A general theory of Wilf-equivalence for permutation classes $\mathrm{Av}(231, \pi)$, and other Catalan structures

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joint work with Michael Albert (University of Otago)

Permutation Patterns 2014

Wilf-equivalence

- $\mathrm{Av}(B)$ is the class of permutations avoiding all patterns in B.
- (a_n) where $a_n = |\mathrm{Av}_n(B)|$ is its enumeration sequence.
- Its generating function is $\sum a_n t^n$.
- Two permutation classes $\mathrm{Av}(B_1)$ and $\mathrm{Av}(B_2)$ are Wilf-equivalent if they have the same enumeration sequence (or generating function).

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More or less famous Wilf-equivalences:

- $\mathrm{Av}(123)$ and $\mathrm{Av}(231)$, enumerated by the Catalan numbers Cat_n
- There are three Wilf-equivalence classes for permutation classes $\mathrm{Av}(\pi)$ with π of size 4, the enumeration of $\mathrm{Av}(1324)$ being open.
- Check all Wilf-equivalences between $\mathrm{Av}(\pi, \tau)$ when π and τ have size 3 or 4 on Wikipedia.
- Some results for arbitrary long patterns:
  $\mathrm{Av}(231 \oplus \pi)$ and $\mathrm{Av}(312 \oplus \pi)$ [West & Stankova 02]

Wilf-equivalences of permutation classes $\mathrm{Av}(231, \pi)$

Harmless assumption: In $\mathrm{Av}(231, \pi)$ throughout the talk, π avoids 231.

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Remark: The generating functions C_n are truncations at level n of the continued fraction defining the generating function of Catalan numbers:

$$C = \cfrac{1}{1 - \cfrac{t}{1 - \cfrac{t}{1 - \cfrac{t}{1 - \cdots}}}}.$$

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New results on this topic:

- Description of all patterns π of size n such that the generating function of $\mathrm{Av}(231, \pi)$ is C_n.
- There are $Motz_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} Cat_k$ such patterns.
- Bijections between $\mathrm{Av}(231, \pi)$ and $\mathrm{Av}(231, \pi')$ for any such patterns.
- For τ of size n, the generating function of $\mathrm{Av}(231, \tau)$ either is C_n or C_n dominates it term by term (and eventually strictly).

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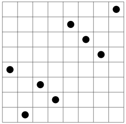
- For τ of size n, the generating function of $\mathrm{Av}(231, \tau)$ either is C_n or C_n dominates it term by term (and eventually strictly).

Main objective: Find all Wilf-equivalences between classes $\mathrm{Av}(231, \pi)$.

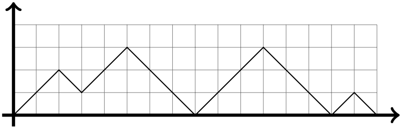
Substructures in Catalan objects

Catalan structures, and their substructures

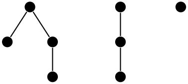
- 231-avoiding permutations

41327658 = 

- Dyck paths



- Plane forests



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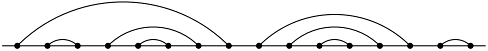
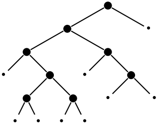


- Complete binary trees



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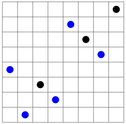
$41327658 =$ 

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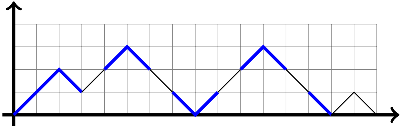

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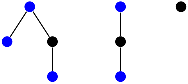

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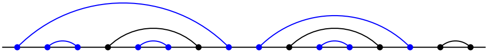

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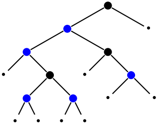

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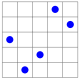
$31254 =$ 

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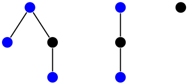

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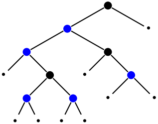

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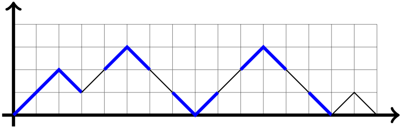

