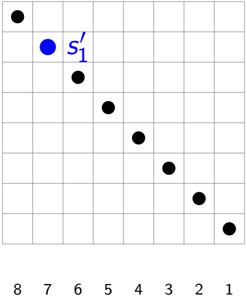

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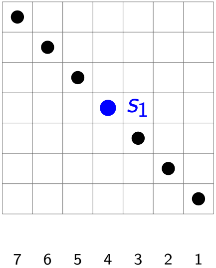

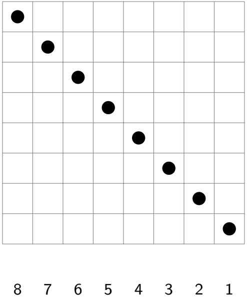

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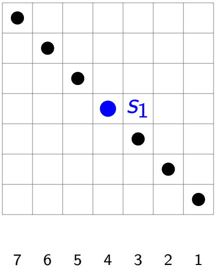

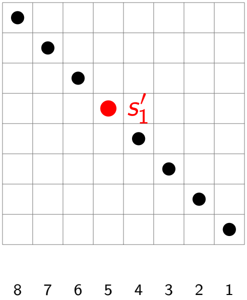

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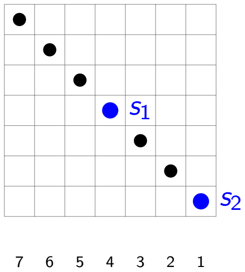

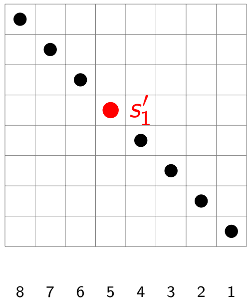

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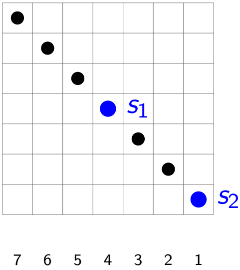

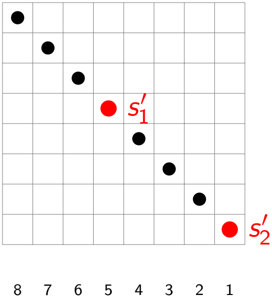

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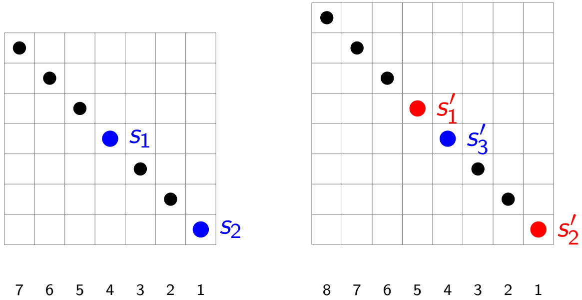

S and D alternate turns.

Inexpressibility of fixed points: Proof

Goal: For each k, exhibit σ and σ' such that

- σ has a fixed point but σ' does not;
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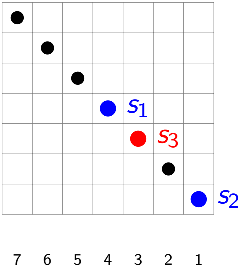

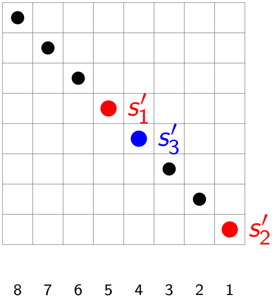

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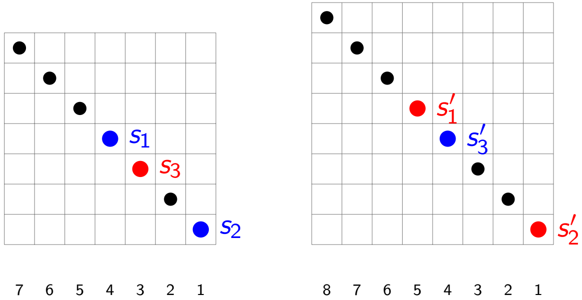

S and D alternate turns. After 3 rounds, D wins!

Intersection of TOTO **and** TOOB

Properties expressible in one/both theories: Examples

Examples of properties expressible in one of TOOB and TOTO only:

- Having a fixed point: expressible in TOOB but not in TOTO;
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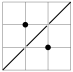
    Express that the diagram of the permutation looks like 

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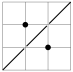
    Express that the diagram of the permutation looks like 
- Extension to larger cycle types $\lambda \cup (1^k)$

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In addition, we have a complete characterization of the properties expressible in both theories.

Exact description of the intersection of TOOB and TOTO

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Rk: This is more precise than the previous theorem. Indeed:

- in $\mathcal{C}_\lambda$ and $\mathcal{D}_\lambda$ there is a bound on the size of the support.
- the property *either E contains all permutations of sufficiently large support, or there is a bound on the size of the support of permutations in E* is stable by union, intersection and complement.

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Tricks/tools in the proof:

- expressing $\mathcal{D}_\lambda$ in TOTO;
- use previous theorem to write E as a finite union of $\mathcal{C}_\lambda$'s and $\mathcal{D}_\lambda$'s;
- and more EF games!

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- Characterization of the permutation classes $\mathcal{C}$ such that "having a fixed point" is expressible in the restriction of TOTO to $\mathcal{C}$.

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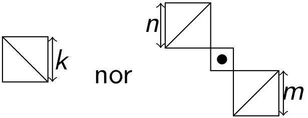

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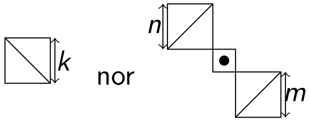

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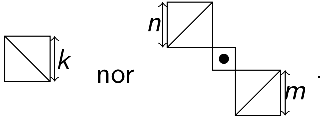

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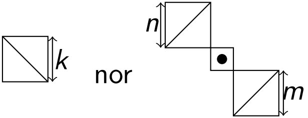

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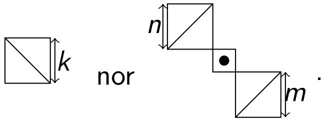

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- Further project with M. Noy: Prove convergence laws in permutation classes (for properties expressible in TOTO).