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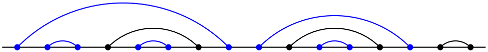

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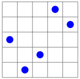
31254 = 

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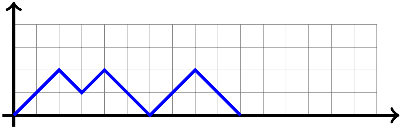

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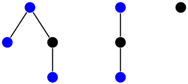

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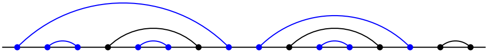

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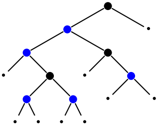

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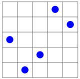
$31254 =$ 

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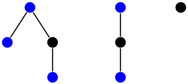

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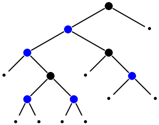

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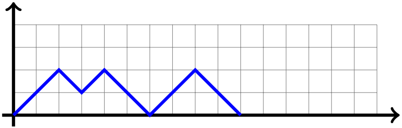

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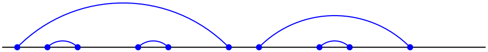

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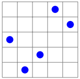
31254 = 

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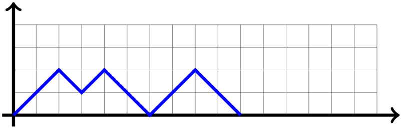

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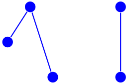

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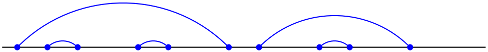

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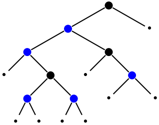

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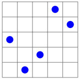
$31254 =$ 

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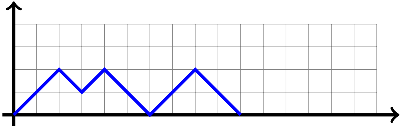

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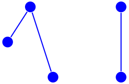

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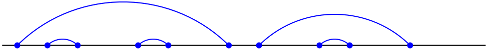

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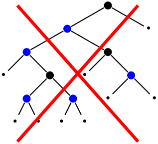

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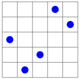
31254 = 

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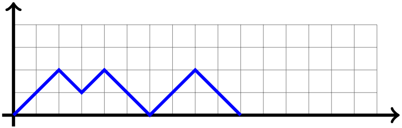

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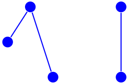

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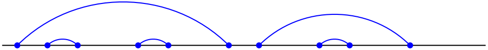

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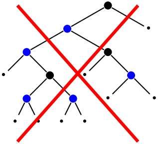
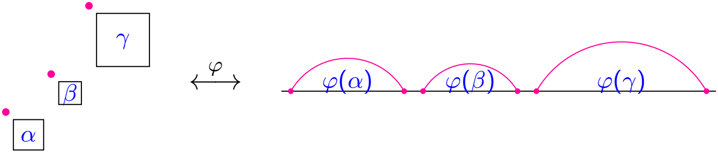


Fact: The usual bijections relating our quartet of Catalan structures preserve the substructure relation.

Classes of arch systems instead of $\mathrm{Av}(231, \pi)$

- The following bijection between 231-avoiding permutations and arch systems preserves the substructure relation:



- Therefore, for all π avoiding 231, $\mathrm{Av}(231, \pi) \stackrel{\varphi}{\longleftrightarrow} \mathrm{Av}(\varphi(\pi))$.

- We will study classes $\mathrm{Av}(A)$ of arch systems avoiding some subsystem A, but all results can be translated to classes $\mathrm{Av}(231, \pi)$ via φ.

Questions addressed in this talk

- Which arch systems A are Wilf-equivalent?
  i.e. which classes $\mathrm{Av}(A)$ have the same enumeration?
- Bijections between $\mathrm{Av}(A)$ and $\mathrm{Av}(B)$ for Wilf-equivalent arch systems A and B?
- How many Wilf-equivalence classes of arch systems are there?

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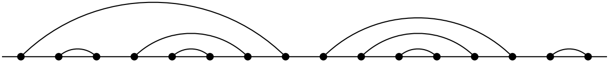
- How many Wilf-equivalence classes of arch systems are there?

Observation and terminology:
An arch system is a concatenation of atoms, i.e. (non-empty) arch systems having a single outermost arch.



An equivalence relation strongly related to Wilf-equivalence

An equivalence relation on arch systems

The binary relation, $\sim$, is the finest equivalence relation that satisfies:

$$
\begin{array}{ll}
(1) & A \sim B \implies \overgroup{A} \sim \overgroup{B} \\
(2) & a \sim b \implies PaQ \sim PbQ \\
(3) & PabQ \sim PbaQ \\
(4) & a\overgroup{bc} \sim \overgroup{ab}c
\end{array}
$$

where A, B, P and Q denote arbitrary arch systems
and a, b and c denote atoms or empty arch systems.

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and a, b and c denote atoms or empty arch systems.

Terminology: The equivalence classes of $\sim$ are called cohorts.

$\sim$ is (a refinement of?) Wilf-equivalence

Main theorem: If A and B are arch systems such that $A \sim B$ then $\mathrm{Av}(A)$ and $\mathrm{Av}(B)$ have the same enumeration, *i.e.* are Wilf-equivalent.

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Conjecture: $\sim$ coincides with Wilf-equivalence.

Data, obtained with PermLab:
The conjecture holds for arch systems of size up to 15 (where $\sim$ has 16,709 equivalence classes on the $Cat_{15} = 9{,}694{,}845$ arch systems).

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Additional results:

- Asymptotic enumeration of the number of cohorts.
- One cohort of arch systems of size n (conjecturally the biggest one) contains $Motz_n$ arch systems, and for A in this cohort $\mathrm{Av}(A)$ is enumerated by C_n.
- Comparison of the enumeration sequences of $\mathrm{Av}(A)$ and $\mathrm{Av}(B)$.

Idea of the proof

Overview of the proof

Main theorem: If A and B are arch systems such that $A \sim B$ then $\mathrm{Av}(A)$ and $\mathrm{Av}(B)$ have the same enumeration, *i.e.* are Wilf-equivalent.

Overview of the proof. . . by induction!

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Base case: If $A = B$ then $\mathrm{Av}(A)$ and $\mathrm{Av}(B)$ are Wilf-equivalent. . .

Inductive case: One case for each rule defining $\sim$.

Rule		bijective proof	analytic proof
(1)	$A \sim B \implies \overset{\frown}{A} \sim \overset{\frown}{B}$	yes	_
(2)	$a \sim b \implies PaQ \sim PbQ$	yes	_
(3)	$PabQ \sim PbaQ$	yes	_
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Having only bijective proofs would allow to "unfold" the induction into a bijective proof that $\mathrm{Av}(A)$ and $\mathrm{Av}(B)$ are Wilf-equivalent, for all $A \sim B$.

Bijective proof in case (2)

$$(2) \qquad a \sim b \implies PaQ \sim PbQ$$

Take $a \sim b$ and suppose that $\mathrm{Av}(a)$ and $\mathrm{Av}(b)$ are Wilf-equivalent.

Take a size-preserving bijection $\sigma : X \mapsto X^\sigma$ from $\mathrm{Av}(a)$ to $\mathrm{Av}(b)$.

Build a size-preserving bijection τ from $\mathrm{Av}(PaQ)$ to $\mathrm{Av}(PbQ)$ as follows:

Bijective proof in case (2)

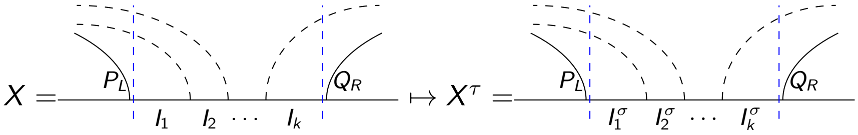
$$(2) \qquad a \sim b \implies PaQ \sim PbQ$$

Take $a \sim b$ and suppose that $\mathrm{Av}(a)$ and $\mathrm{Av}(b)$ are Wilf-equivalent.

Take a size-preserving bijection $\sigma : X \mapsto X^\sigma$ from $\mathrm{Av}(a)$ to $\mathrm{Av}(b)$.

Build a size-preserving bijection τ from $\mathrm{Av}(PaQ)$ to $\mathrm{Av}(PbQ)$ as follows:

- If X avoids PQ, then take $X^\tau = X$.
- Otherwise, apply σ to all intervals determined by the arches having one extremity between the leftmost P and the rightmost Q:



- X^τ avoids PbQ if and only if X avoids PaQ.

Analytic proof in case (4)

$$(4) \quad a\overline{bc} \sim \overline{ab}c$$

Notations: $a = \overline{A}$, $b = \overline{B}$ and $c = \overline{C}$.
$F_X =$ the generating function of $\mathrm{Av}(X)$.

We want that $F_{a\overline{bc}} = F_{\overline{ab}c}$.

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- Compute a system for $F_{a\overparen{bc}}$:

$$F_{a\overparen{bc}} = 1 + tF_A F_{a\overparen{bc}} + t(F_{a\overparen{bc}} - F_A)F_{\overparen{bc}}$$

$$\mathrm{Av}(a\overparen{bc}) = \varepsilon \quad + \underbrace{\overparen{X}\,Y}_{X \text{ avoids } A} + \underbrace{\overparen{Z}\,T}_{Z \text{ contains } A}$$

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F_{\overparen{bc}} &= 1 + tF_{bc}F_{\overparen{bc}} \\
F_{bc} &= 1 + tF_B F_{bc} + t(F_{bc} - F_B)F_c \\
F_c &= 1 + tF_C F_c
\end{aligned}$$

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Analytic proof in case (4)

$$(4) \qquad a\lceil bc\rceil \sim \lceil ab\rceil c$$

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We want that $F_{a\lceil bc\rceil} = F_{\lceil ab\rceil c}$.

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- Consequently, $F_{a\lceil bc\rceil} = F_{c\lceil ab\rceil} = F_{\lceil ab\rceil c}$.
- Using $F_{\lceil X\rceil} = 1/(1 - tF_X)$, we can write:

$$F_{a\lceil bc\rceil} = \frac{1 - t(F_aF_b + F_bF_c + F_cF_a - F_aF_bF_c)}{1 - t(F_a + F_b + F_c - F_aF_bF_c)}$$

Enumeration of cohorts

1, 1, 2, 4, 8, 16, 32, 67, 142, 307, 669, 1 478, 3 290, 7 390, 16 709 . . .

Plane forests, non-plane forests, and cohorts

Plane forests

- Bijection ψ between 231-avoiding permutations and plane forests.
- $Cat_n \sim \frac{1}{\sqrt{\pi}} \cdot 4^n \cdot n^{-3/2}$ distinct permutation classes $\mathrm{Av}(231, \pi)$

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- It follows because the arch systems of π and π' are $\sim$-equivalent, as non-plane isomorphism corresponds to rules (3) $PabQ \sim PbaQ$, (1) $A \sim B \implies \overset{\frown}{A} \sim \overset{\frown}{B}$ and (2) $a \sim b \implies PaQ \sim PbQ$.

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- $=$ equivalence classes of non-plane forest for (4) $a\lceil bc \rceil \sim \lceil ab \rceil c$

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- The generating function of cohorts is $A(t)/t$ where

$$A = t + tA + \frac{1}{t} MSet_{\geq 2}(t^2 MSet_{\geq 3}(A)) + tMSet_{\geq 3}(A)$$

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Moral of the story: Many Wilf-equivalences between classes $\mathrm{Av}(231, \pi)$!

Summary of results and open questions

- $\sim$ refines Wilf-equivalence between permutation classes $\mathrm{Av}(231, \pi)$.
- Conjecture: $\sim$ and Wilf-equivalence coincide.
- Stronger conjecture: Given two arch systems A and B both with n arches, either $A \sim B$ or $|\mathrm{Av}_{2n-2}(A)| \neq |\mathrm{Av}_{2n-2}(B)|$.

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- Extension to other contexts (separable permutations, ...).